STABILITY AND CONSISTENCY OF THE SEMI-IMPLICIT CO-VOLUME SCHEME FOR REGULARIZED MEAN CURVATURE FLOW EQUATION IN LEVEL SET FORMULATION

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Abstract. We show stability and consistency of the linear semi-implicit complementary volume numerical scheme for solving the regularized, in the sense of Evans and Spruck, mean curvature flow equation in the level set formulation. The numerical method is based on the finite volume methodology using so-called complementary volumes to a finite element triangulation. The scheme gives solution in efficient and unconditionaly stable way.

Key words. mean curvature flow, level set equation, numerical solution, semi-implicit scheme, complementary volume method, unconditional stability, consistency.

1. Introduction.

The curvature driven level set equation [30]

$$u_t - |\nabla u| \nabla . \left(\frac{\nabla u}{|\nabla u|} \right) = 0, \tag{1.1}$$

as well as its nontrivial generalizations, is used in the applications as the motion of interfaces (free boundaries) in thermomechanics (solidification, crystal growth) and computational fluid dynamics (free surface flows, multi-phase flows of immiscible fluids, thin films), the smoothing and segmentation of images and the surface reconstructions in the image processing, computer vision and computer graphics (see e.g. [32, 29, 2, 1, 6, 19, 31, 15, 17]), and in many further situations related to the motion of implicit curves or surfaces. On the other hand, the convergence of numerical schemes to unique viscosity solution [9, 14, 7] of equation (1.1) is often an open problem, it is an exception to find an analysis of convergence of the methods used for solving the curvature driven flows in the level set formulation. The level set equation (1.1) represents the so-called Eulerian approach to curve and surface evolutions. It moves level sets (curves in 2D, surfaces in 3D) of the function u in the normal direction with the velocity proportional to the (mean) curvature. The curves and surfaces are represented implicitly and thus the formulation automatically allows topological changes in the interface which yields robustness of the method.

In [10] Deckelnik and Dziuk proved convergence of their finite element numerical scheme to solution of the mean curvature flow of graphs which can be further adjusted, using the Evans and Spruck regularization [14], to the situation of motion of level sets by the mean curvature [11]. Convergence of a particular finite difference scheme has been proved by Oberman in [28] using the technique of Barles and Souganidis [3]. More results are available for schemes based on different than level set formulation. The convergent schemes for the so-called direct (parametric, Lagrangean) approach to curvature driven flows were suggested and studied e.g. in [13, 21]; for further Lagrangean methods we refer e.g. to [12, 25, 26]. Other than the level set, but also Eulerian, approach is represented by the phase field method where the convergence of numerical approximation to the solution of the so-called Allen-Cahn equation (modelling diffused interface evolution) is studied, see e.g. [27, 4].

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In this paper we prove consistency and stability of the semi-implicit fully discrete complementary volume scheme. Our semi-implicit scheme leads to solution of linear systems in every discrete time step (for other semi-implicit approaches to solving nonlinear diffusion see e.g. [18, 22, 16, 17]), so it is much more efficient than a fully implicit nonlinear scheme [33], and it is unconditionally stable without any restriction to time step in spite of many other explicit schemes [30, 32, 29, 31]. Consistency and stability are two properties, in the theory of Barles and Souganidis [3], which are used to show convergence of a numerical scheme to solution of fully nonlinear second order partial differential equations and we discuss them in this paper. A monotonicity is the third one and it remains still an open question regarding our scheme.

The derivation of our numerical method for solving equation (1.1) is based on the finite volume methodology (see e.g. [20, 22]). We construct the so-called complementary volumes (co-volumes) to a finite element triangulation [33, 16]. Integrating equation (1.1) in the co-volume gives the weak (integral) formulation of the problem from which the computational scheme naturally follows. One of our main motivations to solve the curvature driven level set equation and its generalizations comes from image processing applications [15, 16, 17, 23, 24, 8]. The co-volume scheme has been applied to smoothing and segmentation of 2D and 3D medical images in [24, 8] and is based on the original semi-implicit method studied in [16]. In [8] it has been shown experimentally on non-trivial examples of exact solutions that the method converges to true solution, in this paper we show theoretically its consistency and stability. In the proofs we restrict ourselves to 2D situation and only to type of grids which we use in image processing applications, cf. next section, mainly in order to avoid too technical details.

In the next section we present in a detail our numerical scheme, and, in section 3 we prove its properties. For numerical experiments we refer to [16, 24, 23, 8] where co-volume schemes have been applied to problems of interface motion and image smoothing and segmentation.

2. Semi-implicit co-volume scheme. The unknown function $u(t,x)$ in (1.1) is defined in $Q_T = I \times \Omega$, $\Omega \subset I\!R^d$ is a bounded Lipschitz domain, $I = [0,T]$ is a time interval, and the equation is usually accompanied with zero Dirichlet (e.g. in image segmentation) or zero Neumann boundary conditions (e.g. in image smoothing) and by an initial condition

$$u(0,x) = u^0(x). \tag{2.1}$$

To construct the numerical scheme we choose a uniform discrete time step $\tau = \frac{T}{N}$ and replace the time derivative in (1.1) by the backward difference. The nonlinear terms of the equation are treated from the previous time step while the linear ones are considered on the current time level, this means semi-implicitness of the time discretization.

Semi-implicit in time discretization: *Let τ be given time step, and u^0 be a given initial level set function. Then, for $n = 1, \ldots, N$, we look for a function u^n, solution of the equation*

$$\frac{1}{|\nabla u^{n-1}|} \frac{u^n - u^{n-1}}{\tau} = \nabla . \left(\frac{\nabla u^n}{|\nabla u^{n-1}|} \right). \tag{2.2}$$

Let us introduce now the fully discrete scheme. In the image processing applications, a digital image is given on a structure of pixels with rectangular shape in general

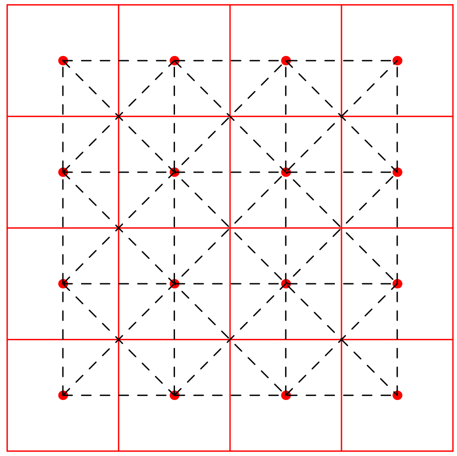

FIG. 2.1. *The co-volumes (pixels, red solid lines), the triangulation for the co-volume method (black dashed lines), and the degree of freedom (DF) nodes (red round points).*

(red rectangles in Figure 2.1). Since in every discrete time step of the method (2.2) we have to evaluate gradient of the level set function at the previous step $|\nabla u^{n-1}|$, we put a triangulation (dashed black lines in Figure 2.1) onto the computational domain and then take a piecewise linear approximation of the level set function on this triangulation. Such approach will give a constant value of the gradient per triangle, allowing simple, fast and clear construction of fully-discrete system of equations.

As can be seen in Figure 2.1, in our method the centers of pixels are connected by a new rectangular mesh and every new rectangle is splitted into four triangles. The centers of pixels will be called degree of freedom (DF) nodes. By this procedure we also get further nodes (at crossing of red lines in Figure 2.1) which, however, will not represent degrees of freedom. We will call them non-degree of freedom (NDF) nodes. Let a function u be given by discrete values in DF nodes. Then in additional NDF nodes we take the average value of the neighboring DF nodal values. By such defined values in NDF nodes, a piecewise linear approximation u_h of u on the triangulation can be built. Let us note, that the computational domain Ω is given by the union of all triangles contained in the triangulation $\mathcal{T}_h$ given by the previous construction. It means Ω is equal to union of all inner pixels and a half-strip of the boundary pixels, cf. Fig. 2.1. For $\mathcal{T}_h$ we construct a complementary (dual) mesh. We modify a basic approach given in [33, 16] in such a way that our co-volume mesh will consist of cells p associated only with DF nodes p of $\mathcal{T}_h$, say $p = 1, \ldots, M$. Since there will be one-to-one correspondence between co-volumes and DF nodes, without any confusion, we use the same notation for them.

For each DF node p of $\mathcal{T}_h$, let $N(p)$ denote the set of all DF nodes q connected to the node p by an edge. We denote cardinality of this set by N_p. The edge connecting p and q will be denoted by σ_{pq} and its length by h_{pq}. Then every *co-volume* p is bounded by the lines (co-edges) e_{pq} that bisect and are perpendicular to the edges $\sigma_{pq}, q \in N(p)$. By this construction, the co-volume mesh corresponds exactly to the pixel structure of the image inside the computational domain Ω. We denote by $\mathcal{E}_{pq}$ the set of triangles having σ_{pq} as an edge. In a situation depicted in Figure 2.1, every $\mathcal{E}_{pq}$ consists of two triangles. For each $T \in \mathcal{E}_{pq}$ let c^T_{pq} be the length of the portion of e_{pq} that is in T, i.e., $c^T_{pq} = m(e_{pq} \cap T)$, where m is a measure in $I\!R^{d-1}$. Let $\mathcal{N}_p$

be the set of triangles that have DF node p as a vertex. Let u_h be a piecewise linear function on triangulation $\mathcal{T}_h$. We will denote a constant value of $|\nabla u_h|$ on $T \in \mathcal{T}_h$ by $|\nabla u_T|$ and define regularized gradients by

$$|\nabla u_T|_\varepsilon = \sqrt{\varepsilon^2 + |\nabla u_T|^2}. \tag{2.3}$$

We will use the notation $u_p = u_h(x_p)$, where x_p is the coordinate of a (DF or NDF) node of triangulation $\mathcal{T}_h$, and also $u_p^n = u_{h,\tau}(x_p, t_n)$ where $u_{h,\tau}$ is our piecewise linear in space and time approximation of the solution to the regularized level set equation. Let u_h^0 be piecewise linear interpolation of the initial function u^0 on triangulation $\mathcal{T}_h$.

