Stability and bifurcation control analysis of a delayed fractional-order eco-epidemiological system

Hao Qi[a] , Wencai Zhao[b]

College of Mathematics and System Science, Shandong University of Science and Technology, Qingdao 266590, People's Republic of China

Abstract Considering the factor of artificial intervention in biological control, a delayed fractional eco-epidemiological system with an extended feedback controller is proposed. By using the digestion delay as bifurcation parameter, the stability and Hopf bifurcation are investigated, and the branching conditions are given. The system undergoes Hopf bifurcation, when the parameter τ passes through the critical value. In addition, it can be pointed out that the negative feedback gain and the feedback delay could affect the bifurcation critical value of the system. Therefore, the Hopf bifurcation can also be induced by taking the feedback delay as a bifurcation parameter. Finally, by plotting the solution curve of the system, the significance of the controller to the stability of the eco-epidemiological system is verified.

1 Introduction

Biological populations thrive in nature, and various organisms are inevitably invaded by various diseases during their lives[1–3]. The COVID-19 pandemic, which started in 2020, still threatens the lives and health of people all over the world. In April 2022, two poultry farms in Hokkaido, Japan, had a highly pathogenic avian influenza outbreak. To prevent the spread of the epidemic, the local government decided to cull more than 500,000 chickens and hundreds of emus. For thousands of years, mankind's struggle against various infectious diseases has never been interrupted. It was not until 1674, Antony van Leeuwenhoek observed the existence of microorganisms with the help of a microscope, which the foundation for human beings to truly understand diseases was laid. In 1840, Jacob Henle used bacterial theory for the first time to elucidate the pathogenesis of diseases. Later, through the work of Louis Pasteur and others, human beings realized that the origin of disease mainly comes from microorganisms, which was a key step toward the conquest of diseases.

In 1927, Kermack et al.[4] proposed the famous Kermack–McKendrick compartment model and successfully studied the law of disease transmission by using differential equation theory. Since then, mathematical models had become an important tool in the research of infectious diseases. In 1986, Anderson and May[5] first combined the infectious disease system and the Lotka–Volterra system to study the invasion, persistence, and spread of infectious diseases in plant and animal communities. Zhou et al.[6] constructed a time-delayed eco-epidemiological model of prey-infected diseases, and studied the stability of the positive equilibria and the existence conditions of the Hopf bifurcation. Saifuddin et al.[7] established a kind of eco-epidemiological model with predators having weak Allee effect and prey populations are infected. They researched the Hopf bifurcation near the equilibrium point and the chaotic dynamic behavior caused by disease. Taking $S(t)$ represents the density of susceptible prey populations, $I(t)$ to be the density of disease-infected prey populations, $P(t)$ to denote the density of predator populations, Moustafa et al.[8] presented the following ecological infectious disease model of prey infection

$$\begin{cases} S^{'}(t) = rS(1 - \frac{S}{k}) - \frac{\beta SI}{1+\eta S}, \\ I^{'}(t) = \frac{\beta SI}{1+\eta S} - \xi I - \varphi IP, \\ P^{'}(t) = \alpha IP - \delta P. \end{cases} \tag{1}$$

Here, $\frac{\beta SI}{1+\eta S}$ is the incidence of nonlinear saturation, and the biological significance of the coefficients in the model is shown in Table 1. Considering the prey is easy to be caught after being infected, model (1) assumed that predators only prey on the diseased prey.

Hao Qi and Wencai Zhao have contributed equally to this work.

[a] e-mail: qihao15864526290@163.com

[b] e-mail: zhaowencai@sdust.edu.cn (corresponding author)

Table 1 Biological significance of symbols

Symbol	Biological significance
r	The intrinsic growth rate of prey population
k	Environmental capacity of prey populations
β	Contact rate factor
η	Half saturation constant of infection
ξ	Mortality of infected prey
φ	Predator attack rate on infected prey
α	Conversion rate of predators on infected prey
δ	Predator mortality

Fractional calculus is an extension of classical calculus theory[9–11]. Since fractional-order system has the property of time memory, establishing complex system using fractional calculus theory can greatly improve the ability of identification, design, and control of dynamic system. According to the needs of modeling, long memory system or short memory system can be used[12–18]. Based on the short memory fractional differential equations, Wu et al.[19] proposed a short memory fractional-order model, and derived the global stability conditions of variable-order neural networks. Because biological populations have memory and genetic characteristics, Kumar et al.[20] used long memory fractional differential equations to establish an eco-epidemiological model, and analyzed the impact of dynamically changing spread and attack rates on system dynamics. Almeida et al.[21] adjusted the order of the fractional derivative of the model to fit the real disease spread data so as to better predict the development of the disease. Mondal et al.[22] deduced that the solution of the fractional-order system converges to several equilibrium points at a slower rate as the order of the differential equation decreases, but the qualitative properties of the solution are the same as those of the integer-order system. Chinnathambi et al.[23] demonstrated that fractional order could enhance the stability of the system and suppress the appearance of oscillations. It should be noted that in most of the previous studies, the biological systems have the same fractional order[8, 24, 25]. In fact, different biological populations have their own unique characteristics[30]. In order to better meet the actual biological background, we construct a fractional-order system with different orders for each variable on the basis of model (1).

To prevent the serious damage or even extinction of animal and plant population caused by infectious diseases, it is particularly important to reasonably intervene and control the ecological epidemiological system[26, 27]. For the treatment of leukemia, Islam et al.[28] designed an adaptive terminal and supertorsional sliding mode controller, and verified the stability of the system using Lyapunov's stability theory. For a nonlinear four-state ODE malignant tumor model, Qaiser et al.[29] designed a fuzzy controller and two kinds of nonlinear controllers: synergetic and state feedback controllers for chemotherapeutic drug control. An extended feedback controller was introduced into the ecological epidemiological model in[30], Wang et al. confirmed that changing the feedback control time delay could affect the stability of the model, thus effectively restrained the occurrence of bifurcation. It should be pointed out that there are few studies on bifurcation control for ecological epidemiological model with time delays. We couple the feedback controller $\Psi(t) = \mu[I(t) - I(t - \upsilon)]$ (where μ denotes the feedback gain and υ represents the feedback control time delay) into the system (1), and consider the fractional derivatives of different orders to obtain the following delay fractional-order ecological epidemiological model,

$$\begin{cases} {}^{C}_{t_0}D^{\theta_1}_t S(t) = rS(1 - \frac{S}{k}) - \frac{\beta SI}{1+\eta S}, \\ {}^{C}_{t_0}D^{\theta_2}_t I(t) = \frac{\beta SI}{1+\eta S} - \xi I - \varphi IP + \mu[I(t) - I(t - \upsilon)], \\ {}^{C}_{t_0}D^{\theta_3}_t P(t) = \alpha I(t - \tau)P(t - \tau) - \delta P, \end{cases} \quad (2)$$

where the initial condition is $S(t) = \psi_1(t) \geq 0$, $I(t) = \psi_2(t) \geq 0$, $P(t) = \psi_3(t) \geq 0$, $t \in [t_0 - max(\tau, \upsilon), t_0]$. Among them, $\theta_1, \theta_2, \theta_3$ are the orders of the fractional derivative of the system variable, and $0 < \theta_1, \theta_2, \theta_3 \leq 1$, ${}^{C}_{t_0}D^{\theta_i}_t$ is the fractional derivative in the sense of Caputo with initial time $t_0 > 0$. τ represents the digestion delay of the predator. The biological significance of other parameters is the same as in Table 1.

This paper mainly investigates the stability of the positive equilibrium of the system (2) and the conditions for the existence of the Hopf bifurcation. The effect of the controller on the stability of the ecological epidemiology model is analyzed. In order to verify the correctness of the theoretical analysis, the L1 scheme (Oldham and Spanier[31]) is used for numerical simulation. The L1 formula is established by a piecewise linear interpolation approximation for the integrand function on each small interval[32–34]. At the same time, the modified Adams–Bashforth–Moulton predictor–corrector scheme is used to solve the numerical calculation problems of fractional-order delay differential equations [35, 36].

