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STABILITY AND COMPACTNESS FOR COMPLETE f-MINIMAL SURFACES

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Abstract. Let $(M, \overline{g}, e^{-f}d\mu)$ be a complete metric measure space with Bakry-Émery Ricci curvature bounded below by a positive constant. We prove that in M there is no complete two-sided L_f-stable immersed f-minimal hypersurface with finite weighted volume. Further, if M is a 3-manifold, we prove a smooth compactness theorem for the space of complete embedded f-minimal surfaces in M with the uniform upper bounds of genus and weighted volume, which generalizes the compactness theorem for complete self-shrinkers in $\mathbb{R}^3$ by Colding-Minicozzi.

1. Introduction

Recall that a self-shrinker (for mean curvature flow in $\mathbb{R}^{n+1}$) is a hypersurface Σ immersed in the Euclidean space $(\mathbb{R}^{n+1}, g_{can})$ satisfying that

$$H = \frac{1}{2}\langle x, \nu\rangle,$$

where x is the position vector in $\mathbb{R}^{n+1}$, ν is the unit normal at x, and H is the mean curvature of Σ at x. Self-shrinkers play an important role in the study of singularity of mean curvature flow and have been studied by many people in recent years. We refer to [4], [5] and the references therein. In particular, Colding-Minicozzi [4] proved the following compactness theorem for self-shrinkers in $\mathbb{R}^3$.

Theorem 1 ([4]). *Given an integer $g \geq 0$ and a constant $V > 0$, the space $\mathcal{S}(g, V)$ of smooth complete embedded self-shrinkers $\Sigma \subset \mathbb{R}^3$ with*

- *genus at most g,*
- $\partial\Sigma = \emptyset$,
- *Area$(B_R(x_0) \cap \Sigma) \leq VR^2$ for all $x_0 \in \mathbb{R}^3$ and $R > 0$*

is compact. Namely, any sequence of these has a subsequence that converges in the topology of C^m convergence on compact subsets for any $m \geq 2$.

In this paper, we extend Theorem 1 to the space of complete embedded f-minimal surfaces in a 3-manifold. A hypersurface Σ immersed in a Riemannian manifold $(M, \overline{g})$ is called an f-minimal hypersurface if its mean curvature H satisfies that, for any $p \in \Sigma$,

$$H = \langle \overline{\nabla} f, \nu\rangle,$$

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where f is a smooth function defined on M and $\overline{\nabla} f$ denotes the gradient of f on M. Here are some examples of f-minimal hypersurfaces:

- $f \equiv C$, an f-minimal hypersurface is just a minimal hypersurface.
- self-shrinker Σ in $\mathbb{R}^{n+1}$. $f = \frac{|x|^2}{4}$.
- Let $(M, \overline{g}, f)$ be a shrinking gradient Ricci solitons; i.e. after a normalization, $(M, \overline{g}, f)$ satisfies the equation $\overline{\mathrm{Ric}} + \overline{\nabla}^2 f = \frac{1}{2}\overline{g}$ or equivalently the Bakry-Émery Ricci curvature $\overline{\mathrm{Ric}}_f := \overline{\mathrm{Ric}} + \overline{\nabla}^2 f = \frac{1}{2}$. We may consider f-minimal hypersurfaces in $(M, \overline{g}, f)$. In particular, the previous example: a self-shrinker Σ in $\mathbb{R}^{n+1}$ is f-minimal in Gauss shrinking soliton $(\mathbb{R}^{n+1}, g_{can}, \frac{|x|^2}{4})$.
- $M = \mathbb{H}^{n+1}(-1)$, the hyperbolic space. Let r denote the distance function from a fixed point $p \in M$ and $f(x) = nar^2(x)$, where $a > 0$ is a constant. Now $\overline{\mathrm{Ric}}_f \geq n(2a-1)$. The geodesic sphere of radius r centered at p is an f-minimal hypersurface if the radius r satisfies $2ar = \coth r$.

An f-minimal hypersurface Σ can be viewed in two ways. One is that Σ is f-minimal if and only if Σ is a critical point of the weighted volume functional $e^{-f}d\sigma$, where $d\sigma$ is the volume element of Σ. The other one is that Σ is f-minimal if and only if Σ is minimal in the new conformal metric $\tilde{g} = e^{-\frac{2f}{n}}\overline{g}$ (see Section 2 and Appendix). f-minimal hypersurfaces have been studied before as even more general stationary hypersurfaces for parametric elliptic functionals; see for instance the work of White [14] and Colding-Minicozzi [7].

We prove the following compactness result:

Theorem 2. *Let $(M^3, \overline{g}, e^{-f}d\mu)$ be a complete smooth metric measure space and $\overline{Ric}_f \geq k$, where k is a positive constant. Given an integer $g \geq 0$ and a constant $V > 0$, the space $S_{g,V}$ of smooth complete embedded f-minimal surfaces $\Sigma \subset M$ with*

- *genus at most g,*
- *$\partial\Sigma = \emptyset$,*
- *$\int_\Sigma e^{-f}d\sigma \leq V$*

is compact in the C^m topology, for any $m \geq 2$. Namely, any sequence of $S_{g,V}$ has a subsequence that converges in the C^m topology on compact subsets to a surface in $S_{g,V}$, for any $m \geq 2$.

Since the existence of the uniform scale-invariant area bound is equivalent to the existence of the uniform bound of the weighted area for self-shrinkers (see Remark 1 in Section 5), Theorem 2 implies Theorem 1. Also, in [2], we will apply Theorem 2 to obtain a compactness theorem for the space of closed embedded f-minimal surfaces with the upper bounds of genus and diameter.

To prove Theorem 2, we need to prove a nonexistence result on L_f-stable f-minimal hypersurfaces, which is of independent interest.

Theorem 3. *Let $(M^{n+1}, \overline{g}, e^{-f}d\mu)$ be a complete smooth metric measure space with $\overline{Ric}_f \geq k$, where k is positive constant. Then there is no complete two-sided L_f-stable f-minimal hypersurface Σ immersed in $(M, \overline{g})$ without boundary and with finite weighted volume (i.e. $\int_\Sigma e^{-f}d\sigma < \infty$), where $d\sigma$ denotes the volume element on Σ determined by the induced metric from $(M, \overline{g})$.*

Here we explain briefly the meaning of L_f stability. For an f-minimal hypersurface Σ, the L_f operator is

$$L_f = \Delta_f + |A|^2 + \overline{Ric}_f(\nu, \nu),$$

where $\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle$ is the weighted Laplacian on Σ. In particular, for self-shrinkers, it is the so-called L operator:

$$L = \Delta + |A|^2 - \frac{1}{2}\langle x, \nabla \cdot \rangle + \frac{1}{2}.$$

L_f-stability of Σ means that its weighted volume $\int_\Sigma e^{-f} d\sigma$ is locally minimal; that is, the second variation of its weighted volume is nonnegative for any compactly supported normal variation. We leave more details about the definition of L_f-stability and some of its properties to Section 2 and the Appendix.

For self-shrinkers in $\mathbb{R}^{n+1}$, Colding-Minicozzi [6] proved that

Theorem 4 ([6]). *There are no L-stable smooth complete self-shrinkers without boundary and with polynomial volume growth in $\mathbb{R}^{n+1}$.*

Since the first and third authors [3] of the present paper proved that for self-shrinkers, properness, the polynomial volume growth, and finite weighted volume are equivalent, Theorem 3 implies Theorem 4.

In this paper, we also discuss the relationship among the properness, polynomial volume growth and finite weighted volume of f-minimal submanifolds (Propositions 3, 4 and 5). We obtain their equivalence when the ambient space $(M, \overline{g}, f)$ is a shrinking gradient Ricci solitons, i.e. $\overline{\text{Ric}} + \overline{\nabla}^2 f = \frac{1}{2}\overline{g}$, with the condition that f is a convex function with $|\overline{\nabla} f|^2 \leq f$ (Corollary 1).

