

Research Article

Stability and Hopf Bifurcation of a Delayed Epidemic Model of Computer Virus with Impact of Antivirus Software

Zizhen Zhang ¹, Ranjit Kumar Upadhyay,² Dianjie Bi,¹ and Ruibin Wei¹

¹School of Management Science and Engineering, Anhui University of Finance and Economics, Bengbu 233030, China

²Department of Applied Mathematics, Indian Institute of Technology (Indian School of Mines), Dhanbad 826004, India

Correspondence should be addressed to Zizhen Zhang; zzzhaida@163.com

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In this paper, we investigate an SLBRS computer virus model with time delay and impact of antivirus software. The proposed model considers the entering rates of all computers since every computer can enter or leave the Internet easily. It has been observed that there is a stability switch and the system becomes unstable due to the effect of the time delay. Conditions under which the system remains locally stable and Hopf bifurcation occurs are found. Sufficient conditions for global stability of endemic equilibrium are derived by constructing a Lyapunov function. Formulae for the direction, stability, and period of the bifurcating periodic solutions are conducted with the aid of the normal form theory and center manifold theorem. Numerical simulations are carried out to analyze the effect of some of the parameters in the system on the dynamic behavior of the system.

1. Introduction

Computer viruses are programs created to carry out activities in a computer without consent of its owner. They not only disrupt the normal functionalities of computer system and damage data files in the computer, but also cause heavy economic losses and tremendous social impacts [1–3]. In recent years, mathematical modeling enjoys popularity in both analyzing and controlling computer viruses based on the similarity between computer viruses and biological viruses. A few works proposing SIR models have appeared in the literatures [4–6]. In [4], Amador studied a stochastic SIRA epidemic model for computer viruses and analyzed the quasi-stationary distribution, the extinction time, and the number of infections in order to understand the spreading mechanism of computer viruses. In [5], Ozturk and Gulsu proposed an approximate solution to a modified SIR computer virus model by using the shifted Chebyshev collocation method. In [6], Khanh studied the stability and approximate iterative solutions of an SIR computer virus model with antidotal component.

Considering the latent period of computer viruses, some models with latency are proposed by some scholars [1, 7–10]. In [7], Yang investigated global stability of a VEISV network

worm attack model by using the Li-Muldowney geometric approach. In [8], Keshri et al. proposed a reduced SEIR scale-free network model and studied its stability. In [9], Hosseini et al. formulated a discrete-time SEIRS model of computer virus propagation in scale-free networks and analyzed the local and global stability of the model. In [1], Guillen et al. proposed an improved SEIRS worm model with considering accurate positions for dysfunctional hosts and their replacements in state transition. In [10], Ren and Xu investigated an SEIR-KS computer virus propagation model based on the kill signals. They studied the local and global stability of the model by applying Routh-Hurwitz criterion and Lyapunov functional method. There are also some other models considering the latency of computer viruses with quarantine [11–14] and vaccination [15–19].

However, as stated in [20], those above models with the exposed compartment neglect the fact that a computer can infect other computers through file copying or file downloading. Therefore, to overcome the above-mentioned defect, computer virus models with infectivity in latent period have received much attention in recent years [21–26]. Unfortunately, most of these models still have some defects. On the one hand, they ignore the effect of time delay. As is known, there are some time delays of one type or another

TABLE 1: Parameters and their meanings in this paper.

Parameter	Description
b_1	Entering rate of susceptible computers
b_2	Entering rate of latent computers
b_3	Entering rate of breaking computers
b_4	Entering rate of recovered computers
α	Rate of latent computers break
β	Infection rate of susceptible computers
γ_1	Recovery rate of all computers
μ_0	Leaving rate of all computers
η	Rate of recovered computers lose immunity
γ_2	Rate of latent and breaking computers reinstall the operating system

in the transmission process of computer viruses due to latent period, temporary immunity period, or other reasons. On the other hand, only the susceptible computers are regarded as the entering computers, but every computer can enter or leave the Internet easily in reality. Finally, they neglect the effect of antivirus software, especially the effect of antivirus software on the susceptible computers. Based on the discussion above, we investigated a delayed SLBRS computer virus model with impact of antivirus software based on the following model proposed in [27]:

$$\begin{aligned}
\frac{dS(t)}{dt} &= b_1 + \gamma_2 L(t) + \gamma_2 B(t) + \eta R(t) - \gamma_1 S(t) \\
&\quad - \mu_0 S(t) - \beta(L(t) + B(t))S(t), \\
\frac{dL(t)}{dt} &= b_2 + \beta(L(t) + B(t))S(t) - \gamma_1 L(t) - \gamma_2 L(t) \\
&\quad - \mu_0 L(t) - \alpha L(t), \\
\frac{dB(t)}{dt} &= b_3 + \alpha L(t) - \gamma_1 B(t) - \gamma_2 B(t) - \mu_0 B(t), \\
\frac{dR(t)}{dt} &= b_4 + \gamma_1 L(t) + \gamma_1 B(t) + \gamma_1 S(t) - \mu_0 R(t) \\
&\quad - \eta R(t),
\end{aligned} \tag{1}$$

where $S(t)$, $L(t)$, $B(t)$, and $R(t)$ denote the numbers of susceptible, latent, breaking, and recovered computers at time t , respectively. More parameters are listed in Table 1 as follows.

Considering the temporary immune period of the recovered computers, we incorporate the time delay due to the temporary immunity period into system (1) and obtain the following delayed model:

$$\begin{aligned}
\frac{dS(t)}{dt} &= b_1 + \gamma_2 L(t) + \gamma_2 B(t) + \eta R(t - \tau) - \gamma_1 S(t) \\
&\quad - \mu_0 S(t) - \beta(L(t) + B(t))S(t), \\
\frac{dL(t)}{dt} &= b_2 + \beta(L(t) + B(t))S(t) - \gamma_1 L(t) - \gamma_2 L(t) \\
&\quad - \mu_0 L(t) - \alpha L(t),
\end{aligned}$$

$$\begin{aligned}
\frac{dB(t)}{dt} &= b_3 + \alpha L(t) - \gamma_1 B(t) - \gamma_2 B(t) - \mu_0 B(t), \\
\frac{dR(t)}{dt} &= b_4 + \gamma_1 L(t) + \gamma_1 B(t) + \gamma_1 S(t) - \mu_0 R(t) \\
&\quad - \eta R(t - \tau),
\end{aligned} \tag{2}$$

where τ is the time delay due to the temporary immunity period.

The remainder of the paper is structured as follows. In Section 2, conditions for local stability of the endemic equilibrium and the existence of Hopf bifurcation are performed. Section 3 deals with global stability of the endemic equilibrium. Section 4 is devoted to establishing the formulae to determine the direction, stability, and period of the bifurcating periodic solutions. Some numerical simulations are presented to illustrate the theoretical results in Section 5. We end the paper with a brief conclusion in Section 6.

