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Stability and guaranteed cost control of uncertain discrete delay systems

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Robust stability and the guaranteed cost control problem are considered for discrete-time systems with time-varying delays from given intervals. A new construction of Lyapunov–Krasovskii functionals (LKFs), which has been recently introduced in the continuous-time case, is applied. To a nominal LKF, which is appropriate to the system with nominal delays, terms are added that correspond to the system with the perturbed delays and that vanish when the delay perturbations approach zero. The nominal LKF is chosen in the form of the descriptor type and is applied either to the original or to the augmented system. The delay-independent result is derived via the Razumikhin approach. Guaranteed cost state-feedback control is designed. The advantage of the new tests is demonstrated via illustrative examples.

1. Introduction

During the last decade, a considerable amount of attention has been payed to stability and control of continuous-time linear systems with delays (see e.g. Li and de Souza 1997, Kolmanovskii and Richard 1999, Fridman 2001, 2004, Niculescu 2001, Fridman and Shaked 2002, and the references therein). Delay-independent and, less conservative, delay-dependent sufficient stability conditions in terms of Riccati or linear matrix inequalities (LMIs) have been derived by using Lyapunov–Krasovskii functionals or Lyapunov–Razumikhin functions. Delay-dependent conditions are based on different model transformations. The most recent one, a descriptor representation of the system (Fridman 2001), minimizes the overdesign that stems from the model transformation used. The conservatism that stems from the bounding of the cross-terms in the derivation of the derivative of the Lyapunov–Krasovskii functional has also been significantly reduced in the past few years. An important result that improves the standard bounding technique of, for exmaple, Li and de Souza (1997) has been proposed in Moon *et al*. (2001).

Less attention has been drawn to the corresponding results for discrete-time delay systems (Verriest and Ivanov 1995, Kapila and Haddad 1998, Song *et al*. 1999, Mahmoud 2000, Lee and Kwom 2002, Chen *et al*. 2003, Gao *et al*. 2004). This is mainly due to the fact that such systems can be transformed into augmented systems without delay. This augmentation of the system is, however, inappropriate for systems with unknown delays or systems with time-varying delays (such systems appear, for example, in the field of communication networks).

For the case of constant 'small' delay from $[0, \mu]$ the delay-dependent conditions were derived in Lee and Kwon (2002), Gao *et al*. (2004) and Chen *et al*. (2003) by applying the discrete counterparts of the method developed in Moon *et al*. (2001) and of the descriptor approach of Fridman and Shaked (2002) correspondingly. There is a difference between the Lyapunov functions V for the descriptor discrete-time system $Ex(k+1) = Ax(k)$, $E = \text{diag}\{I, 0\}$ and the continuous-time system $E\dot{x}(t) = Ax(t)$. Thus, in the discrete-time $V = x^{\mathrm{T}}EPEx$, where $P = P^{\mathrm{T}}$ is a full matrix (Xu and Yang 1999), while in the continuous-time $V = x^{\mathrm{T}}EPx$ with P of block-triangular structure

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(Takaba *et al.* 1995). The method of Chen *et al.* (2003) allows the treatment of the discrete-time case in a continuous-time manner with block triangular P. The case of 'small' time-varying delay has been studied in Fridman and Shaked (2005) via a discrete descriptor Lyapunov function.

The case of uncertain '*non-small*' time-varying delay, where the *nominal delay value is non-zero* and constant, has been recently considered in Xu and Chen (2004). A Lyapunov function has been used there with a 'nominal' part that corresponds to delay-independent stability of the nominal system (i.e. of the systems with a nominal value of the delay). Thus, the necessary condition for the feasibility of the LMIs derived in Xu and Chen (2004) for stability is the delay-independent stability of the nominal system, which is very restrictive.

For continuous-time systems with uncertain non-small delay a new construction of the LKF has been introduced recently in Fridman (2004). To a nominal LKF, which is appropriate to the nominal system (with nominal delays), terms are added which correspond to the perturbed system and which vanish when the delay perturbations approach zero.

In the present paper we apply such a construction of the LKF to the discrete-time systems with time-varying 'non-small' delay, where the descriptor type nominal LKF is applied first to the original system. Further, we augment the system to one with uncertain delay in a segment, starting from zero, and apply the conditions via descriptor nominal LKF to this augmented system. Such an augmentation is impossible in the continuous-time case, where the complete LKF should be used (see, e.g. Kharitonov and Zhabko 2003), which leads to complicated conditions. The augmented system approach essentially improves the results obtained by the direct application of the descriptor nominal LKF (see, Examples 1 and 3). The trade-off is in the higher-dimensional LMIs that are obtained, which require more computational efforts. Moreover, the state-feedback via augmentation depends on the current and the delayed states, while in the direct descriptor approach a memoryless state-feedback is obtained. To derive the reduced-order conditions we apply the descriptor model transformation of the augmented system and the *discrete descriptor* Lyapunov function of the form $V = x^{\mathrm{T}}EPEx$. New delay-independent robust stability conditions are derived in the case of time-varying delay, that are based on the Razumikhin approach. Guaranteed cost state-feedback control is designed via descriptor nominal LKF. Examples are given which show that our conditions are less conservative than those that have appeared in the literature.

2. Robust stability

2.1 *Problem formulation*

We consider the following unforced discrete-time state-delayed system

$$\begin{aligned} x(k+1) &= (A+H\Delta(k)E)x(k) + (A_1+H\Delta(k)E_1)x(k-\tau(k)),\\ x(k) &= \phi(k), \quad -h-\mu_2 \le k \le 0 \end{aligned} \tag{1}$$

where $x(k) \in \mathcal{R}^n$ is the state vector, $\tau(k)$ is a positive number representing the delay $\tau(k) = h + \eta(k)$ with the nominal constant value $h > 0$ and a time-varying perturbation $\eta(k) \in [-\mu_1, \mu_2]$, $h \ge \mu_1 \ge 0$, $\mu_2 \ge 0$. The matrices A, A_1, H, E and E_1 are constant and $\Delta(k) \in \mathcal{R}^{r_1 \times r_2}$ is a time-varying uncertain matrix satisfying the following inequality

$$\Delta^{\mathrm{T}}(k)\Delta(k) \le I. \tag{2}$$

For simplicity only we consider the single delay. The results are easily extended to systems with multiple delay.

2.2 *Lyapunov–Krasovskii method for discrete systems with delays*

Denoting

$$y(k) = x(k+1) - x(k) \tag{3}$$

and taking into account that

$$x(k-\tau(k)) = x(k-h) - \sum_{j=k-h-\eta(k)}^{k-h-1} y(j)$$

we represent (1) in the descriptor form

$$\begin{bmatrix} x(k+1) \\ 0 \end{bmatrix} = \begin{bmatrix} y(k) + x(k) \\ \begin{pmatrix} -y(k) + (A + H\Delta E - I)x(k) \\ +(A_1 + H\Delta E_1)x(k-h) \\ -\sum_{j=k-h-\eta(k)}^{k-h-1}(A_1 + H\Delta E_1)y(k) \end{pmatrix} \end{bmatrix}, \tag{4}$$

$$\left.\begin{aligned} &x(0) = \phi(0),\\ &y(0) = (A + H\Delta E - I)\phi(0) + (A_1 + H\Delta E_1)\phi(-\tau(0)),\\ &y(k) = \phi(k+1) - \phi(k), \quad k = -h-\mu_2, \ldots, -1. \end{aligned}\right\} \tag{5}$$

Thus, if $x(k)$ is a solution of (1), then $\{x(k), y(k)\}$, where $y(k)$ is defined by (3), is a solution of (4) and (5) and vice versa.