With these notations we are ready to derive the co-volume spatial discretization. As it is usual in finite volume methods [20], we integrate (2.2) over every co-volume $p,\ p = 1, \ldots, M$, and then using divergence theorem we get an integral formulation of (2.2)

$$\int_p \frac{1}{|\nabla u^{n-1}|} \frac{u^n - u^{n-1}}{\tau} dx = \sum_{q \in N(p)} \int_{e_{pq}} \frac{1}{|\nabla u^{n-1}|} \frac{\partial u^n}{\partial \nu} ds \tag{2.4}$$

where ν is a unit outer normal to the boundary of p. Now the exact "fluxes" on the right hand side and "capacity function" $\frac{1}{|\nabla u^{n-1}|}$ on the left-hand side will be approximated numerically using piecewise linear reconstruction of u^{n-1} on triangulation $\mathcal{T}_h$. In such a way, for the approximation of the right-hand side of (2.4) we get

$$\sum_{q \in N(p)} \left(\sum_{T \in \mathcal{E}_{pq}} c_{pq}^T \frac{1}{|\nabla u_T^{n-1}|} \right) \frac{u_q^n - u_p^n}{h_{pq}}. \tag{2.5}$$

For the left-hand side of (2.4) we use

$$m(p) \sum_{T \in \mathcal{N}_p} \frac{m(T \cap p)}{m(p)} \frac{1}{|\nabla u_T^{n-1}|} \frac{u_p^n - u_p^{n-1}}{\tau} \tag{2.6}$$

where $m(p)$ is a measure in $I\!R^d$ of co-volume p. In general, we assume for every pair p, q

$$\underline{h} \leq h_{pq} \leq \bar{h}, \quad \frac{\bar{h}}{\underline{h}} \leq h_0$$

and we define

$$d_{pq} := \frac{m(e_{pq})}{h_{pq}} \leq d_0. \tag{2.7}$$

However, we restrict our considerations to uniform rectangular co-volumes with size length h, as plotted in Figure 2.1. Then, e.g.,

$$m(p) = h^2,\ m(e_{pq}) = h_{pq} = h,\ d_{pq} = 1, c_{pq}^T = \frac{1}{2} m(e_{pq}). \tag{2.8}$$

We denote four neighbouring DF nodes of x_p by x_{q_1} (east), x_{q_2} (north), x_{q_3} (west), x_{q_4} (south), and the corners of co-volume p by x_{r_1} (top right), x_{r_2} (top left), x_{r_3}

(bottom left), x_{r_4} (bottom right). The middle point of the edge e_{pq_i} is denoted by x_{m_i}, $i = 1, \dots, 4$.

Now we can define coefficients, where the ε-regularization (2.3) is taken into account, namely,

$$a_{pq}^{n-1} = \frac{1}{|\nabla u_{pq}^{n-1}|_\varepsilon} := \frac{1}{2}\left(\frac{1}{|\nabla u_{T_{pq}^1}^{n-1}|_\varepsilon} + \frac{1}{|\nabla u_{T_{pq}^2}^{n-1}|_\varepsilon}\right), \tag{2.9}$$

$$b_p^{n-1} := \frac{1}{|\nabla u_p^{n-1}|_\varepsilon} = \frac{1}{N_p}\sum_{q\in N(p)} \frac{1}{|\nabla u_{pq}^{n-1}|_\varepsilon}, \tag{2.10}$$

where $T_{pq}^1, T_{pq}^2 \in \mathcal{E}_{pq}$. For example for triangle with points x_p, x_{q_1}, x_{r_1} we have

$$|\nabla u_{T_{pq_1}^1}^{n-1}|_\varepsilon = \sqrt{\frac{(u_{q_1} - u_p)^2}{h^2} + \frac{(2(u_{r_1} - u_{m_1}))^2}{h^2} + \varepsilon^2}. \tag{2.11}$$

Now our computational method can be written as follows.

Fully-discrete semi-implicit co-volume scheme: *Let u_p^0, $p = 1, \dots, M$ be given discrete initial values of the segmentation function. Then, for $n = 1, \dots, N$ we look for $u_p^n, p = 1, \dots, M$, satisfying*

$$b_p^{n-1}\ m(p)\ u_p^n + \tau \sum_{q\in N(p)} a_{pq}^{n-1} d_{pq}(u_p^n - u_q^n) = b_p^{n-1}\ m(p)\ u_p^{n-1}. \tag{2.12}$$

Remark 2.1. Previously studied co-volume algorithms [33, 16] for the level-set-like problems have used either "left oriented" or "right oriented" triangulations and no NDF nodes (see Figure 2.2). But, then the level set curve or surface evolution is influenced by the grid effect. Of course this effect is satisfactory weakened by refining the grid (e.g. in interface motion computations, cf. [16]). In image processing we work with fixed given pixel/voxel structure, and we do not refine this structure, so we want to remove such "non-symmetry" of the method. This can be done by averaging of two, "left" and "right" solutions, or it can be done implicitly by taking the combination of triangulations as plotted in Figure 2.1. Of course, usage of such "symmetric" triangulation can be accompanied also by the linear finite element method of Deckelnick and Dziuk [10, 11], considering also NDF nodes as degrees of freedom. But this would increase the number of unknowns in systems to be solved by factor two, which can be critical in case of image processing applications, usually with huge number of pixels/voxels given. Without any construction of a triangulation, we could also use a bi-linear representation of the level set function on finite elements corresponding to the rectangular grid formed by centers of pixels and build tensor-product finite element method. But then we would face a problem of non-constant gradients in evaluation of nonlinearities. The same problem would arise considering complemetary volume method given by dual grid corresponding to pixels and by a bi-linear representation of the function on the rectangular grid formed by centers of pixels. Again, such technique would require the evaluation and integration of absolute value of gradient of bi-linear functions on the co-volume sides. From the above points of view, our method gives the smallest possible number of unknowns and the most simple (piecewise constant) nonlinear coefficients evaluation.

Such "symmetric" primal-dual grid can be built also in three dimensions. The construction of co-volume mesh in 3D has to use 3D tetrahedral finite element grid

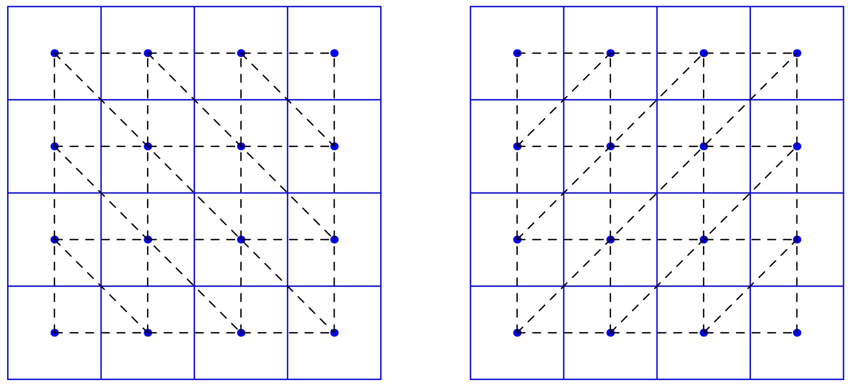

Fig. 2.2. *By dashed lines we plot the "left oriented" triangulation (left) and the "right oriented" triangulation (right). The "symmetric" triangulation corresponding to our method is plotted in* Fig. 2.1.

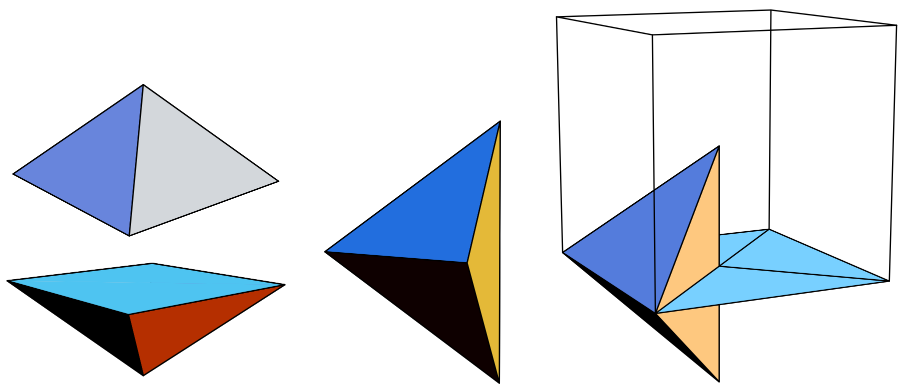

Fig. 2.3. *Neighbouring pyramids which are joined together (left); joining these pyramids and then splitting into four parts give tetrahedron of 3D grid (middle); intersection of the tetrahedron with the bottom face of co-volume (right).*

to which it is complementary. For that goal we use the following approach similar to the so called centered-cubic-lattice method known from computer graphics [5]. First, every cubic voxel is splitted into 6 pyramids with vertex given by the voxel center and base surfaces given by the voxel boundary faces. The neighbouring pyramids of neighbouring voxels are joined together to form octahedron which is then splitted into 4 tetrahedras using diagonals of the voxel boundary face - see Figure 2.3. In such way we get 3D tetrahedral grid. Two nodes of every tetrahedron correspond to centers of neighbouring voxels, and further two nodes correspond to voxel boundary vertices; every tetrahedron intersects common face of neighbouring voxels. Now again only the centers of voxels represent DF nodes, the additional nodes of tetrahedras are NDF nodes which are used only in piecewise linear representation of the level set function. Using such co-volumes one obtains the computational scheme with the same structure as (2.12) but the averages in definitions (2.9)-(2.10) are taken over all tetrahedras crossing the faces and entire co-volume, respectively.

