

University of Groningen

## Stability and controllability of planar bimodal linear complementarity systems

Çamlıbel, M.K.; Heemels, W.P.M.H.; Schumacher, J.M.

*Published in:*  
Proc. of the 42th IEEE Conference on Decision and Control

**IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.**

*Document Version*  
Publisher's PDF, also known as Version of record

*Publication date:*  
2003

[Link to publication in University of Groningen/UMCG research database](#)

*Citation for published version (APA):*  
Çamlıbel, M. K., Heemels, W. P. M. H., & Schumacher, J. M. (2003). Stability and controllability of planar bimodal linear complementarity systems. In *Proc. of the 42th IEEE Conference on Decision and Control* (pp. 1651-1656). IEEE (The Institute of Electrical and Electronics Engineers).

### Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

### Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

*Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.*

## Stability and controllability of planar bimodal linear complementarity systems

M.K. Camlibel

Dogus University, Dept. of ECE  
81010 Acibadem, Istanbul, Turkey  
e-mail: k.camlibel@uvt.nl

Dept. of Econometrics and OR  
Tilburg University, P.O. Box 90153  
5000 LE Tilburg, The Netherlands

Dept. of Electrical Engineering  
Eindhoven Uni. of Tech. P.O. Box 513  
5600 MB Eindhoven, The Netherlands

W.P.M.H. Heemels

Dept. of Electrical Eng.  
Eindhoven Uni. of Tech. P.O. Box 513  
5600 MB Eindhoven, The Netherlands  
e-mail: m.heemels@tue.nl

J.M. Schumacher

Dept. of Econometrics and OR  
Tilburg University, P.O. Box 90153  
5000 LE Tilburg, The Netherlands  
e-mail: jms@uvt.nl

**Abstract**—The object of study of this paper is the class of hybrid systems consisting of so-called linear complementarity (LC) systems, that received a lot of attention recently and has strong connections to piecewise affine (PWA) systems. In addition to PWA systems, some of the linear or affine submodels of the LC systems can ‘live’ at lower-dimensional subspaces and re-initializations of the state variable at mode changes is possible. For LC systems we study the stability and controllability problem. Although these problems received for various classes of hybrid systems ample attention, necessary and sufficient conditions, which are explicit and easily verifiable, are hardly found in the literature. For LC systems with two modes and a state dimension of two such conditions are presented.

**Keywords:** Hybrid systems, stability, controllability, complementarity systems, planar systems.

### I. INTRODUCTION

In this paper we study stability and controllability for the linear complementarity class of hybrid dynamical systems. Linear complementarity systems are composed of linear time-invariant systems in which the usual input and output variables are constrained by complementarity conditions. Complementarity conditions are given by a particular set of equalities and inequalities, which are related to the well-known relations between the constraint variables and Lagrange multipliers in the Karush-Kuhn-Tucker conditions for optimality, the voltage-current relationship of ideal diodes, etc. Moreover, strong links exist to piecewise linear and affine systems [1], [2], [3] and other classes of hybrid models like min-max-plus-scaling systems [4] and mixed logic dynamic systems [5]. Other applications of this framework include mechanical systems subject to unilateral constraints, constrained optimal control problems, variable structure systems, systems with saturation, dead zones or Coulomb friction, projected dynamical systems, relay systems and so on (see [6] for an overview). In view of this wide range of applications, it seems worthwhile to study stability and controllability issues for linear complementarity systems as they form one of the fundamental issues in control and systems theory. However, due to the hybrid nature of the system, these issues are far from being trivial as was pointed out in [7], where it is shown

that for simple classes of hybrid systems these questions turn out to be undecidable or computationally intractable.

For switched systems the stability issue has received considerable attention (see [8] for an overview). The main lines of research deal with the case where we have arbitrary switching and one aims at finding a common Lyapunov function for all dynamics. In case of switched linear systems for which we have commuting vector fields (or other conditions on the Lie algebras generated by the matrices defining the linear vector fields), these conditions are explicit [9]. However in the case one is dealing with linear complementarity systems, which are linked to piecewise linear systems (even with certain linear dynamics ‘living’ at lower-dimensional subspaces), the switching is state-dependent and hence, of a particular form. The approaches above only provide conservative sufficient conditions for stability. For given state-dependent switchings, the literature provides mainly approaches based for the search of suitable Lyapunov functions, where conservatism is reduced by looking for more general forms of Lyapunov functions (e.g. piecewise quadratic types [10], [11], multiple Lyapunov functions [12], etc.) and applying the S-procedure. One obtains then *implicit* tests for the system in the form of feasibility of certain sets of linear matrix inequalities. In this paper we aim at providing *explicit* necessary and sufficient conditions, that are straightforward to check, for bimodal (i.e. consisting of two discrete modes) planar linear complementarity systems (including the case where one of the dynamics is active on a lower-dimensional subspace, which is usually not considered in the piecewise linear case).