2. Local Stability and Existence of Hopf Bifurcation

By a direct computation, it can be concluded that system (2) has the endemic equilibrium $E_*(S_*, L_*, B_*, R_*)$ where

$$\begin{aligned}
S_* &= \frac{(\gamma_1 + \gamma_2 + \mu_0 + \alpha)L_* - b_2}{\beta(L_* + (b_3 + \alpha L_*) / (\gamma_1 + \gamma_2 + \mu_0))}, \\
B_* &= \frac{b_3 + \alpha L_*}{\gamma_1 + \gamma_2 + \mu_0}, \\
R_* &= \frac{(b_4 + (b_1 + b_2 + b_3 + b_4) / \mu_0)}{(\mu_0 + \eta + \gamma_1)},
\end{aligned} \tag{3}$$

where L_* is the positive root of (4)

$$\tilde{a}_2 L^2 + \tilde{a}_1 L + \tilde{a}_0, \tag{4}$$

where

$$\begin{aligned}
\tilde{a}_0 &= a_2 a_5 + a_4, \\
\tilde{a}_1 &= a_1 a_5 + a_2 a_6 - a_3, \\
\tilde{a}_2 &= a_1 a_6,
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
 a_1 &= \beta + \frac{\alpha\beta}{\gamma_1 + \gamma_2 + \mu_0}, \\
 a_2 &= \frac{b_3\beta}{\gamma_1 + \gamma_2 + \mu_0}, \\
 a_3 &= (\gamma_1 + \mu_0)(\gamma_1 + \gamma_2 + \mu_0 + \alpha), \\
 a_4 &= b_2(\gamma_1 + \mu_0), \\
 a_5 &= b_1 + b_2 + \frac{b_3\gamma_2}{\gamma_1 + \gamma_2 + \mu_0} + \eta R_*, \\
 a_6 &= -\frac{(\gamma_1 + \mu_0)(\gamma_1 + \gamma_2 + \mu_0 + \alpha)}{\gamma_1 + \gamma_2 + \mu_0}.
 \end{aligned} \tag{6}$$

The Jacobian matrix of system (2) evaluated at E_* is

$$J_{E_*} = \begin{pmatrix} P_{11} & P_{12} & P_{13} & q_{14}e^{-\lambda\tau} \\ P_{21} & P_{22} & 0 & 0 \\ 0 & P_{32} & P_{33} & 0 \\ P_{41} & P_{42} & P_{43} & P_{44} + q_{44}e^{-\lambda\tau} \end{pmatrix}, \tag{7}$$

where

$$\begin{aligned}
 P_{11} &= -[\gamma_1 + \mu_0 + \beta(L_* + B_*)], \\
 P_{12} &= \gamma_2 - \beta S_*, \\
 P_{13} &= \gamma_2 - \beta S_*, \\
 P_{21} &= \beta(L_* + B_*), \\
 P_{22} &= \beta S_* - (\gamma_1 + \gamma_2 + \mu_0 + \alpha), \\
 P_{32} &= \alpha, \\
 P_{33} &= -(\gamma_1 + \gamma_2 + \mu_0), \\
 P_{41} &= \gamma_1, \\
 P_{42} &= \gamma_1, \\
 P_{43} &= \gamma_1, \\
 P_{44} &= -\mu_0, \\
 q_{14} &= \eta, \\
 q_{44} &= -\eta.
 \end{aligned} \tag{8}$$

The corresponding characteristic equations is

$$\begin{aligned}
 \lambda^4 + P_3\lambda^3 + P_2\lambda^2 + P_1\lambda + P_0 \\
 + (Q_3\lambda^3 + Q_2\lambda^2 + Q_1\lambda + Q_0)e^{-\lambda\tau} = 0,
 \end{aligned} \tag{9}$$

with

$$\begin{aligned}
 P_0 &= P_{33}P_{44}(P_{11}P_{22} - P_{12}P_{21}) + P_{13}P_{32}P_{44}q_{21}, \\
 P_1 &= P_{12}q_{21}(P_{33} + P_{44}) - P_{13}P_{32}q_{21} \\
 &\quad - P_{11}P_{33}(P_{22} + P_{44}) - P_{22}P_{44}(P_{11} + P_{33}), \\
 P_2 &= P_{11}P_{33} + P_{22}P_{44} + (P_{11} + P_{33})(P_{22} + P_{44}) \\
 &\quad - P_{12}q_{21}, \\
 P_3 &= -(P_{11} + P_{22} + P_{33} + P_{44}), \\
 Q_0 &= q_{21}q_{44}(P_{13}P_{32} - P_{12}P_{33}) \\
 &\quad + q_{21}q_{14}(P_{33}P_{42} - P_{32}P_{43}) - P_{22}P_{33}P_{41}q_{14}, \\
 Q_1 &= P_{12}q_{21}q_{44} - q_{44}(P_{11}P_{22} + P_{11}P_{33} + P_{22}P_{33}) \\
 &\quad + P_{41}q_{14}(P_{22} + P_{33}) - P_{21}P_{42}q_{14}, \\
 Q_2 &= q_{44}(P_{11} + P_{22} + P_{33}) - P_{41}q_{14}, \\
 Q_3 &= -q_{44}.
 \end{aligned} \tag{10}$$

For $\tau = 0$, (9) becomes

$$\lambda^4 + (P_3 + Q_3)\lambda^3 + (P_2 + Q_2)\lambda^2 + (P_1 + Q_1)\lambda + P_0 + Q_0 = 0. \tag{11}$$

Clearly, $P_3 + Q_3 = 3\gamma_1 + 4\mu_0 + 2\gamma_2 + \alpha + \eta + \beta(L_* + B_*) > 0$. Hence, it follows from the Hurwitz criterion that all the roots of (11) have negative real parts, if (H_1) : $P_0 + Q_0 > 0$, $(P_2 + Q_2)(P_3 + Q_3) > P_1 + Q_1$ and $(P_1 + Q_1)(P_2 + Q_2)(P_3 + Q_3) > (P_0 + Q_0)(P_3 + Q_3)^2 + (P_1 + Q_1)^2$ holds.

For $\tau > 0$, let $\lambda = i\omega(\omega > 0)$ be the root of (9). Then,

$$\begin{aligned}
 (Q_1\omega - Q_3\omega^3)\sin\tau\omega + (Q_0 - Q_2\omega^2)\cos\tau\omega \\
 = P_2\omega^2 - \omega^4 - P_0, \\
 (Q_1\omega - Q_3\omega^3)\cos\tau\omega - (Q_0 - Q_2\omega^2)\sin\tau\omega \\
 = P_3\omega^3 - P_1\omega.
 \end{aligned} \tag{12}$$

Thus,

$$\omega^8 + \bar{P}_3\omega^6 + \bar{P}_2\omega^4 + \bar{P}_1\omega^2 + \bar{P}_0 = 0, \tag{13}$$

with

$$\begin{aligned}
 \bar{P}_0 &= P_0^2 - Q_0^2, \\
 \bar{P}_1 &= P_1^2 - 2P_0P_2 - Q_1^2 + 2Q_0Q_2, \\
 \bar{P}_2 &= P_2^2 + 2P_0P_3 + 2Q_1Q_3 - Q_2^2, \\
 \bar{P}_3 &= P_3^2 - 2P_2P_3 - Q_3^2.
 \end{aligned} \tag{14}$$

