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STABILITY AND FREE ENERGIES IN LINEAR VISCO-ELASTICITY

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Dedicated to the memory of Gaetano Fichera

In this paper we show that several free energies can be related to a material with fading memory with different domains of definition and topologies. As Fichera has proved, the study of stability can be affected by the space and then by the topology which we chose, while the material must be independent by this one.

In the last part of the work by the semi-group theory, we use the new notion of minimum state to study the asymptotic behavior of the differential system.

1. Introduction

The fundamental work of Gaetano Fichera on the fading memory materials was presented in several papers, whose in particular we remember [14], [15], [16], [17] and [18].

The Fichera viewpoint can be summarize in the following remark.

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Let us consider the two operators

$$T_\infty(\varepsilon^t) = g_0\varepsilon(t) + \int_0^\infty g'(s)\varepsilon(t-s)ds \tag{1}$$

$$T_{t_0}(\varepsilon^t) = g_0\varepsilon(t) + \int_0^{t_0} g'(s)\varepsilon(t-s)ds \tag{2}$$

defined on the history space

$$\mathscr{H} = \left\{ \varepsilon(t-s); \int_0^\infty g'(s)\,|\varepsilon(t-s)|\,ds < \infty \right\}.$$

with g' a real positive function and g_0 a positive constant. He emphasizes as the operator (2) presents a spectrum with only one element g_0, while the first one has a spectrum which can have a continuous domain. He said in [18]: "*L'operatore T_{t_0} che, in ipotesi di modesta regolarità per g', è insensibile alla spazio funzionale nel quale si studia l'equazione*

$$\sigma(t) = g_0\varepsilon(t) + \int_0^{t_0} g'(s)\varepsilon(t-s)ds,$$

diventa sensibilissimo alla scelta di tale spazio se $t_0 = \infty$".

In this paper we show as the sensibility of T_∞ with respect to the space is connected to the different choices of the free energy related to the operator. This occurrence implies that several choices are possible for the notion of stability jointed with fading memory materials.

To this aim, we want to face some problems about the definition of state for linear materials[1] with fading memory, which the stress constitutive equation is

$$\mathbf{T}(t) = \mathbb{G}_0\mathbf{E}(t) + \int_0^\infty \dot{\mathbb{G}}(s)\mathbf{E}^t(s)\,ds,$$

where $\mathbf{E}(t) = \frac{1}{2}[\nabla\mathbf{u}(t) + (\nabla\mathbf{u}(t))^T]$ is the infinitesimal strain tensor and $\mathbf{E}^t(s) = \mathbf{E}(t-s)$ and its history. We stress that it is more natural to describe the state through the actual value of the infinitesimal strain tensor and the stress response $\mathbf{I}_r^t(\cdot)$ associated to the static continuation, instead of using $\mathbf{E}(t)$ and the past history $\mathbf{E}_r^t(\cdot) = \mathbf{E}^t(s) - \mathbf{E}(t)$ of the infinitesimal strain tensor. This new approach is supported by the relation

$$\mathbf{T}(t+\tau) = \mathbb{G}_0\mathbf{E}(t+\tau) + \int_0^\tau \dot{\mathbb{G}}(s)\mathbf{E}^{t+\tau}(s)\,ds - \mathbf{I}_r^t(\tau),$$

[1] Some considerations of this paper can be extend to particular non linear materials with memory, for example to materials with constitutive equation

$$\mathbf{T}(t) = \Phi(\mathbf{E}(t)) + \int_0^\infty \dot{\mathbb{G}}(s)\Phi(\mathbf{E}^t(s))\,ds,$$

where Φ is a suitable regular function.

where

$$\mathbf{I}_r^t(\tau) = -\int_0^\infty \dot{\mathbb{G}}(s+\tau)\mathbf{E}_r^t(s)\,ds = -\int_\tau^\infty \dot{\mathbb{G}}(s)\mathbf{E}_r^{t+\tau}(s)\,ds\,, \quad \tau \in \mathbb{R}^+.$$

Many circumstances make righter to use $\mathbf{I}_r^t(\cdot)$ rather than the past history $\mathbf{E}_r^t(\cdot)$.

The first one follows from the definition of equivalent states given by Noll [21], when the state is defined through the history of the strain. It has been shown in [5], [6] [7] that $\mathbf{E}_1^t(\cdot)$ and $\mathbf{E}_2^t(\cdot)$ are equivalent if $\mathbf{E}_1(t) = \mathbf{E}_2(t)$ and

$$\int_0^\infty \dot{\mathbb{G}}(s+\tau)\mathbf{E}_{r1}^t(s)\,ds = \int_0^\infty \dot{\mathbb{G}}(s+\tau)\mathbf{E}_{r2}^t(s)\,ds \quad \tau \in \mathbb{R}^+.$$

By introducing this equivalence relation in the space of the strain histories there is a bi-univocal correspondence between the pair $(\mathbf{E}(t), \mathbf{I}_r^t(\cdot))$ and the equivalence classes of the histories.

A second reason is related to the definition of minimal free energy, which is directly defined in terms of $(\mathbf{E}(t), \mathbf{I}_r^t(\cdot))$, so that it does not distinguish single histories, but equivalence classes only. We recall that the minimal free energy for a material with fading memory is the unique free energy always defined on whole state space and the real dissipation of the material is associated to this energy.

At last the choose to use $\mathbf{I}_r^t(\cdot)$ has a justification of rigorously mathematical type, because general results can be obtain in natural manner as soon as we use the variable $\mathbf{I}_r^t(\cdot)$ instead of $\mathbf{E}^t(\cdot)$. For example, when the past history of the strain tensor is used, traditionally (see [3], [12]), the Graffi free energy is chosen to study the evolutive problem by using the semi group theory. We show in the last section the same results can be obtained by choosing a new free energy defined in term of $\mathbf{I}_r^t(\cdot)$.

2. Basic notations

The mechanical properties of a material are based in the notion of *state* and *process*. Here we provide these notations following Noll [21] and Coleman and Owen [2], [13].

We consider a body occupying the placement $\mathscr{B}$. For any material point $X \in \mathscr{B}$ and time t, we define the *configuration* $C(X,t)$ as the *deformation gradient* $\mathbf{F}(X,t)$. When the dependence on t is examined, while X is kept fixed, we write $C(t)$ or $\mathbf{F}(t)$. A *mechanical process* P, of duration $d_P > 0$, is a piecewise continuous function on $[0, d_P)$ with values in Lin, given by

$$P(t) = \mathbf{L}^P(t)\,, \quad t \in [0, d_P),$$

where $\mathbf{L} = \nabla\mathbf{v}$ is called *velocity gradient*. The notation $P_{[t_1,t_2)}$ denotes the restriction of P to $[t_1,t_2) \subset [0,d_p)$. In particular, we denote by P_t the restriction of P to the interval $[0,t)$, $t \le d_P$. Let P_1 and P_2 be two processes of duration d_{P_1} and d_{P_2}, the *composition* $P_1 \star P_2$ of P_2 with P_1 is defined as

$$P_1 \star P_2 = \begin{cases} P_1(t), & t \in [0,d_{P_1}) \\ P_2(t-d_{P_1}), & t \in [d_{P_1}, d_{P_1}+d_{P_2}) \end{cases}$$

The response of the material associated to the process P is characterized by a function $\hat{\mathbf{T}} : [0,d_p) \to \mathrm{Sym}$ given by the values of the *stress tensor* $\mathbf{T}$ corresponding to mechanical process P.

Definition 2.1. A simple material, at any $X \in \mathscr{B}$, is a set $\{\Pi, \Sigma, \widetilde{\rho}, \overline{\mathbf{T}}\}$ such that

1. Π is the space of mechanical process P satisfying the following properties:

   i) if $P \in \Pi$, then $P_{[t_1,t_2)} \in \Pi$ for every $[t_1,t_2) \subset [0,d_p)$,

   ii) if $P_1, P_2 \in \Pi$, then $P_1 \star P_2 \in \Pi$,

2. Σ is an abstract set, called *state space*, whose elements σ are called *states*,

3. $\widetilde{\rho} : \Sigma \times \Pi \to \Sigma$ is a map, called *state transition function* which, to each state σ and process P, assigns the state at the end of the process and satisfies the following relation

$$\widetilde{\rho}(\sigma, P_1 \star P_2) = \widetilde{\rho}(\widetilde{\rho}(\sigma, P_1), P_2)\,, \quad \sigma \in \Sigma\,, P_1, P_2 \in \Pi,$$

4. $\overline{\mathbf{T}} : \Sigma \times \Pi \to \mathrm{Sym}$ is the *response function* which, to each state σ and process P, assigns the stress tensor at the end of the process. Namely

$$\overline{\mathbf{T}}(\sigma, P) = \hat{\mathbf{T}}(d_P).$$

When the property 4 can be written

$$\hat{\mathbf{T}}(t) = \widetilde{\mathbf{T}}(\widetilde{\rho}(\sigma, P_t), P(t)) = \widetilde{\mathbf{T}}(\sigma(t), P(t))$$

the system is called causal. Later on we consider only causal systems.

In this framework, following Noll [21], we introduce a definition of equivalence in the state space Σ.

Definition 2.2. Two states σ_1, $\sigma_2 \in \Sigma$ are said *equivalent* if they satisfy

$$\overline{\mathbf{T}}(\sigma_1, P) = \overline{\mathbf{T}}(\sigma_2, P)\,, \quad \forall P \in \Pi. \tag{3}$$

Moreover, we introduce the *minimal state* σ_R as the equivalent class of the states according to Definition 2.2 and denote as Σ_R the set of the minimal states σ_R.