Lemma 1: *If there exist positive numbers* α, β *and a continuous functional*

$$V(k) = V(x(k-h), \dots, x(k), y(k-h-\mu_2), \dots, y(k-1))$$

such that

$$\left.\begin{aligned} &0 \le V(k) \le \beta \max\{\max_{j\in[k-h-\mu_2,k]} |x(j)|^2, \max_{j\in[k-h-\mu_2,k-1]} |y(j)|^2\} \\ &\Delta V(k) \triangleq V(k+1) - V(k) \le -\alpha |x(k)|^2 \end{aligned}\right\} \tag{6a, b}$$

for $x(k)$ *and* $y(k)$ *satisfying* (4), *then* (1) *is asymptotically stable.*

Proof: From (6b) it follows that

$$\sum_{j=0}^{k}(V(j+1) - V(j)) = V(k+1) - V(0) \le -\alpha \sum_{j=0}^{k} |x(j)|^2.$$

Therefore, for $x(k)$ and $y(k)$ satisfying (4) we have due to (6a)

$$|x(k)|^2 \le \sum_{j=0}^{k} |x_j|^2 \le \frac{1}{\alpha} V(0) \le \frac{\beta}{\alpha} \max\left\{\max_{j\in[-h-\mu_2,0]} |x(j)|^2, \max_{j\in[-h-\mu_2,-1]} |y_j|^2\right\}, \quad \forall k \ge 0. \tag{7}$$

Let $x(k)$ be a solution of (1) and $y(k)$ be defined by (3), then $\{x(k), y(k)\}$ satisfies (4), (5) and thus (7). Equation (7) implies that $|x(k)|^2$ is small enough for small enough $\|\phi\|^2 \triangleq \max_{j\in[-\bar{h},0]} |\phi_{-j}|^2$. Moreover, $\sum_{j=0}^{\infty} |x(j)|^2 < \infty$ and, hence, $|x(j)|^2 \to 0$ for $j \to \infty$. □

We suggest to construct the LKF for (4) in the form of

$$V(k) = V_n(k) + V_a(k) \tag{8}$$

where

$$V_a(k) = \sum_{m=-\mu_2}^{\mu_1-1} \sum_{j=k+m-h}^{k-1} y(j)^{\mathrm{T}} R_a y(j), \quad 0 < R_a \tag{9}$$

and V_n is a **nominal** Lyapunov function which corresponds to (4), with $\eta(k) = 0$ and $H = 0$.

We intend to construct V_n in the form of 'descriptor type' (see, e.g. Chen *et al.* 2003) and to apply it either to the original system or to the augmented one.

2.3 *Robust stability via descriptor type nominal LKF*

The nominal LKF (which corresponds to (4) with $\eta(k) = 0$, $H = 0$) is given by (see, e.g. Chen *et al.* 2003)

$$V_n(k) = x^{\mathrm{T}}(k) P_1 x(k) + \sum_{m=-h}^{-1} \sum_{j=k+m}^{k-1} y(j)^{\mathrm{T}} R y(j) + \sum_{j=k-h}^{k-1} x(j)^{\mathrm{T}} S x(j), \quad P_1 > 0,\ R > 0,\ S > 0. \tag{10}$$

The nominal system is asymptotically stable if there exist $n \times n$ matrices $0 < P_1$, P_2, P_3, S, Y, Z_1, Z_2, Z_3, R such that the following LMIs are feasible

$$\Gamma_n = \begin{bmatrix} \Psi_n + hZ & P^{\mathrm{T}} \begin{bmatrix} 0 \\ A_1 \end{bmatrix} - Y^{\mathrm{T}} \\ * & -S \end{bmatrix} < 0, \quad \begin{bmatrix} R & Y \\ * & Z \end{bmatrix} \ge 0 \tag{11a, b}$$

where

$$\begin{aligned} P &= \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix} \\ Y &= [Y_1\ Y_2], \quad Z = \begin{bmatrix} Z_1 & Z_2 \\ * & Z_3 \end{bmatrix}, \quad i = 1, 2 \\ \Psi_n &= P^{\mathrm{T}} \begin{bmatrix} 0 & I \\ A-I & -I \end{bmatrix} + \begin{bmatrix} 0 & I \\ A-I & -I \end{bmatrix}^{\mathrm{T}} P \\ &\quad + \begin{bmatrix} S & 0 \\ 0 & hR + P_1 \end{bmatrix} + \begin{bmatrix} Y \\ 0 \end{bmatrix} + \begin{bmatrix} Y \\ 0 \end{bmatrix}^{\mathrm{T}}, \end{aligned} \tag{12a–d}$$

We obtain the following lemma.

Lemma 2: *Equation* (1) *with* $\Delta \equiv 0$ *is asymptotically stable for* $0 \le h - \mu_1 \le \tau(k) \le h + \mu_2$ *if there exist* $n \times n$ *matrices* $0 < P_1$, P_2, P_3, S, Y_1, Y_2, R *and* $R_a > 0$ *that satisfy the LMI*

$$\Gamma_1 = \begin{bmatrix} \Psi & P^{\mathrm{T}} \begin{bmatrix} 0 \\ A_1 \end{bmatrix} - Y^{\mathrm{T}} & \mu P^{\mathrm{T}} \begin{bmatrix} 0 \\ A_1 \end{bmatrix} & hY^{\mathrm{T}} \\ * & -S & 0 & 0 \\ * & * & -\mu R_a & 0 \\ * & * & * & -hR \end{bmatrix} < 0 \tag{13}$$

where $\mu = \max\{\mu_1, \mu_2\}$, Y *and* Ψ_n *are given by* (12) *and*

$$\Psi = \Psi_n + \begin{bmatrix} 0 & 0 \\ 0 & (\mu_1 + \mu_2) R_a \end{bmatrix}. \tag{14}$$

Proof: We find when $\Delta V(k)$ is strictly negative. The difference $\Delta V_n(k)$ along the trajectories of the nominal system satisfies the inequality (Chen *et al.* 2003)

$$\Delta V_n(k) \leq \xi^{\mathrm{T}}(k)\Gamma_n \xi(k) \quad (15)$$

where Γ_n is given by (11a) and

$$\xi(k) = \mathrm{col}\{x(k),\ y(k),\ x(k-h)\} \quad (16)$$

provided (11b) is satisfied. Note that along the trajectories of (4)

$$\begin{aligned} &x^{\mathrm{T}}(k+1)P_1x(k+1) - x^{\mathrm{T}}(k)P_1x(k) \\ &\quad = 2x^{\mathrm{T}}(k)P_1y(k) + y^{\mathrm{T}}(k)P_1y(k) \\ &\quad = 2\bar{x}^{\mathrm{T}}(k)P^{\mathrm{T}}\begin{bmatrix} y(k) \\ 0 \end{bmatrix} + y^{\mathrm{T}}(k)P_1y(k) \\ &\quad = 2\bar{x}^{\mathrm{T}}(k)P^{\mathrm{T}}\begin{bmatrix} y(k) \\ -y(k) + (A-I)x(k) + A_1x(k-h) \end{bmatrix} \\ &\qquad + y^{\mathrm{T}}(k)P_1y(k) + \delta(k) \end{aligned} \quad (17)$$

where $\bar{x}(k) = \mathrm{col}\{x(k), y(k)\}$

$$\delta(k) = -2\bar{x}^{\mathrm{T}}(k)P^{\mathrm{T}}\sum_{j=k-h-\eta(k)}^{k-h-1}\begin{bmatrix} 0 \\ A_1 \end{bmatrix}y(j)$$

while along the trajectories of the nominal system with $\tau(k) \equiv h$ (17) is obtained with $\delta(k) \equiv 0$.