The rest of this paper is organized as follows: In Section 2 some definitions, notation and facts are given as preliminaries. In Section 3 we prove Propositions 3, 4 and 5. In Section 4 we prove Theorem 3. In Section 5 we prove Theorem 2. In the Appendix we calculate the second variation of the volume functional of f-minimal submanifolds and discuss some properties of L_f-stability for f-minimal submanifolds.

2. Preliminaries

In general, a smooth metric measure space, denoted by $(M^m, \overline{g}, e^{-f} d\mu)$, is an m-dimensional Riemannian manifold $(M^m, \overline{g})$ together with a weighted volume form $e^{-f} d\mu$ on M, where f is a smooth function on M and $d\mu$ is the volume element induced by the metric $\overline{g}$. In this paper, unless otherwise specified, we denote by a bar all quantities on $(M, \overline{g})$, for instance by $\overline{\nabla}$ and $\overline{\text{Ric}}$, the Levi-Civita connection and the Ricci curvature tensor of $(M, \overline{g})$ respectively. For $(M, \overline{g}, e^{-f} d\mu)$, an important and natural tensor is the ∞-Bakry-Émery Ricci curvature tensor $\overline{\text{Ric}}_f$ (for simplicity, Bakry-Émery Ricci curvature), which is defined by

$$\overline{\text{Ric}}_f := \overline{\text{Ric}} + \overline{\nabla}^2 f,$$

where $\overline{\nabla}^2 f$ is the Hessian of f on M. If f is constant, $\overline{\text{Ric}}_f$ is the Ricci curvature $\overline{\text{Ric}}$ on M respectively.

A Riemannian manifold with Bakry-Émery Ricci curvature bounded below by a positive constant has some properties similar to a Riemannian manifold with Ricci curvature bounded below by a positive constant. For instance, see the work of

Wei-Wylie [13], Munteanu-Wang [11,12] and the references therein. In this paper, we will use the following proposition by Morgan [10] (see also its proof in [13]).

Proposition 1. *If a complete smooth metric measure space* $(M, \overline{g}, e^{-f}du)$ *has* $\overline{Ric}_f \geq k$*, where* k *is a positive constant, then* M *has finite weighted volume (i.e.* $\int_M e^{-f}d\mu < \infty$*) and finite fundamental group.*

Now, let $i : \Sigma^n \to M^m, n < m$, be an n-dimensional smooth immersion. Then $i : (\Sigma^n; i^*\overline{g}) \to (M^m, \overline{g})$ is an isometric immersion with the induced metric $i^*\overline{g}$. For simplicity, we still denote $i^*\overline{g}$ by $\overline{g}$ whenever there is no confusion. We will denote for instance by ∇, Ric, Δ and $d\sigma$, the Levi-Civita connection, the Ricci curvature tensor, the Laplacian, and the volume element of $(\Sigma, \overline{g})$ respectively.

The function f induces a weighted measure $e^{-f}d\sigma$ on Σ. Thus we have an induced smooth metric measure space $(\Sigma^n, \overline{g}, e^{-f}d\sigma)$.

The associated weighted Laplacian Δ_f on $(\Sigma, \overline{g})$ is defined by

$$\Delta_f u := \Delta u - \langle \nabla f, \nabla u \rangle.$$

The second order operator Δ_f is a self-adjoint operator on the space of square integrable functions on Σ with respect to the measure $e^{-f}d\sigma$ (however the Laplacian operator in general does not have this property).

The second fundamental form A of $(\Sigma, \overline{g})$ is defined by

$$A(X, Y) = (\overline{\nabla}_X Y)^{\perp}, \quad X, Y \in T_p\Sigma, p \in \Sigma,$$

where $\perp$ denotes the projection to the normal bundle of Σ. The mean curvature vector $\mathbf{H}$ of Σ is defined by $\mathbf{H} = \mathrm{tr}A = \sum_{i=1}^{n} (\overline{\nabla}_{e_i} e_i)^{\perp}$.

Definition 1. The weighted mean curvature vector of Σ with respect to the metric $\overline{g}$ is defined by

$$\mathbf{H}_f = \mathbf{H} + (\overline{\nabla} f)^{\perp}. \tag{1}$$

The immersed submanifold $(\Sigma, \overline{g})$ is called f-minimal if its weighted mean curvature vector $\mathbf{H}_f$ vanishes identically, or equivalently if its mean curvature vector satisfies

$$\mathbf{H} = -(\overline{\nabla} f)^{\perp}. \tag{2}$$

Definition 2. The weighted volume of $(\Sigma, \overline{g})$ is defined by

$$V_f(\Sigma) := \int_{\Sigma} e^{-f} d\sigma. \tag{3}$$

It is well known that Σ is f-minimal if and only if Σ is a critical point of the weighted volume functional. Namely, it holds that

Proposition 2. *If* T *is a compactly supported variational vector field on* Σ*, then the first variation formula of the weighted volume of* $(\Sigma, \overline{g})$ *is given by*

$$\frac{d}{dt} V_f(\Sigma_t)\Big|_{t=0} = -\int_{\Sigma} \langle T^{\perp}, \mathbf{H}_f \rangle_{\overline{g}} e^{-f} d\sigma. \tag{4}$$

On the other hand, an f-minimal submanifold can be viewed as a minimal submanifold under a conformal metric. Precisely, define the new metric $\tilde{g} = e^{-\frac{2}{n}f}\overline{g}$ on M, which is conformal to $\overline{g}$. Then the immersion $i : \Sigma \to M$ induces a metric $i^*\tilde{g}$

on Σ from $(M, \tilde{g})$. In the following, $i^*\tilde{g}$ is still denoted by $\tilde{g}$ for simplicity. The volume of $(\Sigma, \tilde{g})$ is

$$\tilde{V}(\Sigma) := \int_\Sigma d\tilde{\sigma} = \int_\Sigma e^{-f} d\sigma = \mathrm{V}_f(\Sigma). \tag{5}$$

Hence Proposition 2 and (5) imply that

$$\int_\Sigma \langle T^\perp, \tilde{\mathbf{H}} \rangle_{\tilde{g}} d\tilde{\sigma} = \int_\Sigma \langle T^\perp, \mathbf{H}_f \rangle_{\overline{g}} e^{-f} d\sigma, \tag{6}$$

where $d\tilde{\sigma} = e^{-f} d\sigma$ and $\tilde{\mathbf{H}}$ denote the volume element and the mean curvature vector of Σ with respect to the conformal metric $\tilde{g}$ respectively.

Identity (6) implies that $\tilde{\mathbf{H}} = e^{\frac{2f}{n}} \mathbf{H}_f$. Therefore $(\Sigma, \overline{g})$ is f-minimal in $(M, \overline{g})$ if and only if $(\Sigma, \tilde{g})$ is minimal in $(M, \tilde{g})$.

Now suppose that Σ^n is a hypersurface immersed in M^{n+1}. Let $p \in \Sigma$ and ν be a unit normal at p. The second fundamental form A and the mean curvature H of $(\Sigma, \overline{g})$ are as follows:

$$A : T_p\Sigma \to T_p\Sigma, A(X) = \overline{\nabla}_X \nu, X \in T_p\Sigma,$$

$$H = \mathrm{tr} A = -\sum_{i=1}^n \langle \overline{\nabla}_{e_i} e_i, \nu \rangle.$$

Hence the mean curvature vector $\mathbf{H}$ of $(\Sigma, \overline{g})$ satisfies $\mathbf{H} = -H\nu$. Define the weighted mean curvature H_f of $(\Sigma, \overline{g})$ by $\mathbf{H}_f := -H_f \nu$. Then

$$H_f = H - \langle \overline{\nabla} f, \nu \rangle.$$

Definition 3. A hypersurface Σ immersed in $(M^{n+1}, \overline{g}, e^{-f} d\mu)$ with the induced metric $\overline{g}$ is called an f-minimal hypersurface if it satisfies

$$H = \langle \overline{\nabla} f, \nu \rangle. \tag{7}$$

For a hypersurface $(\Sigma, \overline{g})$, the L_f operator is defined by

$$L_f := \Delta_f + |A|^2 + \overline{\mathrm{Ric}}_f(\nu, \nu),$$

where $|A|^2$ denotes the square of the norm of the second fundamental form A of Σ.