This general theory of simple material can be used to describe materials fading memory. The constitutive relation for these material is

$$\mathbf{T}(t) = \hat{\mathbf{T}}(\mathbf{F}^t), \tag{4}$$

where $\mathbf{F}^t$ denotes the history of the deformation gradient $\mathbf{F}$ up to time t, viz $\mathbf{F}^t(s) = \mathbf{F}(t-s)$, $s \in \mathbb{R}^+$. Following Definition 2.1, the state σ can be defined by means of the history $\mathbf{F}^t$, and the domain of the functional (4) will be the state space Σ moreover, the set of all mechanical processes Π is the set of the functions

$$P(t) = \mathbf{L}(t)\ , \quad t \in [t_0, t_0 + d_p).$$

The transition function $\widetilde{\rho}(\sigma, P_t)$ is given by the solution of the Cauchy problem

$$\begin{aligned} \frac{d}{dt}\mathbf{F}(t) &= \mathbf{L}(t)\ , \quad t \in [t_0, t_0 + d_p) \\ \mathbf{F}(t_0) &= \mathbf{F}_0 \end{aligned}$$

Finally the response function is represented by the constitutive equation (4). If any history $\mathbf{F}^t$ is viewed as the pair $(\mathbf{F}(t), \mathbf{F}^t_r(\cdot))$ of the *present value* $\mathbf{F}(t)$ and the *past history* $\mathbf{F}^t_r(s) = \mathbf{F}^t(s) - \mathbf{F}(t)$, $s > 0$, then the domain Σ in given by $\Sigma = \mathrm{Lin} \times \Sigma_r$, where Σ_r is the set of the past histories. Moreover, for any history we define the *static* and the *null continuation* of duration $\tau > 0$ to be the histories $\mathbf{F}^t_\tau$ and ${}_\tau\mathbf{F}^t$ define respectively

$$\mathbf{F}^t_\tau(s) = \begin{cases} \mathbf{F}(t), & s \le \tau \\ \mathbf{F}^t(s-\tau), & s > \tau \end{cases} \quad , \quad {}_\tau\mathbf{F}^t(s) = \begin{cases} \mathbf{0}, & s \le \tau \\ \mathbf{F}^t(s-\tau), & s > \tau \end{cases} .$$

Now we are in a position to characterize the fading memory[2] .

Definition 2.3. A material with memory represented by the constitutive equation (4) satisfies the fading memory property if the function $\hat{\mathbf{T}}(\mathbf{F}^t_\tau)$ is bounded

[2]This property was first considered by Volterra [23], which states :" *the modulus of the variation of the quantity* [given by (4)], *when* $\mathbf{F}^t$ *varies in any way ... in the interval* $(-\infty, t_1)$ *(with* $t_1 < t$*) can be made as small as we please by taking the interval* (t_1, t) *sufficiently large*" (see also [11]). In the Coleman & Noll theory [1], this property is given by the continuity of (4) respect to the

$$\|\mathbf{F}^t_r\|^2_h = \int_0^\infty h(s)|\mathbf{F}^t_r(s)|^2\,ds,$$

where the map $h \in L^1(\mathbb{R}^+)$ is a suitable positive, monotone decreasing function.

for any $\mathbf{F}^t \in \Sigma$, $\tau \in (\mathbb{R}^+$ and there exists an elastic material with constitutive functional $\hat{\mathbf{T}}_{el}$ such that

$$\lim_{\tau\to\infty} \hat{\mathbf{T}}(\mathbf{F}^t_\tau) = \hat{\mathbf{T}}_{el}(\mathbf{F}(t)),$$

where $\mathbf{F}(t) = \mathbf{F}^t(0)$. Moreover

$$\lim_{\tau\to\infty} \hat{\mathbf{T}}({}_\tau\mathbf{F}^t) = \mathbf{0}.$$

Now, let us consider the standard linear model of a viscoelastic solid. In this case the tensor $\mathbf{F}$ is replaced by the infinitesimal strain tensor $\mathbf{E} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$, where $\mathbf{u}$ is the displacement vector, and mechanical process $\mathbf{L}$ by $\dot{\mathbf{E}}$. The constitutive equation (4) takes the form

$$\mathbf{T}(t) = \mathbb{G}_0\mathbf{E}(t) + \int_0^\infty \dot{\mathbb{G}}(s)\mathbf{E}^t(s)\,ds, \tag{5}$$

where $\mathbb{G}_0$ and $\dot{\mathbb{G}}(\cdot)$ are fourth order symmetric tensors, $\dot{\mathbb{G}}(\cdot) \in L^1(\mathbb{R})$, $\mathbb{G}_0$ and $\mathbb{G}_\infty = \mathbb{G}_0 + \int_0^\infty \dot{\mathbb{G}}(s)\,ds$ are positive definite.

A linear viscoelastic material has the fading memory property of Definition 2.3 if

$$\begin{aligned} &\left|\int_0^\infty \dot{\mathbb{G}}(s+\tau)\mathbf{E}^t(s)\,ds\right| < \infty\,, \quad \forall\tau \geq 0 \\ &\lim_{\tau\to\infty}\int_\tau^\infty \dot{\mathbb{G}}(\xi)\mathbf{E}^{t+\tau}(\xi)\,d\xi = \lim_{\tau\to\infty}\int_0^\infty \dot{\mathbb{G}}(s+\tau)\mathbf{E}^t(s)\,ds = 0 \\ &\lim_{\tau\to\infty}\hat{\mathbf{T}}(\mathbf{E}^t_\tau) = \mathbb{G}_\infty\mathbf{E}(t) \end{aligned} \tag{6}$$

Relation $(6)_3$ implies that, as $\tau \to \infty$, the functional with memory $\hat{\mathbf{T}}$ is required to become the response function of an elastic material. Del Piero and Deseri in [6] observe that the linear constitutive equation (5) allows us to rewrite the equivalence relation (3) among states as an equivalence relation among histories (see also [20]), in fact, by using (5) it is easy to show

Proposition 2.4. *Two histories $\mathbf{E}^t_1$ and $\mathbf{E}^t_2$ represent two equivalent states if*

$$\mathbf{E}_1(t) = \mathbf{E}_2(t) \tag{7}$$

and the past histories satisfy

$$\int_0^\infty \dot{\mathbb{G}}(s+\tau)\mathbf{E}^t_1(s)\,ds = \int_0^\infty \dot{\mathbb{G}}(s+\tau)\mathbf{E}^t_2(s)\,ds\,, \quad \forall\tau \in \mathbb{R}^+. \tag{8}$$

It is easy to show that the requirements (7) and (8) are equivalent to

$$\begin{aligned} &\mathbb{G}(t)\mathbf{E}_1(t)+\int_0^{\infty}\dot{\mathbb{G}}(s+\tau)\mathbf{E}_1^t(s)\,ds \\ &=\mathbb{G}(t)\mathbf{E}_2(t)+\int_0^{\infty}\dot{\mathbb{G}}(s+\tau)\mathbf{E}_2^t(s)\,ds\,, \qquad \forall\tau\in\mathbb{R}^+. \end{aligned}$$

As observed in [6] and [20] the equality (8) allows to define the state by means of the function

$$\mathbf{I}_r^t(\tau,\mathbf{E}_r^t)=-\int_0^{\infty}\dot{\mathbb{G}}(s+\tau)\mathbf{E}_r^t(s)\,ds=-\int_{\tau}^{\infty}\dot{\mathbb{G}}(s)\mathbf{E}_r^{t+\tau}(s)\,ds\,,\quad \tau\in\mathbb{R}^+,$$

instead by the past history $\mathbf{E}_r^t$.

Therefore, the state given by the pair $(\mathbf{E}(t),\mathbf{I}_r^t(\cdot,\mathbf{E}_r^t))$ or, equivalently, through the function

$$\mathbf{I}^t(\tau,\mathbf{E}^t)=-\mathbb{G}(\tau)\mathbf{E}(t)-\int_0^{\infty}\dot{\mathbb{G}}(s+\tau)\mathbf{E}^t(s)\,ds=-\mathbb{G}_{\infty}\mathbf{E}(t)+\mathbf{I}_r^t(\tau,\mathbf{E}_r^t), \quad (9)$$

so that, we can obtain $\mathbf{E}(t)$ and $\mathbf{I}_r^t(\cdot,\mathbf{E}_r^t)$ by $\mathbf{I}^t(\cdot,\mathbf{E}^t)$[3].

In [10] and [11] the existence of a bijective map between the set of the minimal states Σ_R and set $\mathscr{H}'$ of the functions $\mathbf{I}^t(\cdot)$ is proved, but only Gentili in [19] frames and explains this result in a right physical mathematical context.