3. Consistency and stability of the numerical scheme. We first give some necessary notations and definitions. Let us assume that $\varepsilon > 0$ is fixed. The Evans and Spruck regularization of the curvature driven level set equation (1.1) can be written in the following form

$$u_t - \text{trace}((\mathbf{I} - \frac{\nabla u \otimes \nabla u}{|\nabla u|_\varepsilon^2})\overline{D}^2 u) = 0 \quad \text{in } I \times \Omega, \tag{3.1}$$

where $\overline{D}^2 u$ denotes a symmetric matrix of second order spatial derivatives of u. If we denote

$$G(\mathbf{X}, p) = \text{trace}((\mathbf{I} - \frac{p \otimes p}{|p|_\varepsilon^2}) \cdot \mathbf{X}),$$

where $(\mathbf{X}, p) \in \mathbf{S}^d \times I\!R^d$, $\mathbf{S}^d$ is the space of $d \times d$ symmetric matrices, then G is an elliptic operator [9]. We denote by $B(Q)$, $Q = I \times \overline{\Omega}$, the set of all uniformly bounded functions in a domain Q. In [14], existence of the unique smooth solution is proved. Let us consider the equation

$$F(D^2 u, Du, u) = 0 \quad \text{in } Q \tag{3.2}$$

where in spatially two dimensional case we define

$$Du = \begin{pmatrix} u_t \\ u_x \\ u_y \end{pmatrix}, \quad D^2 u = \begin{pmatrix} u_{tt} & u_{tx} & u_{ty} \\ u_{tx} & u_{xx} & u_{xy} \\ u_{ty} & u_{xy} & u_{yy} \end{pmatrix}, \quad \overline{\mathcal{I}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \overline{I} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and $F : \mathbf{S}^d \times I\!R^d \times I\!R \to I\!R$ is given by

$$F(D^2 u, Du, u) = \begin{cases} \overline{I} \cdot Du - \text{trace}\left(\left(\overline{\mathcal{I}} - \frac{(\overline{\mathcal{I}} \cdot Du) \otimes (\overline{\mathcal{I}} \cdot Du)}{|\overline{\mathcal{I}} \cdot Du|_\varepsilon^2}\right) D^2 u\right) & \text{in } Q, \\ u(0, x) - u^0(x) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} \text{ or } u \quad \text{on } I \times \partial\Omega, \end{cases}$$

where ν is an outer unit normal to $\partial\Omega$. It is now clear from the properties of G above that F posses the elliptic property, that means for all $(p, u) \in I\!R^d \times I\!R$ and for all $\mathbf{M}, \mathbf{N} \in \mathbf{S}^d$

$$F(\mathbf{M}, p, u) \leq F(\mathbf{N}, p, u) \text{ provided } \mathbf{M} \geq \mathbf{N}.$$

Let us have an approximation scheme of the form

$$S(\rho, Y, u^\rho(Y), u^\rho) = 0 \quad \text{in } Q, \tag{3.3}$$

where $S : I\!R^+ \times Q \times I\!R \times B(Q) \to I\!R$ is a locally uniformly bounded.

Definition 3.1. Approximation scheme S given by (3.3) has the *monotonicity* property if for all $\rho \geq 0$, $Y \in Q$, $\zeta \in I\!R$ and $u, v \in B(Q)$ it holds that, if $u \geq v$ then

$$S(\rho, Y, \zeta, u) \leq S(\rho, Y, \zeta, v). \tag{3.4}$$

Definition 3.2. Approximation scheme S given by (3.3) has the *stability* property, if for all $\rho > 0$ there exists a solution

$$u^\rho \in B(Q) \tag{3.5}$$

of (3.3) with a bound independent of ρ.

Definition 3.3. Approximation scheme S given by (3.3) has the *consistency* property if for all $\Phi \in C^\infty(Q)$ and for all $X \in Q$ it holds that

$$\lim_{\rho\to 0, Y\to X, \xi\to 0} \frac{S(\rho, Y, \Phi(Y)+\xi, \Phi+\xi)}{\rho} = F(D^2\Phi(X), D\Phi(X), \Phi(X)). \tag{3.6}$$

We recall the following important statement:

Theorem 3.1.[3] *Let the approximation scheme S given by* (3.3) *have stability, monotonicity and consistency properties. Then, as $\rho \to 0$, the solution of the scheme converges locally uniformly to the unique continuous solution of* (3.2).

Our aim is to transform numerical scheme (2.12) to the form (3.3) and then prove the stability and consistency properties. The numerical scheme (2.12) can be written in the form (3.3) provided $\rho = \tau$, $Y = (t_n, x_p)$, $u^\rho(Y) = u_p^n$, $u^\rho = u_{h,\tau}$, where $u_{h,\tau}(x, y)$ is a piecewise linear in space and time (i.e. on triangulation $\mathcal{T}_h$ and among discrete time steps) approximation of solution, and

$$S(\rho, (t_n, x_p), u_p^n, u_{h,\tau}) = u_p^n - u_p^{n-1} + \frac{\tau}{b_p^{n-1} m(p)} \sum_{q\in N(p)} a_{pq}^{n-1}(u_p^n - u_q^n) d_{pq} = 0, \tag{3.7}$$

where $u_p^0 = u_0(x_p)$ for all $p = 1, \dots, M$. Let us note that the time step τ is usually coupled with the spatial step h, e.g., by relation $\tau \approx h^2$ which is natural in solving parabolic PDEs.

The zero Neumann boundary conditions are realized using the mirror image extension of the solution values outside the image domain, i.e., adding one outer strip of pixels (co-volumes) q along the boundary pixels p, cf. Fig 2.1, and prescribing $u_q^n = u_p^n$ for these additional pixel values. The result is that the boundary terms $a_{pq}^{n-1}(u_p^n - u_q^n)d_{pq}$ are simply not present in the summation term of the scheme (2.12) or in its equivalent form (3.7), that is also equivalent to prescribing $a_{pq}^{n-1} = 0$ if p is a boundary co-volume and q is an additional one.

Since we consider that computational domain Ω is given by the union of all triangles in $\mathcal{T}_h$, cf. Fig. 2.1, the DF nodes of boundary pixels lie on $\partial\Omega$ and we prescribe 0 values to them in case of Dirichlet boundary condition. The only change in the scheme is that the system contains less number of unknowns (only DF nodes of inner pixels) and that $a_{pq}^{n-1}(u_p^n - u_q^n)d_{pq}$ in the summation term contain the known value $u_q^n = 0$ if q is a boundary pixel.

Theorem 3.2. *There exists unique solution $u_h^n = (u_1^n, \dots, u_M^n)$ of the scheme* (2.12) *for any value of the regularization parameter $\varepsilon > 0$ and for any time step $n = 1, \dots, N$. Moreover, for the fully discrete numerical solution $u_{h,\tau}$ the following estimate holds*

$$\|u_{h,\tau}\|_{L_\infty(Q)} \le \|u_h^0\|_{L_\infty(\overline{\Omega})}, \tag{3.8}$$

which gives the stability property of the scheme.

Proof. From definition (2.9) follows that off-diagonal elements $-\tau a_{pq}^{n-1}$, $q \in N(p)$, of the system (2.12) are symmetric and nonpositive. The positive term b_p^{n-1} given by (2.10) affects only diagonal which is equal to $b_p^{n-1} m(p) + \tau \sum_{q\in N(p)} a_{pq}^{n-1} d_{pq}$. Thus, the matrix of the system (2.12) is symmetric and diagonally dominant M-matrix which imply that it always has unique solution. Let us write (2.12) in the form (3.7)

$$u_p^n + \frac{\tau}{b_p^{n-1} m(p)} \sum_{q \in N(p)} a_{pq}^{n-1}(u_p^n - u_q^n) d_{pq} = u_p^{n-1} \tag{3.9}$$

and let $\max u_h^n = \max(u_1^n, \ldots, u_M^n)$ be achieved in the point p.

In case of the zero Neumann boundary condition, no matter if p is inner or boundary point, the whole second term on the left hand side of (3.9) is nonnegative and thus value $u_p^n \leq u_p^{n-1} \leq \max(u_1^{n-1}, \ldots, u_M^{n-1})$. In the same way we can prove similar relation for minima and together we have

$$\min\ u_p^0 \leq \min\ u_p^n \leq \max\ u_p^n \leq \max\ u_p^0, \quad n \leq N, \tag{3.10}$$

which imply the estimate (3.8).

In case of the zero Dirichlet boundary condition, first let p be a boundary DF node in which the maximum of discrete solution is attained at the nth time step (this maximum is of course equal to 0). It is clear, that it is less or equal to a maximum at the previous time step $n-1$, which can be either positive (if realized in an inner node) or zero (if realized in a boundary node). Secondly, if p is an inner node, similarly to considerations for the Neumann boundary condition above, we have that the whole second term on the left hand side of (3.9) is nonnegative and thus value $u_p^n \leq u_p^{n-1}$ which is less or equal to a maximum at the time step $n-1$. Then recursively we get again the estimate (3.8).

