STABILITY AND GENERALIZED HOPF BIFURCATION THROUGH A REDUCTION PRINCIPLE

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1. INTRODUCTION

We are interested in obtaining an analysis of the bifurcating periodic orbits arising in the generalized Hopf bifurcation problems in R^n. The existence of these periodic orbits has often been obtained by using such techniques as the Liapunov-Schmidt method or topological degree arguments (see [5] and its references). Our approach, on the other hand, is based upon stability properties of the equilibrium point of the unperturbed system. Andronov et. al. [1] showed the fruitfulness of this approach in studying bifurcation problems in R^2 (for more recent papers see Negrini and Salvadori [6] and Bernfeld and Salvadori [2]). In the case of R^2, in contrast to that of R^n, $n > 2$, the stability arguments can be effectively applied because of the Poincaré-Bendixson theory. Bifurcation problems in R^n can be reduced to that of R^2 when two dimensional invariant manifolds are known to exist. The existence of such manifolds occurs, for example when the unperturbed system contains only two purely imaginary eigenvalues.

In this paper we shall be concerned with the general situation in R^n in which the unperturbed system may have several pairs of purely

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In contrast to (a) another property which we consider in this paper is:

(A) For any neighborhood N of f_0, for any integer $j \geq 0$, for any neighborhood U_1 of 0, and for any number $\delta > 0$ there exists $f \in N$ such that (1.2) has j nontrivial periodic orbits lying in U_1 whose period is in $[2\pi-\delta, 2\pi+\delta]$.

In R^2, Andronov et.al. [1] proved that property (a) is a consequence of the origin of (1.1) being h-asymptotically stable or h completely unstable where h is an odd integer and $k = \frac{h-1}{2}$. The origin of (1.1) in R^n is said to be h-asymptotically stable or h-completely unstable if h is the smallest positive integer such that the origin of (1.2) is asymptotically stable (completely unstable) for all f for which $f(p) - f_0(p) = o(\|p\|^h)$ (that is h is the smallest postive integer such that asymptotic stability and complete instability of the origin for (1.1) are recognizable by inspecting the terms up to order h in the Taylor expansion of f_0) (see Negrini and Salvadori [6] for further information on the h-asymptotic stability). In a recent paper Bernfeld and Salvadori [2] in R^2 extended the results of Andronov et.al. [1] by proving property (a) is equivalent to the h-asymptotic stability (h-complete instability) of the origin of (1.1)(where again $k = \frac{h-1}{2}$). It was also shown that property (A) is equivalent to the case in which the origin of (1.1) is neither h-asymptotically stable nor h-completely unstable for any positive integer h.

The problem in R^n was first considered by Chafee [3]. Using the Liapunov-Schmidt method he obtained a determining equation $\psi(\xi, f) = 0$

of a two dimensional system appropriately related to the unperturbed system (1.1). In addition, an algebraic procedure allows for a concrete solution to the problem.

In a forthcoming paper, the authors will apply an extension of the Poincaré procedure [8], given by Salvadori [7] in order to compute in certain cases the number k directly from system (1.1).

2. RESULTS

By an appropriate change of coordinates depending on f we may write systems (1.1) and (1.2) respectively in the form

$$\begin{aligned} \dot{x} &= -y + X_0(x,y,z) \\ \dot{y} &= x + Y_0(x,y,z) \\ \dot{z} &= A_0 z + Z_0(x,y,z), \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \dot{x} &= \alpha x - \beta y + X(x,y,z) \\ \dot{y} &= \alpha y + \beta x + Y(x,y,z) \\ \dot{z} &= Az + Z(x,y,z). \end{aligned} \tag{2.2}$$

Here α,β are constants, A and A_0 are $(n-2) \times (n-2)$ constant matrices, and X, Y, X_0, Y_0 belong to $C^\infty[B^n(r_0),R]$ and Z,Z_0 belong to $C^\infty[B^n(r_0),R^{n-2}]$. Moreover, X,Y,Z,X_0,Y_0,Z_0 are of order greater than one and the eigenvalues of A_0, $\{\lambda_j\}_{j=1}^{n-2}$ satisfy the condition that $\lambda_j \neq mi$, $m = 0, \pm 1, \ldots$. We shall refer to the right hand sides of (2.1) and (2.2) as f_0 and f respectively.