3. Gentili's norm on the process space

In this section we recall the results of Gentili [19] which allow to obtain a self consistent theory founded on the definition of *natural norm* in the space of the processes. This definition is based on the bounded-ness of the mechanical work. Let $\mathbf{E}^t(\cdot)$ be an initial strain history and $\dot{\mathbf{E}}^P$ a process of duration d, we have

$$\left(\mathbf{E}^t\star\dot{\mathbf{E}}_{\tau}^P\right)(s)=\mathbf{E}^{t+\tau}(s)=\begin{cases}\mathbf{E}(t)+\int_0^{\tau-s}\dot{\mathbf{E}}^P(\xi)\,d\xi & \text{for } 0\le s<\tau\\ \mathbf{E}^t(s-\tau) & \text{for } s\ge\tau\end{cases}$$

and the mechanical work along the process $\dot{\mathbf{E}}^P$ as is given by

$$\begin{aligned} w(\mathbf{E}^t,\dot{\mathbf{E}}^P) &= \int_0^d \tilde{\mathbf{T}}\left[\left(\mathbf{E}^t\star\dot{\mathbf{E}}_{\tau}^P\right),\dot{\mathbf{E}}^P(\tau)\right]\cdot\dot{\mathbf{E}}^P(\tau)\,d\tau \\ &=\frac{1}{2}\mathbb{G}_0\mathbf{E}(t+d)\cdot\mathbf{E}(t+d)-\frac{1}{2}\mathbb{G}_0\mathbf{E}(t)\cdot\mathbf{E}(t) \\ &+\int_0^d\int_0^{\infty}\dot{\mathbb{G}}(s)\mathbf{E}^{t+\tau}(s)\cdot\dot{\mathbf{E}}^P(\tau)\,ds\,d\tau. \end{aligned} \quad (10)$$

[3] In the sequel, we denote $\mathbf{I}^t(\cdot,\mathbf{E}^t)$ and $\mathbf{I}_r^t(\cdot,\mathbf{E}^t)$ simply with $\mathbf{I}^t(\cdot)$ by $\mathbf{I}_r^t(\cdot)$.

The Second Law of Thermodynamics for isothermal processes can be formulate in term of Dissipation Principle. To this end, we give the definition of cycle.

Definition 3.1. A pair $(\mathbf{E}^t, \dot{\mathbf{E}}^P)$ is called *cycle* if $\widetilde{\rho}(\mathbf{E}^t, \dot{\mathbf{E}}^P) = \mathbf{E}^t$.

Dissipation principle. For every cycle $(\mathbf{E}^t, \dot{\mathbf{E}}^P)$ we have $w(\mathbf{E}^t, \dot{\mathbf{E}}^P) \geq 0$.

For material with fading memory, cycles are realized only by periodic histories and processes with duration equal to a finite number of period of the history. Moreover cycles are quite rare, because usually the material gets to a state which, although close to initial state, is different from it. For this reason we introduce a[4]

Strong Dissipation Principle. Let $\Sigma_{\mathbf{E}^t} = \{\mathbf{E}^{t\prime}; \exists \dot{\mathbf{E}}^P \in \Pi; \widetilde{\rho}(\mathbf{E}^t, \dot{\mathbf{E}}^P) = \mathbf{E}^{t\prime}\}$. The set $W(\mathbf{E}^t) = \{w(\mathbf{E}^t, \dot{\mathbf{E}}^P); \dot{\mathbf{E}}^P \in \Pi\}$ of the works done to pass from $\mathbf{E}^t$ to any state $\mathbf{E}^{t\prime} \in \Sigma_{\mathbf{E}^t}$ is bounded below. Moreover there exists a state $\mathbf{0}^\dagger$, $\mathbf{0}^\dagger(s) = \mathbf{0}$ for all $s \geq 0$, called *zero state*, such that $\inf W(\mathbf{0}^\dagger) = 0$.

Definition 3.2. A process $\dot{\mathbf{E}}^P \in \Pi$ of duration d is said to be a finite work process if

$$w(\mathbf{0}^\dagger, \dot{\mathbf{E}}^P) = \int_0^d \widetilde{\mathbf{T}}\left[\left(\mathbf{0}^\dagger \star \dot{\mathbf{E}}^P_\tau\right), \dot{\mathbf{E}}^P(\tau)\right] \cdot \dot{\mathbf{E}}^P(t)\, dt < \infty. \tag{11}$$

As a consequence of the Strong Dissipation Principle

$$w(\mathbf{0}^\dagger, \dot{\mathbf{E}}^P) > 0$$

for any process that is not a null process[5].

Afterwards, we require that we require that $\breve{\mathbb{G}}(t) = -\int_t^\infty \dot{\mathbb{G}}(s)\, ds$ belongs to $L^1(\mathbb{R}^+)$ and extend any process $\dot{\mathbf{E}}^P$ of duration d to $\mathbb{R}^+$, by posing $\dot{\mathbf{E}}^P(s) = \mathbf{0}$ for $t \leq d$. Incidentally, we observe that, if we define $\mathbb{G}(t) = \mathbb{G}_0 + \int_0^t \dot{\mathbb{G}}(s)\, ds$, then

$$\breve{\mathbb{G}}(t) = \mathbb{G}(t) - \mathbb{G}_\infty.$$

[4](see [9, 11])

[5]A null process $\dot{\mathbf{E}}^\dagger$ of duration $d > 0$, is defined

$$\dot{\mathbf{E}}^\dagger(s) = \mathbf{0}\,, \quad 0 \leq s < d$$

Under these assumptions we rewrite (11)

$$\begin{aligned} w(\mathbf{0}^\dagger,\dot{\mathbf{E}}^P) &= \int_0^\infty \left[\mathbb{G}_0\mathbf{E}_0(\tau)+\int_0^\tau \dot{\mathbb{G}}(s)\mathbf{E}_(\tau-s)\,ds\right]\cdot\dot{\mathbf{E}}^P(\tau)\,d\tau \\ &= \int_0^\infty\int_0^\tau \mathbb{G}(\tau-s)\dot{\mathbf{E}}^P(s)\cdot\dot{\mathbf{E}}^P(\tau)\,ds\,d\tau \\ &= \frac{1}{2}\int_0^\infty\int_0^\infty \mathbb{G}(|\tau-s|)\dot{\mathbf{E}}^P(s)\cdot\dot{\mathbf{E}}^P(\tau)\,ds\,d\tau \\ &= \frac{1}{2}\mathbb{G}_\infty\mathbf{E}_0(d)\cdot\mathbf{E}_0(d)+\frac{1}{2}\int_0^\infty\int_0^\infty \breve{\mathbb{G}}(|\tau-s|)\dot{\mathbf{E}}^P(s)\cdot\dot{\mathbf{E}}^P(\tau)\,ds\,d\tau \\ &= \frac{1}{2}\mathbb{G}_\infty\mathbf{E}_0(d)\cdot\mathbf{E}_0(d)+\frac{1}{2\pi}\int_{-\infty}^{+\infty} \breve{\mathbb{G}}_c(\omega)\dot{\mathbf{E}}^P_+(\omega)\cdot\left[\dot{\mathbf{E}}^P_+(\omega)\right]^*\,d\omega \end{aligned} \tag{12}$$

where

$$\dot{\mathbf{E}}^P_+(\omega)=\int_0^\infty \dot{\mathbf{E}}^P(\tau)e^{-i\omega\tau}\,d\tau$$

while

$$\breve{\mathbb{G}}_c(\omega)=\int_0^\infty \breve{\mathbb{G}}(\tau)\cos\omega\tau\,d\tau.$$

As a consequence of the dissipation principle, $\breve{\mathbb{G}}_c(\omega)$ is a positive definite tensor for any $\omega\in\mathbb{R}$. It follows from (12) that the set of the finite work processes is

$$\overline{\Pi}_G=\left\{\dot{\mathbf{E}}^P:\mathbb{R}^+\to \mathrm{Sym};\int_{-\infty}^{+\infty}\breve{\mathbb{G}}_c(\omega)\dot{\mathbf{E}}^P_+(\omega)\cdot\left[\dot{\mathbf{E}}^P_+(\omega)\right]^*\,d\omega<\infty\right\},$$

moreover, the positive definiteness of $\breve{\mathbb{G}}_c$ allows us to introduce the following inner product in Π_G

$$\begin{aligned} (\dot{\mathbf{E}}^P_1,\dot{\mathbf{E}}^P_2)_G &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\breve{\mathbb{G}}_c(\omega)\dot{\mathbf{E}}^P_{1+}(\omega)\cdot\left[\dot{\mathbf{E}}^P_{2+}(\omega)\right]^*\,d\omega \\ &= \frac{1}{2}\int_0^\infty\int_0^\infty \breve{\mathbb{G}}(|\tau-s|)\dot{\mathbf{E}}^P_1(s)\cdot\dot{\mathbf{E}}^P_2(\tau)\,ds\,d\tau. \end{aligned}$$

Therefore, we can define the space of the process as the Hilbert space Π_G obtained by the completion of $\overline{\Pi}_G$ with respect to the norm

$$\begin{aligned} \|\dot{\mathbf{E}}^P\|^2_{\Pi_G} &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\breve{\mathbb{G}}_c(\omega)\dot{\mathbf{E}}^P_+(\omega)\cdot\left[\dot{\mathbf{E}}^P_+(\omega)\right]^*\,d\omega \\ &= \frac{1}{2}\int_0^\infty\int_0^\infty \breve{\mathbb{G}}(|\tau-s|)\dot{\mathbf{E}}^P(s)\cdot\dot{\mathbf{E}}^P(\tau)\,ds\,d\tau. \end{aligned} \tag{13}$$

We observe that, extending the processes $\dot{\mathbf{E}}^P$ to $\mathbb{R}^+$, the mechanical work along the process $\dot{\mathbf{E}}^P$ can be rewritten in terms of $\mathbf{I}^t$. In fact, recalling (10), we have

$$\begin{aligned} w(\mathbf{E}^t,\dot{\mathbf{E}}^P) &= \int_0^\infty \left[\mathbb{G}_0\mathbf{E}(t+\tau) + \int_0^\tau \dot{\mathbb{G}}(s)\left(\mathbf{E}(t) + \int_0^{t-s} \dot{\mathbf{E}}^P(\xi)\,d\xi\right)\right. \\ &\qquad \left. + \int_\tau^\infty \dot{\mathbb{G}}(s)\mathbf{E}(t+\tau-s)\,ds\right]\cdot \dot{\mathbf{E}}^P(\tau)\,d\tau \\ &= \int_0^\infty \left[\mathbb{G}(\tau)\mathbf{E}(t) + \int_0^\tau \mathbb{G}(\tau-s)\dot{\mathbf{E}}^P(s) + \right. \\ &\qquad \left. \int_0^\infty \dot{\mathbb{G}}(\tau+s)\mathbf{E}(t-s)\,ds\right]\cdot \dot{\mathbf{E}}^P(\tau)\,d\tau \\ &= \int_0^\infty \left[\frac{1}{2}\int_0^\infty \mathbb{G}(|\tau-s|)\dot{\mathbf{E}}^P(s)\,ds - \mathbf{I}^t(\tau)\right]\cdot \dot{\mathbf{E}}^P(\tau)\,d\tau. \end{aligned} \tag{14}$$

This is the way used by Gentili [19] to obtain in natural manner a definition of space of mechanical processes, so that we call Π_G Gentili's processes space. The space Π_G is important because is the start point to define straightaway a natural topology in the state space. To this end we introduce a new definition of equivalent strain histories (see [19]).