Theorem 3.3. *For any fixed $\varepsilon > 0$ our numerical scheme posses the consistency property.*

Proof. Let $X = (t, x)$ and $\Phi \in C^\infty(Q)$. There exists time step $n \in \{0, 1, \ldots N\}$ such that $t \in \langle t_{n-1}, t_n \rangle$ and co-volume $p \in \{1, \ldots M\}$ such that $x \in p$. We denote $Y = (t_n, x_p)$, and $\Phi_p^n := \Phi(t_n, x_p)$. In order to get consistency, in our case it is sufficient to prove the existence of positive integers k_1, k_2 such that

$$\left|\frac{S(\rho, Y, \Phi(Y), \Phi)}{\rho} - F(D^2\Phi(X), D\Phi(X), \Phi(X))\right| \leq C(||\Phi||_3)(\tau^{k_1} + h^{k_2}),$$

where by $||\Phi||_k$ we denote the norm of the functional space $C^k(Q)$ and $C(||\Phi||_3)$ is a constant which can depend on a $C^3(Q)$ norm of a smooth function Φ. For our scheme it can be written in the following form

$$\left| \frac{\Phi_p^n - \Phi_p^{n-1}}{\tau} - \frac{1}{b_p^{n-1} m(p)} \sum_{q \in N(p)} a_{pq}^{n-1}(\Phi_q^n - \Phi_p^n) - \Phi_t(X) + |\nabla \Phi(X)|_\varepsilon \nabla \cdot \frac{\nabla \Phi(X)}{|\nabla \Phi(X)|_\varepsilon} \right|$$

$$\leq C(||\Phi||_3)(\tau^{k_1} + h^{k_2}). \tag{3.11}$$

We will prove inequality (3.11) subsequently estimating differences of particular terms of the left hand side. Since $\Phi \in C^\infty(Q)$ it is clear that

$$\frac{\Phi_p^n - \Phi_p^{n-1}}{\tau} = \Phi_t(\xi, x_p),$$

where $\xi \in < t_{n-1}, t_n >$. Because $|\xi - t| \leq \tau$ and $|x - x_p| \leq \sqrt{2}h$ we have

$$\left| \frac{\Phi_p^n - \Phi_p^{n-1}}{\tau} - \Phi_t(X) \right| \leq |\Phi_t(\xi, x_p) - \Phi_t(t, x)| \leq C(\|\Phi\|_2)(\tau + h).$$

The second term in (3.11) we can rewrite into the form

$$II = -\frac{1}{b_p^{n-1} m(p)} \sum_{q \in N(p)} \int_{e_{pq}} a_{pq}^{n-1} \frac{\Phi_q^n - \Phi_p^n}{h} ds.$$

Let us omit, for a moment, the upper time index for Φ, and let us use on each edge e_{pq} for the difference term $\frac{\Phi_q - \Phi_p}{h}$ the Taylor expansion in a similar way as it is used to derive usual central difference approximaton. Let $x_p = (x_{1p}, x_{2p})$ and $x_{q_i} = (x_{1q_i}, x_{2q_i})$ for $i = 1, \ldots, 4$. Let $s = (s_1, s_2)$ be a point on the boundary of co-volume p. Then

(3.12) for a point $s \in e_{pq_1}$ we have $s = (x_{1p} + \frac{h}{2}, x_{2p} + t\frac{h}{2}),\ t \in < -1, 1 >$,

(3.13) for a point $s \in e_{pq_2}$ we have $s = (x_{1p} + t\frac{h}{2}, x_{2p} + \frac{h}{2}),\ t \in < -1, 1 >$,

(3.14) for a point $s \in e_{pq_3}$ we have $s = (x_{1p} - \frac{h}{2}, x_{2p} + t\frac{h}{2}),\ t \in < -1, 1 >$,

(3.15) for a point $s \in e_{pq_4}$ we have $s = (x_{1p} + t\frac{h}{2}, x_{2p} - \frac{h}{2}),\ t \in < -1, 1 >$.

Then for e_{pq_1} and e_{pq_3} we have

$$\frac{\Phi_q - \Phi_p}{h} = \frac{\partial \Phi(s)}{\partial \nu} + 2\Phi_{xy}(s) \cdot \text{sgn}(x_{1q} - x_{1p})(x_{2q} - s_2) + O(h^2) \tag{3.16}$$

and for e_{pq_2} and e_{pq_4} similarly

$$\frac{\Phi_q - \Phi_p}{h} = \frac{\partial \Phi(s)}{\partial \nu} + 2\Phi_{xy}(s) \cdot \text{sgn}(x_{2q} - x_{2p})(x_{1q} - s_1) + O(h^2). \tag{3.17}$$

Involving these relations in term II and using

$$\frac{\sum\limits_{q \in N(p)} a_{pq}^{n-1}(w)}{b_p^{n-1}(w)} = 4 \tag{3.18}$$

which holds for any function $w \in B(Q)$ on a uniform rectangular grid due to (2.9)-(2.10), we obtain

$$II = -\frac{1}{b_p^{n-1} m(p)} \sum_{q=1,3} \int_{e_{pq}} a_{pq}^{n-1} \left(\frac{\partial \Phi(s)}{\partial \nu} + 2\Phi_{xy}(s) \cdot \text{sgn}(x_{1q} - x_{1p})(x_{2q} - s_2) \right) ds -$$

$$\frac{1}{b_p^{n-1} m(p)} \sum_{q=2,4} \int_{e_{pq}} a_{pq}^{n-1} \left(\frac{\partial \Phi(s)}{\partial \nu} + 2\Phi_{xy}(s) \cdot \text{sgn}(x_{2q} - x_{2p})(x_{1q} - s_1) \right) ds +$$

$$C(||\Phi||_3)h = -\frac{1}{b_p^{n-1}m(p)} \sum_{q\in N(p)} \int_{e_{pq}} a_{pq}^{n-1} \frac{\partial \Phi(s)}{\partial \nu} ds$$

$$-\frac{1}{b_p^{n-1}m(p)} \sum_{q=1,3} \int_{e_{pq}} a_{pq}^{n-1} 2\Phi_{xy}(s)\cdot \mathrm{sgn}(x_{1q}-x_{1p})(x_{2q}-s_2)ds$$

$$-\frac{1}{b_p^{n-1}m(p)} \sum_{q=2,4} \int_{e_{pq}} a_{pq}^{n-1} 2\Phi_{xy}(s)\cdot \mathrm{sgn}(x_{1q}-x_{1p})(x_{2q}-s_2)ds + C(||\Phi||_3)h =$$

$$= II_1 + II_2 + II_3 + C(||\Phi||_3)h.$$

Using parametrizations (3.12)-(3.15) we can rearrange term II_2 (term II_3 can be estimated analogously) on the edge e_{pq_1} into the following form

$$-\frac{2h}{2b_p^{n-1}m\,(p)} \int_{-1}^{1} a_{pq_1}^{n-1} \Phi_{xy}(x_{1p}+\frac{h}{2}, x_{2p}+t\frac{h}{2})(-t\frac{h}{2})dt$$

and on the edge e_{pq_3} similarly

$$-\frac{2h}{2b_p^{n-1}m\,(p)} \int_{-1}^{1} a_{pq_3}^{n-1} \Phi_{xy}(x_{1p}-\frac{h}{2}, x_{2p}+t\frac{h}{2})(t\frac{h}{2})dt.$$

We can collect these two terms together, and using the fact that $\Phi \in C^\infty(Q)$ we have

$$|II_2| \le \left| \frac{h^2}{2b_p^{n-1}m\,(p)} \int_{-1}^{1} t\left(a_{pq_1}^{n-1} - a_{pq_3}^{n-1}\right) \Phi_{xy}(x_{1p}+\frac{h}{2}, x_{2p}+t\frac{h}{2})dt + \right.$$

$$\left. \frac{h^2}{2b_p^{n-1}m\,(p)} \int_{-1}^{1} t a_{pq_3}^{n-1} \left(\Phi_{xy}(x_{1p}+\frac{h}{2}, x_{2p}+t\frac{h}{2}) - \Phi_{xy}(x_{1p}-\frac{h}{2}, x_{2p}+t\frac{h}{2}) \right) dt \right|$$

$$\le \|\Phi\|_2 \frac{|a_{pq_1}^{n-1} - a_{pq_3}^{n-1}|}{2b_p^{n-1}} + C(\|\Phi\|_3)\frac{a_{pq_3}^{n-1}}{2b_p^{n-1}}h.$$

Putting all together we obtain

$$|II_2| + |II_3| \le C(||\Phi||_2)h\frac{a_{pq_3}^{n-1} + a_{pq_4}^{n-1}}{b_p^{n-1}} + \tag{3.19}$$

$$C(||\Phi||_3)\frac{|a_{pq_1}^{n-1} - a_{pq_3}^{n-1}| + |a_{pq_2}^{n-1} - a_{pq_4}^{n-1}|}{b_p^{n-1}}.$$

First term on the right hand side can be estimated using (3.18) and we obtain $O(h)$ term. In the second term we estimate the difference $a^{n-1}_{pq_1} - a^{n-1}_{pq_3}$ (further part can be treated anagously). We have

$$|a^{n-1}_{pq_1} - a^{n-1}_{pq_3}| = \frac{1}{2}\left|\frac{1}{|\nabla\Phi_{T_{11}}|_\varepsilon} + \frac{1}{|\nabla\Phi_{T_{12}}|_\varepsilon} - \frac{1}{|\nabla\Phi_{T_{31}}|_\varepsilon} - \frac{1}{|\nabla\Phi_{T_{32}}|_\varepsilon}\right| \tag{3.20}$$

where $T_{11} = T^1_{pq_1}$, $T_{12} = T^2_{pq_1}$ are two triangles corresponding to points x_p, x_{q_1} and $T_{31} = T^1_{pq_3}$, $T_{32} = T^2_{pq_3}$ are two triangles corresponding to points x_p, x_{q_3}. We can put together terms with T_{11} and T_{31} (analogously it can be done for terms with T_{12} and T_{32}) and then use our approximation of gradient, cf. (2.11), to get

$$\frac{1}{|\nabla\Phi_{T_{11}}|_\varepsilon} - \frac{1}{|\nabla\Phi_{T_{31}}|_\varepsilon} = \frac{|\nabla\Phi_{T_{11}}|^2 - |\nabla\Phi_{T_{31}}|^2}{|\nabla\Phi_{T_{11}}|_\varepsilon|\nabla\Phi_{T_{31}}|_\varepsilon(|\nabla\Phi_{T_{11}}|_\varepsilon + |\nabla\Phi_{T_{31}}|_\varepsilon)} =$$

$$\frac{(\Phi(x_{q_1}) - \Phi(x_p))^2 - (\Phi(x_{q_3}) - \Phi(x_p))^2}{h^2|\nabla\Phi_{T_{11}}|_\varepsilon|\nabla\Phi_{T_{31}}|_\varepsilon(|\nabla\Phi_{T_{11}}|_\varepsilon + |\nabla\Phi_{T_{31}}|_\varepsilon)} +$$

$$\frac{(2(\Phi(x_{r_1}) - \Phi(x_{m_1})))^2 - (2(\Phi(x_{r_2}) - \Phi(x_{m_3})))^2}{h^2|\nabla\Phi_{T_{11}}|_\varepsilon|\nabla\Phi_{T_{31}}|_\varepsilon(|\nabla\Phi_{T_{11}}|_\varepsilon + |\nabla\Phi_{T_{31}}|_\varepsilon)}.$$