The structure of this article is as follows: Related definitions and lemma are listed in Sect. 2. In Sect. 3, the existence condition of the internal equilibrium point is given. The stability of the system with and without controller is discussed by regarding digestion delay as parameter, and the criteria of Hopf bifurcation are given. In particular, the Hopf bifurcation caused by feedback delay is analyzed. Some numerical simulations and analysis are presented in Sect. 4. Finally, a discussion is presented in Sect. 5.

2 Preliminaries

Definition 1 [9, 11] Let $f(t) \in L^1([t_0, T]; \mathbb{R})$. The Riemann–Liouville integral of order $\theta > 0$ is defined by

$$_{t_0}I_t^\theta f(t) = \frac{1}{\Gamma(\theta)} \int_{t_0}^t (t - s)^{\theta-1} f(s) ds, \quad t \in [t_0, T]$$

where $\Gamma(\theta) = \int_0^\infty t^{\theta-1} e^{-t} dt$.

Definition 2 [9, 11] If $f(t)$ is differentiable, then the Caputo fractional derivative of order $\theta \in (0, 1)$ for $f(t)$ is defined as

$$_{t_0}^C D_t^\theta f(t) = \frac{1}{\Gamma(1 - \theta)} \int_{t_0}^t (t - s)^{-\theta} f'(s) ds, \quad 0 < \theta < 1.$$

Lemma 1 [10] Consider the under n-dimensional linear fractional-order time-delay system:

$$\begin{cases} _{t_0}^C D_t^{\theta_1} x_1(t) = b_{11} x_1(t - \tau_{11}) + b_{12} x_2(t - \tau_{12}) + \cdots + b_{1n} x_n(t - \tau_{1n}), \\ _{t_0}^C D_t^{\theta_2} x_2(t) = b_{21} x_1(t - \tau_{21}) + b_{22} x_2(t - \tau_{22}) + \cdots + b_{2n} x_n(t - \tau_{2n}), \\ \quad \vdots \\ _{t_0}^C D_t^{\theta_n} x_n(t) = b_{n1} x_1(t - \tau_{n1}) + b_{n2} x_2(t - \tau_{n2}) + \cdots + b_{nn} x_n(t - \tau_{nn}), \end{cases} \quad (3)$$

where the initial conditions $x_i(t) = \xi_i(t)$ are given for $t \in [t_0 - \tau_{max}, t_0]$, $\tau_{max} = \max\limits_{1 \le i,j \le n} \tau_{ij}$ and $\theta_i \in (0, 1)(i = 1, 2, \ldots n)$. It is defined as

$$\Lambda(s) = \begin{bmatrix} s^{\theta_1} - b_{11} e^{-s\tau_{11}} & -b_{12} e^{-s\tau_{12}} & \cdots & -b_{1n} e^{-s\tau_{1n}} \\ -b_{21} e^{-s\tau_{21}} & s^{\theta_2} - b_{22} e^{-s\tau_{22}} & \cdots & -b_{2n} e^{-s\tau_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n1} e^{-s\tau_{n1}} & -b_{n2} e^{-s\tau_{n2}} & \cdots & s^{\theta_n} - b_{nn} e^{-s\tau_{nn}} \end{bmatrix}.$$

If all roots of the $\det(\Lambda(s)) = 0$ have negative real parts, then the zero solution of system (3) is Lyapunov globally asymptotically stable.

Corollary 1 [10] In the case of $0 < \theta = \theta_1 = \theta_2 = \cdots = \theta_n < 1$, $\tau_{ij} = 0$, if all the roots of the equation $\det(s^\theta I - (b_{ij})_{n \times n}) = 0$ satisfy $|arg(s^\theta)| > \frac{\theta\pi}{2}$, then the zero solution of system (3) is Lyapunov globally asymptotically stable.

Corollary 2 [10] Assume that all θ_i are rational numbers between 0 and 1, $\tau_{ij} = 0$, m is the smallest of the common multiples of the denominators of $\theta_1, \theta_2, \cdots, \theta_n$. Denote λ by $s^{\frac{1}{m}}$, if all the roots of the equation $\det(diag(\lambda^{m\theta_1}, \lambda^{m\theta_2}, \cdots, \lambda^{m\theta_n}) - (b_{ij})_{n \times n}) = 0$ satisfy $|arg(\lambda_i)| > \frac{\pi}{2m}$, $i = 1, 2, \cdots, k$, $k = m(\theta_1 + \theta_2 + \cdots + \theta_n)$, then the zero solution of system (3) is Lyapunov globally asymptotically stable.

3 Main results

The main objective of this paper is to investigate the existence of periodic solutions by applying the Hopf bifurcation theory. Then, we discuss how the feedback controller affects the bifurcation of the system. Throughout the paper, we make an assumption

[$\mathbb{H}_1$]: $\frac{\beta\delta}{\alpha r} < 1$, $\frac{\beta S^*}{1 + \eta S^*} > \xi$, where S^* is the positive root of the equation $\eta S^2 + (1 - k\eta)S + k(\frac{\beta\delta}{\alpha r} - 1) = 0$. Under [$\mathbb{H}_1$], system (2) admits a positive equilibrium $E^* = (S^*, I^*, P^*)$, where $I^* = \frac{\delta}{\alpha}$, $P^* = \frac{1}{\varphi}(\frac{\beta S^*}{1 + \eta S^*} - \xi)$.

By applying the transformation of variables, let $Q_1(t) = S(t) - S^*$, $Q_2(t) = I(t) - I^*$, $Q_3(t) = P(t) - P^*$. Accordingly, the following system can be obtained from model (2)

$$\begin{cases} _{t_0}^C D_t^{\theta_1} Q_1(t) = r(Q_1(t) + S^*)(1 - \frac{Q_1(t)+S^*}{k}) - \frac{\beta(Q_1(t)+S^*)(Q_2(t)+I^*)}{1+\eta(Q_1(t)+S^*)}, \\ _{t_0}^C D_t^{\theta_2} Q_2(t) = \frac{\beta(Q_1(t)+S^*)(Q_2(t)+I^*)}{1+\eta(Q_1(t)+S^*)} - \varphi(Q_2(t) + I^*)(Q_3(t) + P^*) \\ \qquad\qquad\qquad - \xi(Q_2(t) + I^*) + \mu[Q_2(t) - Q_2(t - \upsilon)], \\ _{t_0}^C D_t^{\theta_3} Q_3(t) = \alpha(Q_2(t - \tau) + I^*)(Q_3(t - \tau) + P^*) - \delta(Q_3(t) + P^*). \end{cases} \quad (4)$$

According to system (4), the corresponding linear system can be expressed as

$$\begin{cases} _{t_0}^C D_t^{\theta_1} Q_1(t) = (r - \frac{2rS^*}{k} - \frac{\beta I^*}{(1+\eta S^*)^2})Q_1(t) - \frac{\beta S^*}{1+\eta S^*} Q_2(t), \\ _{t_0}^C D_t^{\theta_2} Q_2(t) = \frac{\beta I^*}{(1+\eta S^*)^2} Q_1(t) + \mu Q_2(t) - \mu Q_2(t - \upsilon) - \varphi I^* Q_3(t), \\ _{t_0}^C D_t^{\theta_3} Q_3(t) = \alpha P^* Q_2(t - \tau) - \delta Q_3(t) + \alpha I^* Q_3(t - \tau). \end{cases} \quad (5)$$

Consequently, the characteristic matrix of system (5) is

$$\Lambda(s) = \begin{bmatrix} s^{\theta_1} - a_{11} & -a_{12} & 0 \\ -a_{21} & s^{\theta_2} - \mu + \mu e^{-s\upsilon} & -a_{22} \\ 0 & -a_{31}e^{-s\tau} & s^{\theta_3} - a_{32} - a_{33}e^{-s\tau} \end{bmatrix},$$

where $a_{11} = r - \frac{2rS^*}{k} - \frac{\beta I^*}{(1+\eta S^*)^2}$, $a_{12} = -\frac{\beta S^*}{1+\eta S^*}$, $a_{21} = \frac{\beta I^*}{(1+\eta S^*)^2}$, $a_{22} = -\varphi I^*$, $a_{31} = \alpha P^*$, $a_{32} = -\delta$, $a_{33} = \alpha I^*$.

