International Mathematical Forum, Vol. 6, 2011, no. 35, 1725 - 1731

Stability and Bifurcation in a Neural Network Model with Two Delays

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Abstract

This paper is concerned with a two-neurons network model with two discrete delays. By regarding the sum of two discrete time delay as the bifurcation parameter, the stability of the equilibrium and Hopf bifurcations are investigated. Finally, to verify our theoretical predictions, some numerical simulations are also included.

Mathematics Subject Classification: 34K18; 34K20; 92B20

Keywords: Time delay; Neuron network; Stability; Hopf bifurcation

1 Introduction

Recently, a large number of neural networks models have been proposed and studied extensively since Hopfield constructed a simplified neural network. In most networks however, it is usually expected that time delays exist during the processing and transmission of signals. In general, delay-differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could cause a stable equilibrium to become unstable [1]. Recently, time delays have been incorporated into neural network models by many authors [1, 2, 4, 5, 6], there has been great interest in dynamical characteristics of neural network model with delay.

In present paper, we consider a simplified Hopfield-type neural network model with two delays

$$\begin{cases} \dot{u}_1(t) = -u_1(t) + a_{11}f(u_1(t)) + a_{12}f(u_2(t-\tau_1)), \\ \dot{u}_2(t) = -u_2(t) + a_{21}f(u_1(t-\tau_2)) + a_{22}f(u_2(t)), \end{cases} \tag{1}$$

where τ_i $(i = 1, 2)$ are non-negative constants, and $f(x)$ is C^2 function. Throughout this paper we also assume that $f(0) = 0$.

In general, the delays appearing in different terms of a neural network model are not equal each other. Therefore, it is more realistic to consider the dynamics of a dynamical system with different delays. Based on this idea, in this paper, we consider the dynamical behaviors of system (1), that is, by taking $\tau_1 + \tau_2$ as the bifurcation parameter, we investigate the stability and Hopf bifurcations of system (1) induced by the delays.

This paper is organized as follows. In Section 2, we shall consider the stability of the zero equilibrium and the existence of local Hopf bifurcation. In order to verify our theoretical prediction, some numerical simulations are included in Section 3.

2 Stability analysis and Hopf bifurcation

Taking the following variable change: $v_1(t) = u_1(t-\tau_2),\ v_2(t) = u_2(t)$, then the system (1) can be rewritten as

$$\begin{cases} \dot{v}_1(t) = -v_1(t) + a_{11} f(v_1(t)) + a_{12} f(v_2(t-\tau)), \\ \dot{v}_2(t) = -v_2(t) + a_{21} f(v_1(t)) + a_{22} f(v_2(t)), \end{cases} \tag{2}$$

where $\tau = \tau_1 + \tau_2$.

It is obvious that the origin $(0,0)$ is an equilibrium of system (2). Linearizing system (2) about the origin $(0,0)$ yields the following linear system

$$\begin{cases} \dot{v}_1(t) = -v_1(t) + \alpha_{11} v_1(t) + \alpha_{12} v_2(t-\tau), \\ \dot{v}_2(t) = -v_2(t) + \alpha_{21} v_1(t) + \alpha_{22} v_2(t), \end{cases} \tag{3}$$

where $\alpha_{ij} = a_{ij} f'(0)$, i, j=1,2. The associated characteristic equation of system (3) is

$$\lambda^2 + p\lambda + q - \alpha_{12}\alpha_{21} e^{-\lambda\tau} = 0, \tag{4}$$

where

$$p = 2 - \alpha_{11} - \alpha_{22},\ q = (1-\alpha_{11})(1-\alpha_{22}).$$

The stability of the origin $(0,0)$ of system (2) depends on the locations on the complex plane of the roots of the characteristic equation (3). When all roots of Eq. (3) locate on the left half-plane of complex plane, the origin $(0,0)$ of system (2) is stable; otherwise, it is unstable.

Note that when $\tau = 0$, (4) becomes

$$\lambda^2 + p\lambda + q - \alpha_{12}\alpha_{21} = 0, \tag{5}$$

solve Eq. (5), then the roots of (5) are given by

$$\lambda_{1,2} = \frac{-p \pm \sqrt{p^2 - 4(q - \alpha_{12}\alpha_{21})}}{2}.$$

Thus, one can immediately obtain the following result.