Also for controllability similar remarks can be made. Controllability of switched linear systems has received considerable attention, if one has to design the switching sequence (see e.g. [13], [14] and the references therein) or for discrete-time piecewise affine systems [15], where mixed-integer feasibility problems (for finite time controllability) have to be solved to verify the controllability of such systems. Other approaches are used in [16], [17], but they do not come up with easily verifiable and explicit conditions. As in the case of stability, we will provide such necessary and sufficient conditions for a subclass of linear complementarity systems.

The following notational conventions will be in force. The symbol  $\mathbb{R}$  denotes the real numbers,  $\mathbb{C}$  complex numbers. All vector inequalities must be understood componentwise. The notation  $x \geq 0$  for an  $n$ -vector  $x$  means that either  $x = 0$  or  $x_j = 0$  for  $1 \leq j \leq i$  and  $x_i > 0$  where  $1 \leq i \leq n-1$ . Let  $A \in \mathcal{X}^{n \times m}$  be a matrix of the elements of the set  $\mathcal{X}$ . We write  $A_{ij}$  for the  $(i, j)$ th element of  $A$ . The transpose of  $A$  is denoted by  $A^T$ . For the vectors  $x$  and  $y$ , we write  $x \perp y$  if  $x^T y = 0$ . Given two matrices  $A \in \mathcal{X}^{n_a \times m}$  and  $B \in \mathcal{X}^{n_b \times m}$ , the matrix obtained by stacking  $A$  over  $B$  is denoted by  $\text{col}(A, B)$ . The symbol  $\mathcal{L}_2^{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$  stands for all  $n$ -tuples of Lebesgue measurable locally square-integrable functions that are defined on  $\mathbb{R}_+$ .

## II. LINEAR COMPLEMENTARITY SYSTEMS

In this paper, we are interested in the linear complementarity systems (LCSs) of the form

$$\dot{x} = Ax + ez + bu \quad (1a)$$

$$\dot{w} = c^T x + dz \quad (1b)$$

$$0 \leq z \perp w \geq 0 \quad (1c)$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $0 \neq b \in \mathbb{R}^{n \times 1}$ ,  $c \in \mathbb{R}^{n \times 1}$ ,  $d \in \mathbb{R}$ , and  $e \in \mathbb{R}^{n \times 1}$ .

Throughout the paper, we assume that either  $d > 0$  or ( $d = 0$  and  $c^T e > 0$ ). This assumption guarantees that for each input  $u \in \mathcal{L}_2^{\text{loc}}(\mathbb{R}_+, \mathbb{R})$  and initial state  $x_0$  (with  $c^T x_0 \geq 0$  in case  $d = 0$ ), there exist a unique (absolutely) continuous state trajectory  $x$  with  $x(0) = x_0$  and a unique pair  $(z, w) \in \mathcal{L}_2^{\text{loc}}(\mathbb{R}_+, \mathbb{R}^{1+1})$  such that the equations (1) hold almost everywhere. For detailed treatment of the well-posedness of LCSs with external inputs, we refer to [18].

The complementarity conditions (1c) imply that either  $z$  or  $w$  is zero at (almost) each time instant. As a consequence, this gives a system with two modes, i.e. a bimodal system.

To see the bimodal structure explicitly, consider first the case  $d > 0$ . Then, one can rewrite (1) as

$$\dot{x} = \begin{cases} Ax + bu & \text{if } c^T x \geq 0, \\ (A - ed^{-1}c^T)x + bu & \text{if } c^T x \leq 0. \end{cases} \quad (2)$$

Consider now the case:  $d = 0$  and  $c^T e > 0$ . Then, one can rewrite (1) as

$$\dot{x} = \begin{cases} Ax + bu & \text{if } (c^T x, c^T Ax + c^T bu) \geq 0, \\ P(Ax + bu) & \text{if } c^T x = 0 \text{ and } c^T Ax + c^T bu \leq 0 \end{cases} \quad (3)$$

where  $P = I - e(c^T e)^{-1}c^T$ .

## III. STABILITY

In this section, we will be dealing with the linear complementarity systems (without external input  $u$ ) of the form

$$\dot{x} = Ax + ez \quad (4a)$$

$$w = c^T x + dz \quad (4b)$$

$$0 \leq z \perp w \geq 0 \quad (4c)$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $c \in \mathbb{R}^{n \times 1}$ ,  $d \in \mathbb{R}$ , and  $e \in \mathbb{R}^{n \times 1}$ . As usual, the system (4) is said to be *asymptotically stable* if all possible state trajectories  $x$  satisfy  $\lim_{t \rightarrow \infty} x(t) = 0$ . A solution  $(z, x, w)$  of the system is called *periodic* if all three functions are periodic.

**Remark III.1** Normally, one also includes Lyapunov stability in the definition of asymptotic stability. Due to the structure of the system, we get Lyapunov stability for free in case we have asymptotic stability as defined above. Moreover, in that case we even have global exponential stability and asymptotic Lyapunov stability (see, e.g., [19] for the exact definitions).