The L_f-stability of Σ is defined as follows:

Definition 4. A two-sided f-minimal hypersurface Σ is said to be L_f-stable if for any compactly supported smooth function $\varphi \in C_o^\infty(\Sigma)$, it holds that

$$-\int_\Sigma \varphi L_f \varphi e^{-f} d\sigma = \int_\Sigma \left[|\nabla \varphi|^2 - \left(|A|^2 + \overline{\mathrm{Ric}}_f(\nu, \nu) \right) \varphi^2 \right] e^{-f} d\sigma \geq 0. \tag{8}$$

It is known that an f-minimal hypersurface $(\Sigma, \overline{g})$ is L_f-stable if and only if $(\Sigma, \tilde{g})$ is stable as a minimal surface with respect to the conformal metric $\tilde{g} = e^{-f}\overline{g}$. See more details in the Appendix of this paper.

In this paper, for closed hypersurfaces, we choose ν to be the outer unit normal.

3. Properness, polynomial volume growth and finite weighted volume of f-minimal hypersurfaces

In [3], the first and third authors of the present paper proved that the finite weighted volume of a self-shrinker Σ^n immersed in $\mathbb{R}^m$ implies it is properly immersed. In [9], Ding-Xin proved that a properly immersed self-shrinker must have the Euclidean volume growth. Combining these two results, it was proved in [3] that for immersed self-shrinkers, properness, polynomial volume growth and finite weighted volume are equivalent.

In this section we study the relationship among the properness, polynomial volume growth and finite weighted volume of f-minimal submaifolds, some of which will be used later in this paper.

Let Σ be an n-dimensional submanifold in a complete manifold $M^m, n < m$. Σ is said to have polynomial volume growth if, for a $p \in M$ fixed, there exist constants C and d so that for all $r \geq 1$,

$$\mathrm{Vol}(B_r^M(p) \cap \Sigma) \leq Cr^d, \tag{9}$$

where $B_r^M(p)$ is the extrinsic ball of radius r centered at p and $\mathrm{Vol}(B_r^M(p))$ denotes the volume of $B_r^M(p) \cap \Sigma$. When $d = n$ in (9), Σ is said to be of Euclidean volume growth.

Before proving the following Proposition 3, we recall an estimate implied by the Hessian comparison theorem (cf., for instance, [6], Lemma 7.1).

Lemma 1. *Let $(M, \overline{g})$ be a complete Riemannian manifold with bounded geometry, that is, M has sectional curvature bounded by k ($|K_M| \leq k$), and injectivity radius bounded below by $i_0 > 0$. Then the distance function $r(x)$ satisfies*

$$\left|\overline{\nabla}^2 r(V,V) - \frac{1}{r}\left|V - \langle V, \overline{\nabla} r\rangle \overline{\nabla} r\right|^2\right| \leq \sqrt{k},$$

for $r < \min\{i_0, \frac{1}{\sqrt{k}}\}$ and any unit vector $V \in T_xM$.

Using this estimate we will prove

Proposition 3. *Let Σ^n be a complete noncompact f-minimal submanifold immersed in a complete Riemannian manifold M^m. If Σ has finite weighted volume, then Σ is properly immersed.*

Proof. We argue by contradiction. Since the argument is local, we may assume that (M, g) has bounded geometry. Suppose that Σ is not properly immersed. Then there exist a number $2R < \min\{i_0, \frac{1}{\sqrt{k}}\}$ and $o \in M$ so that $\overline{B}_R^M(o) \cap \Sigma$ is not compact in Σ, where $\overline{B}_R^M(o)$ denotes the closure of the (open) ball $B_R^M(o)$ in M of radius R centered at o. Then for any $a > 0$, there is a sequence $\{p_k\}$ of points in $B_R^M(o) \cap \Sigma$ with $\mathrm{dist}_\Sigma(p_k, p_j) \geq a > 0$ for any $k \neq j$. So $B_{\frac{a}{2}}^\Sigma(p_k) \cap B_{\frac{a}{2}}^\Sigma(p_j) = \emptyset$ for any $k \neq j$, where $B_{\frac{a}{2}}^\Sigma(p_k)$ and $B_{\frac{a}{2}}^\Sigma(p_j)$ denote the intrinsic balls in Σ of the radius $\frac{a}{2}$, centered at p_k and p_j respectively. Choose $a < 2R$. Then $B_{\frac{a}{2}}^\Sigma(p_j) \subset B_{2R}^M(o)$. If

$p \in B^{\Sigma}_{\frac{a}{2}}(p_j)$, the extrinsic distance function $r_j(p) = \mathrm{dist}_M(p, p_j)$ from p_j satisfies

$$\begin{aligned}\Delta r_j &= \sum_{i=1}^{n} \overline{\nabla}^2 r_j(e_i, e_i) + \langle \mathbf{H}, \overline{\nabla} r_j \rangle \\ &\geq \frac{n}{r_j} - \frac{1}{r_j}|\nabla r_j|^2 - n\sqrt{k} - \langle \overline{\nabla} f^{\perp}, \overline{\nabla} r_j \rangle \\ &\geq \frac{n}{r_j} - \frac{1}{r_j}|\nabla r_j|^2 - c,\end{aligned}$$

where $c = n\sqrt{k} + \sup_{B^M_{2R}(0)} |\overline{\nabla} f|$. Lemma 1 is used above. Hence

$$\Delta r_j^2 \geq 2n - 2cr_j.$$

Choosing $a \leq \min\{\frac{n}{2c}, 2R\}$, we have for $0 < \mu \leq \frac{a}{2}$,

$$\begin{aligned}\int_{B^{\Sigma}_{\mu}(p_j)} (2n - 2cr_j) d\sigma &\leq \int_{B^{\Sigma}_{\mu}(p_j)} \Delta r_j^2 d\sigma \\ &= \int_{\partial B^{\Sigma}_{\mu}(p_j)} \langle \nabla r_j^2, \nu \rangle d\sigma \\ &\leq 2\mu A(\mu),\end{aligned} \tag{10}$$

where ν denotes the outward unit normal vector of $\partial B^{\Sigma}_{\mu}(p_j)$ and $A(\mu)$ denotes the area of $\partial B^{\Sigma}_{\mu}(p_j)$. Using the co-area formula in (10), we have

$$\int_0^{\mu} (n - cs)A(s) ds \leq \int_0^{\mu} \int_{d_{\Sigma}(p, p_j) = s} (n - cr_j) d\sigma \leq \mu A(\mu). \tag{11}$$

This implies

$$(n - c\mu)V(\mu) \leq V'(\mu),$$

where $V(\mu)$ denotes the volume of $B^{\Sigma}_{\mu}(p_j)$. So

$$\frac{V'(\mu)}{V(\mu)} \geq \frac{n}{\mu} - c. \tag{12}$$

Integrating (12) from $\varepsilon > 0$ to μ, we have

$$\frac{V(\mu)}{V(\varepsilon)} \geq (\frac{\mu}{\varepsilon})^n e^{-c(\mu - \varepsilon)}.$$

Since $\lim_{s \to 0^+} \dfrac{V(s)}{s^n} = \omega_n$,

$$V(\mu) \geq \omega_n \mu^n e^{-c\mu}. \tag{13}$$

Thus we conclude

$$\int_{\Sigma} e^{-f} d\sigma \geq \sum_{j=1}^{\infty} \int_{B^{\Sigma}_{\frac{a}{2}}(p_j)} e^{-f} d\sigma \geq \inf_{B^M_{2R}(o)} (e^{-f}) \sum_{j=1}^{\infty} \int_{B^{\Sigma}_{\frac{a}{2}}(p_j)} d\sigma = \infty.$$

This contradicts the assumption of the finite weighted volume of Σ. □

Proposition 4. *Let $(M^m, \overline{g}, e^{-f} d\mu)$ be a complete smooth metric measure space with $\overline{Ric}_f = k$, where k is a positive constant. Assume that f is a convex function. If Σ^n is a complete noncompact properly immersed f-minimal submanifold in M, then Σ has finite weighted volume and Euclidean (hence polynomial) volume growth.*

Proof. Since $(M, \overline{g}, f)$ is a gradient shrinking Ricci soliton, it is well known that, by a scaling of the metric $\overline{g}$ and a translating of f, still denoted by $\overline{g}$ and f respectively, we may normalize the metric so that $k = \frac{1}{2}$ and the following identities hold:

$$\begin{aligned} \overline{R} + |\overline{\nabla} f|^2 - f = 0, \\ \overline{R} + \overline{\Delta} f = \frac{m}{2}, \\ \overline{R} \geq 0. \end{aligned}$$

From these equations, we have that

$$\overline{\Delta} f - |\overline{\nabla} f|^2 + f = \frac{m}{2} \qquad \text{and} \qquad |\overline{\nabla} f|^2 \leq f.$$

It was proved by Cao and the third author [1] that there is a positive constant c so that

$$\frac{1}{4}(r(x) - c)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c)^2 \tag{14}$$

for any $x \in M$ with $r(x) = \mathrm{dist}_M(p, x) \geq r_0$, where p is a fixed point in M and c, r_0 are positive constants that depend only on m and $f(p)$.