Because of the properties of Φ we have

$$\frac{\Phi(x_{q_1}) - \Phi(x_p)}{h} = \Phi_x(\xi), \quad \frac{\Phi(x_{q_3}) - \Phi(x_p)}{h} = \Phi_x(\eta),$$
$$\frac{2(\Phi(x_{r_1}) - \Phi(x_{m_1}))}{h} = \Phi_y(\zeta), \quad \frac{2(\Phi(x_{r_2}) - \Phi(x_{m_3}))}{h} = \Phi_y(\theta), \tag{3.21}$$

where ξ lays on abscissa with end points x_p, x_{q_1}, η lays on abscissa with end points x_p, x_{q_3}, ζ lays on abscissa with end points x_{m_1}, x_{r_1} and θ lays on abscissa with end points x_{m_3}, x_{r_2}. Employing these facts and again the smoothness properties of Φ we have

$$|\frac{1}{|\nabla\Phi_{T_{11}}|_\varepsilon} - \frac{1}{|\nabla\Phi_{T_{31}}|_\varepsilon}| \le$$

$$\frac{|(\Phi_x(\xi) - \Phi_x(\eta))(\Phi_x(\xi) + \Phi_x(\eta))| + |(\Phi_y(\zeta) - \Phi_y(\theta))(\Phi_y(\zeta) + \Phi_y(\theta))|}{|\nabla\Phi_{T_{11}}|_\varepsilon|\nabla\Phi_{T_{31}}|_\varepsilon(|\nabla\Phi_{T_{11}}|_\varepsilon + |\nabla\Phi_{T_{31}}|_\varepsilon)} \le$$

$$\frac{\sqrt{2}||\Phi||_2 h(|\nabla\Phi_{T_{11}}| + |\nabla\Phi_{T_{31}}|)}{|\nabla\Phi_{T_{11}}|_\varepsilon|\nabla\Phi_{T_{31}}|_\varepsilon(|\nabla\Phi_{T_{11}}|_\varepsilon + |\nabla\Phi_{T_{31}}|_\varepsilon)} \le \frac{C(||\Phi||_3)h}{|\nabla\Phi_{T_{11}}|_\varepsilon|\nabla\Phi_{T_{31}}|_\varepsilon}.$$

If we estimate also the difference for terms with T_{12} and T_{32} in (3.20) and similarly the term $|a^{n-1}_{pq_2} - a^{n-1}_{pq_4}|$ in (3.19) finally we have

$$|II_2| + |II_3| \le C(||\Phi||_3)h + \frac{C(||\Phi||_3)h}{b^{n-1}_p}\left(\frac{1}{|\nabla\Phi_{T_{11}}|_\varepsilon|\nabla\Phi_{T_{31}}|_\varepsilon} + \frac{1}{|\nabla\Phi_{T_{12}}|_\varepsilon|\nabla\Phi_{T_{32}}|_\varepsilon}\right.$$

$$\left.+\frac{1}{|\nabla\Phi_{T_{21}}|_\varepsilon|\nabla\Phi_{T_{41}}|_\varepsilon} + \frac{1}{|\nabla\Phi_{T_{22}}|_\varepsilon|\nabla\Phi_{T_{42}}|_\varepsilon}\right) \le C(||\Phi||_3)h + \frac{C(||\Phi||_3)h}{\varepsilon}.$$

Now, the term II_1 can be written as follows

$$II_1 = -\frac{1}{b_p^{n-1}m(p)} \sum_{q\in N(p)} \int_{e_{pq}} a_{pq}^{n-1} \frac{\partial \Phi(t_n, s)}{\partial \nu} ds$$

$$= -\frac{1}{b_p^{n-1}m(p)} \sum_{q\in N(p)} \int_{e_{pq}} \frac{1}{|\nabla \Phi(t_{n-1}, s)|_\varepsilon} \frac{\partial \Phi(t_n, s)}{\partial \nu} ds$$

$$-\frac{1}{b_p^{n-1}m(p)} \sum_{q\in N(p)} \int_{e_{pq}} \left(a_{pq}^{n-1} - \frac{1}{|\nabla \Phi(t_{n-1}, s)|_\varepsilon} \right) \frac{\partial \Phi(t_n, s)}{\partial \nu} ds = III_1 + III_2.$$

Similar approach as above can also be used to estimate term III_2. We again for a moment omit the time variable in function Φ, and estimate the terms along the oposite sides of the co-volume p boundary. Then for the edge e_{pq_1} we have

$$a_{pq_1}^{n-1} - \frac{1}{|\nabla \Phi(s)|_\varepsilon} = \frac{1}{2} \left(\frac{1}{|\nabla \Phi_{T_{11}}|_\varepsilon} - \frac{1}{|\nabla \Phi(s)|_\varepsilon} \right) + \frac{1}{2} \left(\frac{1}{|\nabla \Phi_{T_{12}}|_\varepsilon} - \frac{1}{|\nabla \Phi(s)|_\varepsilon} \right),$$

and now we rearrange the first term containing T_{11} as follows

$$\frac{1}{|\nabla \Phi(s)|_\varepsilon} - \frac{1}{|\nabla \Phi_{T_{11}}|_\varepsilon} = \frac{|\nabla \Phi_{T_{11}}|^2 - |\nabla \Phi(s)|^2}{|\nabla \Phi_{T_{11}}|_\varepsilon |\nabla \Phi(s)|_\varepsilon (|\nabla \Phi_{T_{11}}|_\varepsilon + |\nabla \Phi(s)|_\varepsilon)} =$$

$$\frac{(\frac{\Phi_{q_1} - \Phi_p}{h})^2 - (\Phi_x(s))^2 + (\frac{2(\Phi_{r_1} - \Phi_{m_1})}{h})^2 - (\Phi_y(s))^2}{|\nabla \Phi_{T_{11}}|_\varepsilon |\nabla \Phi(s)|_\varepsilon (|\nabla \Phi_{T_{11}}|_\varepsilon + |\nabla \Phi(s)|_\varepsilon)} =$$

$$\frac{(\frac{\Phi_{q_1} - \Phi_p}{h}) - \Phi_x(s))(\frac{\Phi_{q_1} - \Phi_p}{h} + \Phi_x(s)) + (\frac{2(\Phi_{r_1} - \Phi_{m_1})}{h} - \Phi_y(s))(\frac{2(\Phi_{r_1} - \Phi_{m_1})}{h} + \Phi_y(s))}{|\nabla \Phi_{T_{11}}|_\varepsilon |\nabla \Phi(s)|_\varepsilon (|\nabla \Phi_{T_{11}}|_\varepsilon + |\nabla \Phi(s)|_\varepsilon)}.$$

We applay again the Taylor expansion using parametrization (3.12) and get

$$\frac{\Phi_{q_1} - \Phi_p}{h} - \Phi_x(s) = 2\Phi_{xy}(s) t \frac{h}{2} + O(h^2),$$

$$\frac{2(\Phi_{r_1} - \Phi_{m_1})}{h} - \Phi_y(s) = \Phi_{yy}(s) \frac{h}{2}(1 - 2t) + O(h^2),$$

and the same can be done also for the second term containing T_{12}. Now we give several notations to simplify integrals in term III_2. For both triangles T_{1i}, $i = 1, 2$ we define

$$n_{1i}(s) = \left(2\Phi_{xy}(s) t \frac{h}{2} + O(h^2) \right) \left(\frac{\Phi_{q_1} - \Phi_p}{h} + \Phi_x(s) \right),$$

$$m_{1i}(s) = \left(\Phi_{yy}(s) \frac{h}{2}(1 - 2t) + O(h^2) \right) \left(\frac{2(\Phi_{r_1} - \Phi_{m_1})}{h} + \Phi_y(s) \right),$$

$$p_{1i}(s) = |\nabla\Phi_{T_{1i}}|_\varepsilon |\nabla\Phi(s)|_\varepsilon \left(|\nabla\Phi_{T_{1i}}|_\varepsilon + |\nabla\Phi(s)|_\varepsilon\right).$$

Using these notations, the parametrization (3.12) and the fact that $\frac{\partial\Phi(s)}{\partial\nu} = \Phi_x(s)$, we get that the integral along e_{pq_1} in term III_2 is equal to

$$\frac{h}{2}\sum_{i=1,2}\int_{-1}^{1}\Phi_x(s)\frac{m_{1i}(s)+n_{1i}(s)}{p_{1i}(s)}dt. \tag{3.22}$$