Therefore ΔV_n along the trajectories of the perturbed system satisfies the inequality

$$\Delta V_n(k) \leq \xi^{\mathrm{T}}(k)\Gamma_n\xi(k) + \delta(k). \quad (18)$$

We have

$$\begin{aligned} \delta(k) &\leq \left| \sum_{j=k-h-\eta(k)}^{k-h-1} \bar{x}^{\mathrm{T}}(k)P^{\mathrm{T}}\begin{bmatrix} 0 \\ A_1 \end{bmatrix}R_a^{-1}[0\ A_1^{\mathrm{T}}]P\bar{x}(k) \right| \\ &\quad + \left| \sum_{j=k-h-\eta(k)}^{k-h-1} y^{\mathrm{T}}(j)R_a y(j) \right| \\ &\leq \mu\bar{x}^{\mathrm{T}}(k)P^{\mathrm{T}}\begin{bmatrix} 0 \\ A_1 \end{bmatrix}R_{ia}^{-1}[0\ A_1^{\mathrm{T}}]P\bar{x}(k) \\ &\quad + \sum_{j=k-h-\mu_2}^{k-h+\mu_1-1} y^{\mathrm{T}}(j)R_a y(j). \end{aligned} \quad (19)$$

From (8), (9), (18), (19) and Schur complements formula we find, provided (11b) is satisfied, the following

$$\Delta V(k) \leq \xi_1(k)^{\mathrm{T}}(\Gamma_1 + \mathrm{diag}\{hZ, 0, 0, 0\})\xi_1(k) \quad (20)$$

where $\xi_1(k) = \mathrm{col}\{\xi(k), y(k), 0\}$. From (11b) it follows that $Z \geq Y^{\mathrm{T}}R^{-1}Y$. Choosing therefore in (20) $Z = Y^{\mathrm{T}}R^{-1}Y$ and applying Schur complements formula it is obtained that (13) implies $\Delta V(k) < 0$ and the asymptotic stability of (1). □

In the case of norm-bounded uncertainties (i.e. $\Delta \neq 0$) we replace A and A_1 in Lemma 2 by $A + H\Delta E$ and $A_1 + H\Delta E_1$, respectively. Applying the bounding (Xie 1996)

$$\alpha\Delta(k)\beta + \beta^{\mathrm{T}}\Delta^{\mathrm{T}}(k)\alpha^{\mathrm{T}} \leq \rho_0^{-1}\alpha\alpha^{\mathrm{T}} + \rho_0\beta^{\mathrm{T}}\beta \quad (21)$$

where ρ_0 is a positive number and where $\alpha^{\mathrm{T}} = [H^{\mathrm{T}}P_2\ \ H^{\mathrm{T}}P_3\ \ 0\ \ 0\ \ 0]$ and $\beta = [E\ \ 0\ \ E_1\ \ \mu E_1\ \ 0]$, we obtain by Schur complements that $\Delta V(k) < 0$ along the trajectories of (4) if the following LMI holds

$$\begin{bmatrix} \Gamma_1 & \begin{bmatrix} P_2^{\mathrm{T}}H \\ P_3^{\mathrm{T}}H \\ 0 \\ 0 \\ 0 \end{bmatrix} & \rho_0\begin{bmatrix} E^{\mathrm{T}} \\ 0 \\ E_1^{\mathrm{T}} \\ \mu E_1^{\mathrm{T}} \\ 0 \end{bmatrix} \\ * & -\rho_0 I & 0 \\ * & * & -\rho_0 I \end{bmatrix} < 0. \quad (22)$$

We have thus proved the following

Theorem 1: *Consider* (1), *where* $0 \leq h - \mu_1 \leq \tau(k) \leq h + \mu_2$. *This system is asymptotically stable if there exist* $n \times n$ *matrices* $0 < P_1,\ P_2,\ P_3,\ S,\ Y_1,\ Y_2,\ R, R_a$ *and a scalar* ρ_0 *that satisfy* (22), *where* $\mu = \max\{\mu_1, \mu_2\}$.

2.4 *Augmentation and descriptor nominal LKF*

In the case when the non-delayed system is not asymptotically stable or $h - \mu_1$ is not large, we represent (1) in the form of the augmented system

$$\begin{aligned} \zeta(k+1) &= (\mathcal{A} + \mathcal{H}\Delta(k)\mathcal{E})\zeta(k) \\ &\quad + (\mathcal{A}_1 + \mathcal{H}\Delta(k)\mathcal{E}_1)\zeta(k - \mu_1 - \eta(k)) \end{aligned} \quad (23)$$

where

$$\zeta(k)=\begin{bmatrix} x(k-h+\mu_1)\\ x(k-h+\mu_1-1)\\ \dots\\ x(k)\end{bmatrix},\quad \mathcal{A}=\begin{bmatrix}0 & I_n & \dots & 0 & 0\\ \dots&\dots&\dots&\dots&\dots\\ 0&0&\dots&0&I_n\\ 0&0&\dots&0&A\end{bmatrix}$$

$$\mathcal{A}_1=\begin{bmatrix}0&0&\dots&0\\ \dots&\dots&\dots&\dots\\ 0&0&\dots&0\\ A_1&0&\dots&0\end{bmatrix},\quad \mathcal{H}=\begin{bmatrix}0\\ \dots\\ 0\\ H\end{bmatrix},$$

$$\mathcal{E}=[0\quad 0\quad \cdots\quad 0\quad E],\qquad \mathcal{E}_1=[E_1\quad 0\quad \cdots\quad 0]. \tag{24}$$

Note that for $\mu_1=0$, the nominal system (23), where $\eta(k)\equiv 0$ and $\Delta\equiv 0$, has no delay and the nominal *exact Lyapunov function* $V_n(k)=\zeta^{\mathrm{T}}(k)P_1\zeta(k)$ should be used. This is different from the continuous case, where the exact (complete) LKF has a complicated form and leads to complicated robust stability conditions (Kharitonov and Zhabko 2003).

In the general case of $\mu_1\geq 0$ we apply Theorem 1 to (23), where $h=\mu_1$, and obtain the following:

Theorem 2: *Consider* (1), *where* $0\leq h-\mu_1\leq\tau(k)\leq h+\mu_2$. *This system is asymptotically stable if there exist* $(h-\mu_1+1)n\times(h-\mu_1+1)n$ *matrices* $0<P_1$, P_2, P_3, S, Y_1, Y_2, R, R_a *and scalars* $\rho_i>0$, $i=0,1$ *that satisfy* (22) *with* $\mu=\max\{\mu_1,\mu_2\}$ *and* $h=\mu_1$, *where* A, A_1, E, E_1 *and* H *should be changed correspondingly to* $\mathcal{A},\mathcal{A}_1,\mathcal{E},\mathcal{E}_1$ *and* $\mathcal{H}$.

Remark 1: The augmentation of the system till some $h_0<h-\mu_1$ with $\zeta^{\mathrm{T}}(k)=[x^{\mathrm{T}}(k-h_0),\dots,x^{\mathrm{T}}(k)]$ can also be applied to obtain less restrictive conditions than those obtained by the descriptor approach. Such augmented system is of lower-order than (23) and has delay $h-h_0+\eta(k)$. Here Theorem 1 should be applied with h substituted by $h-h_0$.