By (14), we know that f is a proper function on M. Since Σ is properly immersed in M and f is proper in M, $f|_\Sigma$ is also a proper smooth function on Σ. Note that with the scaling metric and translating f, Σ is still f-minimal. Hence

$$\begin{aligned} \Delta f - |\nabla f|^2 + f &= (\overline{\Delta} f - \sum_{\alpha=n+1}^{m} f_{\alpha\alpha} - |\overline{\nabla} f^{\perp}|^2) - |\overline{\nabla} f^{\top}|^2 + f \\ &= \overline{\Delta} f - |\overline{\nabla} f|^2 + f - \sum_{\alpha=n+1}^{m} f_{\alpha\alpha} \\ &\leq \frac{m}{2}. \end{aligned}$$

Also we have

$$|\nabla f|^2 = |\overline{\nabla} f^{\top}|^2 \leq |\overline{\nabla} f|^2 \leq f.$$

By Theorem 1.1 of [3], Σ has finite weighted volume and the Euclidean volume growth of the sub-level set of f with respect to the scaling metric and the translating f, and hence with respect to the original metric and f. Moreover, by the estimate (14), we have that Σ has the Euclidean volume growth. □

Next we prove the following:

Proposition 5. *Let $(M^m, \overline{g}, e^{-f} d\mu)$ be a complete smooth metric measure space with $\overline{Ric}_f \geq k$, where k is a positive constant. Assume that $|\overline{\nabla} f|^2 \leq 2kf$. If Σ^n is a complete submanifold (not necessarily f-minimal) with polynomial volume growth, then Σ has finite weighted volume.*

Proof. By a scaling of the metric, we may assume that $k = \frac{1}{2}$. The proof follows from an estimate of f. Munteanu-Wang [11] extended the estimate (14) to $(M^m, \overline{g}, e^{-f} d\mu)$ with $\overline{\mathrm{Ric}}_f \geq \frac{1}{2}$ and $|\overline{\nabla} f|^2 \leq f$. Combining the assumption that Σ

has polynomial volume growth with the estimate (14), we have

$$\begin{aligned}\int_\Sigma e^{-f} d\sigma &= \int_{\Sigma\cap B^M_{r_0}(p)} e^{-f} d\sigma + \sum_{i=0}^{\infty} \int_{\Sigma\cap(B^M_{r_0+i+1}(p)\setminus B^M_{r_0+i}(p))} e^{-f} d\sigma \\ &\le C_1 \mathrm{Vol}(\Sigma\cap B^M_{r_0}(p)) + C\sum_{i=0}^{\infty} e^{-\frac{1}{4}(r_0+i-c)^2} \mathrm{Vol}(\Sigma\cap B^M_{r_0+i+1}(p)) \\ &\le C\big[r_0^d + \sum_{i=0}^{\infty} e^{-\frac{1}{4}(r_0+i-c)^2}(r_0+i+1)^d\big] \\ &< \infty.\end{aligned}$$

□

By Propositions 3, 4 and 5, we have the following.

Corollary 1. *Let $(M^m, \overline{g}, f)$ be a complete shrinking gradient Ricci soliton with $\overline{Ric}_f = \frac{1}{2}$. Assume that f is a convex function. If Σ is a complete f-minimal submanifold immersed in M, then for Σ the properness, polynomial volume growth, and finite weighted volume are equivalent.*

4. Nonexistence of L_f stable f-minimal hypersurfaces

In this section, we prove Theorem 3, which is a key to proving the compactness theorem in Section 5.

Theorem 5 (Theorem 3)**.** *Let $(M, \overline{g}, e^{-f}d\mu)$ be a complete smooth metric measure space with $\overline{Ric}_f \ge k$, where k is a positive constant. Then there is no two-sided L_f-stable complete f-minimal hypersurface Σ immersed in (M, g) without boundary and with finite weighted volume (i.e. $\int_\Sigma e^{-f} d\sigma < \infty$).*

Proof. We argue by contradiction. Suppose that Σ is an L_f-stable complete f-minimal hypersurface immersed in (M, g) without boundary and with finite weighted volume. Recall that a two-sided hypersurface Σ is L_f-stable if the following inequality holds, that is, for any compactly supported smooth function $\varphi \in \mathcal{C}^\infty_o(\Sigma)$,

$$\int_\Sigma \Big[|\nabla\varphi|^2 - \big(|A|^2 + \overline{\mathrm{Ric}}_f(\nu,\nu)\big)\varphi^2\Big] e^{-f} d\sigma \ge 0. \tag{15}$$

Observe that any closed hypersurface cannot be L_f-stable. This is because the assumption $\overline{\mathrm{Ric}}_f \ge k > 0$ implies that (15) cannot hold for $\varphi \equiv c$ on Σ. Hence, Σ must be noncompact.

Let η be a nonnegative smooth function on $[0,\infty)$ satisfying

$$\eta(s) = \begin{cases} 1 & \text{if} \quad s\in[0,1) \\ 0 & \text{if} \quad s\in[2,\infty) \end{cases}$$

and $|\eta'| \le 2$.

Fix a point $p \in \Sigma$ and let $r(x) = \mathrm{dist}_\Sigma(p, x)$ denote the (intrinsic) distance function on Σ. Define a sequence of functions $\varphi_j(x) = \eta(\frac{r(x)}{j})$, $j \ge 1$. Then

$|\nabla\varphi_j|^2 \leq 1$ for $j \geq 2$. Substituting $\varphi_j, j \geq 2$ for φ in (15):

$$\begin{aligned}\int_\Sigma \Big[|\nabla\varphi_j|^2 - (|A|^2 + \overline{\mathrm{Ric}}_f(\nu,\nu))\varphi_j^2\Big] e^{-f} d\sigma \\ &\leq \int_\Sigma \big(|\nabla\varphi_j|^2 - k\varphi_j^2\big) e^{-f} d\sigma \\ &= \int_{B_{2j}^\Sigma(p)\setminus B_j^\Sigma(p)} |\nabla\varphi_j|^2 e^{-f} d\sigma - \int_{B_{2j}^\Sigma(p)} k\varphi_j^2 e^{-f} d\sigma \\ &\leq \int_{B_{2j}^\Sigma(p)\setminus B_j^\Sigma(p)} e^{-f} d\sigma - k\int_{B_{2j}^\Sigma(p)} \varphi_j^2 e^{-f} d\sigma \\ &\leq \int_{B_{2j}^\Sigma(p)\setminus B_j^\Sigma(p)} e^{-f} d\sigma - k\int_{B_2^\Sigma(p)} e^{-f} d\sigma,\end{aligned}$$

where $B_j^\Sigma(p)$ is the intrinsic geodesic ball in M of radius j centered at p. Since Σ has finite weighted volume, we have, when $j \to \infty$,

$$\int_{B_{2j}^\Sigma(p)\setminus B_j^\Sigma(p)} e^{-f} d\sigma \to 0.$$

Choosing j large enough, we have that φ_j satisfies

$$\int_\Sigma \big(|\nabla\varphi_j|^2 - (|A|^2 + \overline{\mathrm{Ric}}_f(\nu,\nu))\varphi_j^2\big) e^{-f} d\sigma < -\frac{k}{2}\int_{B_2^\Sigma(p)} e^{-f} d\sigma < 0.$$

This contradicts the fact that Σ is L_f-stable. □

5. Compactness of complete f-minimal surfaces

Before proving Theorem 2, we give some facts.