The two dimensional surface $z = \phi^{(h)}(x,y)$ is tangent at the origin to the eigenspace corresponding to the eigenvalues $\pm i$. This surface will be called a quasi-invariant manifold of order h.

Given any $h > 0$ define the following two dimensional system

$$(S_h) \qquad \begin{aligned} \dot{x} &= -y + X_0\big(x,y,\phi^{(h)}(x,y)\big) \\ \dot{y} &= x + Y_0\big(x,y,\phi^{(h)}(x,y)\big). \end{aligned}$$

(This is the system referred to in the introduction).

We distinguish the two possible cases:

I. There exists $h > 1$ (and then h must be odd) such that $x \equiv y \equiv 0$ is either h-asymptotically stable or h-completely unstable for (S_h).

II. Case I does not hold.

We are now able to state our main result.

Theorem 1. In case I property (a) holds with $k = \frac{h-1}{2}$. In case II, property (A) holds.

If all the eigenvalues of A_0 have real part not equal to zero, then for every $h > 1$ there exists a C^{h+1} two dimensional center manifold which will be denoted by H_h. We notice that if $z = \phi(x,y)$ is the equation of this center manifold, we can write

$$\phi(x,y) = \phi^{(h)}(x,y) + o(x^2+y^2)^{h/2}.$$

As a corollary of Theorem 1 the following result holds.

Theorem 2. Suppose that all the eigenvalues of A_0 have real part different than zero. Then: (i) if there exists an h (and h must be odd) such that the origin of the unperturbed system (2.1) is either h-asymptotically

Using the transformation

$$\zeta = z - \phi^{(h)}(x,y),$$

we can rewrite the unperturbed system (2.1) as

$$\begin{aligned} \dot{x} &= -y + X_0^{(h)}(x,y,\zeta) \\ \dot{y} &= x + Y_0^{(h)}(x,y,\zeta) \\ \dot{\zeta} &= A_0\zeta + W_0^{(h)}(x,y,\zeta), \end{aligned} \tag{3.1}$$

where $X_0^{(h)}(x,y,0) = X_0\big(x,y,\phi^{(h)}(x,y)\big)$, $Y_0^{(h)}(x,y,0) = Y_0(x,y,\phi^{(h)}(x,y))$. From (2.4) we observe that $W_0^{(h)}(x,y,0)$ is of order greater than h. Analogously, we can rewrite the perturbed system (2.2) as

$$\begin{aligned} \dot{x} &= \alpha x - \beta y + X^{(h)}(x,y,\zeta) \\ \dot{y} &= \alpha y + \beta x + Y^{(h)}(x,y,\zeta) \\ \dot{\zeta} &= A\zeta + W^{(h)}(x,y,\zeta), \end{aligned} \tag{3.2}$$

where $X^{(h)}(x,y,0) = X\big(x,y,\phi^{(h)}(x,y)\big), Y^{(h)}(x,y,0) = Y\big(x,y,\phi^{(h)}(x,y)\big)$ and $X^{(h)}$, $Y^{(h)}$, $W^{(h)}$ are of order ≥ 2. For simplicity, we shall again refer to the right hand sides of (3.1) and (3.2) as f_0 and f respectively.

We now state the following lemma whose proof is based on the implicit function theorem.

<u>Lemma 1</u>. There exists L, ε, $\delta > 0$ and a neighborhood $\overline{N}$ of f_0 such that for every $f \in \overline{N}$ and for every periodic solution $\big(x(t,x_0,y_0,\zeta_0),$ $y(t,x_0,y_0,\zeta_0), z(t,x_0,y_0,\zeta_0)\big)$ of (3.2) lying in $B^n(\varepsilon)$ whose period is in

We now introduce for system (3.4) properties (a') and (A') which corresponds to properties (a) and (A) for system (3.2).