Lemma 2.1 *Assume that* (H1) $p > 0,\ q - \alpha_{12}\alpha_{21} < 0$. *Then all the roots of Eq.* (2) *with* $\tau = 0$ *have always negative real parts.*

Next, we shall investigate the distribution of roots of Eq.(4) with $\tau > 0$. First note that, under condition (H1), Eq.(4) has no zero root. Next we shall look for the possibility of occurrence of a pair pure imaginary roots. Obviously, $i\omega(\omega > 0)$ is a root of (4) if and only if ω satisfies the following equation

$$-\omega^2 + \omega p i + q - \alpha_{12}\alpha_{21}(\cos\omega\tau - i\sin\omega\tau) = 0. \tag{6}$$

Separating the real and imaginary parts of (6) gives the following equations

$$\begin{cases} -\omega^2 + q = \alpha_{12}\alpha_{21}\cos\omega\tau, \\ -\omega p = \alpha_{12}\alpha_{21}\sin\omega\tau. \end{cases} \tag{7}$$

By some simple calculations, it is easy to obtain

$$\omega^4 + (p^2 - 2q)\omega^2 + q^2 - (\alpha_{12}\alpha_{21})^2 = 0, \tag{8}$$

and

$$\tan\omega\tau = \frac{p\omega}{q - \omega^2}. \tag{9}$$

It is easy to see that the first equation of (8) has only one positive root

$$\omega_0 = \left[\frac{2q - p^2 + \sqrt{(2q - p^2)^2 - 4\left[q^2 - (\alpha_{12}\alpha_{21})^2\right]}}{2}\right]^{\frac{1}{2}} \tag{10}$$

provided that the following assumption (H2) $2q - p^2 < 0,\ q^2 - (\alpha_{12}\alpha_{21})^2 < 0$ is satisfied.

From equation (9), we define

$$\tau^j = \tfrac{1}{\omega_0}\left(\arctan\tfrac{p\omega_0}{q - \omega_0^2} + \pi j\right), j = 0, 1, 2, \cdots, \tag{11}$$

then Eq. (4) with $\tau = \tau^j$ has a pair of purely imaginary roots $\pm i\omega_0$.

Since the roots of Eq. (4) continuously depend on the parameter τ, summarizing the above remarks and combining Lemma 2.1, the following result holds.

Lemma 2.2 *Suppose that* (H1) *and* (H2) *hold, then*

(i) *If* $\tau \in [0, \tau^0)$, *then all roots of Eq.* (4) *have strictly negative real parts.*

(ii) *If* $\tau = \tau^0$, *then Eq.* (4) *has a pair of purely imaginary roots* $\pm i\omega_0$ *and other roots have strictly negative real parts.*

(iii) *If $\tau = \tau^j$, then Eq.* (4) *has a simple pair of purely imaginary root $\pm i\omega_0$, where τ^j are defined by* (11) *and ω_0 is defined by* (10).

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the roots of Eq. (4) satisfying

$$\alpha(\tau^j) = 0,\ \omega(\tau^j) = \omega_0,\ j = 0, 1, 2, \cdots .$$

Lemma 2.3 *The following transversality condition is satisfied*

$$\left.\frac{dRe\lambda\,(\tau)}{d\tau}\right|_{\tau=\tau^j} > 0. \tag{12}$$

In fact, differentiating the two sides of (4) with respect to τ, we can obtain

$$2\lambda\lambda' + p\lambda' + \alpha_{12}\alpha_{21}\lambda e^{-\lambda\tau} = 0,$$

which implies

$$\frac{d\lambda}{d\tau} = \frac{-\alpha_{12}\alpha_{21}\lambda e^{-\lambda\tau}}{2\lambda + p}.$$

We can directly compute that

$$\left.\frac{d\mathrm{Re}\lambda(\tau)}{d\tau}\right|_{\tau=\tau^j} = \frac{\alpha_{12}\alpha_{21}(-p\omega_0 \sin\omega_0\tau^j - 2\omega_0^2\cos\omega_0\tau^j)}{p^2 + 4\omega_0^2}.$$

Under the condition (H2), by the equation (7), we have

$$\left.\frac{d\mathrm{Re}\lambda(\tau)}{d\tau}\right|_{\tau=\tau^j} = \frac{\omega_0^2(p^2 - 2q + 2\omega_0^2)}{p^2 + 4\omega_0^2} > 0.$$

As the multiplicities of roots with positive real parts of Eq. (4) can change only if a root appears on or crosses the imaginary axis as time delay τ varies, similar to the proof of the lemma of Wei and Ruan [6], by Lemma 2.3, we have the following result.