Definition 3.3. Two histories $\mathbf{E}_1^t$ and $\mathbf{E}_2^t$ *are w-equivalent* if for any $\dot{\mathbf{E}}^P \in \Pi_G$

$$\int_0^d \widetilde{\mathbf{T}}\left[\left(\mathbf{E}_1^t \star \dot{\mathbf{E}}_\tau^P\right), \dot{\mathbf{E}}^P(\tau)\right]\cdot \dot{\mathbf{E}}^P(\tau)\,d\tau = \int_0^d \widetilde{\mathbf{T}}\left[\left(\mathbf{E}_2^t \star \dot{\mathbf{E}}_\tau^P\right), \dot{\mathbf{E}}^P(\tau)\right]\cdot \dot{\mathbf{E}}^P(\tau)\,d\tau.$$

Theorem 3.4. *Two histories $\mathbf{E}_1^t$ and $\mathbf{E}_2^t$ are w-equivalent if and only if they are equivalent in the sense of* Proposition 2.4

Proof. If $\mathbf{E}_1^t$ and $\mathbf{E}_2^t$ satisfy (7) and (8), then they are w-equivalent. On the other hand, by virtue of (14) and of the arbitrariness of $\dot{\mathbf{E}}^P$, if $\mathbf{E}_1^t$ and $\mathbf{E}_2^t$ are w-equivalent, then $\mathbf{I}_1^t = \mathbf{I}_2^t$. □

The introduction of the norm $\|\cdot\|_G$ in the process space allows to characterize the state space and its natural norm. We prove that the natural norm in the state space coincides with the topology deriving from the minimal free energy.

Afterwards we consider the minimal state σ_R which, for these linear systems can be represented trough the function $\mathbf{I}^t(\cdot)$ or trough the corresponding equivalence class of histories (or trough the process which gives the maximum recoverable work).

In [19] Gentili gives the following

Definition 3.5. The set of the admissible states (histories) is the set of the functions $\mathbf{I}^t(\cdot)$ (histories $\mathbf{E}^t(\cdot)$) such that (14) is finite for any $\dot{\mathbf{E}}^P \in \Pi_G$.

As a consequence of Definition 3.5, the function $\mathbf{I}^t(\cdot)$ represents an admissible state if belongs to the space $\mathfrak{H}_G$, dual space of Π_G.

4. Free energies

4.1. General remarks

It is well known that for materials with fading memory the set of the free energy is not given by a singleton, but by a convex set $\mathscr{C}$ with a minimum and a maximum element.

Any free energy must be satisfies the following

Definition 4.1. A function $\Psi : \mathscr{D}_\Psi \to \mathbb{R}^+$ is a free energy if

1. the domain $\mathscr{D}_\Psi$ is invariant under $\widetilde{\rho}$; namely, for any $\sigma_i \in \mathscr{D}_\Psi$ and $P \in \Pi$, the state $\sigma = \widetilde{\rho}(\sigma_i, P) \in \mathscr{D}_\Psi$,
2. the zero state $\sigma^\dagger \in \mathscr{D}_\Psi$ and $\Psi(\sigma^\dagger) = 0$,
3. for any σ_1, $\sigma_2 \in \mathscr{D}_\Psi$ and process $P \in \Pi$, such that $\sigma_2 = \widetilde{\rho}(\sigma_1, P)$,

$$\Psi(\sigma_2) - \Psi(\sigma_1) \leq w(\sigma_1, P). \tag{15}$$

In order to introduce the *maximum free energy*, we define $\Sigma_{\sigma_0} = \{\sigma \in \Sigma; \sigma = \widetilde{\rho}(\sigma_0, P), P \in \Pi\}$

$$N(\sigma_0; \sigma) = \{w(\sigma_0, P) \text{ for any } P \in \Pi; \sigma = \widetilde{\rho}(\sigma_0, P)\}$$

for any $\sigma \in \Sigma_{\sigma_0}$.

Theorem 4.2. *The functional $\Psi_M : \Sigma_{\sigma^\dagger} \to \mathbb{R}^+$, where $\sigma^\dagger$ is the zero state, defined by*

$$\Psi_M(\sigma) = \inf N(\sigma^\dagger; \sigma)$$

is a free energy, called maximum free energy. Moreover, if $\Psi : \mathscr{D}_\Psi \to \mathbb{R}^+$ is a free energy with $\Psi(\sigma^\dagger) = 0$, we have $\Sigma_{\sigma^\dagger} \subset \mathscr{D}_\Psi$ and

$$\Psi(\sigma) \leq \Psi_M(\sigma), \quad \forall \sigma \in \Sigma_{\sigma^\dagger}$$

Proof. See [9] □

Definition 4.3. A free energy $\Psi_m : \mathscr{D}_{\Psi_m} \to \mathbb{R}^+$ is called *minimum free energy* if

1. the domain $\mathscr{D}_{\Psi_m} = \Sigma$,
2. $\Psi_m(\sigma^\dagger) = 0$
3. for any free energy $\Psi : \mathscr{D}_\Psi \to \mathbb{R}^+$ we have

$$\Psi_m(\sigma) \leq \Psi(\sigma), \quad \forall \sigma \in \mathscr{D}_\Psi. \tag{16}$$

Remark 4.4. Inequality (16) allows us to prove that the minimum free energy (if it exists) is unique

Theorem 4.5. *The functional* $\Psi_m : \Sigma \to \mathbb{R}^+$,

$$\Psi_m(\sigma) = -\inf W(\sigma) = -\inf\{w(\sigma, P); P \in \Pi\} \tag{17}$$

is the minimum free energy.

Proof. See [9] □

We observe that $-\inf W(\sigma)$ represents the maximum recoverable work from the state σ.

4.2. Linear model

In the case of the linear constitutive equation (5), the process $\dot{\mathbf{E}}^m$ from which we obtain the maximum recoverable work from the state $\sigma = \mathbf{E}^0$ is given by the solution of the variational equation

$$\frac{d}{d\varepsilon}[-w(\sigma, P_\varepsilon)] = \int_0^\infty \int_0^\infty \breve{\mathbb{G}}(|t-\tau|)\dot{\mathbf{E}}^m(t) \cdot \dot{\mathbf{e}}(\tau)\, d\tau\, dt + \int_0^\infty \mathbf{I}^0(\tau) \cdot \dot{\mathbf{e}}(\tau)\, d\tau = 0 \tag{18}$$

where $\mathbf{I}^0$ is defined in (9), $P_\varepsilon = \dot{\mathbf{E}}^m + \dot{\mathbf{e}}$, with $\mathbf{e}$ an arbitrary function of Σ such that $\mathbf{e}(0) = \mathbf{0}$.

Since $\dot{\mathbf{e}}$ is arbitrary, (18) yield the Wiener-Hopf equation

$$\int_0^\infty \breve{\mathbb{G}}(|t-\tau|)\dot{\mathbf{E}}^m(t)\, dt = \mathbf{I}^0(\tau), \quad \tau \in \mathbb{R}^+. \tag{19}$$

The properties of the spaces $\mathfrak{H}_G$ and Π_G leave to prove (see [11]) that the equation (19) written as

$$\mathscr{A}\dot{\mathbf{E}}^m = \mathbf{I}^0 \tag{20}$$

is such that the operator $\mathscr{A} : \Pi_G \to \mathfrak{H}_G$ is bounded and coercive. Then the Lax-Milgram theorem gives

Theorem 4.6. *For any* $\mathbf{I}^0 \in \mathfrak{H}_G$ *the equation* (20) *has a unique solution* $\dot{\mathbf{E}}^m \in \Pi_G$, *such that* $\|\dot{\mathbf{E}}^m\|_{\Pi_G}\| \leq \|\mathbf{I}^0\|_{\mathfrak{H}_G}$.

The proof is a trivial consequence of the duality properties of the spaces Π_G and $\mathfrak{H}_G$.

The process $\dot{\mathbf{E}}^m$, solution of (20) is the optimal process for which we obtain the maximum recoverable work from the state $\mathbf{I}^0$.