(a')(i) There exists a neighborhood $\mathcal{N}$ of f_0 and a neighborhood $\mathcal{U}'$ of $r = 0$, $v = 0$ such that for every $f \in \mathcal{N}$ there are at most k nontrivial 2π periodic solutions of (3.4) lying in $\mathcal{U}'$.

(ii) For each integer j, $0 \leq j \leq k$, for each neighborhood N of f_0, $N \subseteq \mathcal{N}$, and for each neighborhood $\mathcal{U}_1'$ of $r = 0$, $v = 0$ there exists $f \in N$ such that (3.4) has exactly j nontrivial 2π periodic solutions lying in $\mathcal{U}_1'$.

(A') For any neighborhood N of f_0, for any integer $j \geq 0$, and for any neighborhood $\mathcal{U}_1'$ of $r = 0$, $v = 0$ there exists $f \in N$ such that (3.4) has j nontrivial 2π periodic orbits lying in $\mathcal{U}_1'$.

We then have:

Lemma 2. Property (a') implies (a).

In order to prove Lemma 2, it is sufficient in view of Lemma 1, to ascertain the following property: (b') the 2π periodic solutions of (3.4) lying in a fixed neighborhood of $r = 0$, $v = 0$ tend to the origin as $f \to f_0$.

A solution $(r(\theta), v(\theta))$ of (3.4) will be called a $(2\pi, v)$ solution if $v(2\pi) = v(0)$. Every 2π periodic solution is a $(2\pi, v)$ solution but the converse is not, in general, true. In order to find the 2π periodic solutions, we only need to analyze the set of $(2\pi, v)$ solutions. Under our assumptions on A_0 we can use the implicit function theorem to derive from the second equation in (3.4) a C^∞ function $\tau(c,f), \tau(0,f) = 0$ such that a solution of (3.4) passing through $(0, c, v_0)$, with $f - f_0, c$, and v_0 sufficiently small, is a $(2\pi, v)$ solution if and only if $v_0 = \tau(c,f)$. Denote for (S_h) we have

$$u_1(\theta,f_0) \equiv 1,\ u_i(2\pi,f_0) = 0,\ i = 2 \ldots h-1,\ u_h(2\pi,f_0) \neq 0,$$

thus implying (3.10) holds (see [6] for more details).

Let us extend the domain of $V(c,f)$ to include negative values of c. Since the origin is a solution of (3.4) for any f, an application of Rolle's Theorem, in view of (3.10), implies that there exists a $\delta > 0$, and a neighborhood N of f_0 such that for any $f \in N$, $V(c,f)$ has at most $h - 1$ nonzero roots lying in $[-\delta,\delta]$. On the other hand, it is easy to recognize that for each positive root of $V(c,f)$ there is a negative root of $V(c,f)$. Thus, there are at most $\frac{h-1}{2}$ 2π periodic solutions of (3.4) lying in a neighborhood U' of $r = 0$, $v = 0$. This proves $(a')(i)$ is satisfied.

Property $(a')(ii)$ can be proved by assuming a particular perturbed system of the form

$$\dot{x} = -y + X_0^{(h)}(x,y,\zeta) + \sum_{i=0}^{(h-3)/2} a_i x(x^2+y^2)^i$$

$$\dot{y} = x + Y_0^{(h)}(x,y,\zeta) + \sum_{i=0}^{(h-3)/2} a_i y(x^2+y^2)^i$$

$$\dot{\zeta} = A_0\zeta + W_0^{(h)}(x,y,\zeta),$$

where a_i are constants depending on j, $0 \le j \le k$, N and U_i.

Since the roots of $V(\cdot,f) = 0$ approach zero as $f \to f_0$, property (b') holds. Lemma 2 then implies (a) holds, proving Theorem 1 for case I.

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