Note that (2) is replaced by

$$\dot{x} = \begin{cases} Ax & \text{if } c^T x \geq 0, \\ (A - ed^{-1}c^T)x & \text{if } c^T x \leq 0. \end{cases} \quad (5)$$

and (3) by

$$\dot{x} = \begin{cases} Ax & \text{if } (c^T x, c^T Ax) \geq 0, \\ PAx & \text{if } c^T x = 0 \text{ and } c^T Ax \leq 0. \end{cases} \quad (6)$$

where  $P = I - e(c^T e)^{-1}c^T$ .

Consider the system (5). Suppose that  $A$  has a real eigenvalue, say  $\rho$ . Let  $v$  be an eigenvector corresponding to this eigenvalue. We can assume that  $c^T v \geq 0$  without loss of generality. The state trajectory of (5) that starts from the initial state  $x_0 = v$  is  $x(t) = \exp(\rho t)v$ . Depending on the sign of the eigenvalue  $\rho$ , this trajectory might be stable or unstable. This argumentation gives the following necessary condition for stability with arbitrary state space dimension  $n$ .

**Lemma III.2** *Suppose that  $d > 0$ . A necessary condition for the asymptotic stability of the system (4) is that neither  $A$  nor  $A - ed^{-1}c^T$  has a real nonnegative eigenvalue.*

When the state space dimension (i.e.,  $n$ ) is 2, one can derive necessary and sufficient conditions as in the following theorem.

**Theorem III.3** *Consider the LCS (4) with  $n = 2$  and  $(c^T, A)$  is an observable pair. The following statements hold.*

- 1) *Suppose that  $d > 0$ . The LCS (4) is asymptotically stable if and only if*
  - a) *neither  $A$  nor  $A - ed^{-1}c^T$  has a real nonnegative eigenvalue, and*
  - b) *if both  $A$  and  $A - ed^{-1}c^T$  have nonreal eigenvalues then  $\sigma_1/\omega_1 + \sigma_2/\omega_2 < 0$  where  $\sigma_1 \pm i\omega_1$  ( $\omega_1 > 0$ ) are the eigenvalues of  $A$  and  $\sigma_2 \pm i\omega_2$  ( $\omega_2 > 0$ ) are the eigenvalues of  $A - ed^{-1}c^T$ .*
- 2) *Suppose that  $d > 0$ . The LCS (4) has a nonconstant periodic solution if and only if both  $A$  and  $A - ed^{-1}c^T$*

have nonreal eigenvalues, and  $\sigma_1/\omega_1 + \sigma_2/\omega_2 = 0$  where  $\sigma_1 \pm i\omega_1$  ( $\omega_1 > 0$ ) are the eigenvalues of  $A$  and  $\sigma_2 \pm i\omega_2$  ( $\omega_2 > 0$ ) are the eigenvalues of  $A - ed^{-1}c^T$ . Moreover, if there is one periodic solution, then all other solutions are also periodic. And,  $\pi/\omega_1 + \pi/\omega_2$  is the period of any solution.

- 3) Suppose that  $d = 0$ . The LCS (4) is asymptotically stable if and only if  $A$  has no real nonnegative eigenvalue and  $[I - e(c^T e)^{-1}c^T]A$  has a real negative eigenvalue (note that one eigenvalue is already zero).

**Remark III.4** Observe that the conditions derived in Theorem III.3 item 1 are connected to the ones obtained in [20], where a stabilizing controller of the type  $\max(0, Fx)$  was designed for a linear system with nonnegative control inputs. As the closed-loop actually becomes a linear complementarity system, the design of the matrix  $F$  must be such that the closed-loop system satisfies the conditions above.

#### IV. CONTROLLABILITY

Consider the LCS (1) with  $n = 2$ , and  $d > 0$ , i.e. the piecewise linear system (2) with  $n = 2$ . We say that LCS (1) is *controllable* if for each pair of states  $(x_-, x_+) \in \mathbb{R}^{2+2}$  there exist an input  $u \in \mathcal{L}_2^{\text{loc}}(\mathbb{R}_+, \mathbb{R})$  and  $T > 0$  such that the state trajectory  $x$  of (1) satisfies  $x(0) = x_-$  and  $x(T) = x_+$ .

Our first aim is to establish necessary conditions for controllability. To do so, we distinguish two cases:  $c^T b \neq 0$  and  $c^T b = 0$ .