2.5 *Augmentation and discrete descriptor Lyapunov function*

We consider $\mu_1=0$ and $\Delta=0$. To reduce the size and the number of the decision variables by the previous augmented method, we consider $h\geq 1$ and the state vector $\zeta=[\zeta_1\dots\zeta_{h+1}]^{\mathrm{T}}$ given by (24). Defining $y(k)=x(k+1-h)-x(k-h)=\zeta_2(k)-\zeta_1(k)$ and representing (1) in the form

$$x(k+1)=Ax(k)+A_1x(k-h)-A_1\sum_{j=k-\eta(k)}^{k-1}y(j) \tag{25}$$

we obtain the descriptor form

$$E\bar{\zeta}(k+1)=\mathcal{A}_d\bar{\zeta}(k)+\mathcal{A}_1\sum_{j=k-\eta(k)}^{k-1}y(j)$$

$$E=\operatorname{diag}\{I_{(h+1)n},\ 0_{n\times n}\},\quad \bar{\zeta}(k)=\begin{bmatrix}\zeta(k)\\ y(k)\end{bmatrix}$$

$$\mathcal{A}_d=\begin{bmatrix}I_n&0&0&\dots&0&I_n\\ 0&0&I_n&\dots&0&0\\ \dots&\dots&\dots&\dots&\dots&\dots\\ 0&0&0&\dots&I_n&0\\ A_1&0&0&\dots&A&0\\ -I_n&I_n&0&\dots&0&-I_n\end{bmatrix},\quad \mathcal{A}_1=-\begin{bmatrix}0\\0\\ \dots\\0\\A_1\\0\end{bmatrix}. \tag{26a–e}$$

We construct the LKF for (4) in the form of $V(k)=V_n(k)+V_a(k)$, where

$$V_a(k)=\mu_2\sum_{m=-\mu_2}^{-1}\sum_{j=k+m}^{k-1}y(j)^{\mathrm{T}}R_ay(j),\quad 0<R_a \tag{27}$$

and V_n is a nominal Lyapunov function which corresponds to (26a), with $\eta(k)=0$

$$V_n=\bar{\zeta}^{\mathrm{T}}(k)EPE\bar{\zeta}(k),\quad P=P^{\mathrm{T}},\ EPE\geq 0. \tag{28}$$

We have

$$\Delta V_n(k)=\xi^{\mathrm{T}}(k)\begin{bmatrix}\mathcal{A}_d^{\mathrm{T}}\\ \mathcal{A}_1^{\mathrm{T}}\end{bmatrix}P\begin{bmatrix}\mathcal{A}_d & \mathcal{A}_1\end{bmatrix}\xi(k)-\bar{\zeta}^{\mathrm{T}}(k)EPE\bar{\zeta}(k)$$

$$\xi^{\mathrm{T}}(k)=\begin{bmatrix}\bar{\zeta}^{\mathrm{T}}(k) & \sum_{j=k-\eta(k)}^{k-1}y^{\mathrm{T}}(j)\end{bmatrix} \tag{29}$$

and

$$\Delta V_a(k)=\mu_2^2y^{\mathrm{T}}(k)R_ay(k)-\mu_2\sum_{j=k-\mu_2}^{k-1}y^{\mathrm{T}}(j)R_ay(j)$$
$$\leq\mu_2^2y^{\mathrm{T}}(k)R_ay(k)-\eta(k)\sum_{j=k-\eta(k)}^{k-1}y^{\mathrm{T}}(j)R_ay(j).$$

By Cauchy–Schwartz inequality

$$\eta(k)\sum_{j=k-\eta(k)}^{k-1}y^{\mathrm{T}}(j)R_ay(j)\geq\left(\sum_{j=k-\eta(k)}^{k-1}y^{\mathrm{T}}(j)R_a\left(\sum_{j=k-\eta(k)}^{k-1}y(j)\right)\right).$$

Hence

$$\Delta V_a(k) \le \mu_2^2 y^{\mathrm{T}}(k) R_a y(k) - \left(\sum_{j=k-\eta(k)}^{k-1} y^{\mathrm{T}}(j)\right) R_a \left(\sum_{j=k-\eta(k)}^{k-1} y(j)\right). \quad (30)$$

Finally we find that

$$\Delta V(k) = \Delta V_n(k) + \Delta V_a(k) \le \xi^{\mathrm{T}}(k) \Gamma_d \xi(k)$$

where

$$\Gamma_d = \begin{bmatrix} \mathcal{A}_d^{\mathrm{T}} P \mathcal{A}_d - EPE + \begin{bmatrix} 0 & 0 \\ 0 & \mu_2^2 R_a \end{bmatrix} & \mathcal{A}_d^{\mathrm{T}} P \mathcal{A}_1 \\ * & -R_a + \mathcal{A}_1^{\mathrm{T}} P \mathcal{A}_1 \end{bmatrix} \quad (31)$$

Therefore, $\Gamma_d < 0$ implies asymptotic stability of (1). We proved the following.

Lemma 3: *Consider* (1), *where* $\Delta \equiv 0$, $1 \le h \le \tau(k) \le h + \mu_2$. *This system is asymptotically stable if there exist a* $(h+2)n \times (h+2)n$ *matrix* $P = P^{\mathrm{T}}$, *such that* $[I_{(h+1)n} \ 0] P [I_{(h+1)n} \ 0]^{\mathrm{T}} > 0$, *and a* $n \times n$ *matrix* R_a *that lead to* $\Gamma_d < 0$, *where* Γ_d, $\mathcal{A}_d$ *and* $\mathcal{A}_1$ *are given by* (31), (26d) *and* (26e), *correspondingly*.

The condition of Lemma 3 can also be written as

$$\Psi = \begin{bmatrix} \begin{pmatrix} \bar{\mathcal{A}}^{\mathrm{T}} P_2 \mathcal{I}^{\mathrm{T}} + \mathcal{I} P_2^{\mathrm{T}} \bar{\mathcal{A}} + \mathcal{I} P_3 \mathcal{I}^{\mathrm{T}} \\ +\mathrm{diag}\{-P_1, \mu_2^2 R_a, -R_a\} \end{pmatrix} & \bar{\mathcal{A}}^{\mathrm{T}} P_1 \\ P_1 \bar{\mathcal{A}} & -P_1 \end{bmatrix} < 0 \quad (32)$$

where

$$\bar{\mathcal{A}} = \begin{bmatrix} [I_{(h+1)n} \quad 0] \mathcal{A}_d & \begin{bmatrix} 0 \\ \dots \\ A_1 \end{bmatrix} \end{bmatrix}, \quad \mathcal{I} = \begin{bmatrix} [0 \quad I_n] \mathcal{A}_d & 0 \end{bmatrix}^{\mathrm{T}}$$

and where we substituted in (31) the structure of

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^{\mathrm{T}} & P_3 \end{bmatrix}$$

and applied the Schur complements formula.

In the case where A and A_1 are replaced by $A + H\Delta(k)E$ and $A_1 + H\Delta(k)E_1$, respectively, we require that

$$\Psi + \begin{bmatrix} \mathcal{I} P_2 \\ P_1 \end{bmatrix} \mathcal{H} \Delta(k) \bar{\mathcal{E}} + \bar{\mathcal{E}}^{\mathrm{T}} \Delta(k)^{\mathrm{T}} \mathcal{H}^{\mathrm{T}} \begin{bmatrix} P_2 \mathcal{I}^{\mathrm{T}} & P_1 \end{bmatrix}$$
$$\le \Psi + \rho^{-1} \begin{bmatrix} \mathcal{I} P_2 \\ P_1 \end{bmatrix} \mathcal{H} \mathcal{H}^{\mathrm{T}} \begin{bmatrix} P_2^{\mathrm{T}} \mathcal{I}^{\mathrm{T}} & P_1 \end{bmatrix} + \rho \bar{\mathcal{E}}^{\mathrm{T}} \bar{\mathcal{E}} < 0$$

for a positive scalar ρ, where $\mathcal{H}$ is defined in (24) and $\bar{\mathcal{E}} = [E_1 \ 0 \ \cdots \ E \ 0 \ E_1 \ 0]$. The latter leads to the following.