Let $\omega^2 = \nu$, then (13) becomes

$$\nu^4 + \bar{P}_3\nu^3 + \bar{P}_2\nu^2 + \bar{P}_1\nu + \bar{P}_0 = 0. \tag{15}$$

Suppose that (H_2) (15) has a positive root ν_0 . Then, (13) has a positive root $\omega_0 = \sqrt{\nu_0}$ such that (9) has a pair of purely imaginary roots $\pm i\omega_0$. For ω_0 ,

$$\tau_0 = \frac{1}{\omega_0} \times \arccos \left\{ \frac{F_1(\omega_0)}{F_2(\omega_0)} \right\}, \quad (16)$$

where

$$\begin{aligned} F_1(\omega_0) &= (Q_2 - P_3Q_3)\omega_0^6 \\ &+ (P_3Q_1 + P_1Q_3 - P_2Q_2 - Q_0)\omega_0^4 \\ &+ (P_2Q_0 + P_0Q_2 - P_1Q_1)\omega_0^2 - P_0Q_0, \quad (17) \\ F_2(\omega_0) &= Q_3^2\omega_0^6 + (Q_2^2 - 2Q_1Q_3)\omega_0^4 \\ &+ (Q_1^2 - 2Q_0Q_2)\omega_0^2 + Q_0^2. \end{aligned}$$

Differentiating on both sides of (9) with respect to τ , one can obtain

$$\begin{aligned} \left[\frac{d\lambda}{d\tau} \right]^{-1} &= -\frac{4\lambda^3 + 3P_3\lambda^2 + 2P_2\lambda + P_1}{\lambda(\lambda^4 + P_3\lambda^3 + P_2\lambda^2 + P_1\lambda + P_0)} \\ &+ \frac{3Q_3\lambda^2 + 2Q_2\lambda + Q_1}{\lambda(Q_3\lambda^3 + Q_2\lambda^2 + Q_1\lambda + Q_0)} - \frac{\tau}{\lambda} \end{aligned} \quad (18)$$

Further, we have

$$\operatorname{Re} \left[\frac{d\lambda}{d\tau} \right]_{\tau=\tau_0}^{-1} = \frac{f'(\nu_0)}{F_2(\omega_0^2)}, \quad (19)$$

where $f(\nu) = \nu^4 + \tilde{P}_3\nu^3 + \tilde{P}_2\nu^2 + \tilde{P}_1\nu + \tilde{P}_0$ and $\nu_0 = \omega_0^2$.

Therefore, if (H_3) : $f'(\nu_0) \neq 0$, then $\operatorname{Re}[d\lambda/d\tau]_{\tau=\tau_0} \neq 0$. Thus, we have the following results based the Hopf bifurcation theorem in [28].

Theorem 1. For system (2), if (H_1) - (H_3) hold, then $E_*(S_*, L_*, B_*, R_*)$ is locally asymptotically stable when $\tau \in [0, \tau_0)$; system (2) undergoes a Hopf bifurcation at $E_*(S_*, L_*, B_*, R_*)$ when $\tau = \tau_0$ and a family of periodic solutions bifurcate from $E_*(S_*, L_*, B_*, R_*)$. τ_0 is defined as in (16).

3. Global Stability Analysis

Theorem 2. If $\min\{l_1, l_2, l_3, l_4\} > 0$, with

$$\begin{aligned} l_1 &= \frac{1}{M_1S_*} (b_1 + \gamma_2L_* + \gamma_2B_* + \eta R_*) - \beta \left(\frac{M_3}{m_2} + 1 \right) \\ &- \frac{\gamma_1}{m_4} \left(1 + \eta M_4 \tau \left(\frac{1}{m_4} + \frac{1}{m_1} \right) \right), \\ l_2 &= \beta + \frac{1}{M_2L_*} (b_2 + \beta B_* S_*) - \frac{\gamma_2}{m_1} - \frac{\alpha}{m_3} \end{aligned}$$

$$\begin{aligned} &- \frac{\gamma_1}{m_4} \left(1 + \eta M_4 \tau \left(\frac{1}{m_4} + \frac{1}{m_1} \right) \right), \\ l_3 &= \beta + \frac{1}{M_3B_*} (b_3 + \alpha L_*) - \frac{\gamma_2}{m_1} - \frac{\beta S_*}{m_2} \\ &- \frac{\gamma_1}{m_4} \left(1 + \eta M_4 \tau \left(\frac{1}{m_4} + \frac{1}{m_1} \right) \right), \\ l_4 &= \eta \left(\frac{1}{M_4} - \left(\frac{1}{m_1} + \frac{1}{m_4} \right) \left(1 + \eta \tau + \frac{\eta M_4 \tau}{m_4} \right) \right) \\ &+ \frac{1}{R_*} \left(\frac{1}{M_4} - \eta \tau \left(\frac{1}{m_1} + \frac{1}{m_4} \right) \right) \\ &\times (b_4 + (S_* + L_* + B_*) \gamma_1), \quad (20) \end{aligned}$$

where $m_1 < S(t) < M_1$, $m_2 < L(t) < M_2$, $m_3 < B(t) < M_3$, and $m_4 < R(t) < M_4$ for $t > 0$, then the endemic equilibrium E_* is globally asymptotically stable.

Proof. Let

$$\begin{aligned} S(t) &= S_* e^{w(t)}, \\ L(t) &= L_* e^{x(t)}, \\ B(t) &= B_* e^{y(t)}, \\ R(t) &= R_* e^{z(t)}. \end{aligned} \quad (21)$$

Then $E_*(S_*, L_*, B_*, R_*)$ becomes the trivial equilibrium $w(t) = x(t) = y(t) = z(t) = 0$ for all $t > 0$, and system (2) can be reduced to the following form:

$$\begin{aligned} \frac{dw}{dt} &= -\frac{1}{S} (b_1 + \gamma_2L_* + \gamma_2B_* + \eta R_*) (e^{w(t)} - 1) \\ &+ L_* \left(\frac{\gamma_2}{S} - \beta \right) (e^{x(t)} - 1) \\ &+ B_* \left(\frac{\gamma_2}{S} - \beta \right) (e^{y(t)} - 1) + \frac{\eta R_*}{S} (e^{z(t-\tau)} - 1), \end{aligned} \quad (22)$$

$$\begin{aligned} \frac{dx}{dt} &= -\frac{1}{L} (b_2 + \beta B_* S_*) (e^{x(t)} - 1) \\ &+ \beta S_* \left(\frac{B}{L} + 1 \right) (e^{w(t)} - 1) \\ &+ \frac{\beta B_* S_*}{L} (e^{y(t)} - 1), \end{aligned} \quad (23)$$

$$\frac{dy}{dt} = \frac{\alpha L_*}{B} (e^{x(t)} - 1) - \frac{1}{B} (b_3 + \alpha L_*) (e^{y(t)} - 1), \quad (24)$$

$$\begin{aligned} \frac{dz}{dt} &= \frac{\gamma_1 S_*}{R} (e^{w(t)} - 1) + \frac{\gamma_1 L_*}{R} (e^{x(t)} - 1) \\ &+ \frac{\gamma_1 B_*}{R} (e^{y(t)} - 1) - \frac{\eta R_*}{R} (e^{z(t-\tau)} - 1) \\ &- \frac{1}{R} (b_4 + \gamma_1 (S_* + L_* + B_*) - \eta R_*) (e^{z(t)} - 1). \end{aligned} \tag{25}$$