Recalling the definition (13) of norm in the process space Π_G we have that the minimum free energy can be represented by the functional

$$\begin{aligned}\Psi_m(\dot{\mathbf{E}}^m) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \breve{\mathbb{G}}_c(\omega)\dot{\mathbf{E}}_+^m(\omega)\cdot\left[\dot{\mathbf{E}}_+^m(\omega)\right]^* d\omega \\ &= \frac{1}{2}\int_0^{\infty}\int_0^{\infty} \breve{\mathbb{G}}(|\tau-s|)\dot{\mathbf{E}}^m(s)\cdot\dot{\mathbf{E}}^m(\tau)\,ds\,d\tau.\end{aligned}$$

Therefore Ψ_m is defined over all Π_G and the Theorem 4.6 assures that there exists an isomorphism between Π_G and $\mathfrak{H}_G$, so that for any $\mathbf{I}^0 \in \mathfrak{H}_G$ the minimum free energy is well defined.

In the case of the linear constitutive equation (5) the maximum free energy has the following representation (see [9])

$$\Psi_M(\mathbf{E}^t) = \int_0^t \widetilde{\mathbf{T}}\left[\left(\mathbf{0}^\dagger \star \dot{\mathbf{E}}_\tau\right), \dot{\mathbf{E}}(\tau)\right]\cdot\dot{\mathbf{E}}(\tau)\,d\tau, \tag{21}$$

where

$$\mathbf{E}^t(\tau) = \begin{cases} \displaystyle\int_0^{t-\tau} \dot{\mathbf{E}}(s)\,ds = & \text{if } 0 \le \tau < t \\ \mathbf{0} & \text{if } \tau \ge t \end{cases}.$$

It is easy to observe that (21) can be written in the form

$$\begin{aligned}\Psi_M(\mathbf{E}^t) &= \frac{1}{2}\mathbb{G}_\infty\mathbf{E}(t)\cdot\mathbf{E}(t) \\ &\quad + \frac{1}{2}\int_0^{\infty}\int_0^{\infty} \mathbb{G}_{12}(|s_1-s_2|)\mathbf{E}_r^t(s_1)\cdot\mathbf{E}_r^t(s_2)\,ds_1\,ds_2 \\ &= \frac{1}{2}\mathbb{G}_\infty\mathbf{E}(t)\cdot\mathbf{E}(t) \\ &\quad - \frac{1}{\pi}\int_0^{\infty}\Big[\omega\dot{\mathbb{G}}_s(\omega)(\mathbf{E}_s^t(\omega) - \frac{\mathbf{E}(t)}{\omega})\cdot(\mathbf{E}_s^t(\omega) - \frac{\mathbf{E}(t)}{\omega}) \\ &\qquad + \omega\dot{\mathbb{G}}_s(\omega)\mathbf{E}_c^t(\omega)\cdot(\mathbf{E}_c^t(\omega)\Big]\,d\omega,\end{aligned}$$

where

$$\mathbb{G}_{12}(|s_1-s_2|) = \frac{\partial^2}{\partial s_1 \partial s_2}\mathbb{G}(|s_1-s_2|)$$

and the Dissipation Principle assures that

$$-\omega\dot{\mathbb{G}}_s(\omega) > 0 \forall\omega \neq 0. \tag{22}$$

The domain $\mathscr{D}_{\Psi_M}$ of the maximum free energy Ψ_M is given by the history space

$$\mathscr{D}_{\Psi_M} = \left\{\mathbf{E}^t; \Psi_M(\mathbf{E}^t;) < \infty\right\}.$$

When the kernel $\dot{\mathbb{G}} \in C^1(\mathbb{R}^+)$ and

$$\dot{\mathbb{G}}(s) < 0\,, \quad \ddot{\mathbb{G}}(s) \geq 0\,, \quad \forall s \geq 0, \tag{23}$$

it is easy to prove that this kernel satisfies (22) and it is possible to introduce a new free energy Ψ_G defined by

$$\Psi_G(\mathbf{E}^t) = \frac{1}{2}\mathbb{G}_\infty \mathbf{E}(t)\cdot\mathbf{E}(t) - \frac{1}{2}\int_0^\infty \dot{\mathbb{G}}(s)\mathbf{E}_r^t(s)\cdot\mathbf{E}_r^t(s)\,ds, \tag{24}$$

called Graffi-Volterra free energy. Recently, in [7, 8, 11] it is shown that it is possible to define another free energy using the definition of state in terms of $\mathbf{I}^t = (\mathbf{E}(t), \mathbf{I}_r^t)$i the following manner:

$$\Psi_F(\mathbf{I}^t) = \frac{1}{2}\mathbb{G}_\infty \mathbf{E}(t)\cdot\mathbf{E}(t) - \frac{1}{2}\int_0^\infty \dot{\mathbb{G}}^{-1}(s)\frac{d}{ds}\mathbf{I}_r^t(s)\cdot\frac{d}{ds}\mathbf{I}_r^t(s)\,ds. \tag{25}$$

Actually, we consider the time derivative of Ψ_F

$$\dot{\Psi}_F(\mathbf{I}^t) = \mathbb{G}_\infty \mathbf{E}(t)\cdot\dot{\mathbf{E}}(t) - \int_0^\infty \dot{\mathbb{G}}^{-1}(s)\frac{d}{ds}\mathbf{I}_r^t(s)\cdot\frac{d}{ds}\dot{\mathbf{I}}_r^t(s)\,ds$$

because of the equality

$$\dot{\mathbf{I}}_r^t(s) = -\frac{d}{dt}\int_0^\infty \dot{\mathbb{G}}(s+\tau)\mathbf{E}_r(\tau)\,d\tau = -\breve{\mathbb{G}}(s)\mathbf{E}(t) + \frac{d}{ds}\mathbf{I}_r^t(s) \tag{26}$$

we have

$$\begin{aligned}\dot{\Psi}_F(\mathbf{I}^t) &= \widehat{\mathbf{T}}(\mathbf{I}^t)\cdot\dot{\mathbf{E}}(t) - \int_0^\infty \dot{\mathbb{G}}^{-1}(\tau)\frac{d^2}{d\tau^2}\mathbf{I}_r^t(\tau)\cdot\frac{d}{d\tau}\mathbf{I}_r^t(\tau)\,d\tau \\ &= \widehat{\mathbf{T}}(\mathbf{I}^t)\cdot\dot{\mathbf{E}}(t) + \frac{1}{2}\dot{\mathbb{G}}^{-1}(0)\frac{d}{d\tau}\mathbf{I}_r^t(0)\cdot\frac{d}{d\tau}\mathbf{I}_r^t(0) \\ &\quad -\frac{1}{2}\int_0^\infty \ddot{\mathbb{G}}(\tau)\dot{\mathbb{G}}^{-1}(\tau)\frac{d}{d\tau}\mathbf{I}_r^t(\tau)\cdot\dot{\mathbb{G}}^{-1}(\tau)\frac{d}{d\tau}\mathbf{I}_r^t(\tau)\,d\tau\end{aligned}\;. \tag{27}$$

On the basis of (23), equality (27) yield the dissipation condition

$$\dot{\Psi}_F(\mathbf{I}^t) \leq \widehat{\mathbf{T}}(\mathbf{I}^t)\cdot\dot{\mathbf{E}}(t).$$

5. Domains of definition and topologies

Any free energy is defined on a different subset of the state space, which we denote with $\mathscr{D}_\Psi$. As we have already prove, the domain $\mathscr{D}_{\Psi_m}$ is $\mathfrak{H}_G$ and theorems 4.2 and 4.5 yield that any free energy satisfies the following inequality

$$\Psi_m(\sigma) \leq \Psi(\sigma) \leq \Psi_M(\sigma) \quad \forall\sigma \in \mathscr{D}_{\Psi_M}.$$

From here, we may claim that there exists the following condition for the free energy domains

$$\mathscr{D}_{\Psi_M} \subseteq \mathscr{D}_{\Psi} \subseteq \mathscr{D}_{\Psi_m}. \tag{28}$$

Moreover, when the kernel $\dot{\mathbb{G}}$ satisfies the conditions (23), we shall prove that

$$\Psi_F(\sigma) \leq \Psi_G(\sigma) \quad \forall \sigma \in \mathscr{D}_{\Psi_G}. \tag{29}$$

In fact, let $\mathbf{E}^t(\cdot)$ be a history belongs to $\mathscr{D}_{\Psi_G}$ and $\mathbf{I}^t(\cdot)$ the minimal state associated to $\mathbf{E}^t(\cdot)$, recalling (25), we have:

$$\begin{aligned}
2\Psi_F(\mathbf{I}^t) &= \mathbb{G}_\infty \mathbf{E}(t) \cdot \mathbf{E}(t) \\
&\quad - \int_0^\infty \dot{\mathbb{G}}^{-1}(s) \int_0^\infty \ddot{\mathbb{G}}(s+\tau_1)\mathbf{E}_r^t(\tau_1)\, d\tau_1 \cdot \int_0^\infty \ddot{\mathbb{G}}(s+\tau_2)\mathbf{E}_r^t(\tau_2)\, d\tau_2\, ds \\
&\leq \mathbb{G}_\infty \mathbf{E}(t) \cdot \mathbf{E}(t) + \int_0^\infty \left|\dot{\mathbb{G}}^{-1}(s)\right| \left|\int_0^\infty \ddot{\mathbb{G}}(s+\tau)\mathbf{E}_r^t(\tau)\, d\tau\right|^2 ds \\
&\leq \mathbb{G}_\infty \mathbf{E}(t) \cdot \mathbf{E}(t) \\
&\quad + \int_0^\infty \left|\dot{\mathbb{G}}^{-1}(s)\right| \left|\int_0^\infty \ddot{\mathbb{G}}(s+\tau)\, d\tau\right| \int_0^\infty \ddot{\mathbb{G}}(s+\tau)\mathbf{E}_r^t(\tau) \cdot \mathbf{E}_r^t(\tau)\, d\tau\, ds \\
&= \mathbb{G}_\infty \mathbf{E}(t) \cdot \mathbf{E}(t) - \int_0^\infty \dot{\mathbb{G}}(\tau)\mathbf{E}_r^t(\tau) \cdot \mathbf{E}_r^t(\tau)\, d\tau = 2\Psi_G(\mathbf{E}^t).
\end{aligned}$$

Then, from (28) and (29), we obtain

$$\mathscr{D}_{\Psi_M} \subseteq \mathscr{D}_{\Psi_G} \subseteq \mathscr{D}_{\Psi_F} \subseteq \mathscr{D}_{\Psi_m}. \tag{30}$$

In the linear case it is possible to associate a norm to any free energy trough the following identity:

$$\|\sigma\|_\Psi^2 = 2\Psi(\sigma)\,, \quad \sigma \in \mathscr{D}_\Psi$$

and the domain $\mathscr{D}_\Psi$ of the energy Ψ is the Banach space completed with the norm $\|\cdot\|_\Psi$.