Theorem 3: *Consider* (1), *where* $1 \le h \le \tau(k) \le h + \mu_2$. *This system is asymptotically stable if there exist:* $(h+1)n \times (h+1)n$ *matrix* $0 < P_1$, $(h+1)n \times n$ *matrix* P_2, $n \times n$ *matrix* P_3, $n \times n$ *matrix* $0 < R_a$ *and a positive scalar* ρ *that satisfy the following LMI*

$$\begin{bmatrix} \Psi & \begin{bmatrix} \mathcal{I} P_2^{\mathrm{T}} \\ P_1 \end{bmatrix} \mathcal{H} & \rho \bar{\mathcal{E}}^{\mathrm{T}} \\ * & -\rho I & 0 \\ * & * & -\rho I \end{bmatrix} < 0 \quad (33)$$

where Ψ *is defined in* (32).

Remark 2: Considering $\mu_1 > 0$ and combining V_n of the form of discrete descriptor LKF (i.e. V_n of (10), where the first term should be changed to $x^{\mathrm{T}}(k) EPEx(k)$ with V_a of (30) may lead to further improvement of the results by Lemma 3 and Theorem 3.

We next consider the case of the 'small' delay $\tau(k) \in [0, \mu_2]$ with $h = 0$, $\Delta = 0$ and representing (1) in the descriptor form

$$E\bar{x}(k+1) = \tilde{\mathcal{A}} \bar{x}(k) + \tilde{\mathcal{A}}_1 \sum_{j=k-\eta(k)}^{k-1} y(j), \quad E = \mathrm{diag}\{I_n, 0_{n\times n}\}$$
$$\bar{x}(k) = \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}, \quad \tilde{\mathcal{A}} = \begin{bmatrix} I_n & I_n \\ A_1 + A - I_n & -I_n \end{bmatrix}$$
$$\tilde{\mathcal{A}}_1 = -\begin{bmatrix} 0 \\ A_1 \end{bmatrix}. \quad (34\mathrm{a-e})$$

By applying the above derivations to (34) we obtain a new stability criterion:

Corollary 1: *Consider* (1), *where* $\Delta \equiv 0$, $0 \le \tau(k) \le \mu_2$. *This system is asymptotically stable if there exist* $2n \times 2n$ *matrix* $P = P^{\mathrm{T}}$, *satisfying* $[I_n \ 0] P [I_n \ 0]^{\mathrm{T}} > 0$ and $n \times n$ *matrix* R_a *such that* $\Gamma_d < 0$, *where* Γ_d *is given by* (31) *and where* $\mathcal{A}_d$ *and* $\mathcal{A}_1$ *should be substituted by* $\tilde{\mathcal{A}}$ *and* $\tilde{\mathcal{A}}_1$ *correspondingly*.

2.6 *Delay-independent conditions in the case of time-varying delays*

As in the continuous-time situation, this case is treated adopting the Lyapunov–Razumikhin approach (see Zhang and Chen 2001).

Lemma 4: *Consider the system* (1), *where* $\Delta(k) \equiv 0$, *with time-varying delay. This system is asymptotically stable if there exist* $0 < P \in \mathcal{R}^{n\times n}$ *and scalars* $\alpha \in (0, 1)$ *and* $q > 1$ *that satisfy the LMI*

$$\bar{\Gamma}_{\mathrm{ind}} \triangleq \begin{bmatrix} A^{\mathrm{T}} P A - \alpha P & A^{\mathrm{T}} P A_1 \\ * & A_1^{\mathrm{T}} P A_1 - ((1-\alpha)/q) P \end{bmatrix} < 0. \quad (35)$$

Proof: Choosing the Lyapunov–Razumikhin function $V(k) = x(k)^T Px(k)$ and assuming that for some $q > 1$

$$V(k-i) \le qV(k), \quad -\bar{h} \le i \le -1, \quad k \ge 0$$

we find

$$\begin{aligned} V(k+1) - V(k) &= (x(k)^T A^T + x(k-\tau(k))A_1^T)P(Ax(k) \\ &\quad + A_1 x(k-\tau(k))) - x(k)^T Px(k) \\ &= x(k)^T(A^T PA - \alpha P)x(k) \\ &\quad + 2x^T(k-\tau(k))A_1^T PAx(k) \\ &\quad + x^T(k-\tau(k))A_1^T PA_1 x(k-\tau(k)) \\ &\quad - (1-\alpha)x(k)^T Px(k) \\ &\le \left[x(k)^T \; x^T(k-\tau(k))\right]\bar{\Gamma}_{\text{ind}} \begin{bmatrix} x(k) \\ x(k-\tau(k)) \end{bmatrix} \end{aligned}$$

and thus due to (35) $V(k+1) - V(k) < 0$, which implies the asymptotic stability of (1) (see Zhang and Chen 1998). □

By Schur complements (35) is equivalent to

$$\Omega \triangleq \begin{bmatrix} -\alpha P & 0 & A^T P \\ * & -((1-\alpha)/q)P & A_1^T P \\ * & * & -P \end{bmatrix} < 0. \tag{36}$$

We replace A with $A + H\Delta(k)E$ and A_1 with $A_1 + H\Delta(k)E_1$ and obtain, applying Lemma 2 to the uncertain system (1), that the stability of the system is guaranteed if the following inequality holds

$$\begin{aligned} &\Omega + \begin{bmatrix} 0 & \begin{bmatrix} E^T \\ E_1^T \end{bmatrix} \Delta^T H^T P \\ * & 0 \end{bmatrix} \\ &\le \Omega + \varepsilon \begin{bmatrix} \begin{bmatrix} E^T \\ E_1^T \end{bmatrix} [E \; E_1] & 0 \\ * & 0 \end{bmatrix} + \varepsilon^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & PHH^T P \end{bmatrix} < 0. \end{aligned} \tag{37}$$

In the latter inequality we used the bounding (21) where $\rho_0 = \varepsilon$ and

$$\alpha^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & H^T P \end{bmatrix}, \quad \beta^T = \begin{bmatrix} 0 & 0 & E^T \\ 0 & 0 & E_1^T \\ 0 & 0 & 0 \end{bmatrix}.$$

We proved the following.

Theorem 4: *Consider the system* (1) *with* $\Delta(k)$ *that satisfies* (2). *This system is asymptotically stable for all delays* $\tau(k)$ *if there exist* $P = P^T \in \mathcal{R}^{n\times n}$, $\alpha \in (0,1)$, $q > 1$ *and* $\varepsilon > 0$ *that satisfy the LMI*

$$\begin{bmatrix} -\alpha P + \varepsilon E^T E & \varepsilon E^T E_1 & A^T P & 0 \\ * & -((1-\alpha)/q)P + \varepsilon E_1^T E_1 & A_1^T P & 0 \\ * & * & -P & PH \\ * & * & * & -\varepsilon I \end{bmatrix} < 0. \tag{38}$$

2.7 *Examples*

Example 1: We consider the system (1) where

$$A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.97 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix} \quad \text{and} \quad H = 0. \tag{39}$$

Assuming that h is constant, we seek the maximum value of $\bar{h}$ for which the asymptotic stability of the system is guaranteed. We compare three methods: the criterion of Song *et al.* (1999), Theorem 1 in Lee and Kwon (2002) and Theorem 1 above. It is found that the method of Song *et al.* (1999) does not provide a solution even for $\bar{h} = 1$. The maximum value of $\bar{h}$, achievable by the method of Lee and Kwon (2002), is 12, whereas a value of $\bar{h} = 16$ was obtained by applying Chen *et al.* (2003). Using augmentation it is found that the system considered is asymptotically stable for all $h \le 18$. The criterion of Lemma 3 did not provide a solution, so that no delay-independent solution has been found.