Now, we have

$$e^{z(t-\tau)} = e^{z(t)} - \int_{t-\tau}^t e^{z(s)} \frac{dz}{ds} ds. \tag{26}$$

Now, (22) can be rewritten as follows by using above relation,

$$\begin{aligned} \frac{dw}{dt} &= -\frac{1}{S} (b_1 + \gamma_2 L_* + \gamma_2 B_* + \eta R_*) (e^{w(t)} - 1) \\ &+ L_* \left(\frac{\gamma_2}{S} - \beta \right) (e^{x(t)} - 1) + B_* \left(\frac{\gamma_2}{S} - \beta \right) (e^{y(t)} \\ &- 1) + \frac{\eta R_*}{S} (e^{z(t)} - 1) - \frac{\eta R_*}{S} \int_{t-\tau}^t e^{z(s)} \frac{dz}{ds} ds, \\ &= -\frac{1}{S} (b_1 + \gamma_2 L_* + \gamma_2 B_* + \eta R_*) (e^{w(t)} - 1) \\ &+ L_* \left(\frac{\gamma_2}{S} - \beta \right) (e^{x(t)} - 1) + B_* \left(\frac{\gamma_2}{S} - \beta \right) (e^{y(t)} \\ &- 1) + \frac{\eta R_*}{S} (e^{z(t)} - 1) - \frac{\eta R_*}{S} \\ &\cdot \int_{t-\tau}^t e^{z(s)} \left\{ \frac{\gamma_1 S_*}{R} (e^{w(s)} - 1) + \frac{\gamma_1 L_*}{R} (e^{x(s)} - 1) \right. \\ &+ \frac{\gamma_1 B_*}{R} (e^{y(s)} - 1) \\ &- \frac{1}{R} (b_4 + \gamma_1 (S_* + L_* + B_*) - \eta R_*) (e^{z(s)} - 1) \\ &\left. - \frac{\eta R_*}{R} (e^{z(s-\tau)} - 1) \right\} ds, \end{aligned} \tag{27}$$

Let $V_1(t) = |w(t)|$. It follows from the above equation

$$\begin{aligned} D^+ V_1 &\leq -\frac{1}{M_1} (b_1 + \gamma_2 L_* + \gamma_2 B_* + \eta R_*) |e^{w(t)} - 1| \\ &+ L_* \left(\frac{\gamma_2}{m_1} - \beta \right) |e^{x(t)} - 1| + B_* \left(\frac{\gamma_2}{m_1} - \beta \right) |e^{y(t)} \\ &- 1| + \frac{\eta R_*}{m_1} |e^{z(t)} - 1| + \frac{\eta R_*}{m_1} \\ &\cdot \int_{t-\tau}^t e^{z(s)} \left\{ \frac{\gamma_1 S_*}{m_4} |e^{w(s)} - 1| + \frac{\gamma_1 L_*}{m_4} |e^{x(s)} - 1| \right. \end{aligned}$$

$$\begin{aligned} &+ \frac{\gamma_1 B_*}{m_4} |e^{y(s)} - 1| \\ &+ \frac{1}{M_4} (b_4 + (S_* + L_* + B_*) \gamma_1) |e^{z(s)} - 1| \\ &\left. + \frac{1}{m_4} \eta R_* |e^{z(s)} - 1| + \frac{\eta R_*}{M_4} |e^{z(s-\tau)} - 1| \right\} ds, \end{aligned} \tag{28}$$

We find that there exists a $t_1 > 0$, such that $R^* e^{z(t)} < M_4$ for all $t > t_1$ and for $t > t_1 + \tau$, we have

$$\begin{aligned} D^+ V_1 &\leq -\frac{1}{M_1} (b_1 + \gamma_2 L_* + \gamma_2 B_* + \eta R_*) |e^{w(t)} - 1| \\ &+ L_* \left(\frac{\gamma_2}{m_1} - \beta \right) |e^{x(t)} - 1| + B_* \left(\frac{\gamma_2}{m_1} - \beta \right) |e^{y(t)} \\ &- 1| + \frac{\eta R_*}{m_1} |e^{z(t)} - 1| + \frac{\eta M_4}{m_1} \int_{t-\tau}^t \left\{ \frac{\gamma_1 S_*}{m_4} |e^{w(s)} - 1| \right. \\ &+ \frac{\gamma_1 L_*}{m_4} |e^{x(s)} - 1| + \frac{\gamma_1 B_*}{m_4} |e^{y(s)} - 1| \\ &+ \frac{1}{M_4} (b_4 + (S_* + L_* + B_*) \gamma_1) |e^{z(s)} - 1| \\ &\left. + \frac{1}{m_4} \eta R_* |e^{z(s)} - 1| + \frac{\eta R_*}{M_4} |e^{z(s-\tau)} - 1| \right\} ds \end{aligned} \tag{29}$$

Again due to form of (29) we consider the following functional:

$$\begin{aligned} V_{11}(t) &= V_1(t) + \frac{\eta M_4}{m_1} \int_{t-\tau}^t \int_v^t \left\{ \frac{\gamma_1 S_*}{m_4} |e^{w(s)} - 1| \right. \\ &+ \frac{\gamma_1 L_*}{m_4} |e^{x(s)} - 1| + \frac{\gamma_1 B_*}{m_4} |e^{y(s)} - 1| \\ &+ \frac{1}{M_4} (b_4 + (S_* + L_* + B_*) \gamma_1) |e^{z(s)} - 1| \\ &+ \frac{1}{m_4} \eta R_* |e^{z(s)} - 1| + \frac{\eta R_*}{M_4} |e^{z(s-\tau)} - 1| \left. \right\} ds dv \\ &+ \frac{\eta^2 R_* \tau}{m_1} \int_{t-\tau}^t |e^{z(s)} - 1| ds, \end{aligned} \tag{30}$$