Finally, we shall prove that the stress tensor $\hat{\mathbf{T}}$ is a continuous function on any space $\mathscr{D}_\Psi$ considered in (30).

Let us begin considering the continuity of $\hat{\mathbf{T}}$ on the space $\mathscr{D}_{\Psi_M}$. By using the Parseval relation, stress tensor (5) can be written as

$$\mathbf{T}(\mathbf{E}^t) = \mathbb{G}_\infty \mathbf{E}(t) - \frac{1}{\pi}\int_0^\infty \dot{\mathbb{G}}_s(\omega)\mathbf{E}_s^t(\omega)\, d\omega,$$

where by

$$\begin{aligned}|\mathbf{T}(\mathbf{E}^t)|^2 \;&\leq\; |\mathbb{G}_\infty|\mathbb{G}_\infty\mathbf{E}(t)\cdot\mathbf{E}(t)\\ &\quad -\frac{1}{\pi}\int_0^\infty \frac{\dot{\mathbb{G}}_s(\omega)}{\omega}\,d\omega\left[\int_0^\infty \omega\dot{\mathbb{G}}_s(\omega)\mathbf{E}_c^t(\omega)\cdot\mathbf{E}_c^t(\omega)\,d\omega\right.\\ &\qquad\left.+\int_0^\infty \omega\dot{\mathbb{G}}_s(\omega)\left(\mathbf{E}_s^t(\omega)-\frac{\mathbf{E}(t)}{\omega}\right)\cdot\left(\mathbf{E}_s^t(\omega)-\frac{\mathbf{E}(t)}{\omega}\right)d\omega\right]\\ &=\; |\mathbb{G}_\infty|\mathbb{G}_\infty\mathbf{E}(t)\cdot\mathbf{E}(t)\\ &\quad -\frac{1}{\pi}|\mathbb{G}_0-\mathbb{G}_\infty|\left[\int_0^\infty \omega\dot{\mathbb{G}}_s(\omega)\mathbf{E}_c^t(\omega)\cdot\mathbf{E}_c^t(\omega)\,d\omega\right.\\ &\qquad\left.+\int_0^\infty \omega\dot{\mathbb{G}}_s(\omega)\left(\mathbf{E}_s^t(\omega)-\frac{\mathbf{E}(t)}{\omega}\right)\cdot\left(\mathbf{E}_s^t(\omega)-\frac{\mathbf{E}(t)}{\omega}\right)d\omega\right].\end{aligned}$$

Hence

$$|\mathbf{T}(\mathbf{E}^t)|^2 \leq 2\sup\{|\mathbb{G}_\infty|,|\mathbb{G}_0-\mathbb{G}_\infty|\}\Psi_M(\mathbf{E}^t). \tag{31}$$

By using the classical representation (5)

$$\mathbf{T}(\mathbf{E}^t)=\mathbb{G}_\infty\mathbf{E}(t)+\int_0^\infty\left(-\dot{\mathbb{G}}(s)\right)^{\frac12}\left[\left(-\dot{\mathbb{G}}(s)\right)^{\frac12}\left(\mathbf{E}^t(s)-\mathbf{E}(t)\right)\right]ds,$$

and the Schwartz inequality, we obtain

$$\begin{aligned}|\mathbf{T}(\mathbf{E}^t)|^2 \leq\quad & |\mathbb{G}_\infty|\mathbb{G}_\infty\mathbf{E}(t)\cdot\mathbf{E}(t)-\\ & |\mathbb{G}_0-\mathbb{G}_\infty|\int_0^\infty \dot{\mathbb{G}}(s)\left(\mathbf{E}^t(s)-\mathbf{E}(t)\right)\cdot\left(\mathbf{E}^t(s)-\mathbf{E}(t)\right)ds,\end{aligned}$$

from which

$$|\mathbf{T}(\mathbf{E}^t)|^2 \leq 2\sup\{|\mathbb{G}_\infty|,|\mathbb{G}_0-\mathbb{G}_\infty|\}\Psi_G(\mathbf{E}^t). \tag{32}$$

To obtain the continuity $\hat{\mathbf{T}}$ with respect to free energy Ψ_F, we use the following representation of $\mathbf{T}$:

$$\begin{aligned}\mathbf{T}(\mathbf{E}^t)\;&=\mathbb{G}_\infty\mathbf{E}(t)-\int_0^\infty\frac{d}{d\tau}\mathbf{I}_r^t(\tau)\,d\tau\\ &=\;\mathbb{G}_\infty\mathbf{E}(t)-\int_0^\infty\left(-\dot{\mathbb{G}}(s)\right)^{\frac12}\left(-\dot{\mathbb{G}}(s)\right)^{-\frac12}\frac{d}{d\tau}\mathbf{I}_r^t(\tau)\,d\tau\end{aligned}$$

and, the Schwartz inequality yields

$$|\mathbf{T}(\mathbf{I}^t)|^2 \leq |\mathbb{G}_\infty|\mathbb{G}_\infty\mathbf{E}(t)\cdot\mathbf{E}(t)+|\mathbb{G}_0-\mathbb{G}_\infty|\int_0^\infty \dot{\mathbb{G}}^{-1}(s)\frac{d}{d\tau}\mathbf{I}_r^t(\tau)\cdot\frac{d}{d\tau}\mathbf{I}_r^t(\tau)\,d\tau. \tag{33}$$

Hence (33) gives for Ψ_F an inequality completely similar to (31) and (32).

Finally, we consider the minimum free energy Ψ_m. Equation (17) gives

$$\begin{aligned}\mathbf{T}(\mathbf{I}^t)\;&=-\,\mathbf{I}^t(0)=-\mathbb{G}_\infty\mathbf{E}(t)-\int_0^\infty\check{\mathbb{G}}(r)\dot{\mathbf{E}}^m(r)\,dr\\ &=-\mathbb{G}_\infty\mathbf{E}(t)-\frac{1}{\pi}\int_0^\infty\check{\mathbb{G}}_c(\omega)\dot{\mathbf{E}}_c^m(\omega)\,d\omega,\end{aligned}$$

then

$$|\mathbf{T}(\mathbf{I}^t)|^2 \leq |\mathbb{G}_\infty|\mathbb{G}_\infty\mathbf{E}(t)\cdot\mathbf{E}(t)+\frac{1}{\pi}|\mathbb{G}_0-\mathbb{G}_\infty|\int_0^\infty \check{\mathbb{G}}_c(\omega)\dot{\mathbf{E}}_c^m(\omega)\cdot\dot{\mathbf{E}}_c^m(\omega)\,d\omega.$$

6. Semigroup theory

Let us consider a linear viscoelastic material with relaxation function

$$\mathbb{G}(\mathbf{x},s) = g(s)\mathbb{A}(\mathbf{x})$$

where $\mathbb{A}$ is a symmetric fourth order tensor positive definite and g is a scalar function which satisfies the following properties:

(h_1) $\check{g}(s) = g(s) - g_\infty > 0$, $s \in \mathbb{R}^+$ and $\check{g}(s) \in L^1(\mathbb{R}^+)$,

(h_2) $g'(s) < 0$, $g''(s) \geq 0$, $s \in \mathbb{R}^+$,

(h_3) there exists a positive constant k such that

$$kg'(s) + g''(s) \geq 0\,, \quad s \in \mathbb{R}^+. \tag{34}$$

Under the previous hypotheses we have

$$\mathbf{I}_r^t(\tau) = -\mathbb{A}\nabla\int_0^\infty g'(s+\tau)\mathbf{u}_r^t(s)\,ds = -\mathbb{A}\nabla\mathbf{b}_r^t(\tau)$$

with $\mathbf{b}_r^t(\tau) = \int_0^\infty g'(s+\tau)\mathbf{u}_r^t(s)\,ds$.