Allowing τ to be time-varying we apply Lemma 2, where $h = \mu_1 = 1$ and $\mu_2 = 7$. We obtain thus that asymptotic stability is guaranteed for all $0 \le \tau(k) \le 8$. The same result is obtained by Corollary 1 via discrete descriptor Lyapunov function. Choosing $h = 8$, $\mu_1 = \mu_2 = 3$; $h = 10$, $\mu_1 = \mu_2 = 2$ and $h = 11$, $\mu_1 = 1$, $\mu_2 = 2$ we verified that conditions of Lemma 2 are feasible. Hence the system is asymptotically stable for all $\tau(k)$ from the following intervals: [3, 10], [5, 11], [8, 12] and [10, 13]. Note that conditions of Xu and Chen (2004) are not feasible even for $0 \le \tau(k) \le 1$.

By augmentation via the discrete descriptor Lyapunov function we verify that the conditions of Lemma 3 are feasible for $h = 3, \mu_2 = 7$; $h = 5, \mu_2 = 6$; $h = 7, \mu_2 = 5$ and for $h = 9, \mu_2 = 4$ and thus the stability intervals are larger for $h \ge 7$: [3, 10], [5, 11], [7, 12] and [9, 13]. The augmented approach via descriptor LKF of Lemma 2 leads to the same stability intervals as Lemma 3, but needs essentially more time for computation.

Next considering the case where the system parameters are uncertain, with A and A_1 given in (39) and with

$$H = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.02 \end{bmatrix}, \quad E = I_2 \quad \text{and} \quad E_1 = 0.5I_2,$$

we apply Theorem 1 for $h = 0, 3$ and 5 and verify that the system (1) is asymptotically stable for all $\Delta(k)$ that satisfy (19) and for $\tau(k)$ from the following segments: $[0, 4]$, $[3, 5]$ and $[5, 6]$.

By the augmented system approach via descriptor LKF, we find that the conditions of Theorem 2 are feasible for $h = 3$, $\mu_1 = 1$, $\mu_2 = 2$ and for $h = 5$, $\mu_1 = 1$, and $\mu_2 = 1$. Thus the stability intervals, starting from non-zero values, are larger $[2, 5]$ and $[4, 6]$. By augmentation via discrete descriptor approach we find the following intervals: $[3, 5]$ and $[4, 6]$.

The augmented system approach improves the results, but it takes essentially more time for computations due to high dimensional LMIs. The conditions by Theorem 3 need less time for verification than those by Theorem 2.

Example 2 (Wu and Hong 1994)**:** We consider the system (1) where

$$A = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0.2 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -0.4 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad H = 0.$$

In the case of constant delay, this system is delay-independently stable by the conditions of Wu and Hong (1994). In the case of time-varying delay, by conditions of Song *et al.* (1999) the system is asymptotically stable for $0 < \tau(k) \le 2$. By Theorem 4, it is verified that also in the case of time-varying delay the system is delay-independently stable. This is achieved by taking $\alpha = 0.5$ and $q = 1.01$.

3. Guaranteed cost control

Extending the description of (1) to include a control input $u(k) \in \mathcal{R}^m$, we consider the system

$$\begin{aligned} x(k+1) &= (A + H\Delta(k)E)x(k) + (A_1 + H\Delta(k)E_1) \\ &\quad \times x(k - \tau(k)) + (B + H\Delta(k)E_2)u(k), \\ x(k) &= \phi(k), \quad -h - \mu_2 \le k \le 0 \end{aligned} \tag{40}$$

where $x(k) \in \mathcal{R}^n$, $\tau(k)$, A, A_1, H, E, E_1 and $\Delta(k)$ are as in (1) and (2) and B and E_2 are constant matrices of the appropriate dimensions. We also consider the cost function

$$J = \sum_{j=0}^{\infty} z^{\mathrm{T}}(k)z(k) \tag{41}$$

where the objective vector $z(k) \in \mathcal{R}^p$ is defined by

$$z(k) = Lx(k) + Du(k) \tag{42}$$

for matrices L and D of the appropriate dimensions. A control law

$$u(k) = Kx(k) \tag{43}$$

is sought that for a given $\phi(k)$, $0 \le h - \mu_1 \le \tau(k) \le h + \mu_2$ leads to a minimum guaranteed cost δ for $J(\phi)$, namely, $J(\phi) \le \delta$ for the delay described in (1) and for all Δ that satisfy (2).

3.1 *Guaranteed cost via descriptor nominal LKF*

Denoting

$$\Delta\Phi(k) = V(k+1) - V(k) + z^{\mathrm{T}}(k)z(k) \tag{44}$$

where $V(k)$ is defined in (8) and (9), it is obtained, similarly to the proof of Lemma 2, that in the case where $H = 0$ and $u(k) \equiv 0$ the following holds

$$\Delta\Phi(k) \le \bar{\xi}^{\mathrm{T}}(k)\Gamma_z\bar{\xi}(k) \tag{45}$$

where $\bar{\xi}(k) = \mathrm{col}\{x(k), y(k), x(k-h), y(k), z(k)\}$ and

$$\Gamma_z = \begin{bmatrix} \Gamma_1 & \begin{bmatrix} L^{\mathrm{T}} \\ 0 \end{bmatrix} \\ * & -I_p \end{bmatrix}. \tag{46}$$

Requiring that

$$\Gamma_z < 0 \tag{47}$$

we take the sum of the two sides of (45), from 0 to N, and obtain that

$$\sum_{k=0}^{N} z^{\mathrm{T}}(k)z(k) \le V(0) - V(N+1) \le V(0) \tag{48}$$

and thus

$$\begin{aligned} J \le V(0) &= \phi^{\mathrm{T}}(0)P_1\phi(0) + \sum_{m=-h}^{-1}\sum_{j=m}^{-1}(\phi^{\mathrm{T}}(j+1) - \phi(j)) \\ &\quad \times R(\phi(j+1) - \phi(j)) + \sum_{j=-h}^{-1}\phi^{\mathrm{T}}(j)S\phi(j) \\ &\quad + \sum_{m=-\mu_2}^{\mu_1-1}\sum_{j=m-h}^{-1}((\phi^{\mathrm{T}}(j+1) \\ &\quad - \phi^{\mathrm{T}}(j))R_a(\phi(j+1) - \phi(j)). \end{aligned} \tag{49}$$

Admitting the control law (43) the following result is thus obtained.

Lemma 5: *Consider the system* (40) *where* $H=0$ *and the cost function* (41). *The control law* (43) *stabilizes the system and achieves a prescribed guaranteed cost* $0<\bar{\delta}$, *namely* $J\le\bar{\delta}$, *if there exit* $n\times n$ *matrices* $0<P_1$, P_2, P_3, S, Y_1, Y_2, R *and* R_a *and a* $m\times n$ *matrix* K *that satisfy the following two inequalities*

$$\hat{\Gamma}_z=\begin{bmatrix}\hat{\Gamma}_1 & \begin{bmatrix}L^{\mathrm{T}}+K^{\mathrm{T}}D^{\mathrm{T}}\\0\end{bmatrix}\\ * & -I_p\end{bmatrix}<0 \quad (50a)$$

$$V(0)\le\bar{\delta} \quad (50b)$$

where

$$\hat{\Gamma}_1=\begin{bmatrix}\hat{\Psi} & P^{\mathrm{T}}\begin{bmatrix}0\\A_1\end{bmatrix}-Y^{\mathrm{T}} & \mu P^{\mathrm{T}}\begin{bmatrix}0\\A_1\end{bmatrix} & hY^{\mathrm{T}}\\ * & -S & 0 & 0\\ * & * & -\mu R_a & 0\\ * & * & * & -hR\end{bmatrix}<0 \quad (50c)$$

$$\hat{\Psi}=P^{\mathrm{T}}\begin{bmatrix}0 & I\\A+BK-I & -I\end{bmatrix}+\begin{bmatrix}0 & I\\A+BK-I & -I\end{bmatrix}^{\mathrm{T}}P+\begin{bmatrix}S & 0\\0 & hR+P_1+(\mu_1+\mu_2)R_a\end{bmatrix}+\begin{bmatrix}Y\\0\end{bmatrix}+\begin{bmatrix}Y\\0\end{bmatrix}^{\mathrm{T}} \quad (50d)$$

and where $V(0)$ *is given in* (49) *and* P *has the structure of* (12a).