Case 1:  $c^T b \neq 0$

Let  $f, g \in \mathbb{R}^2$  be such that  $f^T b = 0$ ,  $c^T g = 0$ , and  $f^T g = 1$ . Then, we have

$$\begin{bmatrix} f^T \\ c^T/c^T b \end{bmatrix}^{-1} = \begin{bmatrix} g & b \end{bmatrix}. \quad (7)$$

Define  $\xi_1 = f^T x$  and  $\xi_2 = c^T x/c^T b$ . In these new  $\xi$  coordinates (2) can be written as

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} K & L_1 \\ M & N_1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u & \text{if } c^T b \xi_2 \geq 0 \\ \begin{bmatrix} K & L_2 \\ M & N_2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u & \text{if } c^T b \xi_2 \leq 0 \end{cases} \quad (8)$$

where  $K := f^T Ag$ ,  $L_1 := f^T Ab$ ,  $L_2 := f^T (A - ed^{-1}c^T)b$ ,  $M = c^T Ag/c^T b$ ,  $N_1 = c^T Ab/c^T b$ , and  $N_2 = c^T (A - ed^{-1}c^T)b/c^T b$ . Let  $(\xi, u)$  satisfy (8). Then, one gets

$$\xi_1(t) = \int_0^t \exp(K(t-s))\eta(s) ds \quad (9)$$

where

$$\eta(t) = \begin{cases} L_1 \xi_2(t) & \text{if } c^T b \xi_2(t) \geq 0 \\ L_2 \xi_2(t) & \text{if } c^T b \xi_2(t) \leq 0 \end{cases} \quad (10)$$

Note that if

$$L_1 L_2 = f^T Ab \cdot f^T (A - ed^{-1}c^T)b \leq 0 \quad (11)$$

then  $\xi_1$ , given by (9), is either nonnegative or nonpositive. However, this would mean that zero initial state cannot be steered to certain final states, and hence lack of controllability.

Case 2:  $c^T b = 0$

Suppose now that  $c^T b = 0$ . In this case, one can take  $f = c$ . As a consequence, (11) holds if and only if  $c^T Ab = 0$ , i.e.  $(c^T, A)$  is not observable.

Therefore, if  $(c^T, A)$  is observable then

$$f^T Ab \cdot f^T (A - ed^{-1}c^T)b > 0 \quad (12)$$

is a necessary condition for controllability of the LCS (1). This shows that the piecewise linear system (2) is controllable only if the linear dynamics of both sides are controllable. Indeed, lack of controllability of one of the linear dynamics would mean  $f^T Ab = 0$  or  $f^T (A - ed^{-1}c^T)b = 0$  and thus violation of (12).

It turns out that the necessary condition of (12) is also sufficient as the following theorem states.

**Theorem IV.1** Consider the system (1) with  $n = 2$ ,  $(c^T, A)$  is observable, and  $d > 0$ . It is controllable if and only if (12) holds where  $f$  is such that  $f^T e = 0$ .

**Remark IV.2** The condition (12) is equivalent to saying that the determinants of the controllability matrices of  $(A, e)$  and  $(A - bd^{-1}c^T, e)$  should have the same sign. To see this, take a vector  $f$  such that  $f^T e = 0$  and  $f^T Ae = 1$ . This can be achieved as the pair  $(A, e)$  is necessarily controllable. Let the matrix  $\text{col}(g^T, f^T)$  be the inverse of the controllability matrix  $\begin{bmatrix} e & Ae \end{bmatrix}$ . Then, we have  $g^T e = 1$ . So, we obtain

$$\begin{bmatrix} g^T \\ f^T \end{bmatrix} \begin{bmatrix} e & (A - bd^{-1}c^T)e \end{bmatrix} = \begin{bmatrix} 1 & g^T (A - bd^{-1}c^T)e \\ 0 & f^T (A - bd^{-1}c^T)e \end{bmatrix}.$$

Note that the determinant of the right hand side is  $f^T (A - bd^{-1}c^T)e$  and it is positive if and only if the determinants of  $\begin{bmatrix} e & Ae \end{bmatrix}$  and  $\begin{bmatrix} e & (A - bd^{-1}c^T)e \end{bmatrix}$  have the same sign.

#### V. CONCLUSIONS

In this paper we studied the stability and controllability problem for the linear complementarity class of hybrid systems with state dimension two and two modes. Easily verifiable and explicit necessary and sufficient conditions were derived for this case and some necessary conditions for the stability of higher order bimodal linear complementarity systems have been presented. Of course, it would be nice to generalize these conditions to higher order and multi-modal systems. However, a direct generalization of the proofs seems

to be hard as they use a lot of insight related to the two-dimensionality of state space (e.g. phase portraits, explicit computation of trajectories, etc.). It is a very interesting topic of further research to see in which directions extensions are possible.

## VI. APPENDIX: PROOFS

### A. Proof of Theorem III.3

1: The following lemma will clear the way to the proof of the theorem for the case  $d > 0$ .

**Lemma VI.1** Consider the LCS (4) with  $n = 2$ ,  $d > 0$ , and  $(c^T, A)$  is an observable pair. The following statements hold.

- i) If one of the matrices  $A$  and  $A - ed^{-1}c^T$  has only real negative eigenvalues, then the LCS (4) is asymptotically stable if and only if the other does not have real nonnegative eigenvalues.
- ii) If both matrices  $A$  and  $A - ed^{-1}c^T$  have nonreal eigenvalues, then the LCS (4) is asymptotically stable if and only if  $\sigma_1/\omega_1 + \sigma_2/\omega_2 < 0$  where  $\sigma_1 \pm i\omega_1$  ( $\omega_1 > 0$ ) are the eigenvalues of  $A$  and  $\sigma_2 \pm i\omega_2$  ( $\omega_2 > 0$ ) are the eigenvalues of  $A - ed^{-1}c^T$ .