3.1 Influence of time delay on bifurcation dynamics of the uncontrolled system

Let's consider the Hopf bifurcation conditions of the uncontrolled system with regarding time delay as a bifurcation parameter. When $\upsilon = 0$ or $\mu = 0$, system (2) is a fractional system with no controller, the system is as follows,

$$\begin{cases} {}^{C}_{t_0}D^{\theta_1}_t S(t) = rS(1 - \frac{S}{k}) - \frac{\beta SI}{1+\eta S}, \\ {}^{C}_{t_0}D^{\theta_2}_t I(t) = \frac{\beta SI}{1+\eta S} - \xi I - \varphi I P, \\ {}^{C}_{t_0}D^{\theta_3}_t P(t) = \alpha I(t - \tau)P(t - \tau) - \delta P. \end{cases} \tag{6}$$

The characteristic Eq. of (6) is

$$W_1(s) + W_2(s)e^{-s\tau} = 0, \tag{7}$$

where

$$W_1(s) = s^{\theta_1+\theta_2+\theta_3} - a_{32}s^{\theta_1+\theta_2} - a_{11}s^{\theta_2+\theta_3} + a_{11}a_{32}s^{\theta_2} - a_{12}a_{21}s^{\theta_3} + a_{12}a_{21}a_{32},$$

$$W_2(s) = -a_{33}s^{\theta_1+\theta_2} - a_{22}a_{31}s^{\theta_1} + a_{11}a_{33}s^{\theta_2} + a_{11}a_{22}a_{31} + a_{12}a_{21}a_{33}.$$

Firstly, we consider the stability of equilibrium point E^* in the case of $\tau = 0$ in system (6).

If $\theta_1 = \theta_2 = \theta_3 = \theta$, we have the following equality

$$\sigma^3 + \mu_1\sigma^2 + \mu_2\sigma + \mu_3 = 0, \tag{8}$$

where $\sigma = s^\theta$, $\mu_1 = S^*(\frac{r}{k} - \frac{\beta\delta\eta}{\alpha(1+\eta S^*)^2})$, $\mu_2 = \frac{\delta\beta^2 S^*}{\alpha(1+\eta S^*)^3} + \varphi\delta P^*$, $\mu_3 = \varphi\delta S^* P^*(\frac{r}{k} - \frac{\beta\delta\eta}{\alpha(1+\eta S^*)^2})$. Under hypothesis [$\mathbb{H}_1$], we get $\mu_1 > 0$, $\mu_1\mu_2 - \mu_3 > 0$, $\mu_3 > 0$. By the Routh–Hurwitz criterion, we can find all the roots of Eq.(8) have negative real parts. That is $|arg(\sigma)| = |arg(s^\theta)| > \frac{\theta\pi}{2}$. From Corollary 1, one can deduce E^* of system (6) is locally asymptotically stable.

If θ_i, $i = 1, 2, 3$ are rational numbers between 0 and 1, m is the smallest of the common multiples of the denominators of θ_i. Denote λ by $s^{\frac{1}{m}}$, Eq.(7) can be written as $\lambda^{m(\theta_1+\theta_2+\theta_3)} - a_{11}\lambda^{m(\theta_2+\theta_3)} - a_{22}a_{31}\lambda^{m\theta_1} - a_{12}a_{21}\lambda^{m\theta_3} + a_{11}a_{22}a_{31} = 0$. If hypothesis [$\mathbb{H}_2$]: $|arg(\lambda_i)| > \frac{\pi}{2m}$, $i = 1, 2, \cdots, k$, $k = m(\theta_1 + \theta_2 + \theta_3)$ is satisfied, then E^* of system (6) is locally asymptotically stable by Corollary 2.

Based on the above discussion, Theorem 1 can be derived.

Theorem 1 Suppose the conditions [$\mathbb{H}_1$] and [$\mathbb{H}_2$] hold, then the positive equilibria $E^* = (S^*, I^*, P^*)$ of the fractional-order system (6) is locally asymptotically stable if $\tau = 0$.

Next, we consider the stability of equilibrium point E^* in the case of $\tau > 0$ in system (6). Let $s = r_1(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})(r_1 > 0)$ be a purely imaginary root of (7), so it follows from (7) that

$$\begin{cases} W_2^r \cos r_1\tau + W_2^i \sin r_1\tau = -W_1^r, \\ W_2^i \cos r_1\tau - W_2^r \sin r_1\tau = -W_1^i, \end{cases} \tag{9}$$

where

$$W_1^r = r_1^{\theta_1+\theta_2+\theta_3} \cos\frac{\theta_1+\theta_2+\theta_3}{2}\pi - a_{32}r_1^{\theta_1+\theta_2}\cos\frac{\theta_1+\theta_2}{2}\pi - a_{11}r_1^{\theta_2+\theta_3}\cos\frac{\theta_2+\theta_3}{2}\pi$$
$$+ a_{11}a_{32}r_1^{\theta_2}\cos\frac{\theta_2}{2}\pi - a_{12}a_{21}r_1^{\theta_3}\cos\frac{\theta_3}{2}\pi + a_{12}a_{21}a_{32},$$

$$W_1^i = r_1^{\theta_1+\theta_2+\theta_3}\sin\frac{\theta_1+\theta_2+\theta_3}{2}\pi - a_{32}r_1^{\theta_1+\theta_2}\sin\frac{\theta_1+\theta_2}{2}\pi - a_{11}r_1^{\theta_2+\theta_3}\sin\frac{\theta_2+\theta_3}{2}\pi$$
$$+ a_{11}a_{32}r_1^{\theta_2}\sin\frac{\theta_2}{2}\pi - a_{12}a_{21}r_1^{\theta_3}\sin\frac{\theta_3}{2}\pi,$$

$$W_2^r = -a_{33}r_1^{\theta_1+\theta_2}\cos\frac{\theta_1+\theta_2}{2}\pi - a_{22}a_{31}r_1^{\theta_1}\cos\frac{\theta_1}{2}\pi + a_{11}a_{33}r_1^{\theta_2}\cos\frac{\theta_2}{2}\pi$$
$$+ a_{11}a_{22}a_{31} + a_{12}a_{21}a_{33},$$

$$W_2^i = -a_{33}r_1^{\theta_1+\theta_2}\sin\frac{\theta_1+\theta_2}{2}\pi - a_{22}a_{31}r_1^{\theta_1}\sin\frac{\theta_1}{2}\pi + a_{11}a_{33}r_1^{\theta_2}\sin\frac{\theta_2}{2}\pi.$$

As far as Eq. (9), one yields

$$\begin{cases} \cos r_1\tau = -\dfrac{\kappa_1(r_1)}{\kappa_3(r_1)}, \\ \sin r_1\tau = -\dfrac{\kappa_2(r_1)}{\kappa_3(r_1)}, \end{cases} \tag{10}$$

where $\kappa_1(r_1) = W_1^r W_2^r + W_1^i W_2^i$, $\kappa_2(r_1) = W_1^r W_2^i - W_2^r W_1^i$, $\kappa_3(r_1) = (W_2^r)^2 + (W_2^i)^2$. It is obtained from Eq. (10) that

$$\kappa_1^2(r_1) + \kappa_2^2(r_1) - \kappa_3^2(r_1) = 0. \tag{11}$$

Suppose Eq. (11) has a positive real root r_{10}, we get $\tau^k = \frac{1}{r_{10}}\left[\arccos(-\frac{\kappa_1(r_{10})}{\kappa_3(r_{10})}) + 2k\pi\right], k = 0, 1, 2....$. At the same time, we apply the notation $\tau_0 = min\{\tau^k, k = 0, 1, 2...\}$. Then, in order to better search for the criterion of the occurrence for bifurcation, differentiating Eq.(7) with respect to τ, we have

$$W_1'(s)\frac{ds}{d\tau} + W_2'(s)\frac{ds}{d\tau}e^{-s\tau} + W_2(s)e^{-s\tau}(-\tau\frac{ds}{d\tau} - s) = 0.$$