Wei-Wylie ([13], Theorem 7.3) used the mean curvature comparison theorem to give a distance estimate for two compact hypersurfaces Σ_1 and Σ_2 in a smooth metric measure space $(M, \overline{g}, e^{-f}d\mu)$ with $\overline{\mathrm{Ric}}_f \geq k$, where k is a positive constant. Observe that for two complete properly immersed hypersurfaces Σ_1 and Σ_2, if at least one of them is compact, there is a minimizing geodesic segment joining Σ_1 and Σ_2 and realizing their distance. Hence the proof of Theorem 7.3 [13] can be applied to obtain the following.

Proposition 6. *Let $(M, \overline{g}, e^{-f}d\mu)$ be an $(n+1)$-dimensional complete smooth metric measure space with $\overline{Ric}_f \geq k$, where k is a positive constant. If Σ_1 and Σ_2 are two complete properly immersed hypersurfaces, at least one of which is compact, then the distance $d(\Sigma_1, \Sigma_2)$ satisfies*

$$d(\Sigma_1,\Sigma_2) \leq \frac{1}{k}\big(\max_{x\in\Sigma_1}|H_f^{\Sigma_1}(x)| + \max_{x\in\Sigma_2}|H_f^{\Sigma_2}(x)|\big), \tag{16}$$

where $H_f^{\Sigma_i}, i = 1, 2$, denotes the weighted mean curvatures of Σ_i respectively.

Corollary 2. *Let $(M, \overline{g}, e^{-f}d\mu)$ be as in Proposition 6. Then there is a closed ball $\overline{B}^M$ of M satisfying that any complete properly immersed f-minimal hypersurface Σ must intersect it.*

Proof. Fix $p \in M$ and a geodesic sphere $S_r^M(p)$ of M. By Proposition 6,

$$d(S_r^M(p), \Sigma) \leq \frac{1}{k}\max_x |H_f^{S_r^M(p)}(x)| = C,$$

where C is independent of Σ. Therefore there is a closed ball $\overline{B}^M$ of M with radius big enough so that any Σ must intersect it. $\square$

We need the following fact:

Proposition 7. *Let M be a simply connected Riemannian manifold. If a hypersurface Σ is complete, not necessarily connected, properly embedded, and has no boundary, then every component of Σ separates M into two components and thus is two-sided. Therefore Σ has a globally defined unit normal.*

Proof. Suppose Σ_j is a component of Σ. By contrast, $M\backslash\Sigma_j$ has one component. Since Σ is a properly embedded f-minimal hypersurface, for any $p \in \Sigma_j$ there is a neighborhood W of p in M so that $W \cap \Sigma_j = W \cap \Sigma$ only has one piece (i.e. it is a graph above a connected domain in the tangent plane of p). Thus we have a simply closed curve γ passing p, transversal to Σ_j at p, and $\Sigma_j \cap \gamma = p$. Since M is simply connected, we have a disk D with the boundary γ. Again since Σ is proper, the intersection of Σ_j with $\partial D = \gamma$ cannot be one point, which is a contradiction. $\square$

Combining Proposition 3 in Section 3 with Proposition 7, we obtain

Proposition 8. *Let $(M, \overline{g}, e^{-f}d\mu)$ be a simply connected complete smooth metric measure space. If a complete f-minimal hypersurface has finite weighted volume, then every component of Σ separates M into two components and thus is two-sided. Therefore Σ has a globally defined unit normal.*

We will take the same approach as in Colding-Minicozzi's paper [4] to prove Theorem 2, a smooth compactness theorem for complete f-minimal surfaces. First we recall a well known local singular compactness theorem for embedded minimal surfaces in a Riemannian 3-manifold.

Proposition 9 (cf. [4], Proposition 2.1)**.** *Given a point p in a Riemannian 3-manifold M, there exists an $R > 0$ such that the following holds: Let Σ_j be embedded minimal surfaces in $B_{2R}(p) \subset M$ with $\partial\Sigma_j \subset \partial B_{2R}(p)$. If each Σ_j has area at most V and genus at most g for some fixed V, g, then there exist a finite collection of points x_k, a smooth embedded minimal surface $\Sigma \subset B_R(p)$ with $\partial\Sigma \subset \partial B_R(p)$ and a subsequence of $\{\Sigma_j\}$ that converges in $B_R(p)$ (with finite multiplicity) to Σ away from the set $\{x_k\}$.*

Here and in the following, we denote by B_R the ball B_R^M in M for simplicity.

It is known that Σ is f-minimal with respect to metric $\overline{g}$ if and only if Σ is minimal with the conformal metric $\tilde{g} = e^{-f}g$ (see Appendix). Using this fact and applying Proposition 9, we may prove a global singular compactness theorem for f-minimal surfaces.

Proposition 10. *Let M be a complete 3-manifold and $(M, \overline{g}, e^{-f}d\mu)$ a smooth metric measure space. Suppose that $\Sigma_i \subset M$ is a sequence of smooth complete embedded f-minimal surfaces with genus at most g, without boundary, and with weighted area at most V, i.e.*

$$\int_{\Sigma_i} e^{-f} d\sigma \leq V < \infty. \tag{17}$$

Then there are a subsequence, still denoted by Σ_i, a smooth embedded complete non-trivial f-minimal surface $\Sigma \subset M$ without boundary, and a locally finite collection *of points $\mathcal{S} \subset \Sigma$ so that Σ_i converges smoothly (possibly with multiplicity) to Σ off of $\mathcal{S}$. Moreover, Σ satisfies $\int_\Sigma e^{-f} d\sigma \leq V$ and is properly embedded.*

Here a set $\mathcal{S} \subset M$ is said to be locally finite if $B_R(p) \cap \mathcal{S}$ is finite for every $p \in M$ and for all $R > 0$.

Proof. Consider the conformal metric $\tilde{g} = e^{-f}\overline{g}$ on M. For a point $p \in M$, let $\tilde{B}_{2R}(p) \subset M$ denote the ball in $(M, \tilde{g})$ of radius $2R$ centered at p. Then the area of $\tilde{B}_{2R}(p) \cap \Sigma_j$ satisfies

$$\widetilde{\mathrm{Area}}(\tilde{B}_{2R}(p) \cap \Sigma_j) \leq \int_{\Sigma_j} d\tilde{\sigma} = \int_{\Sigma_j} e^{-f} d\sigma \leq V. \tag{18}$$

Also, it is clear that the genus of $\tilde{B}_{2R}(p) \cap \Sigma_j$ remains at most g. Then by Proposition 9, there exist an $R > 0$ and a finite collection of points x_k, a smooth embedded minimal surface $\Sigma \subset \tilde{B}_R(p)$, with $\partial\Sigma \subset \partial\tilde{B}_R$ and a subsequence of $\{\Sigma_j\}$ that converges in $\tilde{B}_R(p)$ (with finite multiplicity) to Σ away from the set $\{x_k\}$.

Let $\{\tilde{B}_{R_i}(p_i)\}$ be a countable cover of $(M, \tilde{g})$ of small balls such that $\{\tilde{B}_{2R_i}(p_i)\}$ is still a cover of $(M, \tilde{g})$. On each $\tilde{B}_{2R_i}(p_i)$, applying the previous local convergence and then passing to a diagonal subsequence, we obtain that there are a subsequence of Σ_i, still denoted by Σ_i, a smooth embedded minimal surface Σ (with respect to the metric $\tilde{g}$) without boundary, and a locally finite collection of points $\mathcal{S} \subset \Sigma$ so that Σ_i converges smoothly (possibly with multiplicity) to Σ off of $\mathcal{S}$. Since Σ has no boundary, it is complete in the original metric $\overline{g}$. Thus we obtain the smooth convergence of the subsequence to the smooth embedded complete f-minimal surface Σ off of $\mathcal{S}$.