whose derivative along the solution of system (2) is given by

$$\begin{aligned} D^+ V_{11}(t) &\leq D^+ V_1(t) + \frac{\eta M_4 \tau}{m_1} \left\{ \frac{\gamma_1 S_*}{m_4} |e^{w(t)} - 1| \right. \\ &+ \frac{\gamma_1 L_*}{m_4} |e^{x(t)} - 1| + \frac{\gamma_1 B_*}{m_4} |e^{y(t)} - 1| + \frac{1}{M_4} (b_4 \\ &+ (S_* + L_* + B_*) \gamma_1) |e^{z(t)} - 1| + \frac{\eta R_*}{m_4} |e^{z(t)} - 1| \left. \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{\eta^2 R_* \tau}{m_1} |e^{z(t)} - 1| - \frac{\eta M_4}{m_1} \int_{t-\tau}^t \left\{ \frac{\gamma_1 S_*}{m_4} |e^{w(s)} - 1| \right. \\
& + \frac{\gamma_1 L_*}{m_4} |e^{x(s)} - 1| + \frac{\gamma_1 B_*}{m_4} |e^{y(s)} - 1| \\
& + \frac{1}{M_4} (b_4 + (S_* + L_* + B_*) \gamma_1) |e^{z(s)} - 1| \\
& + \left. \frac{\eta R_*}{m_4} |e^{z(s)} - 1| + \frac{\eta R_*}{M_4} |e^{z(s-\tau)} - 1| \right\} ds \\
& \leq \left\{ -\frac{1}{M_1} (b_1 + \gamma_2 L_* + \gamma_2 B_* + \eta R_*) + \frac{\eta M_4 \gamma_1 S_* \tau}{m_1 m_4} \right\} \\
& \cdot |e^{w(t)} - 1| + L_* \left\{ \frac{\gamma_2}{m_1} - \beta + \frac{\eta M_4 \tau \gamma_1}{m_1 m_4} \right\} |e^{x(t)} - 1| \\
& + B_* \left\{ \frac{\gamma_2}{m_1} - \beta + \frac{\eta M_4 \tau \gamma_1}{m_1 m_4} \right\} |e^{y(t)} - 1| + \left\{ \frac{\eta R_*}{m_1} \left(1 \right. \right. \\
& + \left. \left. \eta \tau + \frac{\eta M_4 \tau}{m_4} \right) + \frac{\eta \tau}{m_1} (b_4 + (S_* + L_* + B_*) \gamma_1) \right\} \\
& \cdot |e^{z(t)} - 1| \tag{31}
\end{aligned}$$

Again let $V_2(t) = |x(t)|$ and $V_3(t) = |y(t)|$. Now calculate the derivative of $V_2(t)$ and $V_3(t)$ with the solution of (2), it follows from, respectively, (23) and (24)

$$\begin{aligned}
D^+ V_2 & \leq \beta S_* \left(\frac{M_3}{m_2} + 1 \right) |e^{w(t)} - 1| \\
& - \frac{1}{M_2} (b_2 + \beta B_* S_*) |e^{x(t)} - 1| \\
& + \frac{\beta B_* S_*}{m_2} |e^{y(t)} - 1|, \tag{32}
\end{aligned}$$

$$D^+ V_3 \leq \frac{\alpha L_*}{m_3} |e^{x(t)} - 1| - \frac{1}{M_3} (b_3 + \alpha L_*) |e^{y(t)} - 1|, \tag{33}$$

Now, (25) can be rewritten as follows by using (26),

$$\begin{aligned}
\frac{dz}{dt} & = \frac{\gamma_1 S_*}{R} (e^{w(t)} - 1) + \frac{\gamma_1 L_*}{R} (e^{x(t)} - 1) \\
& + \frac{\gamma_1 B_*}{R} (e^{y(t)} - 1) - \frac{\eta R_*}{R} (e^{z(t)} - 1) - \frac{1}{R} (b_4 \\
& + \gamma_1 (S_* + L_* + B_*) - \eta R_*) (e^{z(t)} - 1) + \frac{\eta R_*}{R} \\
& \cdot \int_{t-\tau}^t e^{z(s)} \frac{dz}{ds} ds. \\
& = \frac{\gamma_1 S_*}{R} (e^{w(t)} - 1) + \frac{\gamma_1 L_*}{R} (e^{x(t)} - 1) + \frac{\gamma_1 B_*}{R} (e^{y(t)}
\end{aligned}$$

$$\begin{aligned}
& - 1) - \frac{\eta R_*}{R} (e^{z(t)} - 1) - \frac{1}{R} (b_4 + \gamma_1 (S_* + L_* \\
& + B_*) - \eta R_*) (e^{z(t)} - 1) + \frac{\eta R_*}{R} \\
& \cdot \int_{t-\tau}^t e^{z(s)} \left\{ \frac{\gamma_1 S_*}{R} (e^{w(s)} - 1) + \frac{\gamma_1 L_*}{R} (e^{x(s)} - 1) \right. \\
& + \frac{\gamma_1 B_*}{R} (e^{y(s)} - 1) - \left. \frac{\eta R_*}{R} (e^{z(s-\tau)} - 1) \right. \\
& - \left. \frac{1}{R} (b_4 + \gamma_1 (S_* + L_* + B_*) - \eta R_*) \right. \\
& \cdot \left. (e^{z(s)} - 1) \right\} ds. \tag{34}
\end{aligned}$$

Again let $V_4(t) = |z(t)|$. It follows from the above equation

$$\begin{aligned}
D^+ V_4 & \leq \frac{\gamma_1 S_*}{m_4} |e^{w(t)} - 1| + \frac{\gamma_1 L_*}{m_4} |e^{x(t)} - 1| \\
& + \frac{\gamma_1 B_*}{m_4} |e^{y(t)} - 1| - \frac{\eta R_*}{M_4} |e^{z(t)} - 1| \\
& + \left(-\frac{b_4 + \gamma_1 (S_* + L_* + B_*)}{M_4} + \frac{\eta R_*}{m_4} \right) |e^{z(t)} - 1| \\
& + \frac{\eta R_*}{m_4} \int_{t-\tau}^t e^{z(s)} \left\{ \frac{\gamma_1 S_*}{m_4} |e^{w(s)} - 1| + \frac{\gamma_1 L_*}{m_4} |e^{x(s)} - 1| \right. \\
& + \frac{\gamma_1 B_*}{m_4} |e^{y(s)} - 1| + \left. \frac{\eta R_*}{M_4} |e^{z(s-\tau)} - 1| \right. \\
& + \left. \left(\frac{b_4 + \gamma_1 (S_* + L_* + B_*)}{M_4} + \frac{\eta R_*}{m_4} \right) \right. \\
& \cdot \left. |e^{z(s)} - 1| \right\} ds. \tag{35}
\end{aligned}$$

We find that there exists a $t_1 > 0$, such that $R^* e^{z(t)} < M_4$ for all $t > t_1$ and for $t > t_1 + \tau$, we have

$$\begin{aligned}
D^+ V_4 & \leq \frac{\gamma_1 S_*}{m_4} |e^{w(t)} - 1| + \frac{\gamma_1 L_*}{m_4} |e^{x(t)} - 1| \\
& + \frac{\gamma_1 B_*}{m_4} |e^{y(t)} - 1| - \frac{\eta R_*}{M_4} |e^{z(t)} - 1| \\
& + \left(-\frac{b_4 + \gamma_1 (S_* + L_* + B_*)}{M_4} + \frac{\eta R_*}{m_4} \right) |e^{z(t)} - 1| \\
& + \frac{\eta M_4}{m_4} \int_{t-\tau}^t \left\{ \frac{\gamma_1 S_*}{m_4} |e^{w(s)} - 1| + \frac{\gamma_1 L_*}{m_4} |e^{x(s)} - 1| \right. \\
& + \left. \frac{\gamma_1 B_*}{m_4} |e^{y(s)} - 1| + \frac{\eta R_*}{M_4} |e^{z(s-\tau)} - 1| \right.
\end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{b_4 + \gamma_1 (S_* + L_* + B_*)}{M_4} + \frac{\eta R_*}{m_4} \right) \\
 & \cdot |e^{z(s)} - 1| \Big\} ds.
 \end{aligned} \tag{36}$$