The initial boundary value problem connected with a visco-elastic solid material can be written in the form

$$\begin{array}{l} \rho\dot{\mathbf{v}}(t) = \nabla\cdot[\mathbb{A}\nabla(g_\infty\mathbf{u}(t) - \mathbf{b}_r^t(0))] + \mathbf{f}(t) \\ \mathbf{u}(0) = \mathbf{u}_0\,, \quad \mathbf{v}(0) = \mathbf{v_0}\,, \quad \mathbf{b_r^0}(\tau) = \mathbf{b_0}(\tau)\,, \tau \in \mathbb{R}^+ \\ \mathbf{u}_{|\partial\mathscr{B}} = 0\,, \quad \mathbf{b}^t_{r|\partial\mathscr{B}}(\cdot) = 0\,, t \in \mathbb{R}^+ \end{array} \tag{35}$$

To use the semigroups theory, we introduce the triple

$$\chi = (\mathbf{v}, \nabla\mathbf{u}, \mathbf{b}_r^t)$$

and rewritten the problem (35) as an abstract first order Cauchy problem

$$\begin{array}{l} \dot{\chi}(t) = \mathrm{A}\chi(t) + \mathrm{f}(t) \\ \chi(0) = \chi_0 \end{array} \tag{36}$$

with

$$A\chi(t) = \left(\frac{1}{\rho}\nabla\cdot[\mathbb{A}\nabla(g_\infty\mathbf{u}(t)-\mathbf{b}_r^t(0))], \nabla\mathbf{v}(t), \frac{d}{d\tau}\mathbf{b}_r^t(\tau)-\breve{g}(\tau)\mathbf{v}(t)\right),$$
$$\mathrm{f}(t) = (\mathbf{f}(t),0,0), \quad \chi_0 = (\mathbf{v}_0, \nabla\mathbf{u}_0, \breve{\mathbf{b}}^0(\cdot)).$$

The natural set in which to look for the solution of the problem (36) is the space $\mathscr{K}$ of the triples χ for which the total energy

$$\begin{aligned}\mathscr{E}(\chi(t)) &= \frac{1}{2}\int_{\mathscr{B}}\left[\rho|\mathbf{v}(t)|^2 + g_\infty\mathbb{A}\nabla\mathbf{u}(t)\cdot\nabla\mathbf{u}(t)\right]d\mathbf{x}\\ &-\frac{1}{2}\int_{\mathscr{B}}\int_0^\infty\frac{1}{g'(s)}\mathbb{A}\frac{d}{ds}\nabla\mathbf{b}_r^t(s)\cdot\frac{d}{ds}\nabla\mathbf{b}_r^t(s)\,ds\,d\mathbf{x}\end{aligned}$$

is finite.

Remark 6.1. In [3, 12] the Graffi-Volterra free energy Ψ_G is used and the exponential decay of solutions is proved for initial conditions belong to $\mathscr{D}_{\Psi_G}$. Here, we use the free energy Ψ_F, whose domain of definition is larger of $\mathscr{D}_{\Psi_G}$, as we have proved in Section 5 (see (30)). Moreover, an other advantage is given by the use of the topology of $\mathscr{D}_{\Psi_F}$, which does not distinguish equivalent histories.

We endowed $\mathscr{K}$ with the inner product

$$\begin{aligned}\langle\chi_1(t),\chi_2(t)\rangle &= \langle(\mathbf{v}_1(t),\nabla\mathbf{u}_1(t),\mathbf{b}_{r1}^t),(\mathbf{v}_2(t),\nabla\mathbf{u}_2(t),\mathbf{b}_{r2}^t)\rangle\\ &= \int_{\mathscr{B}}(\ (\rho\mathbf{v}_1(t)\cdot\mathbf{v}_2(t) + g_\infty\mathbb{A}\nabla\mathbf{u}_1(t)\cdot\nabla\mathbf{u}_2(t))\,d\mathbf{x}\\ &-\int_{\mathscr{B}}\int_0^\infty\frac{1}{\dot{g}'(\tau)}\mathbb{A}\frac{d}{ds}\nabla\mathbf{b}_{r1}^t(\mathbf{x},\tau)\cdot\frac{d}{ds}\nabla\mathbf{b}_{r2}^t(\mathbf{x},\tau)\,d\tau\,d\mathbf{x},\end{aligned}\tag{37}$$

so that $\langle\chi,\chi\rangle = \|\chi\|^2 = 2\mathscr{E}(\chi)$.

We denote with $\mathscr{D}(A)$ the domain of the operator A, namely

$$\mathscr{D}(A) = \{\chi\in\mathscr{K};\chi\in\mathscr{K} \text{ and the boundary conditions } (35)_2 \text{ hold}\}$$

and claim that the operator A is dissipative.

In fact, if $\chi\in\mathscr{D}(A)$, we have

$$\begin{aligned}&\langle A\chi(t),\chi(t)\rangle\\ &= \int_{\mathscr{B}}(\ \left\{\nabla\cdot[\mathbb{A}\nabla(g_\infty\mathbf{u}(t)-\mathbf{b}_r^t(0))]\cdot\mathbf{v}(t)) + g_\infty\mathbb{A}\nabla\mathbf{v}(t)\cdot\nabla\mathbf{u}(t)\right\}d\mathbf{x}\\ &-\int_{\mathscr{B}}\int_0^\infty\frac{1}{g'(\tau)}\frac{d}{d\tau}\left(\mathbb{A}\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)-\breve{g}(\tau)\nabla\mathbf{v}(t)\right)\cdot\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)\,d\tau\,d\mathbf{x}\\ &= \frac{1}{2}\int_{\mathscr{B}}\frac{1}{g'(0)}\mathbb{A}\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)_{|\tau=0}\cdot\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)_{|\tau=0}\,d\mathbf{x}\\ &-\frac{1}{2}\int_{\mathscr{B}}\int_0^\infty\frac{g''(\tau)}{[g'^2}\mathbb{A}\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)\cdot\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)\,d\tau\,d\mathbf{x}\le 0\end{aligned}\tag{38}$$

We now proceed to show that also $\widetilde{A}$ the adjoint of A is dissipative, so that, thanks to Lumer-Phillips theorem, A generates a C_0-semigroup.

Let $\widetilde{\chi} = (\widetilde{\mathbf{v}}, \nabla\widetilde{\mathbf{u}}, \widetilde{\mathbf{b}}^t_r) \in \mathscr{K}$, denoting by H the Heaviside function and introducing a function $\mathbf{J}(\widetilde{\mathbf{b}}^t_r)$ such that

$$\frac{d}{d\tau}\mathbf{J}(\widetilde{\mathbf{b}}^t_r)(\tau) = \frac{d}{d\tau}\left(\frac{H(\tau)}{g'(\tau)}\right)\frac{d}{d\tau}\mathbf{b}^t_r(\tau),$$

we claim that $\widetilde{A}\widetilde{\chi}(t)$ is equal to

$$\left(-\frac{1}{\rho}\nabla\cdot[\mathbb{A}\nabla(g_\infty\widetilde{\mathbf{u}}(t) - \widetilde{\mathbf{b}}^t_r(0))], -\nabla\widetilde{\mathbf{v}}(t), -\frac{d}{d\tau}\widetilde{\mathbf{b}}^t_r(\tau) + \breve{g}(\tau)\nabla\widetilde{\mathbf{v}}(t) + \mathbf{J}(\widetilde{\mathbf{b}}^t_r)\right),$$

and $\mathscr{D}(A) = \mathscr{D}(\widetilde{A})$.

In fact, if χ and $\widetilde{\chi}$ are in $\mathscr{D}(A)$, a direct calculus gives

$$\begin{aligned}
&\langle A\chi(t), \widetilde{\chi}(t)\rangle = -\langle A\widetilde{\chi}(t), \chi(t)\rangle \\
&\quad -\frac{1}{2}\int_{\mathscr{B}}\frac{1}{g'(0)}\mathbb{A}\frac{d}{d\tau}\nabla\mathbf{b}^t_r(\tau)_{|\tau=0}\cdot\frac{d}{d\tau}\nabla\widetilde{\mathbf{b}}^t_r(\tau)_{|\tau=0}\,d\mathbf{x} \\
&\quad -\frac{1}{2}\int_{\mathscr{B}}\int_0^\infty\frac{g''(\tau)}{|g'^2}\mathbb{A}\frac{d}{d\tau}\nabla\mathbf{b}^t_r(\tau)\cdot\frac{d}{d\tau}\nabla\widetilde{\mathbf{b}}^t_r(\tau)\,d\tau\,d\mathbf{x}.
\end{aligned}$$

Moreover

$$\begin{aligned}
&\langle\widetilde{A}\widetilde{\chi}(t), \widetilde{\chi}(t)\rangle \\
&\quad = \frac{1}{2}\int_{\mathscr{B}}\frac{1}{g'(0)}\mathbb{A}\frac{d}{d\tau}\nabla\widetilde{\mathbf{b}}^t_r(\tau)_{|\tau=0}\cdot\frac{d}{d\tau}\nabla\widetilde{\mathbf{b}}^t_r(\tau)_{|\tau=0}\,d\mathbf{x} \\
&\quad -\frac{1}{2}\int_{\mathscr{B}}\int_0^\infty\frac{g''(\tau)}{|g'^2}\mathbb{A}\frac{d}{d\tau}\nabla\widetilde{\mathbf{b}}^t_r(\tau)\cdot\frac{d}{d\tau}\nabla\widetilde{\mathbf{b}}^t_r(\tau)\,d\tau\,d\mathbf{x} \le 0.
\end{aligned}$$

Remark 6.2. If A generates a C_0-semigroup, making use of the results obtained by Da Prato and Sinestrari [4], it is possible to state the well posedness of the problem (36), i.e. if $\mathrm{f} \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^+, L^2(\mathscr{B}))$ and $\chi_0 \in \mathscr{D}(A)$, then the problem (36) admits one and only one strict solution $\chi \in C^1(\mathbb{R}^+, \mathscr{K}) \cap C(\mathbb{R}^+, \mathscr{D}(A))$.

It is also possible to prove existence and uniqueness of strict solutions under other assumptions of regularity for the source term. Furthermore, under weaker assumptions for the regularity of both source and initial data, it is possible to obtain results of existence and uniqueness of 'mild' solutions (see, for istance, [4]).