For the edge e_{pq_3} we similarly obtain (denoting variable on this edge by z)

$$a_{pq_3}^{n-1} - \frac{1}{|\nabla\Phi(z)|_\varepsilon} = \frac{1}{2}\left(\frac{1}{|\nabla\Phi_{T_{31}}|_\varepsilon} - \frac{1}{|\nabla\Phi(z)|_\varepsilon}\right) + \frac{1}{2}\left(\frac{1}{|\nabla\Phi_{T_{32}}|_\varepsilon} - \frac{1}{|\nabla\Phi(z)|_\varepsilon}\right),$$

$$\frac{1}{|\nabla\Phi(z)|_\varepsilon} - \frac{1}{|\nabla\Phi_{T_{31}}|_\varepsilon} =$$

$$\frac{(\frac{\Phi_p-\Phi_{q_3}}{h} - \Phi_x(z))(\frac{\Phi_p-\Phi_{q_3}}{h} + \Phi_x(z)) + (\frac{2(\Phi_{r_2}-\Phi_{m_3})}{h} - \Phi_y(z))(\frac{2(\Phi_{r_2}-\Phi_{m_3})}{h} + \Phi_y(z))}{|\nabla\Phi_{T_{31}}|_\varepsilon|\nabla\Phi(z)|_\varepsilon(|\nabla\Phi_{T_{31}}|_\varepsilon + |\nabla\Phi(z)|_\varepsilon)}$$

and using again

$$\frac{\Phi_p - \Phi_{q_3}}{h} - \Phi_x(z) = 2\Phi_{xy}(z)t\frac{h}{2} + O(h^2),$$

$$\frac{2(\Phi_{r_2} - \Phi_{m_3})}{h} - \Phi_y(z) = \Phi_{yy}(z)\frac{h}{2}(1-2t) + O(h^2),$$

we can define

$$n_{3i}(z) = (2\Phi_{xy}(z)t\frac{h}{2} + O(h^2))(\frac{\Phi_p - \Phi_{q_3}}{h}) + \Phi_x(z),$$

$$m_{3i}(z) = (\Phi_{yy}(z)\frac{h}{2}(1-2t) + O(h^2))(\frac{2(\Phi_{r_2} - \Phi_{m_3})}{h} + \Phi_y(z)),$$

$$p_{3i}(z) = |\nabla\Phi_{T_{3i}}|_\varepsilon|\nabla\Phi(z)|_\varepsilon(|\nabla\Phi_{T_{3i}}|_\varepsilon + |\nabla\Phi(z)|_\varepsilon).$$

Now we get (notice that $\frac{\partial\Phi(z)}{\partial\nu} = -\Phi_x(z)$) that the integral along e_{pq_3} in term III_2 is equal to

$$-\frac{h}{2}\sum_{i=1,2}\int_{-1}^{1}\Phi_x(z)\frac{n_{3i}(z)+m_{3i}(z)}{p_{3i}(z)}dt. \tag{3.23}$$

We can put together terms in (3.22) and (3.23) to obtain

$$\frac{h}{2}\sum_{i=1,2}\int_{-1}^{1}\Phi_x(s)\frac{m_{1i}(s)+n_{1i}(s)}{p_{1i}(s)} - \Phi_x(z)\frac{m_{3i}(z)+n_{3i}(z)}{p_{3i}(z)}dt$$

$$= \frac{h}{2} \sum_{i=1,2} \int_{-1}^{1} \left((\Phi_x(s) - \Phi_x(z)) \, \frac{m_{1i}(s) + n_{1i}(s)}{p_{1i}(s)} \right.$$

$$+ \frac{h}{2} \sum_{i=1,2} \int_{-1}^{1} \Phi_x(z) \frac{m_{1i}(s) + n_{1i}(s) - (m_{3i}(z) + n_{3i}(z))}{p_{3i}(z)} dt$$

$$+ \frac{h}{2} \sum_{i=1,2} \int_{-1}^{1} \Phi_x(z) \left((m_{1i}(s) + n_{1i}(s)) \left(\frac{1}{p_{1i}(s)} - \frac{1}{p_{3i}(z)} \right) dt = IV_1 + IV_2 + IV_3. \right.$$

In term IV_1 we can see that

$$\left| \frac{m_{1i}(s) + n_{1i}(s)}{p_{1i}(s)} \right| \leq \frac{C(||\Phi||_3) h}{|\nabla \Phi_{T_{1i}}|_\varepsilon |\nabla \Phi(s)|_\varepsilon} \leq \frac{C(||\Phi||_3) h}{\varepsilon |\nabla \Phi_{T_{1i}}|_\varepsilon}. \tag{3.24}$$

Since ε is fixed in our model and numerical scheme we get

$$|IV_1| \leq C(||\Phi||_3) h^3 \left(\frac{1}{|\nabla \Phi_{T_{11}}|_\varepsilon} + \frac{1}{|\nabla \Phi_{T_{12}}|_\varepsilon} \right),$$

where C depedns on ε. This dependence will not be explicitely stated in further estimates.
Term IV_2 we can estimate similarly. First we have

$$|m_{1i}(s) - m_{3i}(z)| \leq$$

$$\left| \left(\Phi_{yy}(s) \frac{h}{2}(1 - 2t) + O(h^2) \right) \left(\frac{2(\Phi_{r_1} - \Phi_{m_1})}{h} + \Phi_y(s) \right) - \right.$$

$$\left. \left(\Phi_{yy}(z) \frac{h}{2}(1 - 2t) + O(h^2) \right) \left(\frac{2(\Phi_{r_2} - \Phi_{m_3})}{h} + \Phi_y(z) \right) \right| \leq$$

$$\left| (\Phi_{yy}(z) - \Phi_{yy}(s)) \, \frac{h}{2}(1 - 2t) + O(h^2) \right| \left| \frac{2(\Phi_{r_2} - \Phi_{m_3})}{h} + \Phi_y(z) \right| +$$

$$\left| \Phi_{yy}(s) \frac{h}{2}(1 - 2t) + O(h^2) \right| \left| \left(\frac{2(\Phi_{r_2} - \Phi_{m_3})}{h} + \Phi_y(z) \right) - \left(\frac{2(\Phi_{r_1} - \Phi_{m_1})}{h} + \Phi_y(s) \right) \right|$$

and analogously we can do it for term $|n_{1i} - n_{3i}|$. Then we get

$$|IV_2| \leq C(||\Phi||_3) h^3 \sum_{i=1,2} \int_{-1}^{1} |\Phi_x(z)| \left(\frac{|\nabla \Phi_{T_{3i}}|_\varepsilon + |\nabla \Phi(z)|_\varepsilon}{p_{3i}(z)} \right)$$

$$+ C(||\Phi||_2) h^3 \sum_{i=1,2} \int_{-1}^{1} \frac{|\Phi_x(z)|}{p_{3i}(z)}$$

$$\leq C(||\Phi||_3)h^3\left(\frac{1}{|\nabla\Phi_{T_{31}}|_\varepsilon}+\frac{1}{|\nabla\Phi_{T_{32}}|_\varepsilon}\right)+\frac{C(||\Phi||_3)h^3}{\varepsilon}\left(\frac{1}{|\nabla\Phi_{T_{31}}|_\varepsilon}+\frac{1}{|\nabla\Phi_{T_{32}}|_\varepsilon}\right)$$

$$\leq C(||\Phi||_3)h^3\left(\frac{1}{|\nabla\Phi_{T_{31}}|_\varepsilon}+\frac{1}{|\nabla\Phi_{T_{32}}|_\varepsilon}\right).$$

For term IV_3 we get due to (3.24)

$$|IV_3|\leq\frac{h}{2}\sum_{i=1,2}\int_{-1}^{1}|\Phi_x(z)|\frac{|(m_{1i}(s)+n_{1i}(s)|\,|p_{3i}(z)-p_{1i}(s)|}{p_{1i}(s)p_{3i}(z)}dt$$

$$\leq C(||\Phi||_3)h^2\sum_{i=1,2}\int_{-1}^{1}|\Phi_x(z)|\frac{|p_{3i}(z)-p_{1i}(s)|}{p_{3i}(z)|\nabla\Phi(s)|_\varepsilon|\nabla\Phi_{T_{1i}}|_\varepsilon}dt.$$

Now we first estimate

$$|p_{3i}(z)-p_{1i}(s)|=\big||\nabla\Phi_{T_{3i}}|_\varepsilon^2|\nabla\Phi(z)|_\varepsilon+|\nabla\Phi_{T_{3i}}|_\varepsilon|\nabla\Phi(z)|_\varepsilon^2)$$

$$-\left(|\nabla\Phi_{T_{1i}}|_\varepsilon^2|\nabla\Phi(s)|_\varepsilon+|\nabla\Phi_{T_{1i}}|_\varepsilon|\nabla\Phi(s)|_\varepsilon^2\right)\big|$$

$$\leq\left||\nabla\Phi_{T_{3i}}|_\varepsilon^2-|\nabla\Phi_{T_{1i}}|_\varepsilon^2\right||\nabla\Phi(s)|_\varepsilon+|\nabla\Phi_{T_{1i}}|_\varepsilon\left||\nabla\Phi(s)|_\varepsilon^2-|\nabla\Phi(z)|_\varepsilon^2\right|$$

$$+|\nabla\Phi_{T_{3i}}|_\varepsilon^2\,||\nabla\Phi(z)|_\varepsilon-|\nabla\Phi(s)|_\varepsilon|+|\nabla\Phi(z)|_\varepsilon^2\,||\nabla\Phi_{T_{3i}}|_\varepsilon-|\nabla\Phi_{T_{1i}}|_\varepsilon|$$

$$\leq C(||\Phi||_3)h\,||\nabla\Phi(s)|_\varepsilon+|\nabla\Phi_{T_{1i}}|_\varepsilon|+C(||\Phi||_3)h\left(|\nabla\Phi_{T_{3i}}|_\varepsilon^2+|\nabla\Phi(z)|_\varepsilon^2\right).$$