Again due to the above form of (36) we consider the following functional:

$$\begin{aligned}
 V_{44}(t) = & V_4(t) + \frac{\eta M_4}{m_4} \int_{t-\tau}^t \int_{\nu}^t \left\{ \frac{\gamma_1 S_*}{m_4} |e^{w(s)} - 1| \right. \\
 & + \frac{\gamma_1 L_*}{m_4} |e^{x(s)} - 1| + \frac{\gamma_1 B_*}{m_4} |e^{y(s)} - 1| \\
 & + \frac{\eta R_*}{M_4} |e^{z(s-\tau)} - 1| \\
 & + \left(\frac{b_4 + \gamma_1 (S_* + L_* + B_*)}{M_4} + \frac{\eta R_*}{m_4} \right) \\
 & \cdot |e^{z(s)} - 1| \Big\} ds dv + \frac{\eta^2 R_* \tau}{m_4} \int_{t-\tau}^t |e^{z(s)} \\
 & - 1| ds,
 \end{aligned} \tag{37}$$

whose right derivative along the solution of the system (2) is given by

$$\begin{aligned}
 D^+ V_{44}(t) \leq & D^+ V_4(t) + \frac{\eta M_4 \tau}{m_4} \left\{ \frac{\gamma_1 S_*}{m_4} |e^{w(t)} - 1| \right. \\
 & + \frac{\gamma_1 L_*}{m_4} |e^{x(t)} - 1| + \frac{\gamma_1 B_*}{m_4} |e^{y(t)} - 1| \\
 & + \left(\frac{b_4 + \gamma_1 (S_* + L_* + B_*)}{M_4} + \frac{\eta R_*}{m_4} \right) |e^{z(t)} - 1| \Big\} \\
 & + \frac{\eta^2 R_* \tau}{m_4} |e^{z(t)} - 1| - \frac{\eta M_4}{m_4} \int_{t-\tau}^t \left\{ \frac{\gamma_1 S_*}{m_4} |e^{w(s)} - 1| \right. \\
 & + \frac{\gamma_1 L_*}{m_4} |e^{x(s)} - 1| + \frac{\gamma_1 B_*}{m_4} |e^{y(s)} - 1| \\
 & + \frac{\eta R_*}{M_4} |e^{z(s-\tau)} - 1| \\
 & + \left(\frac{b_4 + \gamma_1 (S_* + L_* + B_*)}{M_4} + \frac{\eta R_*}{m_4} \right) \\
 & \cdot |e^{z(s)} - 1| \Big\} ds, \\
 \leq & \frac{\gamma_1 S_*}{m_4} \left\{ 1 + \frac{\eta M_4 \tau}{m_4} \right\} |e^{w(t)} - 1| + \frac{\gamma_1 L_*}{m_4} \left\{ 1 \right. \\
 & + \frac{\eta M_4 \tau}{m_4} \Big\} |e^{x(t)} - 1| + \frac{\gamma_1 B_*}{m_4} \left\{ 1 + \frac{\eta M_4 \tau}{m_4} \right\} |e^{y(t)}
 \end{aligned}$$

$$\begin{aligned}
 & - 1| + \left\{ \frac{\eta R_*}{m_4} \left(1 + \eta \tau - \frac{m_4}{M_4} + \frac{\eta M_4 \tau}{m_4} \right) + \left(\frac{\eta \tau}{m_4} \right. \right. \\
 & \left. \left. - \frac{1}{M_4} \right) (b_4 + \gamma_1 (S_* + L_* + B_*)) \right\} |e^{z(t)} - 1|
 \end{aligned} \tag{38}$$

Let us define a Lyapunov functional $V(t)$ as

$$\begin{aligned}
 V(t) = & V_{11}(t) + V_2(t) + V_3 + V_{44}(t) \\
 & > |w(t)| + |x(t)| + |y(t)| + |z(t)|.
 \end{aligned} \tag{39}$$

Computing the upper right derivative of $V(t)$ along the solution of system (2) and by using (31)-(33) and (38), we obtain

$$\begin{aligned}
 D^+ V(t) = & D^+ V_{11}(t) + D^+ V_2(t) + D^+ V_3(t) \\
 & + D^+ V_{44}(t) \leq \left\{ -\frac{1}{M_1} (b_1 + \gamma_2 L_* + \gamma_2 B_* + \eta R_*) \right. \\
 & + \frac{\eta M_4 \gamma_1 S_* \tau}{m_1 m_4} \Big\} |e^{w(t)} - 1| + L_* \left\{ \frac{\gamma_2}{m_1} - \beta \right. \\
 & + \frac{\eta M_4 \tau \gamma_1}{m_1 m_4} \Big\} |e^{x(t)} - 1| + B_* \left\{ \frac{\gamma_2}{m_1} - \beta + \frac{\eta M_4 \tau \gamma_1}{m_1 m_4} \right\} \\
 & \cdot |e^{y(t)} - 1| + \left\{ \frac{\eta R_*}{m_1} \left(1 + \eta \tau + \frac{\eta M_4 \tau}{m_4} \right) + \frac{\eta \tau}{m_1} (b_4 \right. \\
 & + (S_* + L_* + B_*) \gamma_1) \Big\} |e^{z(t)} - 1| + \beta S_* \left(\frac{M_3}{m_2} + 1 \right) \\
 & \cdot |e^{w(t)} - 1| - \frac{1}{M_2} (b_2 + \beta B_* S_*) |e^{x(t)} - 1| \\
 & + \frac{\beta B_* S_*}{m_2} |e^{y(t)} - 1| + \frac{\alpha L_*}{m_3} |e^{x(t)} - 1| - \frac{1}{M_3} (b_3 \\
 & + \alpha L_*) |e^{y(t)} - 1| + \frac{\gamma_1 S_*}{m_4} \left\{ 1 + \frac{\eta M_4 \tau}{m_4} \right\} |e^{w(t)} - 1| \\
 & + \frac{\gamma_1 L_*}{m_4} \left\{ 1 + \frac{\eta M_4 \tau}{m_4} \right\} |e^{x(t)} - 1| + \frac{\gamma_1 B_*}{m_4} \left\{ 1 \right. \\
 & + \frac{\eta M_4 \tau}{m_4} \Big\} |e^{y(t)} - 1| + \left\{ \frac{\eta R_*}{m_4} \left(1 + \eta \tau - \frac{m_4}{M_4} \right. \right. \\
 & + \frac{\eta M_4 \tau}{m_4} \Big\} + \left(\frac{\eta \tau}{m_4} - \frac{1}{M_4} \right) (b_4 \\
 & + \gamma_1 (S_* + L_* + B_*)) \Big\} \times |e^{z(t)} - 1| \\
 = & -S_* \left\{ \frac{1}{M_1 S_*} (b_1 + \gamma_2 L_* + \gamma_2 B_* + \eta R_*) \right. \\
 & - \beta \left(\frac{M_3}{m_2} + 1 \right) - \frac{\gamma_1}{m_4} \left(1 + \eta M_4 \tau \left(\frac{1}{m_4} + \frac{1}{m_1} \right) \right) \Big\} \\
 & \cdot |e^{w(t)} - 1| - L_* \left\{ \beta + \frac{1}{M_2 L_*} (b_2 + \beta B_* S_*) - \frac{\gamma_2}{m_1} \right.
 \end{aligned}$$