So we can get

$$\frac{ds}{d\tau} = \frac{\varepsilon(s)}{\zeta(s)}, \tag{12}$$

where $\varepsilon(s) = sW_2(s)e^{-s\tau}$, $\zeta(s) = W_1'(s) + \left[W_2'(s) - \tau W_2(s)\right]e^{-s\tau}$. Define $\varepsilon_1, \varepsilon_2$ be the real and imaginary parts of $\varepsilon(s)$ individually. ζ_1, ζ_2 be the real and imaginary parts of $\zeta(s)$ individually. Based on algebraic analysis, we can obtain from Eq.(12) that

$$Re\left[\frac{ds}{d\tau}\right]\bigg|_{(r_1=r_{10},\tau=\tau_0)} = \frac{\varepsilon_1\zeta_1 + \varepsilon_2\zeta_2}{\zeta_1^2 + \zeta_2^2}, \tag{13}$$

where

$$\varepsilon_1 = r_{10}\left(W_2^r\sin r_{10}\tau_0 - W_2^i\cos r_{10}\tau_0\right),$$
$$\varepsilon_2 = r_{10}\left(W_2^r\cos r_{10}\tau_0 + W_2^i\sin r_{10}\tau_0\right),$$
$$\zeta_1 = (W_1')^r + \left((W_2')^r - \tau_0 W_2^r\right)\cos r_{10}\tau_0 + \left((W_2')^i - \tau_0 W_2^i\right)\sin r_{10}\tau_0,$$
$$\zeta_2 = (W_1')^i + \left((W_2')^i - \tau_0 W_2^i\right)\cos r_{10}\tau_0 - \left((W_2')^r - \tau_0 W_2^r\right)\sin r_{10}\tau_0.$$

Now, if we have $[\mathbb{H}_3] : \frac{\varepsilon_1\zeta_1+\varepsilon_2\zeta_2}{\zeta_1^2+\zeta_2^2} > 0$, then the transversality condition $Re\left[\frac{ds}{d\tau}\right]\bigg|_{(r_1=r_{10},\tau=\tau_0)} > 0$ is true, so we can draw a conclusion as following.

Theorem 2 If assumptions $[\mathbb{H}_1]$ and $[\mathbb{H}_3]$ hold, then

1 If $\tau \in [0, \tau_0)$, all the roots of Eq. (7) have negative real parts, and the positive equilibria E^* of system (6) is locally asymptotically stable.
2 If $\tau > \tau_0$, the roots of Eq. (7) have at least one root with positive real part, and the positive equilibria E^* of system (6) is unstable.

3 If $\tau = \tau_0$, the roots of Eq. (7) have a purely imaginary root, and system (6) exhibits a Hopf bifurcation at the positive equilibria, which imply it has a branch of periodic solution bifurcating from E^* near $\tau = \tau_0$.

Remark 1 According to Hopf bifurcation theory [37], if the bifurcation parameter passes through the critical value, making the eigenvalue pass through the imaginary axis, Hopf bifurcation will occur in the system. At this point, the critical value is called the bifurcation point. Therefore, if $Re\left[\dfrac{ds}{d\tau}\right]\Big|_{(r_1=r_{10},\tau=\tau_0)} > 0$, system (6) will change from the initial stable state to an unstable state, and Hopf bifurcation will occur.

3.2 Influence of time delay on bifurcation dynamics of controlled system (2)

In what follows, the influence of the controller $\Psi(t)$ will be investigated. For the given $\Psi(t) = \mu[I(t) - I(t - \upsilon)]$, the corresponding characteristic Eq. of (2) is

$$P_1(s) + P_2(s)e^{-s\tau} = 0, \tag{14}$$

where

$$P_1(s) = W_1(s) + \mu[-s^{\theta_1+\theta_3} + a_{32}s^{\theta_1} + a_{11}s^{\theta_3} - a_{11}a_{32} + (s^{\theta_1+\theta_3} - a_{32}s^{\theta_1}$$
$$- a_{11}s^{\theta_3} + a_{11}a_{32})e^{-s\upsilon}],$$
$$P_2(s) = W_2(s) + \mu[a_{33}s^{\theta_1} - a_{11}a_{33} + (-a_{33}s^{\theta_1} + a_{11}a_{33})e^{-s\upsilon}].$$

Suppose that $s = r_2(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})(r_2 > 0)$ is a purely imaginary root of (14), it follows from (14) that

$$\begin{cases} P_2^r \cos r_2\tau + P_2^i \sin r_2\tau = -P_1^r, \\ P_2^i \cos r_2\tau - P_2^r \sin r_2\tau = -P_1^i, \end{cases} \tag{15}$$

where

$$P_1^r = W_1^r + \mu[-r_2^{\theta_1+\theta_3} \cos \frac{\theta_1+\theta_3}{2}\pi + a_{32}r_2^{\theta_1} \cos \frac{\theta_1}{2}\pi + a_{11}r_2^{\theta_3} \cos \frac{\theta_3}{2}\pi - a_{11}a_{32}$$
$$+ (r_2^{\theta_1+\theta_3} \cos \frac{\theta_1+\theta_3}{2}\pi - a_{32}r_2^{\theta_1} \cos \frac{\theta_1}{2}\pi - a_{11}r_2^{\theta_3} \cos \frac{\theta_3}{2}\pi + a_{11}a_{32}) \cos r_2\upsilon$$
$$+ (r_2^{\theta_1+\theta_3} \sin \frac{\theta_1+\theta_3}{2}\pi - a_{32}r_2^{\theta_1} \sin \frac{\theta_1}{2}\pi - a_{11}r_2^{\theta_3} \sin \frac{\theta_3}{2}\pi) \sin r_2\upsilon],$$

$$P_1^i = W_1^i - \mu[-r_2^{\theta_1+\theta_3} \sin \frac{\theta_1+\theta_3}{2}\pi + a_{32}r_2^{\theta_1} \sin \frac{\theta_1}{2}\pi + a_{11}r_2^{\theta_3} \sin \frac{\theta_3}{2}\pi$$
$$+ (r_2^{\theta_1+\theta_3} \sin \frac{\theta_1+\theta_3}{2}\pi - a_{32}r_2^{\theta_1} \sin \frac{\theta_1}{2}\pi - a_{11}r_2^{\theta_3} \sin \frac{\theta_3}{2}\pi) \cos r_2\upsilon$$
$$+ (r_2^{\theta_1+\theta_3} \sin \frac{\theta_1+\theta_3}{2}\pi - a_{32}r_2^{\theta_1} \sin \frac{\theta_1}{2}\pi - a_{11}r_2^{\theta_3} \sin \frac{\theta_3}{2}\pi + a_{11}a_{32}) \sin r_2\upsilon],$$

$$P_2^r = W_2^r + \mu[a_{33}r_2^{\theta_1} \cos \frac{\theta_1}{2}\pi - a_{11}a_{33} + (a_{11}a_{33} - a_{33}r_2^{\theta_1} \cos \frac{\theta_1}{2}\pi) \cos r_2\upsilon$$
$$- a_{33}r_2^{\theta_1} \sin \frac{\theta_1}{2}\pi \sin r_2\upsilon],$$

$$P_2^i = W_2^i + \mu[a_{33}r_2^{\theta_1} \sin \frac{\theta_1}{2}\pi + (a_{33}r_2^{\theta_1} \cos \frac{\theta_1}{2}\pi - a_{11}a_{33}) \cos r_2\upsilon$$
$$- a_{33}r_2^{\theta_1} \sin \frac{\theta_1}{2}\pi \cos r_2\upsilon].$$