Using this estimate we have

$$|IV_3|\leq C(||\Phi||_3)h^4\sum_{i=1,2}\int_{-1}^{1}\frac{|\Phi_x(z)|\,||\nabla\Phi(s)|_\varepsilon+|\nabla\Phi_{T_{1i}}|_\varepsilon|}{p_{3i}(z)|\nabla\Phi(s)|_\varepsilon|\nabla\Phi_{T_{1i}}|_\varepsilon}dt$$

$$+C(||\Phi||_3)h^3\sum_{i=1,2}\int_{-1}^{1}\frac{|\Phi_x(z)|\left(|\nabla\Phi_{T_{3i}}|_\varepsilon^2+|\nabla\Phi(z)|_\varepsilon^2\right)}{p_{3i}(z)|\nabla\Phi(s)|_\varepsilon|\nabla\Phi_{T_{1i}}|_\varepsilon}dt$$

$$\leq C(||\Phi||_3)h^4\sum_{i=1,2}\int_{-1}^{1}\frac{|\Phi_x(z)|}{p_{3i}(z)}\left(\frac{1}{|\nabla\Phi_{T_{1i}}|_\varepsilon}+\frac{1}{|\nabla\Phi(s)|_\varepsilon}\right)+$$

$$C(||\Phi||_3)h^3\sum_{i=1,2}\int_{-1}^{1}\frac{|\Phi_x(z)-\Phi_x(s)|(|\nabla\Phi_{T_{3i}}|_\varepsilon+|\nabla\Phi(z)|_\varepsilon)^2}{p_{3i}(z)|\nabla\Phi(s)|_\varepsilon|\nabla\Phi_{T_{1i}}|_\varepsilon}dt+$$

$$C(||\Phi||_3)h^3\sum_{i=1,2}\int_{-1}^{1}\frac{|\Phi_x(s)|(|\nabla\Phi_{T_{3i}}|_\varepsilon+|\nabla\Phi(z)|_\varepsilon)^2}{p_{3i}(z)|\nabla\Phi(s)|_\varepsilon|\nabla\Phi_{T_{1i}}|_\varepsilon}dt$$

$$\leq \frac{C(||\Phi||_3)h^4}{\varepsilon}\sum_{i=1,2}\int_{-1}^{1}\frac{|\Phi_x(z)|}{p_{3i}(z)}+\frac{C(||\Phi||_3)h^4}{\varepsilon}\sum_{i=1,2}\int_{-1}^{1}\frac{|\nabla\Phi_{T_{3i}}|_\varepsilon+|\nabla\Phi(z)|_\varepsilon}{|\nabla\Phi_{T_{3i}}|_\varepsilon|\nabla\Phi(z)|_\varepsilon|\nabla\Phi_{T_{1i}}|_\varepsilon}dt+$$

$$C(||\Phi||_3)h^3\sum_{i=1,2}\int_{-1}^{1}\frac{(|\nabla\Phi_{T_{3i}}|_\varepsilon+|\nabla\Phi(z)|_\varepsilon)}{|\nabla\Phi_{T_{3i}}|_\varepsilon|\nabla\Phi(z)|_\varepsilon|\nabla\Phi_{T_{1i}}|_\varepsilon}dt\leq$$

$$\frac{C(||\Phi||_3)h^4}{\varepsilon^2}\left(\frac{1}{|\nabla\Phi_{T_{31}}|_\varepsilon}+\frac{1}{|\nabla\Phi_{T_{32}}|_\varepsilon}\right)+$$

$$\frac{C(||\Phi||_3)h^4}{\varepsilon^2}\left(\frac{1}{|\nabla\Phi_{T_{11}}|_\varepsilon}+\frac{1}{|\nabla\Phi_{T_{12}}|_\varepsilon}\right)+\frac{C(||\Phi||_3)h^3}{\varepsilon}\left(\frac{1}{|\nabla\Phi_{T_{11}}|_\varepsilon}+\frac{1}{|\nabla\Phi_{T_{12}}|_\varepsilon}\right)\leq$$

$$C(||\Phi||_3)h^4\left(\frac{1}{|\nabla\Phi_{T_{11}}|_\varepsilon}+\frac{1}{|\nabla\Phi_{T_{12}}|_\varepsilon}\right)+C(||\Phi||_3)h^3\left(\frac{1}{|\nabla\Phi_{T_{11}}|_\varepsilon}+\frac{1}{|\nabla\Phi_{T_{12}}|_\varepsilon}\right).$$

If we use all these estimates for all edges in III_2 and use the relation (3.18) between b_p^{n-1} and a_{pq}^{n-1} finally we obtain

$$|III_2|\leq C(||\Phi||_3)h+C(||\Phi||_3)h^2$$

In term III_1 we can use Green's theorem to obtain

$$III_1=-\frac{1}{b_p^{n-1}m(p)}\int\limits_p\nabla\cdot\left(\frac{\nabla\Phi(t_n,w)}{|\nabla\Phi(t_{n-1},w)|_\varepsilon}\right)dw$$

$$=-\frac{1}{m(p)}\int\limits_p\left(\frac{1}{b_p^{n-1}}-|\nabla\Phi(t_{n-1},w)|_\varepsilon\right)\nabla\cdot\left(\frac{\nabla\Phi(t_n,w)}{|\nabla\Phi(t_{n-1},w)|_\varepsilon}\right)dw$$

$$-\frac{1}{m(p)}\int\limits_p|\nabla\Phi(t_{n-1},w)|_\varepsilon\nabla\cdot\left(\frac{\nabla\Phi(t_n,w)}{|\nabla\Phi(t_{n-1},w)|_\varepsilon}\right)dw$$

$$=-\frac{1}{m(p)}\int\limits_p\left(\frac{1}{b_p^{n-1}}-|\nabla\Phi(t_{n-1},w)|_\varepsilon\right)\nabla\cdot\left(\frac{\nabla\Phi(t_n,w)}{|\nabla\Phi(t_{n-1},w)|_\varepsilon}\right)dw$$

$$-|\nabla\Phi(t_{n-1},\xi)|_\varepsilon\nabla\cdot\left(\frac{\nabla\Phi(t_n,\xi)}{|\nabla\Phi(t_{n-1},\xi)|_\varepsilon}\right)=V_1+V_2,$$

where ξ is some point in co-volume p from the mean value theorem. First we estimate the difference (we omit again for a moment the variable t_{n-1})

$$\left|\frac{1}{b_p^{n-1}}-|\nabla\Phi(w)|_\varepsilon\right|=\left|\frac{\frac{1}{|\nabla\Phi(w)|_\varepsilon}-b_p^{n-1}}{b_p^{n-1}\frac{1}{|\nabla\Phi(w)|_\varepsilon}}\right|. \tag{3.25}$$

We can use (2.9) and (2.10) in the numerator of (3.25) and get

$$\frac{1}{|\nabla\Phi(w)|_\varepsilon} - b_p^{n-1} =$$

$$= \frac{1}{N_p}\sum_{q\in N(p)} \frac{1}{2}\left(\left(\frac{1}{|\nabla\Phi(w)|_\varepsilon} - \frac{1}{|\nabla\Phi^{n-1}_{T^1_{pq}}|_\varepsilon}\right) + \left(\frac{1}{|\nabla\Phi(w)|_\varepsilon} - \frac{1}{|\nabla\Phi^{n-1}_{T^2_{pq}}|_\varepsilon}\right)\right).$$

From all terms in the sum we present the estimation of only one (concerning the triangle $T_{11} = T^1_{pq_1}$). Other terms can be treated in an analogous way. We use (2.11) and (3.21) to obtain

$$\left|\frac{1}{|\nabla\Phi_{T_{11}}|_\varepsilon} - \frac{1}{|\nabla\Phi(w)|_\varepsilon}\right| = \left|\frac{|\nabla\Phi_{T^1_{pq_1}}|^2 - |\nabla\Phi(w)|^2}{|\nabla\Phi_{T_{11}}|_\varepsilon|\nabla\Phi(w)|_\varepsilon(|\nabla\Phi_{T_{11}}|_\varepsilon + |\nabla\Phi(w)|_\varepsilon)}\right|$$

$$= \left|\frac{(\frac{\Phi_{q_1}-\Phi_p}{h})^2 - (\Phi_x(w))^2 + (\frac{2(\Phi_{r_1}-\Phi_{m_1})}{h})^2 - (\Phi_y(w))^2}{|\nabla\Phi_{T_{11}}|_\varepsilon|\nabla\Phi(w)|_\varepsilon(|\nabla\Phi_{T_{11}}|_\varepsilon + |\nabla\Phi(w)|_\varepsilon)}\right|$$

$$= \left|\frac{(\Phi_x(\xi) - \Phi_x(w))(\Phi_x(\xi) + \Phi_x(w)) + (\Phi_y(\zeta) - \Phi_y(w))(\Phi_y(\zeta) + \Phi_y(w))}{|\nabla\Phi_{T_{11}}|_\varepsilon|\nabla\Phi(w)|_\varepsilon(|\nabla\Phi_{T_{11}}|_\varepsilon + |\nabla\Phi(w)|_\varepsilon)}\right|.$$

Now using the properties of Φ and the inequality $a + b \le \sqrt{2}\sqrt{a^2 + b^2 + \varepsilon^2}$ holding for all $a \ge 0, b \ge 0$, we conclude

$$\left|\frac{1}{|\nabla\Phi_{T_{11}}|_\varepsilon} - \frac{1}{|\nabla\Phi(w)|_\varepsilon}\right| \le \frac{C(||\Phi||_3)h}{|\nabla\Phi_{T_{11}}|_\varepsilon|\nabla\Phi(w)|_\varepsilon}.$$

Employing this type of estimates in (3.25) we have

$$\left|\frac{1}{b_p^{n-1}} - |\nabla\Phi(y)|_\varepsilon\right| \le C(||\Phi||_3)h. \tag{3.26}$$