*Proof:* By means of a state space transformation  $\xi = Sx$ , we can always bring the pair  $(c^T, A)$  in the observability canonical form. In other words, (5) can be taken as

$$\dot{\xi} = \begin{cases} \begin{bmatrix} a_1 & 1 \\ a_2 & 0 \end{bmatrix} \xi & \text{if } \xi_1 \geq 0, \\ \begin{bmatrix} a_1 - e_1 d^{-1} & 1 \\ a_2 - e_2 d^{-1} & 0 \end{bmatrix} \xi & \text{if } \xi_1 \leq 0. \end{cases} \quad (13)$$

Let  $A_1$  be the first one and  $A_2$  be the second one of the above matrices.

i: Suppose, for the moment, that  $A_1$  has real negative eigenvalues. Therefore, the statement that we want to prove is that LCS (4) is asymptotically stable if and only if  $A_2$  does not have real nonnegative eigenvalues. The 'only if' part follows from Lemma III.2. Suppose now that  $A_2$  does not have real nonnegative eigenvalues. Any trajectory of the system  $\dot{\xi} = A_2\xi$  with an initial state  $\xi(0)$  such that  $\xi_1(0) \leq 0$

- either satisfies  $\xi_1(t) \leq 0$  for all  $t$ ,
- or there exists a  $\tau > 0$  such that  $\xi_1(\tau) > 0$ .

In the former case, both  $\xi_1$  and  $\xi_2$  must converge to zero as this corresponds to the case for which both eigenvalues of  $A_2$  are real negative. In the latter, the dynamics  $\dot{\xi} = A_1\xi$  with the constraint  $\xi_1 \geq 0$  and an initial condition  $\xi(0)$  such that  $\xi_1(0) = 0$  and  $\xi_2(0) > 0$  becomes active. In this case, we would get  $\xi_1(t) = c(\exp(\lambda_2 t) - \exp(\lambda_1 t))$  for some  $c > 0$  if  $A_1$  has two distinct eigenvalues  $\lambda_1 < \lambda_2 < 0$  or  $\xi_1(t) = ct \exp(\lambda t)$  for some  $c > 0$  if  $A_1$  has one eigenvalue  $\lambda$  with multiplicity two. It can be verified, in either case, that  $\xi_1(t) > 0$  for all  $t$ . Consequently, there cannot be mode

changes anymore. This means that the trajectories of the system converge to zero as time tends to infinity since  $A_1$  has real negative eigenvalues. If we swap  $A_1$  and  $A_2$ , the above argumentation is still valid with sign modifications.

ii: Consider the dynamics  $\dot{\xi} = A_1\xi$  with the constraint  $\xi_1 \geq 0$ . Suppose that  $\xi_1(0) = 0$  and  $\xi_2(0) > 0$ . Since both eigenvalues of  $A_1$  are nonreal, the  $\xi_1$ - $\xi_2$  trajectories are elliptical and hence they cross the  $\xi_2$  axis again. In other words, there exists a  $T$  such that  $\exp(A_1 T)\xi(0) = \rho\xi(0)$ . This means that  $\rho$  is an eigenvalue of  $\exp(A_1 T)$ , i.e., either  $\sigma_1 + i\omega_1$  or  $\sigma_1 - i\omega_1$ . The constraint  $\xi_1 \geq 0$  yields that  $\rho = \exp((\sigma_1 + i\omega_1)T)$ . However,  $\rho$  should be real. Then, we get  $\omega_1 T = \pi$  and thus  $\rho = -\exp(\sigma_1 \pi / \omega_1)$ . This means that we have a Poincaré mapping  $\Pi_1 : \{\zeta \mid \zeta > 0\} \rightarrow \{\zeta \mid \zeta < 0\}$  given by  $\Pi_1(\zeta) = -\zeta \exp(\sigma_1 \pi / \omega_1)$ . In a similar fashion, for the dynamics corresponding to  $A_2$ , one can find another Poincaré mapping  $\Pi_2 : \{\zeta \mid \zeta < 0\} \rightarrow \{\zeta \mid \zeta > 0\}$  given by  $\Pi_2(\zeta) = -\zeta \exp(\sigma_2 \pi / \omega_2)$ . Clearly, the LCS is asymptotically stable if and only if  $\Pi_2(\Pi_1(\zeta)) < \zeta$ , and this holds if and only if  $\sigma_1/\omega_1 + \sigma_2/\omega_2 < 0$ . ■

The proof of Theorem III.3 item 1 follows from Lemma III.2 and Lemma VI.1. For a proof of Theorem III.3 item 2, it is enough to consider the above defined Poincaré mappings and to note that existence of a periodic solution is equivalent to saying that  $\Pi_2(\Pi_1(\zeta)) = \zeta$  for some  $\zeta$ , and this is equivalent to saying that  $\sigma_1/\omega_1 + \sigma_2/\omega_2 = 0$ . Furthermore, all solutions are periodic as soon as  $\sigma_1/\omega_1 + \sigma_2/\omega_2 = 0$  holds.