The inequality (50a) is non-linear in P and K. In order to obtain a LMI we consider the case where $Y=\varepsilon\begin{bmatrix}0 & A_1^{\mathrm{T}}\end{bmatrix}P$, for some tuning parameter ε. Realizing that the second block on the diagonal of $\hat{\Psi}$ in $\hat{\Gamma}_1$ is $-P_3-P_3^{\mathrm{T}}+hR+P_1+(\mu_1+\mu_2)R_a$ it is found that P is invertible. Denoting

$$P^{-1}=Q=\begin{bmatrix}Q_1 & 0\\Q_2 & Q_3\end{bmatrix}$$

we obtain the following.

Lemma 6: *Consider the system* (40) *where* $H=0$ *and the cost function* (41). *The control law* (43) *stabilizes the system and achieves a prescribed guaranteed cost* $0<\bar{\delta}$ *if for some tuning scalar parameter* ε *there exit* $n\times n$ *matrices* Q_1, Q_2, Q_3, $\bar{S}$, $\bar{R}$, $\bar{R}_a$, M_R, M_{R_a} *and* M_S, *a scalar* M_Q, *and a* $m\times n$ *matrix* $\bar{Y}$ *that satisfy the following six inequalities*

$$\begin{bmatrix}\hat{\Gamma}_k & \begin{bmatrix}Q_1L^{\mathrm{T}}+\bar{Y}^{\mathrm{T}}D^{\mathrm{T}}\\0\end{bmatrix}\\ * & -I_p\end{bmatrix}<0 \quad (51a)$$

$$\begin{bmatrix}M_Q & \phi^{\mathrm{T}}(0)\\ * & Q_1\end{bmatrix}>0,\quad \begin{bmatrix}M_R & I_n\\ * & \bar{R}\end{bmatrix}>0 \quad (51b, c)$$

$$\begin{bmatrix}M_{R_a} & I_n\\ * & \bar{R}_a\end{bmatrix}>0,\quad \begin{bmatrix}M_S & I_n\\ * & \bar{S}\end{bmatrix}>0 \quad (51d, e)$$

and

$$\bar{J}(\phi)=M_Q+\sum_{m=-h}^{-1}\sum_{j=m}^{-1}\left(\phi^{\mathrm{T}}(j+1)-\phi^{\mathrm{T}}(j)\right)M_R(\phi(j+1)-\phi(j))+\sum_{j=-h}^{-1}\phi^{\mathrm{T}}(j)M_S\phi(j)+\sum_{m=-\mu_2}^{\mu_1-1}\sum_{j=m-h}^{-1}\left(\phi^{\mathrm{T}}(j+1)-\phi^{\mathrm{T}}(j)\right)\times M_{R_a}(\phi(j+1)-\phi(j))\le\bar{\delta} \quad (51f)$$

where

$$\hat{\Gamma}_k=\begin{bmatrix}\Psi_k & (1-\varepsilon)\begin{bmatrix}0\\A_1\end{bmatrix}\bar{S} & \mu\begin{bmatrix}0\\A_1\end{bmatrix}\bar{R}_a & \varepsilon h\begin{bmatrix}0\\A_1\end{bmatrix}\bar{R}\\ * & -\bar{S} & 0 & 0\\ * & * & -\mu\bar{R}_a & 0\\ * & * & * & -h\bar{R}\end{bmatrix} \quad (51g)$$

and

$$\Psi_k=\begin{bmatrix}Q_2+Q_2^{\mathrm{T}} & \begin{pmatrix}-Q_1+Q_3-Q_2^{\mathrm{T}}\\+Q_1(A^{\mathrm{T}}+\varepsilon A_1^{\mathrm{T}})+\bar{Y}^{\mathrm{T}}B^{\mathrm{T}}\end{pmatrix} & Q_1 & hQ_2^{\mathrm{T}} & Q_2^{\mathrm{T}} & (\mu_1+\mu_2)Q_2^{\mathrm{T}}\\ * & \left[-Q_3-Q_3^{\mathrm{T}}\right] & 0 & hQ_3^{\mathrm{T}} & Q_3^{\mathrm{T}} & (\mu_1+\mu_2)Q_3^{\mathrm{T}}\\ * & * & -\bar{S} & 0 & 0 & 0\\ * & * & * & -h\bar{R} & 0 & 0\\ * & * & * & * & -Q_1 & 0\\ * & * & * & * & * & -(\mu_1+\mu_2)\bar{R}_a\end{bmatrix}. \quad (51h)$$

If a solution to the above exists, the feedback gain that achieves the guaranteed cost $\bar{\delta}$ is given by

$$K = \bar{Y} Q_1^{-1}. \tag{52}$$

Proof: We first multiply (50a) by $diag\{Q^{\mathrm{T}}, 0, 0, 0\}$ and $diag\{Q, 0, 0, 0\}$, from the left and the right, respectively. Using the fact that:

$$Q^{\mathrm{T}}\begin{bmatrix} 0 & 0 \\ 0 & hR \end{bmatrix} Q = \begin{bmatrix} Q_2^{\mathrm{T}} \\ Q_3^{\mathrm{T}} \end{bmatrix} hR \begin{bmatrix} Q_2 & Q_3 \end{bmatrix} \text{ and}$$
$$Q^{\mathrm{T}}\begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} Q = \begin{bmatrix} Q_1 \\ 0 \end{bmatrix} S \begin{bmatrix} Q_1 & 0 \end{bmatrix}$$

and denoting $\bar{Y} = KQ_1$, $\bar{R} = R^{-1}$, $\bar{R}_a = R_a^{-1}$, $\bar{S} = S^{-1}$, the requirement of (51a) follows applying Schur complements arguments. The requirement (51b) follows from the fact that M_Q, M_R, M_{R_a} and M_S are upper-bounds on Q_1, R, R_a and S, respectively, and thus $V(0) < \bar{J}(\phi)$. □

Lemma 5 addresses the case where $H = 0$. Its result can be readily extended to the case where $H \neq 0$. Replacing A, A_1 and B in the LMI of Lemma 5 by $A + H\Delta E$, $A_1 + H\Delta E_1$ and $B + H\Delta E_2$, respectively, and applying Schur complements arguments we readily obtain the following result for the case with norm-bounded uncertainties.