By Corollary 2, Σ is nontrivial. The convergence of Σ_i to Σ and (17) imply $\int_\Sigma e^{-f} d\sigma \leq V$. By Proposition 3, Σ is properly embedded. □

We need to show that the convergence is smooth across the points in $\mathcal{S}$. To prove this, we need the following.

Proposition 11. *Assume that the ambient manifold M in Proposition* 10 *is simply connected. If the convergence of the sequence $\{\Sigma_i\}$ has multiplicity greater than one, then Σ is L_f-stable.*

Proof. By Proposition 8, we know that Σ_i and Σ are orientable. We may have two ways to prove the proposition. The first is to use the known fact on minimal surfaces. It is known that (cf. [6], Appendix A) if the multiplicity of the convergence of a sequence of embedded orientable minimal surfaces in a simply connected 3-manifold is not one, then the limit minimal surface is stable. Under the conformal metric $\tilde{g}$, a sequence $\{\Sigma_i\}$ of minimal surfaces converges to a smooth embedded orientable minimal surface Σ and thus Σ is stable. Also, the conclusion that Σ is stable with respect to the conformal metric $\tilde{g}$ is equivalent to saying that Σ is L_f-stable under the original metric $\overline{g}$ (see Appendix).

The second way is to prove it directly. We may prove that L_f is the linearization of the f-minimal equation by a proof similar to the one in [4], Appendix A. By arguing as in Proposition 3.2 in [4], we can find a smooth positive function u on Σ satisfying

$$L_f u = 0. \tag{19}$$

This implies that Σ is L_f-stable. □

Proof of Theorem 2. By the assumption on $\overline{\mathrm{Ric}}_f$ and Proposition 1, M has finite fundamental group. After passing to the universal covering, we may assume that M is simply connected. Given a sequence of smooth complete embedded f-minimal surfaces $\{\Sigma_i\}$ with genus g, $\partial\Sigma_i = \emptyset$, and the weighted area at most V, by Proposition 10 there is a subsequence, still denoted by $\{\Sigma_i\}$, that converges in the topology of smooth convergence on compact subsets to a smooth embedded complete f-minimal surface Σ away from a locally finite set $\mathcal{S} \subset \Sigma$ (possibly with multiplicity). Moreover, the limit surface $\Sigma \subset M$ is complete, properly embedded, $\int_\Sigma e^{-f} d\sigma \leq V$, has no boundary and has a well-defined unit normal ν. We also have the equivalent convergence under the conformal metric $\overline{g}$.

If $\mathcal{S}$ is not empty, Allard's regularity theorem implies that the convergence has multiplicity greater than one. Then by Proposition 11, we conclude that Σ is L_f-stable. But Proposition 5 says that there is no such Σ. This contradiction implies that $\mathcal{S}$ must be empty. We complete the proof of the theorem. □

Remark 1. For self-shrinkers, the condition that the scale-invariant uniform area bound exists (i.e. there is a uniform bound V_1: $\mathrm{Area}(B_R(x_0) \cap \Sigma) \leq V_1 R^2$ for all $x_0 \in \mathbb{R}^3$ and $R > 0$) implies that the uniform bound V of weighted area (i.e. $\int_\Sigma e^{-f} d\sigma < V$) exists (cf. the proof of Proposition 5). The converse is also true by the conclusion that the entropy of a self-shrinker can be achieved by $F_{0,1}$ for self-shrinkers with polynomial volume growth (see Section 7 of [5]). Therefore Theorem 2 generalizes the result of Colding-Minicozzi (Theorem 1) for self-shrinkers.

Remark 2. Combining Theorem 2 with the upper bound estimate of weighted area for closed embedded f-minimal surfaces of fixed genus in a complete 3-manifold with $\overline{\mathrm{Ric}}_f \geq k > 0$, we may obtain the smooth compactness theorem for the space of closed embedded f-minimal surfaces of fixed topological type and with diameter bound. We discuss it in [2].

Appendix

In this appendix, we discuss the L_f-stability properties of f-submanifolds. With the same notation as in Section 2, let $(M^m, \overline{g})$ be an m-dimensional Riemannian manifold and $i : \Sigma^n \to M^m, n < m$, be an immersion. Let $\tilde{g} = e^{-\frac{2}{n}f}\overline{g}$ denote the new conformal metric on M. Therefore i may induce two isometric immersions of Σ: $(\Sigma, \overline{g}) \to (M, \overline{g})$ and $(\Sigma, \tilde{g}) \to (M, \tilde{g})$ respectively.

When $(\Sigma, \tilde{g})$ is minimal, it is well known that the second variation of the volume of $(\Sigma, \tilde{g})$ is given by

Proposition 12 (cf. [6])**.** *Let $(\Sigma, \tilde{g})$ be a minimal submanifold in $(M, \tilde{g})$. If T is a normal compactly supported variational vector field on Σ (that is, $T = T^\perp$), then the second variational formula of the volume $\tilde{V}$ of $(\Sigma, \tilde{g})$ is given by*

$$\frac{d^2}{dt^2}\tilde{V}(\Sigma_t)\bigg|_{t=0} = -\int_\Sigma \langle T, JT\rangle_{\tilde{g}} d\tilde{\sigma}, \tag{20}$$

where the stability operator (or Jacobi operator) J is defined on a normal vector field T to Σ by

$$JT = \Delta^\perp_{(\Sigma,\tilde{g})} T + tr_{(\Sigma,\tilde{g})}[\widetilde{Rm}(\cdot, T)\cdot]^\perp + \tilde{B}(T). \tag{21}$$

Here $\Delta^{\perp}_{(\Sigma,\tilde{g})}T=\sum_{i=1}^{n}(\nabla^{\perp}_{\tilde{e}_i}\nabla^{\perp}_{\tilde{e}_i}T-\nabla^{\perp}_{\nabla_{\tilde{e}_i}\tilde{e}_i}T)$ *is the Laplacian determined by the normal connection* $\nabla^{\perp}$ *of* $(\Sigma,\tilde{g})$, $\widetilde{Rm}$ *is the curvature tensor on* $(M,\tilde{g})$, $tr_{(\Sigma,\tilde{g})}[\widetilde{Rm}(\cdot,T)\cdot]^{\perp}=\sum_{i=1}^{n}[\widetilde{Rm}(\tilde{e}_i,T)\tilde{e}_i]^{\perp}$, $\tilde{A}$ *denotes the second fundamental form of* $(\Sigma,\tilde{g})$, $\tilde{B}(T)=\sum_{i,j=1}^{n}\langle\tilde{A}(\tilde{e}_i,\tilde{e}_j),T\rangle\tilde{A}(\tilde{e}_i,\tilde{e}_j)$, *and* $\{\tilde{e}_i\}$, $i=1,\cdots,n$, *is a local orthonormal base of* $(\Sigma,\tilde{g})$.

Recall that the weighted volume of $(\Sigma,\overline{g})$ is defined by

$$V_f(\Sigma)=\int_{\Sigma}e^{-f}d\sigma. \tag{22}$$

By a direct computation similar to that of (20), we may prove the second variation formula of the weighted volume of f-minimal submanifold $(\Sigma,\overline{g})$.