Lemma 2.4 *If $\tau \in (\tau^j, \tau^{j+1})$, then Eq.* (4) *has $2(j+1)(j = 0, 1, 2, \cdots\cdots)$ roots with positive real part.*

By Lemmas 2.1-2.4, we have the following result on stability and bifurcation for system (2).

Theorem 2.5 *Assume that* (H1) *and* (H2) *hold.*

(i) *If $\tau \in [0, \tau^0)$, then the zero solution of system* (2) *is asymptotically stable.*

(ii) *If $\tau > \tau^0$, then the zero solution of system* (2) *is unstable.*

(iii) $\tau = \tau^j (j = 0, 1, 2, \cdots)$ *are Hopf bifurcation values for system* (2).

3 A numerical example

In this section, we give some numerical simulations to illustrate our results. As an example, we consider system (1) with $f(\cdot) = \tanh(\cdot)$, $a_{11} = -0.5$, $a_{12} = -1.8$, $a_{21} = 1.3$, $a_{22} = 1.7$, then (1) becomes the following system

$$\begin{cases} \dot{u}_1(t) = -u_1(t) - 0.5 \tanh u_1(t) - 1.8 \tanh(u_2(t-\tau_1)), \\ \dot{u}_2(t) = -u_2(t) + 1.3 \tanh(u_1(t-\tau_2)) + 1.7 \tanh u_2(t). \end{cases} \tag{13}$$

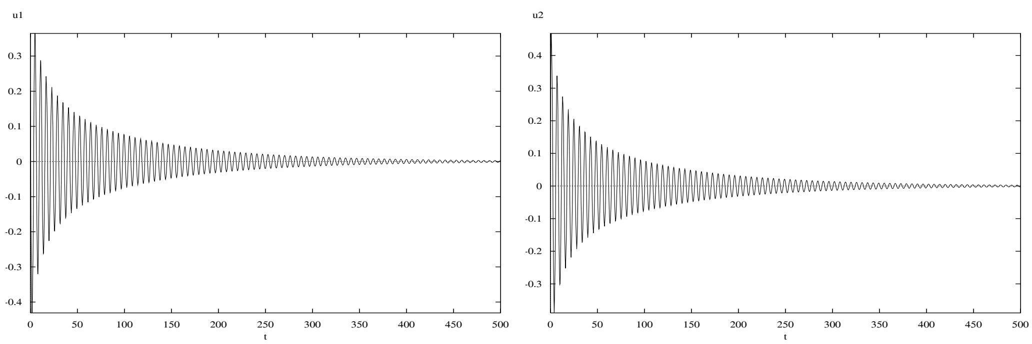

Fig. 1. The trajectory graph of (13) with $\tau_1 = 0.03$, $\tau_2 = 0.08$ and $u_1(t) = u_2(t) = 0.2, t \in [-0.11, 0]$.

By directly calculating, we may verify that hypotheses (H1) and (H2) hold, and $\tau_0 = 0.12$. Thus from Theorem 2.5 we know that the zero solution of system (13) is asymptotically stable when $0 < \tau < \tau_0 = 0.12$, (see Fig.1-Fig.2). The system (13) also undergoes a Hopf bifurcation at the origin $(0, 0)$ when τ crosses through increasingly the critical value $\tau_0 = 0.12$ (see Fig.3-Fig.5).