Theorem III.3 item 3 can be proven as follows. Suppose that  $A$  has a real nonnegative eigenvalue. It follows from the discussion preceding Lemma III.2 that the LCS cannot be stable in this case. Suppose now that  $PA$ , where  $P = I - e(c^T e)^{-1}c^T$ , does not have a real negative eigenvalue. Since  $Pe = 0$ ,  $P$  is of at most rank 1. So, is  $PA$ . Then, there can be only two possibilities: either  $PA$  has a zero eigenvalue and a real positive eigenvalue or it has two zero eigenvalues. In the former case, we can show that the LCS is unstable as follows. Let  $v$  be an eigenvector corresponding to the positive eigenvalue  $\rho$ . Then, we have  $PAv = \rho v$ . By pre-multiplying both sides by  $c^T$ , we get  $c^T v = 0$  since  $\rho \neq 0$  and  $c^T P = 0$ . Due to the observability of  $(c^T, A)$ , we know that  $c^T Av \neq 0$ . Therefore, we can assume that  $c^T Av < 0$  without loss of generality. It is easy to check that  $x(t) = \exp(\rho t)v$  satisfies  $\dot{x} = PAx$ ,  $c^T x(t) = 0$ , and  $c^T Ax(t) < 0$ . Positivity of  $\rho$  destroys stability of the LCS. In the other case,  $PA$  has only zero eigenvalues. Again, there are only two possibilities: either  $PA = 0$ , which would immediately lead to instability, or the geometric multiplicity of the zero eigenvalue is 1, i.e., there exist  $v$  and  $w$  such that  $PAv = 0$  and  $PAw = v$ . From the last equality, we get  $c^T v = 0$  and the constant trajectory starting from the initial state  $v$  destroys asymptotical stability. This concludes the proof of 'only if' part. For the 'if part', consider the dynamics of the mode  $\dot{x} = Ax$  with  $(c^T x, c^T Ax) \neq 0$ .

Since  $A$  does not have any real nonnegative eigenvalue, state trajectories either converge to zero or hit the boundary  $\{x \mid c^T x = 0 \text{ and } c^T A x < 0\}$ . Then, the dynamics  $\dot{x} = PAx$  with  $c^T x = 0$  and  $c^T A x \leq 0$  starting from an initial state  $x_0$  such that  $c^T x_0 = 0$  and  $c^T A x_0 < 0$  becomes active. We claim that  $x_0$  is an eigenvector of  $PA$ . To see this, consider any eigenvector  $v$  of  $PA$  corresponding to the real negative eigenvalue  $\rho$ . Then, we have  $PAv = \rho v$ . By pre-multiplying both sides by  $c^T$ , we get  $c^T v = 0$  since  $\rho \neq 0$  and  $c^T P = 0$ . Therefore,  $c^T x_0 = 0$  implies that  $x_0$  is an eigenvector. Then, the solution of  $\dot{x} = PAx$  with  $x(0) = x_0$  is  $x(t) = \exp(\rho t)x_0$ . Further, we have  $c^T x(t) = 0$  and  $c^T A x(t) < 0$ . This means that there will be no more mode changes. Negativity of  $\rho$  implies asymptotic stability of the LCS. ■

### B. Proof of Theorem IV.1

Necessity of (12) has already been proved in the paragraphs preceding the statement of the theorem. The rest of the proof is inspired by [21]. To prove sufficiency, we distinguish two cases,  $c^T b \neq 0$  and  $c^T b = 0$ .

#### Case 1: $c^T b \neq 0$

Without loss of generality, we can assume that we are dealing with the piecewise linear system (8). Consider the system

$$\dot{\xi}_1 = K\xi_1 + \eta \quad (14)$$

where

$$\eta(t) = Q(\xi_2(t)) = \begin{cases} L_1 \xi_2(t) & \text{if } c^T b \xi_2 \geq 0 \\ L_2 \xi_2(t) & \text{if } c^T b \xi_2 \leq 0. \end{cases} \quad (15)$$

Suppose that for any pair of  $(\xi^0, \xi^F) \in \mathbb{R}^{2+2}$  we can find a real number  $T > 0$  and an absolutely continuous  $\xi_2$  such that  $\xi_2(0) = \xi_2^0$ ,  $\xi_2(T) = \xi_2^F$ , and there exists a solution to (14) with  $\xi_1(0) = \xi_1^0$  and  $\xi_1(T) = \xi_1^F$ . Then, the input

$$u = \begin{cases} \dot{\xi}_2 - M\xi_1 - N_1\xi_2 & \text{if } c^T b \xi_2 \geq 0 \\ \dot{\xi}_2 - M\xi_1 - N_2\xi_2 & \text{if } c^T b \xi_2 \leq 0 \end{cases} \quad (16)$$

would steer the initial state  $\xi^0$  to the final state  $\xi^F$  under the dynamics of (8). This means that (8) is controllable if (14) is controllable with  $\eta \in \Omega_{(\alpha, \beta, T)}$  for all  $(\alpha, \beta) \in \mathbb{R}^{1+1}$  and  $T > 0$  where

$$\Omega_{(\alpha, \beta, T)} = \{\eta \mid \eta = Q(\xi_2) \text{ for some abs. cont. } \xi_2 \text{ with } \xi_2(0) = \alpha, \xi_2(T) = \beta\}. \quad (17)$$