Now, the term V_1 we can rearrange as follows

$$V_1 = -\frac{1}{m(p)}\int_p \left(\frac{1}{b_p^{n-1}} - |\nabla\Phi(t_{n-1}, w|_\varepsilon\right)\frac{\Delta\Phi(t_n, w)}{|\nabla\Phi(t_{n-1}, w)|_\varepsilon}dw$$

$$-\frac{1}{m(p)}\int_p \left(\frac{1}{b_p^{n-1}} - |\nabla\Phi(t_{n-1}, w|_\varepsilon\right)\nabla\Phi(t_n, w)\cdot\nabla\left(\frac{1}{|\nabla\Phi(t_{n-1}, w)|_\varepsilon}\right)dw = V_{12} + V_{13}$$

Estimation of term V_{12} is straightforward, due to the properties of Φ and the inequality (3.26) we get

$$|V_{12}| \le C(||\Phi||_2)\frac{h}{\varepsilon} \le C(||\Phi||_2)h.$$

In term V_{13} we use

$$\nabla\left(\frac{1}{|\nabla\Phi(t_{n-1},w)|_\varepsilon}\right) = -\frac{1}{|\nabla\Phi(t_{n-1},w)|_\varepsilon^3}\boldsymbol{\Psi}(t_{n-1},w),$$

where for two dimensional problem if $Z=(t_{n-1},w)$ it holds

$$\boldsymbol{\Psi}(Z) = \begin{pmatrix} \Phi_x(Z)\Phi_{xx}(Z)+\Phi_y(Z)\Phi_{xy}(Z) \\ \Phi_x(Z)\Phi_{xy}(Z)+\Phi_y(Z)\Phi_{yy}(Z) \end{pmatrix}$$

with the property

$$|\boldsymbol{\Psi}(Z)| \le C(\|\Phi\|_2)|\nabla\Phi(Z)|. \tag{3.27}$$

Now for V_{13} again taking into account the estimate (3.26) and the properties of Φ, we have

$$|V_{13}| \le C(||\Phi||_2)h\frac{1}{m(p)}\int\limits_p |\nabla\Phi(t_n,w)|\frac{1}{|\nabla\Phi(t_{n-1},w)|_\varepsilon^2}dw$$

$$\le C(||\Phi||_2)h\frac{1}{m(p)}\int\limits_p |\nabla\Phi(t_n,w)\pm|\nabla\Phi(t_{n-1},w)||\frac{1}{|\nabla\Phi(t_{n-1},w)|_\varepsilon^2}dw$$

$$\le C(||\Phi||_2)h\left(\frac{\tau}{\varepsilon^2}+\frac{1}{\varepsilon}\right) \le C(||\Phi||_2)h + C(||\Phi||_2,||\Phi||_1)h\tau.$$

Finally we couple together the term V_2 and the last term of the left hand side of the inequality (3.11) and define

$$VI = -|\nabla\Phi(t_{n-1},\xi)|_\varepsilon\nabla\cdot\left(\frac{\nabla\Phi(t_n,\xi)}{|\nabla\Phi(t_{n-1},\xi)|_\varepsilon}\right) + |\nabla\Phi(X)|_\varepsilon\nabla\cdot\left(\frac{\nabla\Phi(X)}{|\nabla\Phi(X)|_\varepsilon}\right)$$

where $X=(t,x)$ and the points x and ξ belong to co-volume p and $t \in< t_{n-1},t_n >$. Since

$$|\nabla\Phi(X)|_\varepsilon\nabla\cdot\left(\frac{\nabla\Phi(X)}{|\nabla\Phi(X)|_\varepsilon}\right) = \Delta\Phi(X) - \frac{\nabla\Phi(X)\cdot\boldsymbol{\Psi}(X)}{|\nabla\Phi(X)|_\varepsilon^2},$$

where the vector Ψ is defined as above, we obtain

$$|VI| \le \left|-\Delta\Phi(t_n,\xi) + \frac{\nabla\Phi(t_n,\xi)\cdot\boldsymbol{\Psi}(t_{n-1},\xi)}{|\nabla\Phi(t_{n-1},\xi)|_\varepsilon^2} + \Delta\Phi(X) - \frac{\nabla\Phi(X)\cdot\boldsymbol{\Psi}(X)}{|\nabla\Phi(X)|_\varepsilon^2}\right|.$$

Because $|t-t_n| \le \tau$ and $|x-\xi| \le \sqrt{2}h$, we immediately have

$$|\Delta\Phi(X) - \Delta\Phi(t_n,\xi)| \le C(\|\Phi\|_3)(\tau+h)$$

The rest terms we rearrange as follows

$$\frac{\nabla\Phi(t_n,\xi)\cdot\boldsymbol{\Psi}(t_{n-1},\xi)}{|\nabla\Phi(t_{n-1},\xi)|_\varepsilon^2} \pm \frac{\nabla\Phi(X)\cdot\boldsymbol{\Psi}(t_{n-1},\xi)}{|\nabla\Phi(t_{n-1},\xi)|_\varepsilon^2}$$

$$\pm\frac{\nabla\Phi(X)\cdot\boldsymbol{\Psi}(X)}{|\nabla\Phi(t_{n-1},\xi)|_\varepsilon^2}-\frac{\nabla\Phi(X)\cdot\boldsymbol{\Psi}(X)}{|\nabla\Phi(X))|_\varepsilon^2}.$$

Using the properties of Φ, $\boldsymbol{\Psi}$ and (3.27) we have

$$\left|\frac{\nabla\Phi(t_n,\xi)\cdot\boldsymbol{\Psi}(t_{n-1},\xi)}{|\nabla\Phi(t_{n-1},\xi)|_\varepsilon^2}-\frac{\nabla\Phi(X)\cdot\boldsymbol{\Psi}(t_{n-1},\xi)}{|\nabla\Phi(t_{n-1},\xi)|_\varepsilon^2}\right|\leq$$

$$\frac{C(||\Phi||_3)(h+\tau)|\nabla\Phi(t_{n-1},\xi|}{|\nabla\Phi(t_{n-1},\xi)|_\varepsilon^2}\leq C(||\Phi||_3)(h+\tau).$$

Now we denote by $W=(t_{n-1},\xi)$ and we can use

$$|\boldsymbol{\Psi}(W)-\boldsymbol{\Psi}(X)|$$

$$=\left|\begin{array}{c}\Phi_x(W)\Phi_{xx}(W)+\Phi_y(W)\Phi_{xy}(W)-\Phi_x(X)\Phi_{xx}(X)-\Phi_y(X)\Phi_{xy}(X)\\ \Phi_x(W)\Phi_{xy}(W)+\Phi_y(W)\Phi_{yy}(W)-\Phi_x(X)\Phi_{xy}(X)-\Phi_y(X)\Phi_{yy}(X)\end{array}\right|$$

$$\leq C\left(\Phi_x(W)\Phi_{xx}(W)+\Phi_y(W)\Phi_{xy}(W)-\Phi_x(X)\Phi_{xx}(X)-\Phi_y(X)\Phi_{xy}(X)\right|+$$

$$+|\Phi_x(W)\Phi_{xy}(W)+\Phi_y(W)\Phi_{yy}(W)-\Phi_x(X)\Phi_{xy}(X)-\Phi_y(X)\Phi_{yy}(X)|)$$

$$\leq C\left(||\Phi||_3\right)(h+\tau).$$

Then we have

$$\left|\frac{\nabla\Phi(X)\cdot\boldsymbol{\Psi}(t_{n-1},\xi)}{|\nabla\Phi(t_{n-1},\xi)|_\varepsilon^2}-\frac{\nabla\Phi(X)\cdot\boldsymbol{\Psi}(X)}{|\nabla\Phi(t_{n-1},\xi)|_\varepsilon^2}\right|\leq\frac{C(||\Phi||_3)(h+\tau)|\nabla\Phi(X)|}{|\nabla\Phi(t_{n-1},\xi)|_\varepsilon^2}$$

$$\leq C(||\Phi||_3)(h+\tau)\frac{|\nabla\Phi(X)\pm|\nabla\Phi(t_{n-1},\xi)|}{|\nabla\Phi(t_{n-1},\xi)|_\varepsilon^2}\leq C(||\Phi||_3)\left(\frac{(h+\tau)^2}{\varepsilon^2}+\frac{h+\tau}{\varepsilon}\right).$$

Finally using (3.27) we subsequently get

$$\left|\frac{\nabla\Phi(X)\cdot\boldsymbol{\Psi}(X)}{|\nabla\Phi(t_{n-1},\xi)|_\varepsilon^2}-\frac{\nabla\Phi(X)\cdot\boldsymbol{\Psi}(X)}{|\nabla\Phi(X))|_\varepsilon^2}\right|$$

$$\leq|\nabla\Phi(X)\cdot\boldsymbol{\Psi}(X)|\left|\frac{|\nabla\Phi(X)|^2-|\nabla\Phi(t_{n-1},\xi)|^2}{|\nabla\Phi(X)|_\varepsilon^2|\nabla\Phi(t_{n-1},\xi)|_\varepsilon^2}\right|$$

$$\leq C(||\Phi||_2)(h+\tau)|\nabla\Phi(X)|^2\left(\frac{1}{|\nabla\Phi(X)|_\varepsilon|\nabla\Phi(t_{n-1},\xi)|_\varepsilon^2}+\frac{1}{|\nabla\Phi(X)|_\varepsilon^2|\nabla\Phi(t_{n-1},\xi)|_\varepsilon}\right)$$

$$\leq C(||\Phi||_2)(h+\tau)\left(\frac{|\nabla\Phi(X)\pm\nabla\Phi(t_{n-1},\xi)|}{|\nabla\Phi(t_{n-1},\xi)|_\varepsilon^2}+\frac{1}{|\nabla\Phi(t_{n-1},\xi)|_\varepsilon}\right)$$

$$\leq C(||\Phi||_3)\left(\frac{(h+\tau)^2}{\varepsilon^2}+\frac{h+\tau}{\varepsilon}\right)\leq C(||\Phi||_3)((h+\tau)^2+(h+\tau)),$$

what ends the proof.

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