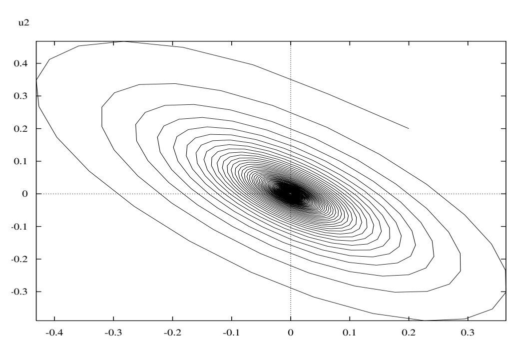

Fig. 2. The phase graph of (13) with $\tau_1 = 0.03$, $\tau_2 = 0.08$ and $u_1(t) = u_2(t) = 0.2, t \in [-0.11, 0]$.

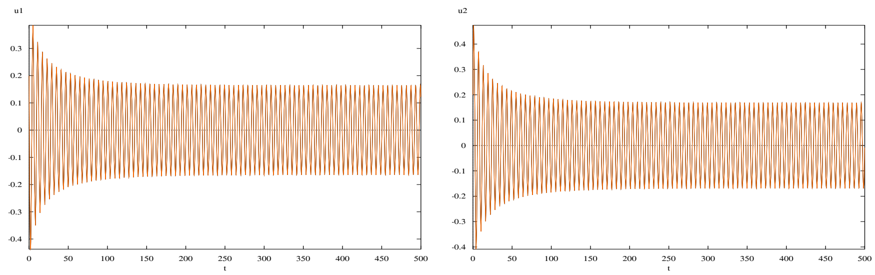

Fig. 3. The trajectory graph of (13) with $\tau_1 = 0.04,\ \tau_2 = 0.09$ and $u_1(t) = u_2(t) = 0.2, t \in [-0.13, 0]$.

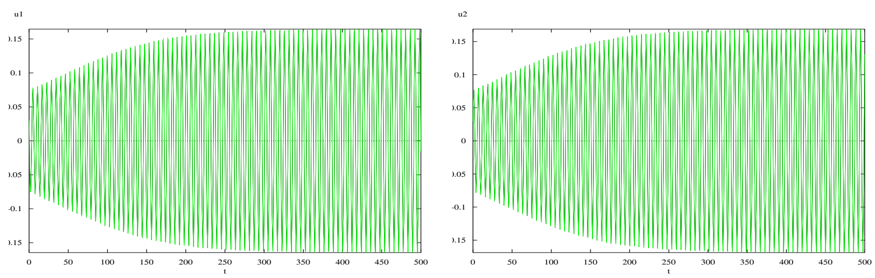

Fig. 4. The trajectory graph of (13) with $\tau_1 = 0.04,\ \tau_2 = 0.09$ and $u_1(t) = u_2(t) = 0.03, t \in [-0.13, 0]$.

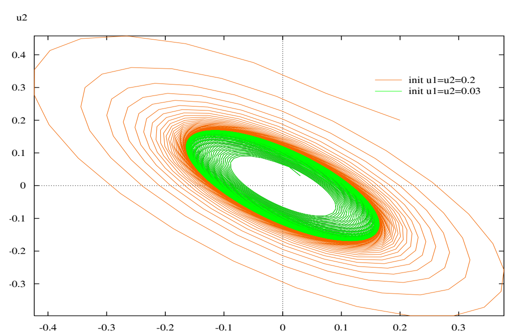

Fig. 5. The phase graph of system (13) with $\tau_1 = 0.04,\ \tau_2 = 0.09$.

4 Conclusions

In present paper, we have already obtained that, under certain conditions, the system (1) can undergo a Hopf bifurcation at the zero equilibrium when $\tau_1 + \tau_2$ takes some critical values $\tau^j (j = 0, 1, 2, \dots)$. The dynamics of systems similar to (1) have been investigated extensively and many interesting results have been obtained (e.g. [3, 5, 7, 8, 9]). Differ from these papers mentioned, our result in this paper is general since we do not limit the values of τ_1 and τ_2. In fact, for the system (1), the change of the values of τ_1 and τ_2 will not affect its topological structure. For example, the phase graph of (13) with $\tau_1 = 0.01,\ \tau_2 = 0.10$ and $\tau_1 = 0.07,\ \tau_2 = 0.06$ are same as Fig. 2 and Fig. 5 respectively.

ACKNOWLEDGEMENTS. This work was supported by the National Natural Science Foundation of China (11026212), and the Foundation of Nanjing University of Information and Technology.

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Received: December, 2010