Theorem 5: *Consider the system* (40) *and the cost function* (41). *The control law* (43) *stabilizes the system and achieves a prescribed guaranteed cost* $0 < \bar{\delta}$ *if for some tuning scalar parameter* ε *there exit* $n \times n$ *matrices* Q_1, Q_2, Q_3, $\bar{S}$, $\bar{R}$, $\bar{R}_a$, M_Q, M_R, M_{R_a} *and* M_S, *a* $m \times n$ *matrix* $\bar{Y}$ *and a scalar* ρ *that satisfy* (51b–f) *and*

$$\begin{bmatrix} \hat{\Gamma}_k & \rho \begin{bmatrix} 0 \\ H \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \begin{pmatrix} Q_1(E+\varepsilon E_1)^{\mathrm{T}} \\ +\bar{Y}^{\mathrm{T}} E_2^{\mathrm{T}} \end{pmatrix} \\ 0 \\ 0 \\ (1-\varepsilon)\bar{S}E_1^{\mathrm{T}} \\ \mu \bar{R}_a E_1^{\mathrm{T}} \\ \varepsilon h \bar{R} E_1^{\mathrm{T}} \end{bmatrix} & \begin{bmatrix} \begin{pmatrix} Q_1 L^{\mathrm{T}} \\ +\bar{Y}^{\mathrm{T}} D^{\mathrm{T}} \end{pmatrix} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ * & -\rho I & 0 & 0 \\ * & * & -\rho I & 0 \\ * & * & * & -I \end{bmatrix} < 0 \tag{53}$$

where $\hat{\Gamma}_k$ *is defined in* (51g). *If a solution to the latter inequalities exists, the state-feedback gain K that achieved the guaranteed cost is given by* (52).

3.2 *Guaranteed cost via augmentation and descriptor nominal LKF*

The stability result of §2.4 can be extended to solve the guaranteed cost problem. However, dissimilar to the solution of the stability verification problem there, when attempting to obtain a state-feedback control law that stabilizes the system and achieves a given bound on its cost, using the augmented system of (23), the problem becomes one of finding a static output-feedback controller. The latter task is known to be non-convex. If, on the other hand, a state-feedback control law of the form

$$u(k) = \sum_{j=0}^{h-\mu_1} K_j x(k-j) \tag{54}$$

is sought the problem can be solved using the augmented system representation of (23). We have

$$\begin{aligned} \zeta(k+1) &= (\mathcal{A} + \mathcal{H}\Delta(k)\mathcal{E})\zeta(k) \\ &\quad + (\mathcal{A}_1 + \mathcal{H}\Delta(k)\mathcal{E}_1)\zeta(k-\mu_1-\eta(k)) + \mathcal{B}u(k) \\ \zeta^{\mathrm{T}}(k) &= \left[\phi^{\mathrm{T}}(k-h+\mu_1)\ \phi^{\mathrm{T}}(k-h+\mu_1+1)\cdots \phi^{\mathrm{T}}(k)\right] \\ &\triangleq \psi^{\mathrm{T}}(k), \quad -\mu_2-\mu_1 \le k \le 0 \\ z(k) &= \mathcal{L}\zeta(k) + Du(k) \end{aligned} \tag{55}$$

where $\mathcal{B} = [0 \ldots 0 \quad B^{\mathrm{T}}]^{\mathrm{T}}$, $\mathcal{L} = [0, \ldots, 0 \quad L]$ and

$$u(k) = [K_{h-\mu_1} \ldots K_0]\zeta(k).$$

Then Theorem 4 implies the following result.

Theorem 6: *Consider the system* (40) *and the cost function* (41). *The control law* (54) *stabilizes the system and achieves a prescribed guaranteed cost* $0 < \bar{\delta}$ *if for some tuning scalar parameter* ε *there exist* $n(h-\mu_1+1)\times n(h-\mu_1+1)$ *matrices* Q_1, Q_2, Q_3, $\bar{S}$, $\bar{R}$, $\bar{R}_a$, M_Q, M_R M_R, M_{R_a}, M_S, *a* $m \times n(h-\mu_1+1)$ *matrix* $\bar{Y}$ *and scalars* $\bar{\rho}_0$ *and* $\bar{\rho}_1$ *that satisfy* (51b–e),(53a, b) *with* $\hat{\Gamma}_k$ *and* $\bar{J}(\phi)$ *given by* (51g) *and* (51f), *where* h, A, A_1, E, E_1, H, L *and* ϕ *should be substituted by* μ_1, $\mathcal{A}$, $\mathcal{A}_1$, $\mathcal{E}$, $\mathcal{E}_1$, $\mathcal{H}$, $\mathcal{L}$ *and* ψ *correspondingly. If a solution to the above inequalities exists, the state-feedback gain* $K = [K_{h-\mu_1} \ldots K_0]$ *that achieves the guaranteed cost is given by* (52).

Remark 3: Guaranteed cost control via the augmented discrete descriptor approach may be designed by an iterative method similarly to Fridman and Shaked (2005) and will not be considered here. Razumikhin approach seems to be inapplicable to guaranteed cost and to H_∞ control.

3.3. *Example 3*

We consider the problem of Chen *et al.* (2003). Given the system (40) with

$$A = \begin{bmatrix} 1.01 & 0 \\ 0 & 1.2 \end{bmatrix}, \; A_1 = \begin{bmatrix} -0.25 & 0.1 \\ 0 & 0.1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$H = I_2, \; E = E_1 = 0.08 I_2, \; E_2 = 0$$

$$L = \sqrt{0.1}\begin{bmatrix} I_2 \\ 0 \end{bmatrix}, \; D = \sqrt{0.1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; \phi(-k) = \begin{bmatrix} e^{-k} \\ 0 \end{bmatrix}.$$

For $h = 2$, $\mu_1 = \mu_2 = 0$ a state-feedback control law (43) is sought that stabilizes the system and achieves a minimum cost bound. In Chen *et al.* (2003) such a feedback control has been found which achieves the cost bound of $\bar{\delta} = 1.2387$. Applying the result of Theorem 5 for $\varepsilon = 1$ we obtain a cost bound of $\bar{\delta} = 0.1392$ for $K = -[0.2763 \quad 1.3471]$.

For a smaller uncertainty, where $H = I_2$, $E = E_1 = 0.02 I_2$ and $E_2 = 0$, application of Theorem 5 yields a cost bound of $\bar{\delta} = 0.05$ for $K = -[0.2231 \; 1.034]$. For this uncertainty we also consider the case where $h = 2$, $\mu_1 = 0$ and $\mu_2 = 1$. For $x(-3) = [e^{-3} \; 0]^T$, the application of Theorem 5 yields a minimum cost bound of $\bar{\delta} = 0.1282$ for $K = -[0.0737 \; 1.2503]$.

Applying the exact LKF result of Theorem 6 to the system with the latter uncertainty (where H is $0.02 I_2$, $h = 2$, $\mu_1 = 0$ and $\mu_2 = 1$) we obtain $\bar{\delta} = 0.0891$, for $k = [0.1438 \quad -0.1520 \quad 0.3422 \quad -0.1661 \quad -0.2766 \quad -1.135]$. The difference between the latter result and the corresponding result that was obtained via descriptor V_n is accentuated in the case where $H = 0$. Then, the descriptor based result of Theorem 4 yields a bound of $\bar{\delta} = 0.0661$ compared to $\bar{\delta} = 0.0383$ obtained by the exact LKF result of Theorem 6.

4. Conclusions

Delay-dependent and delay-independent criteria have been derived for determining the asymptotic stability of discrete-time systems with uncertain delay and norm-bounded uncertainties. The delay is assumed to be time-varying either from a given interval or unbounded. In the first case the Lyapunov–Krasovskii method is applied via descriptor model transformation, while the second (delay-independent) case is treated by Laypunov–Razumikhin technique. The Lyapunov–Krasovskii method is applied to guaranteed cost control.

For the first time, augmentation is applied to the case of time-varying delay in order to reduce the nominal value of the delay which appears in the augmented system. Such an augmentation leads to less conservative conditions. However, the resulting LMIs possess high-dimensional decision variables which require longer computational time.

Another possibility for reducing the conservatism is to apply the discrete-time counterpart of the complete LKF that corresponds to the necessary and sufficient conditions for the stability of the nominal system and which was applied to robust stability of continuous-time systems by Kharitonov and Zhabko (2003). The latter LMIs may have computational advantages over the high-order ones that are based on the augmentation. This issue is currently under study.

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