$$\begin{aligned}
& -\frac{\alpha}{m_3} - \frac{\gamma_1}{m_4} \left(1 + \eta M_4 \tau \left(\frac{1}{m_4} + \frac{1}{m_1} \right) \right) \left| e^{x(t)} - 1 \right| \\
& - B_* \left\{ \beta + \frac{1}{M_3 B_*} (b_3 + \alpha L_*) - \frac{\gamma_2}{m_1} - \frac{\beta S_*}{m_2} \right. \\
& - \frac{\gamma_1}{m_4} \left(1 + \eta M_4 \tau \left(\frac{1}{m_4} + \frac{1}{m_1} \right) \right) \left. \right\} \times \left| e^{y(t)} - 1 \right| \\
& - R_* \left\{ \eta \left(\frac{1}{M_4} \right. \right. \\
& - \left. \left(\frac{1}{m_1} + \frac{1}{m_4} \right) \left(1 + \eta \tau + \frac{\eta M_4 \tau}{m_4} \right) \right. + \frac{1}{R_*} \left(\frac{1}{M_4} \right. \\
& - \left. \left. \eta \tau \left(\frac{1}{m_1} + \frac{1}{m_4} \right) \right) (b_4 + (S_* + L_* + B_*) \gamma_1) \right\} \left| e^{z(t)} \right. \\
& - 1 \left| = -S_* l_1 \left| e^{w(t)} - 1 \right| - L_* l_2 \left| e^{x(t)} - 1 \right| \right. \\
& \left. - B_* l_3 \left| e^{y(t)} - 1 \right| - R_* l_4 \left| e^{z(t)} - 1 \right|, \tag{40}
\end{aligned}$$

where l_1, l_2, l_3 , and l_4 are defined above in (20).

Since the model system (2) is positive invariant, therefore, for all $t > t_1^*$, we have

$$\begin{aligned}
S_* e^{w(t)} &= S(t) > \underline{S}, \\
L_* e^{x(t)} &= L(t) > \underline{L}, \\
B_* e^{y(t)} &= B(t) > \underline{B}, \\
R_* e^{z(t)} &= R(t) > \underline{R}. \tag{41}
\end{aligned}$$

Using the mean value theorem, we have

$$\begin{aligned}
S_* \left| e^{w(t)} - 1 \right| &= S_* e^{\theta_1(t)} |w(t)| > m_1 |w(t)|, \\
L_* \left| e^{x(t)} - 1 \right| &= L_* e^{\theta_2(t)} |x(t)| > m_2 |x(t)|, \\
B_* \left| e^{y(t)} - 1 \right| &= B_* e^{\theta_3(t)} |y(t)| > m_3 |y(t)|, \\
R_* \left| e^{z(t)} - 1 \right| &= R_* e^{\theta_4(t)} |z(t)| > m_4 |z(t)|, \tag{42}
\end{aligned}$$

where $S_* e^{\theta_1(t)}$ lies between S_* and $S(t)$, $L_* e^{\theta_2(t)}$ lies between L_* and $L(t)$, $B_* e^{\theta_3(t)}$ lies between B_* and $B(t)$, and $R_* e^{\theta_4(t)}$ lies between R_* and $R(t)$. Therefore,

$$\begin{aligned}
D^+ V(t) &\leq -l_1 \underline{S} |w(t)| - l_2 \underline{L} |x(t)| - l_3 \underline{B} |y(t)| \\
&\quad - l_4 \underline{R} |z(t)| \\
&\leq -l (|w(t)| + |x(t)| + |y(t)| + |z(t)|), \tag{43}
\end{aligned}$$

$$\text{where } l = \min \{l_1 \underline{S}, l_2 \underline{L}, l_3 \underline{B}, l_4 \underline{R}\}.$$

Note that $V(t) > |w(t)| + |x(t)| + |y(t)| + |z(t)|$. Hence, from theory of global stability and (43), we conclude that the zero solution of the reduced system (22)-(25) is globally asymptotically stable. Therefore, the endemic equilibrium E^* of model system (2) is globally asymptotically stable. \square

4. Direction and Stability of Hopf Bifurcation

Let $\tau = \tau_0 + \zeta$ ($\zeta \in \mathbb{R}$), $u_1 = S(\tau t)$, $u_2 = L(\tau t)$, $u_3 = B(\tau t)$, and $u_4 = R(\tau t)$. System (2) becomes

$$\dot{u}(t) = L_\zeta(u_t) + F(\zeta, u_t), \tag{44}$$

where $u(t) = (u_1, u_2, u_3, u_4)^T \in C = C([-1, 0], \mathbb{R}^4)$ and $L_\zeta: C \rightarrow \mathbb{R}^4$ and $F: \mathbb{R} \times C \rightarrow \mathbb{R}^4$ are defined as follows:

$$L_\zeta \phi = (\tau_0 + \zeta) (P_{max} \phi(0) + Q_{max} \phi(-1)), \tag{45}$$

and

$$F(\zeta, \phi) = (\tau_0 + \zeta) \begin{bmatrix} -\beta \phi_1(0) (\phi_2(0) + \phi_3(0)) \\ \beta \phi_1(0) (\phi_2(0) + \phi_3(0)) \\ 0 \\ 0 \end{bmatrix} \tag{46}$$

with

$$\begin{aligned}
P_{max} &= \begin{pmatrix} p_{11} & p_{12} & p_{13} & 0 \\ p_{21} & p_{22} & 0 & 0 \\ 0 & p_{32} & p_{33} & 0 \\ p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix}, \\
Q_{max} &= \begin{pmatrix} 0 & 0 & 0 & q_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{44} \end{pmatrix}, \tag{47}
\end{aligned}$$

Thus, there exists $\eta(\theta, \zeta)$ such that

$$L_\zeta \phi = \int_{-1}^0 d\eta(\theta, \zeta) \phi(\theta), \quad \text{for } \phi \in C. \tag{48}$$

In fact,

$$\eta(\theta, \zeta) = (\tau_0 + \zeta) (P_{max} \delta(\theta) + Q_{max} \delta(\theta + 1)), \tag{49}$$

where $\delta(\theta)$ is the Dirac delta function.