Solving Eq.(15) yields

$$\begin{cases} \cos r_2\tau = -\dfrac{\alpha_1(r_2)}{\alpha_3(r_2)}, \\ \sin r_2\tau = -\dfrac{\alpha_2(r_2)}{\alpha_3(r_2)}, \end{cases} \tag{16}$$

where $\alpha_1(r_2) = P_1^r P_2^r + P_1^i P_2^i$, $\alpha_2(r_2) = P_1^r P_2^i - P_2^r P_1^i$, $\alpha_3(r_2) = (P_2^r)^2 + (P_2^i)^2$. It is apparent from Eq.(16) that

$$\alpha_1^2(r_2) + \alpha_2^2(r_2) - \alpha_3^2(r_2) = 0. \tag{17}$$

Suppose Eq.(17) has a positive real root r_{20}, we can get $\tau^k = \dfrac{1}{r_{20}}\left[\arccos(-\dfrac{\alpha_1(r_{20})}{\alpha_3(r_{20})}) + 2k\pi\right], k = 0, 1, 2....$ Meanwhile, we denote $\tau_1 = min\{\tau^k, k = 0, 1, 2...\}$. For the sake of searching for the criterion of the occurrence for bifurcation, differentiating Eq.(14) with respect to τ, we have

$$P_1^{'}(s)\frac{ds}{d\tau} + P_2^{'}(s)\frac{ds}{d\tau}e^{-s\tau} + P_2(s)e^{-s\tau}(-\tau\frac{ds}{d\tau} - s) = 0.$$

So we can obtain

$$\frac{ds}{d\tau} = \frac{\rho(s)}{\varsigma(s)}, \tag{18}$$

where $\rho(s) = sP_2(s)e^{-s\tau}$, $\varsigma(s) = P_1^{'}(s) + \left[P_2^{'}(s) - \tau P_2(s)\right]e^{-s\tau}$. Note ρ_1, ρ_2 be the real and imaginary parts of $\rho(s)$ individually. ς_1, ς_2 be the real and imaginary parts of $\varsigma(s)$ individually. By mathematical manipulation, we conclude from Eq. (18) that

$$Re\left[\frac{ds}{d\tau}\right]\Big|_{(r_2=r_{20}, \tau=\tau_1)} = \frac{\rho_1\varsigma_1 + \rho_2\varsigma_2}{\varsigma_1^2 + \varsigma_2^2}, \tag{19}$$

where

$$\rho_1 = r_{20}\left(P_2^r \sin r_{20}\tau_1 - P_2^i \cos r_{20}\tau_1\right),$$
$$\rho_2 = r_{20}\left(P_2^r \cos r_{20}\tau_1 + P_2^i \sin r_{20}\tau_1\right),$$
$$\varsigma_1 = (P_1^{'})^r + \left((P_2^{'})^r - \tau_1 P_2^r\right)\cos r_{20}\tau_1 + \left((P_2^{'})^i - \tau_1 P_2^i\right)\sin r_{20}\tau_1,$$
$$\varsigma_2 = (P_1^{'})^i + \left((P_2^{'})^i - \tau_1 P_2^i\right)\cos r_{20}\tau_1 - \left((P_2^{'})^r - \tau_1 P_2^r\right)\sin r_{20}\tau_1.$$

As a result, if the suppose $[\mathbb{H}_4]$: $\frac{\rho_1\varsigma_1 + \rho_2\varsigma_2}{\varsigma_1^2 + \varsigma_2^2} > 0$ is true, then we can deduce the transversality criteria $Re\left[\frac{ds}{d\tau}\right]\Big|_{(r_2=r_{20}, \tau=\tau_1)} > 0$, so we can summarize what we have proved as the following results.

Theorem 3 In the case of $[\mathbb{H}_1]$ and $[\mathbb{H}_4]$, we have

1. If $\tau \in [0, \tau_1)$, all the roots of Eq. (14) have negative real parts, and the positive equilibria E^* of system (2) is locally asymptotically stable.
2. If $\tau > \tau_1$, the roots of Eq. (14) have at least one root with positive real part, and the positive equilibria E^* of system (2) is unstable.
3. If $\tau = \tau_1$, the roots of Eq. (14) have a purely imaginary root, and system (2) exhibits a Hopf bifurcation at the positive equilibria, which imply it has a branch of periodic solution bifurcating from E^* near $\tau = \tau_1$.

3.3 The Hopf bifurcation of system (2) caused by feedback delay

In the first two subsections, we discussed the stability and bifurcation of the system with regarding gestation period τ as the parameter. In fact, when the time delay τ is given, feedback control delay υ can also affect the stability of the system and induce Hopf bifurcation of the system. Hence, we give the following results.

Theorem 4 Suppose conditions $[\mathbb{H}_1]$ and $[\mathbb{H}_5]$ satisfy, we have

1. If $\upsilon \in [0, \upsilon_0)$, the positive equilibria E^* of system (2) is locally asymptotically stable.
2. If $\upsilon > \upsilon_0$, the positive equilibria E^* of system (2) is unstable.
3. If $\upsilon = \upsilon_0$, system (2) exhibits a Hopf bifurcation at the positive equilibria, which implies it has a branch of periodic solution bifurcating from E^* near $\upsilon = \upsilon_0$.

For the detailed proof of Theorem 4, we can refer to Appendix.

Remark 2 In the previous studies, many scholars ignored the importance of delay parameters in feedback controllers[38, 39]. In fact, as stated in Theorem 4, the feedback control delay parameter can also affect the stability of the system and induce Hopf bifurcation phenomenon.

4 Numerical results

For verify the feasibility of the theoretical analysis on system stability and bifurcation control, we consider the following system by using the same coefficients as in [8]:

$$\begin{cases} {}^{C}_{0}D^{\theta_1}_t S(t) = 2.5S(1 - \frac{S}{4}) - \frac{3SI}{1+S}, \\ {}^{C}_{0}D^{\theta_2}_t I(t) = \frac{3SI}{1+S} - I - IP + \mu[I(t) - I(t - \upsilon)], \\ {}^{C}_{0}D^{\theta_3}_t P(t) = 1.5I(t - \tau)P(t - \tau) - P. \end{cases} \tag{20}$$

At the same time, we define initial conditions $S(t) = 3$, $I(t) = 1$, $P(t) = 1$, $t \in [-max(\tau, \upsilon), 0]$. By calculation, system (20) has a unique positive equilibria $E^* = (3.2468, 0.6667, 1.2935)$.

**Case I**. First of all, let's talk about the controller-free system, which is $\mu = 0$ or $\upsilon = 0$.

(i) Fix $\theta_1 = 0.95$, $\theta_2 = 0.96$, $\theta_3 = 0.97$, note that Theorem 1 is satisfied for the parameters selected when $\tau = 0$, the trajectory diagram and phase diagram of the uncontrol system solution are shown in Fig. 1. From this figure, we observe that the species density of susceptible prey, infected prey, and predator changes correspondingly over time from the initial state, but eventually converges to the equilibria $E^* = (3.2468, 0.6667, 1.2935)$. Namely, the positive equilibrium E^* of the controller-free system is locally asymptotically stable if $\tau = 0$.

(ii) Fix $\theta_1 = 0.95$, $\theta_2 = 0.96$, $\theta_3 = 0.97$. By numerical calculation, we can infer $\tau_0 = 0.2145$ and transversality condition $[\mathbb{H}_3]$ is satisfied. With the help of Theorem 2, the positive equilibria E^* is asymptotically stable for $\tau < \tau_0$, and unstable for $\tau > \tau_0$. Furthermore, the uncontrol system has a branch of periodic solution bifurcating from E^* near $\tau = \tau_0$. Fig. 2 depicts the solution curves and phase diagrams of the system at $\tau = 0.15$ and $\tau = 0.22$, respectively.

(iii) In particular, we find that the fractional order also affects the stability of the system. Just for comparison purposes, we simply change $\theta_3 = 0.97$ in **Case I**(ii) to $\theta_3 = 0.9$. After operation, we acquire that $\tau_0 = 0.3308$. Therefore, if we fix $\tau = 0.22$, the positive equilibria E^* becomes stable from unstable when θ_3 changes from 0.97 to 0.9(see Fig. 3).