We even claim that controllability of (14) with  $\eta \in \Omega_{(0,0,T)}$  for some  $T > 0$  would suffice for controllability of (8). To see this, note that any initial state  $(\xi_1^0, \xi_2^0)$  with  $\xi_2^0 \neq 0$  can be steered to a state of the form  $(\xi_1^i, 0)$ . Indeed, an input that does the job can be obtained from (16) by taking  $\xi_2(t) = -\xi_2(0)t + \xi_2(0)$  and solving (14). A similar argument on the time-reversed version of (14) shows that for any state  $(\xi_1^F, \xi_2^F)$  there exists a state of the form  $(\xi_1^i, 0)$  such that

$(\xi_1^i, 0)$  can be steered to  $(\xi_1^F, \xi_2^F)$ . As a consequence of the above analysis, we concentrate on the system

$$\dot{\xi}_1 = K\xi_1 + \eta \quad \eta \in \Omega_{(0,0,T)} \quad (18)$$

for some  $T > 0$ . Define

$$\mathcal{R}_T := \{x \mid x = \int_0^T \exp(K(T-s))\eta(s) ds \text{ for some } \eta \in \Omega_{(0,0,T)}\}.$$

Basically,  $\mathcal{R}_T$  is the set of all states which can be reached from zero at time  $t = T$  under the dynamics of (18). Note that  $L_1 L_2 > 0$  implies that there exists  $\eta_-, \eta_+ \in \Omega_{(0,0,T)}$  such that  $0 \geq \eta_- \neq 0$  and  $0 \neq \eta_+ \geq 0$ . Therefore,  $\mathcal{R}_T$  contains a neighborhood of zero as  $\Omega_{(0,0,T)}$  is a cone. This means that we can reach any point from the origin since  $\Omega_{(0,0,T)}$  is a cone. Now, reverse the time in (18) and apply the same argumentation as above. This would show that any point can be steered to zero. Consequently, the system (18) is controllable and so is LCS (1).

#### Case 2: $c^T b = 0$

Take  $f = c$ . Let  $g \in \mathbb{R}^2$  be such that  $c^T g = 1$  and  $b^T g = 0$ . Then, we have

$$\begin{bmatrix} c^T \\ b^T / \|b\| \end{bmatrix}^{-1} = \begin{bmatrix} g & b \end{bmatrix}. \quad (19)$$

Define  $\xi_1 = c^T x$  and  $\xi_2 = b^T x / \|b\|$ . In these new coordinates (2) can be written as

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} K_1 & L \\ M_1 & N \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u & \text{if } \xi_1 \geq 0 \\ \begin{bmatrix} K_2 & L \\ M_2 & N \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u & \text{if } \xi_1 \leq 0 \end{cases} \quad (20)$$

where  $K_1 := c^T A g$ ,  $K_2 := c^T (A - ed^{-1}c^T)g$ ,  $L := c^T A b$ ,  $M_1 = b^T A g / \|b\|$ ,  $M_2 = b^T (A - ed^{-1}c^T)g / \|b\|$ , and  $N = b^T A b / \|b\|$ . Note that  $L \neq 0$  as  $(c^T, A)$  is observable. Suppose that the initial state  $\xi^0 \in \mathbb{R}^2$  is desired to be steered to the final state  $\xi^F \in \mathbb{R}^2$ . Let  $\xi_1(t)$  be a third order polynomial in the indeterminate  $t$  such that

$$\begin{aligned} 1) & \xi_1(0) = \xi_1^0 \text{ and } \xi_1(T) = \xi_1^F, \\ 2) & \dot{\xi}_1(0) = \begin{cases} K_1 \xi_1^0 + L \xi_2^0 & \text{if } \xi_1^0 \geq 0, \\ K_2 \xi_1^0 + L \xi_2^0 & \text{if } \xi_1^0 \leq 0. \end{cases} \\ 3) & \dot{\xi}_1(T) = \begin{cases} K_1 \xi_1^F + L \xi_2^F & \text{if } \xi_1^F \geq 0, \\ K_2 \xi_1^F + L \xi_2^F & \text{if } \xi_1^F \leq 0. \end{cases} \end{aligned}$$

Define  $\xi_2$  as

$$\xi_2(t) = \begin{cases} L^{-1}(\dot{\xi}_1(t) - K_1 \xi_1(t)) & \text{if } \xi_1(t) \geq 0, \\ L^{-1}(\dot{\xi}_1(t) - K_2 \xi_1(t)) & \text{if } \xi_1(t) \leq 0. \end{cases} \quad (21a)$$