Definition 5. For any normal vector field T on $(\Sigma,\overline{g})$, the second order operator $\Delta^{\perp}_f$ is defined by

$$\begin{aligned}\Delta^{\perp}_f T&:=\Delta^{\perp}T-\mathrm{tr}[\nabla f\otimes\nabla^{\perp}T(\cdot,\cdot)]\\&=\sum_{i=1}^{n}(\nabla^{\perp}_{e_i}\nabla^{\perp}_{e_i}T-\nabla^{\perp}_{\nabla_{e_i}e_i}T)-\sum_{i=1}^{n}(e_if)(\nabla^{\perp}_{e_i}T).\end{aligned}$$

The operator L_f on $(\Sigma,\overline{g})$ is defined by

$$L_fT=\Delta^{\perp}_fT+R(T)+B(T)+F(T). \tag{23}$$

In the above, $\nabla^{\perp}$ denotes the normal connection of $(\Sigma,\overline{g})$; $\{e_i\}$, $i=1,\ldots,n$, is a local orthonormal base of $(\Sigma,\overline{g})$; $B(T)=\sum_{i,j=1}^{n}\langle A(e_i,e_j),T\rangle A(e_i,e_j)$, where A denotes the second fundamental form of $(\Sigma,\overline{g})$; $R(T)=\mathrm{tr}_{(\Sigma,\overline{g})}[\overline{Rm}(\cdot,T)\cdot]^{\perp}=\sum_{i=1}^{n}[\overline{Rm}(e_i,T)e_i]^{\perp}$, where $\overline{Rm}$ denotes the Riemannian curvature tensor of $(M,\overline{g})$; and $F(T)=[\overline{\nabla}^2f(T)]^{\perp}=\sum_{\alpha=n+1}^{m}\overline{\nabla}^2f(T,e_\alpha)e_\alpha$, where $\{e_\alpha\}$, $\alpha=n+1,\cdots,m$, is a local orthonormal normal vector field on $(\Sigma,\overline{g})$.

Proposition 13. *Let* $(\Sigma,\overline{g})$ *be an* f*-minimal submanifold in* $(M,\overline{g})$. *If* T *is a normal compactly supported variational vector field on* Σ *(that is,* $T=T^{\perp}$*), then the second variation of the weighted volume of* $(\Sigma,\overline{g})$ *is given by*

$$\frac{d^2}{dt^2}V_f(\Sigma_t)\bigg|_{t=0}=-\int_{\Sigma}\langle T,L_fT\rangle_{\overline{g}}e^{-f}d\sigma. \tag{24}$$

Proof. Let $\psi(\cdot,t),t\in(-\varepsilon,\varepsilon)$ be a compactly supported variation of Σ so that $T=d\psi(\frac{\partial}{\partial t})$ is the variational vector field, $\Sigma_t=\psi(\Sigma,t),\Sigma_0=\Sigma$. Choose a normal coordinate system $\{x_1,\ldots,x_n\}$ at a point $p\in\Sigma$. We can consider $\{x_1,\ldots,x_n,t\}$ to be a coordinate system of $\Sigma\times(-\varepsilon,\varepsilon)$ near the point $(p,0)$. Denote $e_i=d\psi(\frac{\partial}{\partial x_i})$ for $i=1,\ldots,n$. The induced metric on Σ_t from $(M,\overline{g})$ is given for $g_{ij}=\langle e_i,e_j\rangle$.

Hence $g_{ij}(p,0)=\delta_{ij}$ and $\nabla_{e_i}e_j(p,0)=0$. Denote by $d\sigma_t$ the volume element of Σ_t. Then $d\sigma_t=J(x,t)d\sigma_0$, where $d\sigma_0=d\sigma$ and the function $J(x,t)$ is given by

$$J(x,t)=\frac{\sqrt{G(x,t)}}{\sqrt{G(x,0)}},$$

with $G(x,t)=\det(g_{ij}(x,t))$. Denote by $d(\sigma_f)_t$ the weighted volume element of Σ_t. Then $d(\sigma_f)_t=J_f(x,t)d\sigma_0$, where $J_f(x,t)=J(x,t)e^{-f(x,t)}$, $f(x,t)=f(\psi(x,t))$.

Since $\frac{\partial J}{\partial t}=\sum_{i,j=1}^n g^{ij}\langle\overline{\nabla}_{e_i}T,e_j\rangle J$, $\frac{\partial J_f}{\partial t}=\big(\sum_{i,j=1}^n g^{ij}\langle\overline{\nabla}_{e_i}T,e_j\rangle-\langle\overline{\nabla}f,T\rangle\big)J_f$.

Note that T is a normal vector field. A direct computation gives, at $(p,0)$,

$$\begin{aligned}\frac{\partial^2 J_f}{\partial^2 t}\bigg|_{t=0}=&\Big[-2\sum_{i,j=1}^n\langle A_{ij},T\rangle^2+\langle\overline{R}(e_i,T)T,e_i\rangle\\&+\sum_{i=1}^n\langle\overline{\nabla}_{e_i}\overline{\nabla}_T T,e_i\rangle+\sum_{i=1}^n\langle\overline{\nabla}_{e_i}T,\overline{\nabla}_{e_i}T\rangle\\&-\overline{\nabla}^2 f(T,T)-\langle\overline{\nabla}f,\overline{\nabla}_T T\rangle\\&+\big(\sum_{i=1}^n\langle\overline{\nabla}_{e_i}T,e_i\rangle-\langle\overline{\nabla}f,T\rangle\big)\big(\sum_{j=1}^n\langle\overline{\nabla}_{e_j}T,e_j\rangle-\langle\overline{\nabla}f,T\rangle\big)\Big]J_f.\end{aligned}$$

By

$$\begin{aligned}\sum_{i=1}^n\langle\overline{\nabla}_{e_i}T,\overline{\nabla}_{e_i}T\rangle&=\sum_{i,j=1}^n\langle\overline{\nabla}_{e_i}T,e_j\rangle^2+\sum_{i=1}^n\sum_{\alpha=n+1}^m\langle\overline{\nabla}_{e_i}T,e_\alpha\rangle^2\\&=\sum_{i,j=1}^n\langle A_{ij},T\rangle^2+\sum_{i=1}^n\langle\nabla^\perp_{e_i}T,\nabla^\perp_{e_i}T\rangle\\&=|\langle A(\cdot,\cdot),T\rangle|^2+|\nabla^\perp T|^2\end{aligned}$$

and $\sum_{i=1}^n\langle\overline{\nabla}_{e_i}\overline{\nabla}_T T,e_i\rangle=\operatorname{div}(\overline{\nabla}_T T)^\top-\langle(\overline{\nabla}_T T)^\perp,\mathbf{H}\rangle$ we have that, at p,

$$\begin{aligned}\frac{\partial^2 J_f}{\partial t^2}\bigg|_{t=0}=&\Big[-|\langle A(\cdot,\cdot),T\rangle|^2-\sum_{i=1}^n\langle\overline{R}(e_i,T)e_i,T\rangle+|\nabla^\perp T|^2+\operatorname{div}(\overline{\nabla}_T T)^\top\\&-\langle(\overline{\nabla}_T T)^\perp,\vec{H}\rangle-\overline{\nabla}^2 f(T,T)-\langle\overline{\nabla}f,\overline{\nabla}_T T\rangle+\langle T,\mathbf{H}_f\rangle^2\Big]e^{-f}.\end{aligned}$$

Using $\operatorname{div}\big(e^{-f}(\overline{\nabla}_T T)^\top\big)=e^{-f}\operatorname{div}(\overline{\nabla}_T T)^\top-e^{-f}\langle(\overline{\nabla}_T T)^\top,\nabla f\rangle$, we have at p:

$$\begin{aligned}\frac{\partial^2 J_f}{\partial t^2}\bigg|_{t=0}=&\Big[|\nabla^\perp T|^2-|\langle A(\cdot,\cdot),T\rangle|^2-\sum_{i=1}^n\langle\overline{R}(e_i,T)e_i,T\rangle-\overline{\nabla}^2 f(T,T)\\&-\langle(\overline{\nabla}_T T)^\perp,\mathbf{H}_f\rangle+\langle T,\mathbf{H}_f\rangle^2\Big]e^{-f}+\operatorname{div}\big(e^{-f}(\overline{\nabla}_T T)^\top\big).\end{aligned}\tag{25}$$