Remark 3 In order to describe the influence of fractional order on system stability more clearly, the bifurcation critical values τ_0 corresponding to parameter θ_3 from 0.9 to 1 are fitted in Fig. 4. As some researchers have found, the fractional-order system has a wider stability region, which inhibits the periodic oscillation behavior of the system[40, 41].

Case II. Next, we begin to discuss fractional control system. For the sake of comparison, we take $\theta_1 = 0.95$, $\theta_2 = 0.96$, $\theta_3 = 0.97$ again.

(i) We select $\mu = -1$, $\upsilon = 0.5$, then $\tau_1 = 0.4199$ can be obtained by numerical calculation, and transversality condition $[\mathbb{H}_4]$

Fig. 1 Time series and phase-portrait of the uncontrolled system with $\theta_1 = 0.95$, $\theta_2 = 0.96$, $\theta_3 = 0.97$ and $\tau = 0$
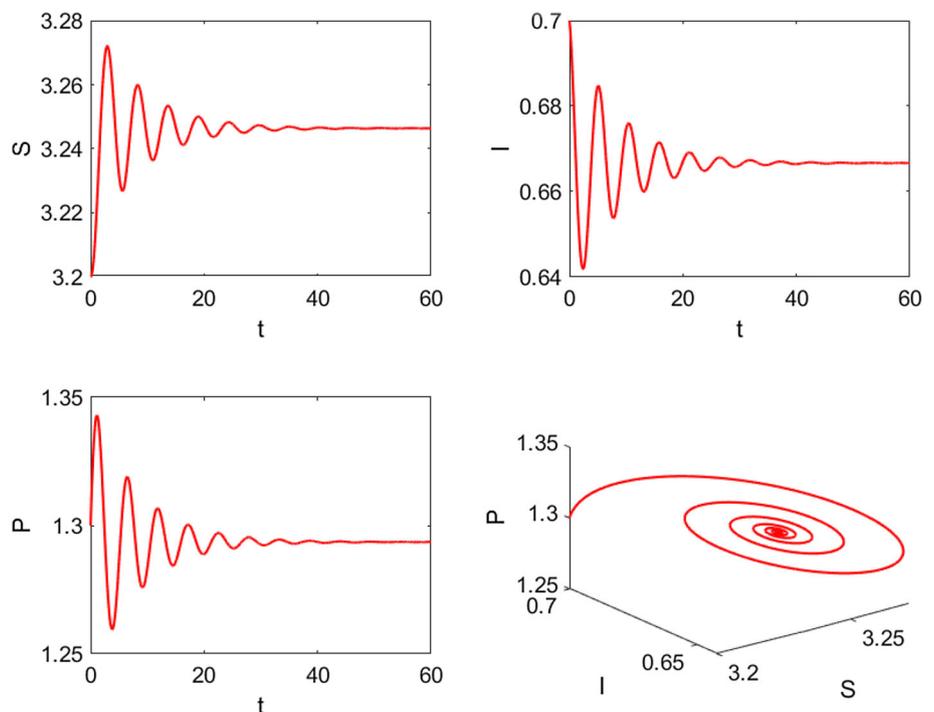
Fig. 2 Time series and phase-portraits of the uncontrolled system with $\theta_1 = 0.95, \theta_2 = 0.96, \theta_3 = 0.97$
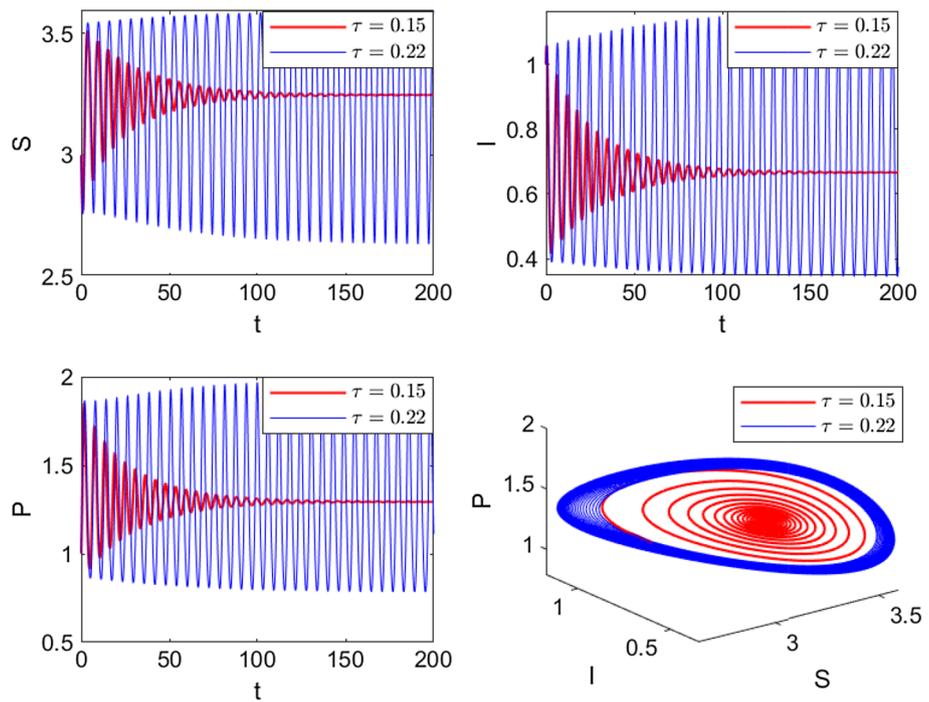
Fig. 3 Time series and phase-portraits of the uncontrolled system with $\theta_1 = 0.95, \theta_2 = 0.96$ and $\tau = 0.22$
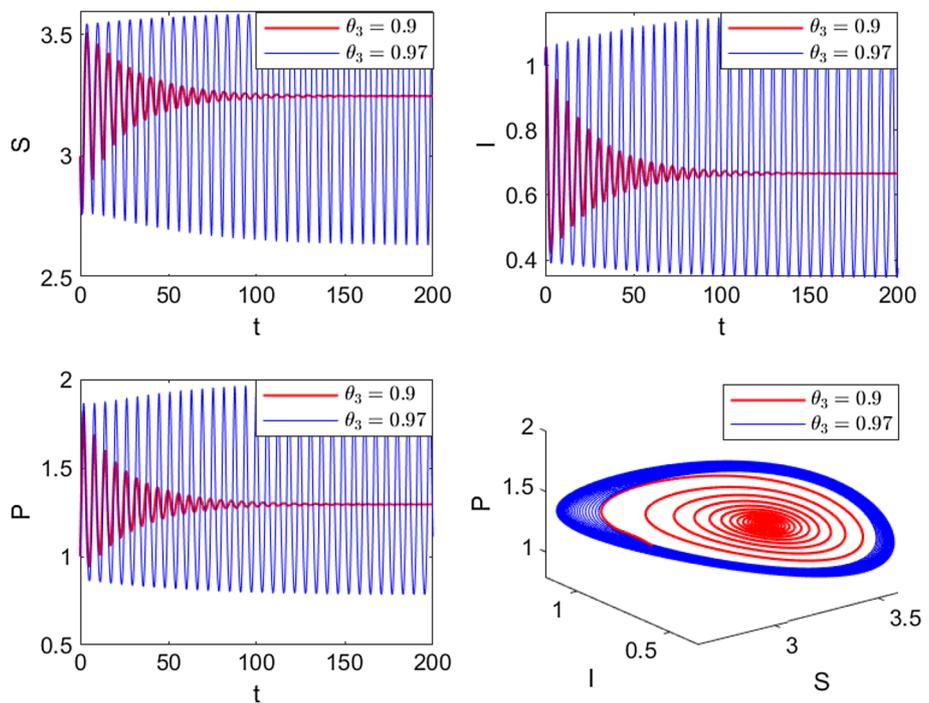
is hold. In terms of Theorem 3, the positive equilibria E^* is asymptotically stable for $\tau < \tau_1$, and unstable for $\tau > \tau_1$. In the meantime, system (20) has a branch of periodic solution bifurcating from E^* near $\tau = \tau_1$. Therefore, the equilibria E^* is asymptotically stable when $\tau = 0.22 < \tau_1$. Compared with the control free system, it is obvious that the extended feedback controller can affect the stability of the system (20), which are simulated in Fig. 5. In particular, if we fix $\upsilon = 0.5$ beforehand, the bifurcation critical value τ_1 of system (20) can be changed by changing the value of the extended feedback gain parameter μ within a certain range, which is shown in Fig. 6.