Note that  $\xi_2$  is continuous and 'piecewise polynomial', thus almost everywhere differentiable. Define  $u$  as

$$u(t) = \begin{cases} \dot{\xi}_2(t) - M_1\xi_2(t) - N\xi_1(t) & \text{if } \xi_1(t) \geq 0, \\ \dot{\xi}_2(t) - M_2\xi_2(t) - N\xi_1(t) & \text{if } \xi_1(t) \leq 0. \end{cases} \quad (21b)$$

It can be verified that the triple  $(u, \xi_1, \xi_2)$  satisfy (20). Moreover,  $\xi_2(0) = \xi_2^0$  and  $\xi_2(T) = \xi_2^F$ . ■

## VII. REFERENCES

- [1] M.K. Çamlıbel and J.M. Schumacher, "Existence and uniqueness of solutions for a class of piecewise linear dynamical systems", *Linear Algebra and its Applications*, vol. 351-352, pp. 147-184, 2002.
- [2] B. C. Eaves and C. E. Lemke, "Equivalence of LCP and PLS", *Mathematics of Operations Research*, vol. 6, pp. 475-484, 1981.
- [3] W.P.M.H. Heemels, B. De Schutter and A. Bemporad, "Equivalence of Hybrid Dynamical Models", *Automatica*, vol. 37, n. 7, pp. 1085-1091, 2001.
- [4] B. De Schutter and T. van den Boom, "On model predictive control for max-min-plus-scaling discrete event systems", *Automatica*, vol. 37, n. 7, pp. 1049-1056, 2001.
- [5] A. Bemporad and M. Morari, "Control of systems integrating logic, dynamics, and constraints", *Automatica*, vol. 35, n. 3, pp. 407-427, 1999.
- [6] W.P.M.H. Heemels, J.M. Schumacher and S. Weiland, "Applications of complementarity systems", in *Proceedings of the European Control Conference in Karlsruhe (Germany)*, 1999.
- [7] V.D. Blondel and J.N. Tsitsiklis, "Complexity of stability and controllability of elementary hybrid systems", *Automatica*, vol. 35, n. 3, pp. 479-490, 1999.
- [8] D. Liberzon and A.S. Morse, "Basic problems in stability and design of switched systems", *IEEE Control Systems Magazine*, pp. 59-70, 1999.
- [9] D. Liberzon, J.P. Hespanha and A.S. Morse, "Stability of switched systems: a Lie-algebraic condition", *System and Control Letters*, vol. 37, pp. 117, 1999.
- [10] M. Johansson and A. Rantzer, "Computation of piecewise quadratic Lyapunov functions for hybrid systems", *IEEE Transactions on Automatic Control*, vol. 43, n. 4, pp. 555-559, 1998.
- [11] D. Mignone, G. Ferrari-Trecate and M. Morari, "Stability and stabilization of piecewise affine and hybrid systems: an LMI approach", in *Proceedings of the 39th IEEE Conference on Decision and Control*, pp. 504-509, 2000.
- [12] M.S. Branicky, "Multiple Lyapunov theory and other analysis tools for switched and hybrid systems", *IEEE Transactions on Automatic Control*, vol. 43, n. 4, pp. 475-482, 1998.
- [13] Z. Sun, S.S. Ge and T.H. Lee, "Controllability and reachability criteria for switched linear systems", *Automatica*, vol. 38, pp. 775-786, 2002.
- [14] G. Xie, D. Zheng and L. Wang, "Controllability of switched linear systems", *IEEE Trans. Automatic Control*, vol. 47, n. 8, pp. 1401-1405, 2002.
- [15] A. Bemporad, G. Ferrari-Trecate and M. Morari, "Observability and controllability of piecewise affine and hybrid systems", *IEEE Transactions on Automatic Control*, vol. 45, n. 10, pp. 1864-1876, 2000.
- [16] V.M. Veliov and M.I. Krastanov, "Controllability of piecewise linear systems", *Systems & Control Letters*, vol. 7, pp. 335-341, 1986.
- [17] K.K. Lee and A. Arapostathis, "On the controllability of piecewise-linear hypersurface systems", *Systems & Control Letters*, vol. 9, pp. 89-96, 1987.
- [18] M.K. Çamlıbel, W.P.M.H. Heemels and J.M. Schumacher, "On linear passive complementarity systems", special issue 'Dissipativity of Dynamical Systems: Applications in Control' (dedicated to V.M. Popov) of *European Journal of Control*, vol. 8, n. 3, 2002.
- [19] H.K. Khalil, *Nonlinear Systems*, MacMillan, New York, 1992.
- [20] F. Willems, W.P.M.H. Heemels, B. de Jager and A.A. Stoorvogel, "Positive feedback stabilization of centrifugal compressor surge".
- [21] R.F. Brammer, "Controllability in linear autonomous systems with positive controllers", *SIAM J. Control*, vol. 10, n. 2, pp. 329-353, 1972.