Observe that the right-hand side of (25) is independent of the choice of coordinates. Hence (25) holds on Σ. By integrating (25) and using the fact that Σ is f-minimal (i.e. $\mathbf{H}_f = 0$), we obtain

$$\begin{aligned}\frac{d^2}{dt^2}V_f(\Sigma_t)\Big|_{t=0} &= \int_\Sigma \big(|\nabla^\perp T|^2 - |\langle A(\cdot,\cdot),T\rangle|^2 - \langle R(T),T\rangle - \overline{\nabla}^2 f(T,T)\big)e^{-f}\,d\sigma \\ &= -\int_\Sigma \langle T, \Delta_f^\perp T + A(T) + R(T) + F(T)\rangle e^{-f}\,d\sigma \\ &= -\int_\Sigma \langle T, L_f T\rangle e^{-f}\,d\sigma.\end{aligned}$$

Substituting $e^{-f}T$ for T in the identity $\int_\Sigma |\nabla^\perp T|^2 d\sigma = -\int_\Sigma \langle T, \Delta^\perp T\rangle d\sigma$, we have

$$\int_\Sigma |\nabla^\perp T|^2 e^{-f}\,d\sigma = -\int_\Sigma \langle T, \Delta_f^\perp T\rangle e^{-f}\,d\sigma.$$

Thus we have the second variation formula of the weighted volume of Σ:

$$\begin{aligned}\frac{d^2}{dt^2}V_f(\Sigma_t)\Big|_{t=0} &= -\int_\Sigma \langle T, \Delta_f^\perp T + A(T) + R(T) + F(T)\rangle e^{-f}\,d\sigma \\ &= -\int_\Sigma \langle T, L_f T\rangle e^{-f}\,d\sigma.\end{aligned}$$

□

Definition 6. An f-minimal submanifold $(\Sigma, \overline{g})$ is called L_f-stable if the second variation of the weighted volume of Σ given by (24) is nonnegative for any normal compactly supported variational vector field T on Σ.

Observe that for an f-minimal submanifold Σ and its normal compactly supported variation, it holds that $V_f(\Sigma_t) = \tilde{V}(\Sigma_t)$. Then

$$\frac{d^2}{dt^2}\tilde{V}(\Sigma_t)\Big|_{t=0} = \frac{d^2}{dt^2}V_f(\Sigma_t)\Big|_{t=0}. \tag{26}$$

By (20), (24), and (26), we have

$$\int_\Sigma \langle T, JT\rangle_{\tilde{g}}\,d\tilde{\sigma} = \int_\Sigma \langle T, L_f T\rangle_{\overline{g}} e^{-f}\,d\sigma. \tag{27}$$

This implies that

$$\int_\Sigma e^{-\frac{2f}{n}}\langle T, JT\rangle_{\overline{g}} e^{-f}\,d\sigma = \int_\Sigma \langle T, L_f T\rangle_{\overline{g}} e^{-f}\,d\sigma. \tag{28}$$

By (28), the following equality holds.

Corollary 3. *For any normal vector field T on Σ,*

$$JT = e^{\frac{2f}{n}} L_f T.$$

The operator L_f corresponds to a symmetric bilinear form $B_f(T,T)$ for the space of normal compactly supported vector fields on Σ:

$$B_f(T,T) := -\int_\Sigma \langle T, L_f T\rangle_{\overline{g}} e^{-f}\,d\sigma. \tag{29}$$

We define the L_f-index, denoted by L_f-ind, of $(\Sigma, \overline{g})$ by the maximum of the dimensions of negative definite subspaces of B_f. Hence $(\Sigma, \overline{g})$ is L_f-stable if and only if its L_f-ind $= 0$.

On the other hand, for minimal $(\Sigma, \tilde{g})$, it is well known that the stability operator J also defines a symmetric bilinear form $\tilde{B}(T,T)$,

$$\tilde{B}(T,T) := -\int_\Sigma \langle T, JT\rangle_{\tilde{g}} d\tilde{\sigma}. \tag{30}$$

There are also the concepts of index and stability of $(\Sigma, \tilde{g})$. In particular, $(\Sigma, \tilde{g})$ is stable if and only if the index $\mathrm{ind}(\Sigma, \tilde{g}) = 0$. Since $B_f(T,T) = \tilde{B}(T,T)$, it holds that

Proposition 14. *L_f-ind of $(\Sigma, \overline{g})$ is equal to the index of $(\Sigma, \tilde{g})$. In particular, $(\Sigma, \overline{g})$ is L_f-stable if and only if $(\Sigma, \tilde{g})$ is stable in $(M, \tilde{g})$.*

Now if Σ is a two-sided hypersurface, that is, if there is a globally-defined unit normal ν on $(\Sigma, \overline{g})$, take $T = \varphi\nu$. Then the second variation (24) implies that

Proposition 15. *Let Σ be a two-sided f-minimal hypersurface in $(M^{n+1}, \overline{g})$. If φ is a compactly supported smooth function on Σ, then the second variation of the weighted volume of $(\Sigma, \overline{g})$ is given by*

$$\frac{d^2}{dt^2} V_f(\Sigma_t)\bigg|_{t=0} = -\int_\Sigma \varphi L_f(\varphi) e^{-f} d\sigma, \tag{31}$$

where ν denotes the unit normal of $(\Sigma, \overline{g})$ and the operator L_f is defined by $L_f = \Delta_f + |A|^2_g + \overline{Ric}_f(\nu, \nu)$.

Definition 7. The operator $L_f = \Delta_f + |A|^2_g + \overline{\mathrm{Ric}}_f(\nu, \nu)$ is called the L_f-stability operator of hypersurface$(\Sigma, \overline{g})$.

A bilinear form on space $C^\infty_o(\Sigma)$ of compactly supported smooth functions on Σ is defined by

$$\begin{aligned} B_f(\varphi, \varphi) :&= -\int_\Sigma \varphi L_f \varphi e^{-f} d\sigma \\ &= \int_\Sigma [|\nabla\varphi|^2 - (|A|^2_g + \overline{\mathrm{Ric}}_f(\nu,\nu))\varphi^2] e^{-f} d\sigma. \end{aligned} \tag{32}$$

The L_f-index, denoted by L_f-ind, of $(\Sigma, \overline{g})$ is defined to be the maximum of the dimensions of negative definite subspaces of B_f. Hence $(\Sigma, \overline{g})$ is L_f-stable if and only if L_f-ind $= 0$. Clearly the definition of L_f-index is equivalent to the corresponding definition using the variational vector field T as before.

Also, for minimal hypersurface $i : (\Sigma, \tilde{g}) \to (M^{n+1}, \tilde{g})$, it is well known that if ψ is a compactly supported smooth function on Σ, then the second variation of the volume $\tilde{V}$ of $(\Sigma, i^*\tilde{g})$ is given by

$$\frac{d^2}{dt^2} \tilde{V}(\Sigma_t)\bigg|_{t=0} = -\int_\Sigma \psi J(\psi) d\tilde{\sigma}, \tag{33}$$

where $\tilde{A}$ denotes the second fundamental form of $(\Sigma, \tilde{g})$, $\tilde{\nu}$ denotes the unit normal of $(\Sigma, \tilde{g})$, and $J = \triangle_{\tilde{g}} + |\tilde{A}|^2_{\tilde{g}} + \widetilde{\mathrm{Ric}}(\tilde{\nu}, \tilde{\nu})$ is the stability operator (or the Jacobi operator) of $(\Sigma, \tilde{g})$.

The following holds, from (28).

Proposition 16. *Let (Σ^n, g) be an f-minimal hypersurface immersed in $(M, \overline{g})$. Then for all $\varphi \in \mathcal{C}_o^\infty(\Sigma)$,*

$$\int_\Sigma (e^{-\frac{f}{n}}\varphi) J(e^{-\frac{f}{n}}\varphi) e^{-f} d\sigma = \int_\Sigma \varphi L_f(\varphi) e^{-f} d\sigma. \tag{34}$$

Corollary 4. *For $\varphi \in C^\infty(\Sigma)$, $J(e^{-\frac{f}{n}}\varphi) = e^{\frac{f}{n}} L_f(\varphi)$.*

Corollary 5. *L_f-ind of $(\Sigma, \overline{g})$ is equal to the index of $(\Sigma, \tilde{g})$. In particular, $(\Sigma, \overline{g})$ is L_f-stable if and only if $(\Sigma, \tilde{g})$ is stable in $(M, \tilde{g})$.*

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