(ii) For test whether Hopf bifurcation can be induced by feedback control delay, let's choose $\tau = 0.2$ and $\mu = 0.1$. After calculation, we can get $\upsilon_0 = 0.5555$ and transversality condition $[\mathbb{H}_5]$ is satisfied. Based on Theorem 4, it implies that the positive equilibria

Fig. 4 The effect of θ_3 on bifurcation point τ_0
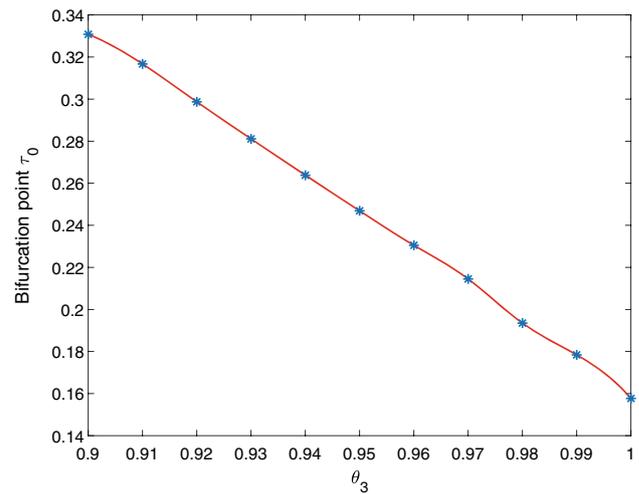
**Fig. 5** Time series and phase-portraits of system (20) with $\theta_1 = 0.95, \theta_2 = 0.96, \theta_3 = 0.97$ and $\tau = 0.22$
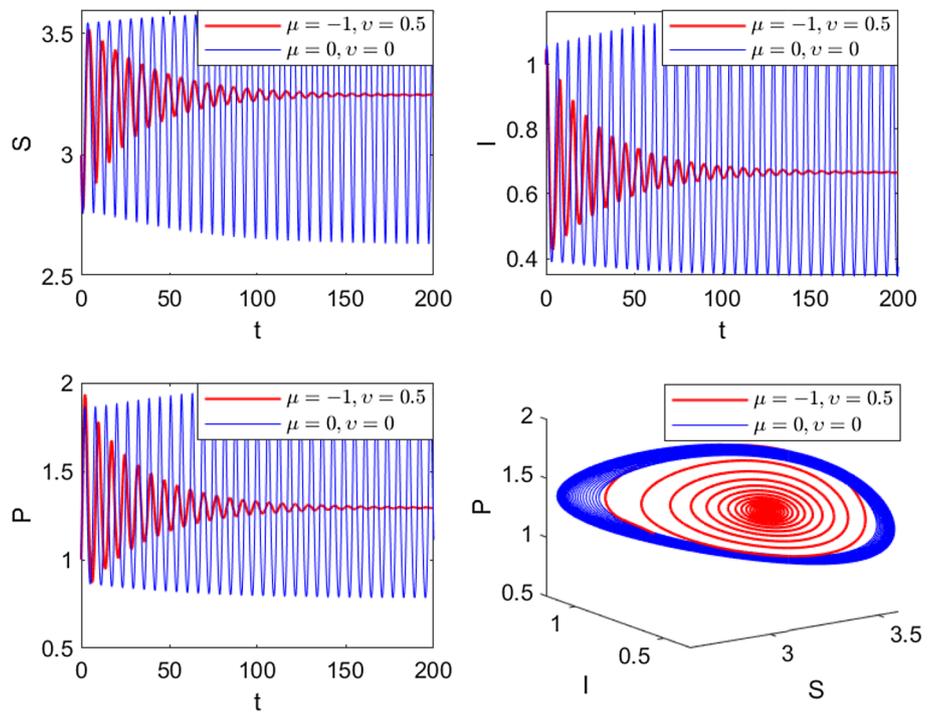
E^* is asymptotically stable for $\upsilon < \upsilon_0$, unstable for $\upsilon > \upsilon_0$, and system (20) has a branch of periodic solution bifurcating from E^* near $\upsilon = \upsilon_0$. Time series and phase-portraits of system (20) with $\upsilon = 0.2, 0.56$ are depicted in Fig. 7.

Remark 4 After the detailed comparison above, it is not difficult to find that the desired behavior can be obtained by appropriately adjusting the extended feedback gain parameters and feedback delay parameters. Therefore, the control system proposed in this paper could better meet the needs of real life.

Fig. 6 The effect of μ on bifurcation point τ_1
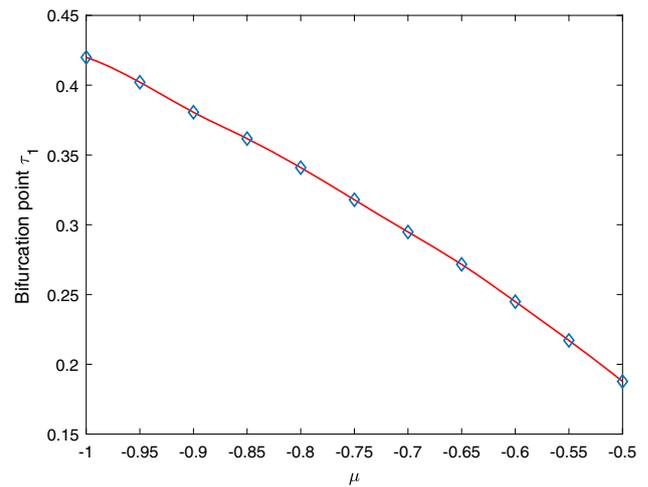
Fig. 7 Time series and phase-portraits of system (20) with $\mu = 0.1, \tau = 0.2$
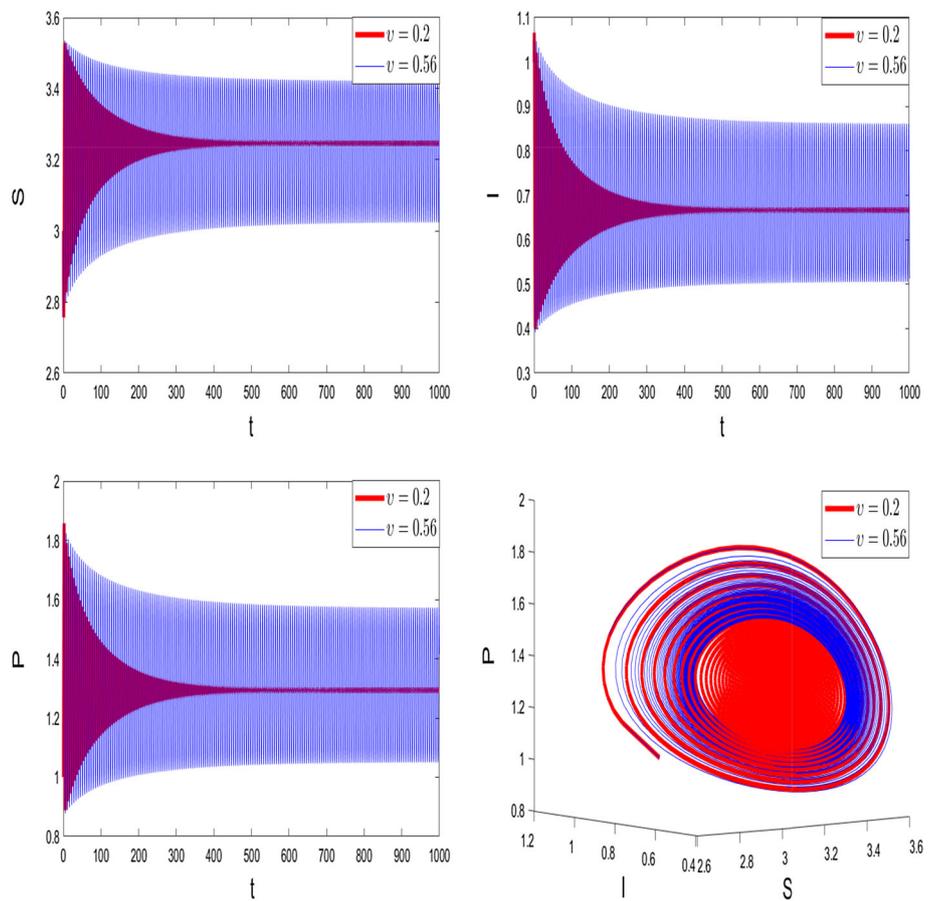
5 Conclusion