In order to show that, in absence of sources, an exponential decay of the energy occurs in time, we introduce the functional

$$\mathscr{L}_{t_0}(t) = (t+t_0)\mathscr{E}(t) + \int_{\mathscr{B}}\rho\mathbf{v}(t)\cdot\left(\mathbf{u}(t) + \frac{2}{\breve{g}(0)}\mathbf{b}^t_r(0)\right)d\mathbf{x} \tag{39}$$

and starting by proving the following lemma

Lemma 6.3. *If the kernel g satisfies the hypotheses (h_1), (h_2) and (h_3), then the following inequality holds for t_0 sufficiently large*

$$\frac{d}{dt}\mathscr{L}_{t_0}(t) \leq 0, \tag{40}$$

moreover there exists a positive constant δ, such that

$$\mathscr{L}_{t_0}(T) - \mathscr{L}_{t_0}(0) \geq (T + t_0 - \delta)\mathscr{E}(T) - (t_0 + \delta)\mathscr{E}(0), \quad T > 0. \tag{41}$$

Proof. In ordet to prove (40), we observe that, if χ is a solution of (36), then it is easy to show that

$$\begin{aligned} \frac{d}{dt}\mathscr{E}(t) = \; & \frac{1}{2}\int_{\mathscr{B}} \frac{1}{g'(0)}\mathbb{A}\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)_{|\tau=0} \cdot \frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)_{|\tau=0}\, d\mathbf{x} \\ & -\frac{1}{2}\int_{\mathscr{B}}\int_0^\infty \frac{g''(\tau)}{[g'(\tau)]^2}\mathbb{A}\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau) \cdot \frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)\, d\tau\, d\mathbf{x} \leq 0 \end{aligned} \tag{42}$$

and

$$\begin{aligned} & \frac{d}{dt}\int_{\mathscr{B}} \rho\mathbf{v}(t)\left(\mathbf{u}(t) + \frac{2}{\breve{g}(0)}\mathbf{b}_r^t(0)\right) d\mathbf{x} = -\rho\int_{\mathscr{B}} |\mathbf{v}(t)|^2\, d\mathbf{x} \\ & \quad -g_\infty\int_{\mathscr{B}} \mathbb{A}\nabla\mathbf{u}(t)\cdot\nabla\mathbf{u}(t)\, d\mathbf{x} + \left(1 - \frac{2}{g_\infty}\breve{g}(0)\right)\int_{\mathscr{B}} \mathbb{A}\nabla\mathbf{u}(t)\cdot\nabla\mathbf{b}_r^t(0)\, d\mathbf{x} \\ & \quad +\frac{2}{\breve{g}(0)}\int_{\mathscr{B}} \mathbb{A}\nabla\mathbf{b}_r^t(0)\cdot\nabla\mathbf{b}_r^t(0)\, d\mathbf{x} + \frac{2\rho}{\breve{g}(0)}\int_{\mathscr{B}} \mathbf{v}(t)\cdot\frac{d}{d\tau}\mathbf{b}_r^t(\tau)_{|\tau=0}\, d\mathbf{x} \end{aligned}$$

so that

$$\begin{aligned} & \frac{d}{dt}\mathscr{L}_{t_0}(t) = (t + t_0)\frac{d}{dt}\mathscr{E}(t) - \mathscr{E}(t) + \left(1 - \frac{2}{g_\infty}\breve{g}(0)\right)\int_{\mathscr{B}} \mathbb{A}\nabla\mathbf{u}(t)\cdot\nabla\mathbf{b}_r^t(0)\, d\mathbf{x} \\ & \quad +\frac{2}{\breve{g}(0)}\int_{\mathscr{B}} \mathbb{A}\nabla\mathbf{b}_r^t(0)\cdot\nabla\mathbf{b}_r^t(0)\, d\mathbf{x} + \frac{2\rho}{\breve{g}(0)}\int_{\mathscr{B}} \mathbf{v}(t)\cdot\frac{d}{d\tau}\mathbf{b}_r^t(\tau)_{|\tau=0}\, d\mathbf{x} \\ & \quad -\frac{1}{2}\int_{\mathscr{B}}\int_0^\infty \frac{1}{g'(\tau)}\mathbb{A}\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)\cdot\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)\, d\tau\, d\mathbf{x} \end{aligned} \tag{43}$$

We now proceed to estimate the various terms in (43).

Observing that

$$\mathbf{b}_r^t(0) = -\int_0^\infty \frac{d}{d\tau}\mathbf{b}_r^t(\tau)\, d\tau,$$

we have

$$\begin{aligned} |\nabla\mathbf{b}_r^t(0)|^2 & \leq \int_0^\infty \left|\sqrt{-g'(\tau)}\frac{1}{\sqrt{-g'(\tau)}}\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)\right|^2 d\tau \\ & \leq -\breve{g}(0)\int_0^\infty \frac{1}{g'(\tau)}\left|\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)\right|^2 d\tau. \end{aligned} \tag{44}$$

Recalling that $\mathbb{A}$ is bounded and positive define, we obtain

$$\frac{2}{\breve{g}(0)}\int_{\mathscr{B}} \mathbb{A}\nabla\mathbf{b}_r^t(0)\cdot\nabla\mathbf{b}_r^t(0)\,d\mathbf{x} \leq -\gamma_1\int_{\mathscr{B}}\int_0^\infty \frac{1}{g'(\tau)}\mathbb{A}\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)\cdot\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)\,d\tau\,d\mathbf{x} \tag{45}$$

with γ_1 a suitable positive constant.

Using (45) and classical inequalities we have

$$\begin{aligned}\left(1-\frac{2}{g_\infty}\breve{g}(0)\right)\int_{\mathscr{B}} \mathbb{A}\nabla\mathbf{u}(t)\cdot\nabla\mathbf{b}_r^t(0)\,d\mathbf{x} &\leq \frac{1}{4}g_\infty\int_{\mathscr{B}}\mathbb{A}\nabla\mathbf{u}(t)\cdot\nabla\mathbf{u}(t)\,d\mathbf{x}\\ &-\gamma_2\int_{\mathscr{B}}\int_0^\infty\frac{1}{g'(\tau)}\mathbb{A}\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)\cdot\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)\,d\tau\,d\mathbf{x},\end{aligned} \tag{46}$$

with $\gamma_2 > 0$.

Finally, using the Poincaré inequality, we have

$$\begin{aligned}\frac{2\rho}{\breve{g}(0)}\int_{\mathscr{B}}\mathbf{v}(t)\cdot\frac{d}{d\tau}\mathbf{b}_r^t(\tau)_{|\tau=0}\,d\mathbf{x} &\leq \frac{1}{4}\rho\int_{\mathscr{B}}|\mathbf{v}(t)|^2\,d\mathbf{x}\\ &-\gamma_3\int_{\mathscr{B}}\frac{1}{g'(0)}\mathbb{A}\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)_{|\tau=0}\cdot\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)_{|\tau=0}\,d\mathbf{x},\end{aligned} \tag{47}$$

with $\gamma_3 > 0$.

Moreover, the condition (34) yields

$$\begin{aligned}&-\int_{\mathscr{B}}\int_0^\infty\frac{1}{g'(\tau)}\mathbb{A}\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)\cdot\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)\,d\tau\,d\mathbf{x}\\ &\leq\frac{1}{k}\int_{\mathscr{B}}\int_0^\infty\frac{g''(\tau)}{[g'(\tau)]^2}\mathbb{A}\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)\cdot\frac{d}{d\tau}\nabla\mathbf{b}_r^t(\tau)\,d\tau\,d\mathbf{x}\end{aligned} \tag{48}$$

Therefore, subsituing (45)-(47) in (43) and using (42) and (48), we have

$$\frac{d}{dt}\mathscr{L}_{t_0}(t) \leq (t+t_0-\gamma)\frac{d}{dt}\mathscr{E}(t)-\frac{1}{2}\mathscr{E}(t), \tag{49}$$

with $2\gamma = \max\{\frac{1}{k}(\gamma_1+\gamma_2),\gamma_3\}$. Choosing $t_0 > \gamma$, we obtain (40).

To prove (41) it is suffices to observe that, applying the Poincaré inequality together with (44) and using the positive definiteness of $\mathbb{A}$, we have

$$\left|\int_{\mathscr{B}}\rho\mathbf{v}(t)\cdot\left(\mathbf{u}(t)+\frac{2}{\breve{g}(0)}\mathbf{b}_r^t(0)\right)\,d\mathbf{x}\right| \leq \delta\mathscr{E}(t).$$

□

We are now able to give the main result of this section.

Theorem 6.4. *Let χ a solution of (36) with initial data $\chi_0 \in \overline{\mathscr{D}}(A)$. If the kernel g satisfies the hypotheses (h_1), (h_2) and (h_3), then*

$$\mathscr{E}(t) \leq Me^{-\mu t}\mathscr{E}(0)$$

where M and μ are positive constants.

Proof. Taking t_0 sufficiently large, (40) and (41) yield

$$0 \geq \mathscr{L}_{t_0}(T) - \mathscr{L}_{t_0}(0) \geq (T + t_0 - \delta)\mathscr{E}(T) - (t_0 + \delta)\mathscr{E}(0),$$

so that

$$\mathscr{E}(T) \leq \frac{t_0 + \delta}{T + t_0 - \delta}\mathscr{E}(0). \tag{50}$$

Estimate (50) ensures the exponential decay of $\mathscr{E}$ thanks to semigroup properties proved (see, for instance, Th 4.1 in [22]). □

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