In this research, a delayed fractional ecological epidemiological model with extended feedback controller is presented. First, the stability and Hopf bifurcation of the system in the controller-free and controller states are discussed with regarding the digestion delay as a parameter, respectively, and the existence conditions of periodic solution are given. Secondly, this paper confirms that the Hopf bifurcation can be induced by taking the feedback delay as a bifurcation parameter. Finally, by drawing the solution curve of the system, the influence of the controller on the stability of the system is verified in detail. That is, the stability of the system could be maintained by adjusting the negative feedback gain parameters and feedback time delay reasonably, thereby suppressing the occurrence of periodic solutions. Furthermore, we interestingly find that the periodic oscillatory behavior of the system solution can

be suppressed by fractional order, which implies that the fractional-order system has a wider stability region than the integer-order system.

Author contributions All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Declarations

Conflict of interest The authors declare no conflict of interest.

Appendix

In this section, we give the proof of Theorem 4. If the delay τ is given, the corresponding characteristic Eq. of (2) is

$$G_1(s) + G_2(s)e^{-s\upsilon} = 0, \tag{21}$$

where

$$
\begin{aligned}
G_1(s) =& W_1(s) + W_2(s)e^{-s\tau} - G_2(s), \\
G_2(s) =& \mu[s^{\theta_1+\theta_2} - a_{22}s^{\theta_1} - a_{11}s^{\theta_2} + a_{11}a_{22} + (a_{11}a_{23} - a_{23}s^{\theta_1} - a_{12}a_{21})e^{-s\tau}].
\end{aligned}
$$

Suppose that s is a purely imaginary root of (21), where $s = r_3(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})(r_3 > 0)$, it procures that

$$
\begin{cases}
G_2^r \cos r_3\upsilon + G_2^i \sin r_3\upsilon = -G_1^r, \\
G_2^i \cos r_3\upsilon - G_2^r \sin r_3\upsilon = -G_1^i,
\end{cases} \tag{22}
$$

where

$$
\begin{aligned}
G_1^r =& W_1^r + W_2^r \cos r_3\tau + W_2^i \sin r_3\tau - G_2^r, \\
G_1^i =& W_1^i + W_2^i \cos r_3\tau - W_2^r \sin r_3\tau - G_2^i, \\
G_2^r =& \mu[r_3^{\theta_1+\theta_2} \cos\frac{\theta_1+\theta_2}{2}\pi - a_{22}r_3^{\theta_1}\cos\frac{\theta_1}{2}\pi - a_{11}r_3^{\theta_2}\cos\frac{\theta_2}{2}\pi + a_{11}a_{22} \\
& - a_{23}r_3^{\theta_1}\sin\frac{\theta_1}{2}\pi\sin r_3\tau + (a_{11}a_{23} - a_{23}r_3^{\theta_1}\cos\frac{\theta_1}{2}\pi - a_{12}a_{21})\cos r_3\tau], \\
G_2^i =& \mu[r_3^{\theta_1+\theta_2}\sin\frac{\theta_1+\theta_2}{2}\pi - a_{22}r_3^{\theta_1}\sin\frac{\theta_1}{2}\pi - a_{11}r_3^{\theta_2}\sin\frac{\theta_2}{2}\pi - a_{23}r_3^{\theta_1}\sin\frac{\theta_1}{2}\pi\cos r_3\tau \\
& + (a_{23}r_3^{\theta_1}\cos\frac{\theta_1}{2}\pi + a_{12}a_{21} - a_{11}a_{23})\sin r_3\tau].
\end{aligned}
$$

Taking account of Eq.(22), we can get

$$
\begin{cases}
\cos r_3\upsilon = -\dfrac{\gamma_1(r_3)}{\gamma_3(r_3)}, \\
\sin r_3\upsilon = -\dfrac{\gamma_2(r_3)}{\gamma_3(r_3)},
\end{cases} \tag{23}
$$

where $\gamma_1(r_3) = G_1^r G_2^r + G_1^i G_2^i$, $\gamma_2(r_3) = G_1^r G_2^i - G_2^r G_1^i$, $\gamma_3(r_3) = (G_2^r)^2 + (G_2^i)^2$. Draw support from Eq.(23), we have

$$\gamma_1^2(r_3) + \gamma_2^2(r_3) - \gamma_3^2(r_3) = 0. \tag{24}$$

Let's assume that Eq.(24) has a positive real root r_{30}, we obtain $\upsilon^k = \dfrac{1}{r_{30}}\left[\arccos(-\dfrac{\gamma_1(r_{30})}{\gamma_3(r_{30})}) + 2k\pi\right]$, $k = 0, 1, 2....$ Now, we make $\upsilon_0 = min\{\upsilon^k, k = 0, 1, 2...\}$. Similarly, in order to better study the Hopf bifurcation, differentiating Eq.(21) with respect to υ, one can infer that

$$G_1^{'}(s)\frac{ds}{d\upsilon} + G_2^{'}(s)\frac{ds}{d\upsilon}e^{-s\upsilon} + G_2(s)e^{-s\upsilon}(-\upsilon\frac{ds}{d\upsilon} - s) = 0.$$

Accordingly, we have

$$\frac{ds}{d\upsilon} = \frac{\iota(s)}{\varrho(s)}, \tag{25}$$

where $\iota(s) = sG_2(s)e^{-sv}$, $\varrho(s) = G_1'(s) + \left[G_2'(s) - vG_2(s)\right]e^{-sv}$. Put ι_1, ι_2 be the real and imaginary parts of $\iota(s)$ individually. ϱ_1, ϱ_2 be the real and imaginary parts of $\varrho(s)$ individually. After several algebraic calculations, we can further obtain from Eq.(25) that

$$Re\left[\frac{ds}{dv}\right]\Bigg|_{(r_3=r_{30}, v=v_0)} = \frac{\iota_1\varrho_1 + \iota_2\varrho_2}{\varrho_1^2 + \varrho_2^2}, \qquad (26)$$

where

$$\iota_1 = r_{30}\left(G_2^r \sin r_{30}v_0 - G_2^i \cos r_{30}v_0\right),$$
$$\iota_2 = r_{30}\left(G_2^r \cos r_{30}v_0 + G_2^i \sin r_{30}v_0\right),$$
$$\varrho_1 = (G_1')^r + \left((G_2')^r - v_0 G_2^r\right)\cos r_{30}v_0 + \left((G_2')^i - v_0 G_2^i\right)\sin r_{30}v_0,$$
$$\varrho_2 = (G_1')^i + \left((G_2')^i - v_0 G_2^i\right)\cos r_{30}v_0 - \left((G_2')^r - v_0 G_2^r\right)\sin r_{30}v_0.$$

In consequence, if the suppose $[\mathbb{H}_5]$: $\frac{\iota_1\varrho_1 + \iota_2\varrho_2}{\varrho_1^2 + \varrho_2^2} > 0$ is true, then the transversality criteria $Re\left[\frac{ds}{dv}\right]\Big|_{(r_3=r_{30}, v=v_0)} > 0$ can be concluded, so Theorem 4 can be obtained.

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