For $\phi \in C([-1, 0], \mathbb{R}^4)$, define

$$A(\zeta) \phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \leq \theta < 0, \\ \int_{-1}^0 d\eta(\theta, \zeta) \phi(\theta), & \theta = 0, \end{cases} \tag{50}$$

and

$$R(\zeta) \phi = \begin{cases} 0, & -1 \leq \theta < 0, \\ F(\zeta, \phi), & \theta = 0. \end{cases} \tag{51}$$

Then system (44) is equivalent to

$$\dot{u}(t) = A(\zeta) u_t + R(\zeta) u_t. \tag{52}$$

For $\varphi \in C^1([0, 1], (R^4)^*)$, define

$$A^*(\varphi) = \begin{cases} -\frac{d\varphi(s)}{ds}, & 0 < s \leq 1, \\ \int_{-1}^0 d\eta^T(s, 0)\varphi(-s), & s = 0, \end{cases} \quad (53)$$

and the bilinear inner form for A and A^*

$$\langle \varphi(s), \phi(\theta) \rangle = \bar{\varphi}(0)\phi(0) - \int_{\theta=-1}^0 \int_{\xi=0}^{\theta} \bar{\varphi}(\xi - \theta) d\eta(\theta)\phi(\xi) d\xi, \quad (54)$$

where $\eta(\theta) = \eta(\theta, 0)$.

Let $v(\theta) = (1, v_2, v_3, v_4)^T e^{i\tau_0\omega_0\theta}$ be the eigenvector of $A(0)$ corresponding to $+i\tau_0\omega_0$ and $v^*(s) = D(1, v_2^*, v_3^*, v_4^*)^T e^{i\tau_0\omega_0 s}$ be the eigenvector of $A^*(0)$ corresponding to $-i\tau_0\omega_0$, respectively. Based on the definition of $A(0)$ and A^* , we obtain

$$\begin{aligned} v_2 &= \frac{p_{21}}{i\omega_0 - p_{22}}, \\ v_3 &= \frac{p_{32}v_2}{i\omega_0 - p_{33}}, \\ v_4 &= \frac{i\omega_0 - p_{11} - p_{12}v_2 - p_{13}v_3}{q_{14}e^{-i\tau_0\omega_0}}, \\ v_2^* &= \frac{i\omega_0 + p_{11} + p_{41}v_4}{p_{21}}, \\ v_3^* &= -\frac{p_{13} + p_{43}v_4}{i\omega_0 + p_{33}}, \\ v_4^* &= -\frac{q_{14}e^{\tau_0\omega_0}}{i\omega_0 + p_{44} + q_{44}e^{\tau_0\omega_0}}. \end{aligned} \quad (55)$$

From (54), the expression of Q can be obtain as follows:

$$\begin{aligned} \bar{D} &= [1 + v_2\bar{v}_2^* + v_3\bar{v}_3^* + v_4\bar{v}_4^* \\ &\quad + \tau_0 e^{-\tau_0\omega_0} v_4 (q_{14} + q_{44}\bar{v}_4^*)]^{-1}, \end{aligned} \quad (56)$$

such that $\langle v^*, v \rangle = 1$ and $\langle v^*, \bar{v} \rangle = 0$.

Next, we can obtain the expressions of g_{20}, g_{11}, g_{02} , and g_{21} by the algorithms in [28] and the computation process in [29–31]:

$$\begin{aligned} g_{20} &= 2\tau_0\bar{D}\beta(\bar{v}_2^* - 1)(v_2 + v_3), \\ g_{11} &= \tau_0\bar{D}\beta(\bar{v}_2^* - 1)(\text{Re}\{v_2\} + \text{Re}\{v_3\}), \\ g_{02} &= 2\tau_0\bar{D}\beta(\bar{v}_2^* - 1)(\bar{v}_2 + \bar{v}_3), \\ g_{21} &= 2\beta\tau_0\bar{D}(\bar{v}_2^* - 1)\left(W_{11}^{(1)}(0)v_2 + \frac{1}{2}W_{20}^{(1)}(0)\bar{v}_2 \right. \\ &\quad \left. + W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0) + W_{11}^{(1)}(0)v_3 \right. \\ &\quad \left. + \frac{1}{2}W_{20}^{(1)}(0)\bar{v}_3 + W_{11}^{(3)}(0) + \frac{1}{2}W_{20}^{(3)}(0)\right), \end{aligned} \quad (57)$$

with

$$\begin{aligned} W_{20}(\theta) &= \frac{i g_{20} v(0)}{\tau_0 \omega_0} e^{i\tau_0 \omega_0 \theta} + \frac{i \bar{g}_{02} \bar{v}(0)}{3\tau_0 \omega_0} e^{-i\tau_0 \omega_0 \theta} \\ &\quad + E_1 e^{2i\tau_0 \omega_0 \theta}, \end{aligned} \quad (58)$$

$$W_{11}(\theta) = -\frac{i g_{11} v(0)}{\tau_0 \omega_0} e^{i\tau_0 \omega_0 \theta} + \frac{i \bar{g}_{11} \bar{v}(0)}{\tau_0 \omega_0} e^{-i\tau_0 \omega_0 \theta} + E_2.$$

E_1 and E_2 can be obtained by the following two equations:

$$\begin{aligned} E_1 &= 2 \begin{pmatrix} 2i\omega_0 - p_{11} & -p_{12} & -p_{13} & -q_{14}e^{-2i\tau_0\omega_0} \\ -p_{21} & 2i\omega_0 - p_{22} & 0 & 0 \\ 0 & -p_{32} & 2i\omega_0 - p_{33} & 0 \\ -p_{41} & -p_{42} & -p_{43} & 2i\omega_0 - p_{44} - q_{44}e^{-2i\tau_0\omega_0} \end{pmatrix}^{-1} \times \begin{pmatrix} -\beta(v_2 + v_3) \\ \beta(v_2 + v_3) \\ 0 \\ 0 \end{pmatrix}, \\ E_2 &= - \begin{pmatrix} p_{11} & p_{12} & p_{13} & q_{14} \\ p_{21} & p_{22} & 0 & 0 \\ 0 & p_{32} & p_{33} & 0 \\ p_{41} & p_{42} & p_{43} & p_{44} + q_{44} \end{pmatrix}^{-1} \times \begin{pmatrix} -\beta(\text{Re}\{v_2\} + \text{Re}\{v_3\}) \\ \beta(\text{Re}\{v_2\} + \text{Re}\{v_3\}) \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \quad (59)$$

Then, one can obtain

$$\begin{aligned} C_1(0) &= \frac{i}{2\tau_0\omega_0} \left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2} \\ \mu_2 &= -\frac{\text{Re}\{C_1(0)\}}{\text{Re}\{\lambda'(\tau_0)\}}, \end{aligned}$$

$$\beta_2 = 2 \text{Re}\{C_1(0)\},$$

$$T_2 = -\frac{\text{Im}\{C_1(0)\} + \mu_2 \text{Im}\{\lambda'(\tau_0)\}}{\tau_0\omega_0}, \quad (60)$$

In conclusion, we have the following results.

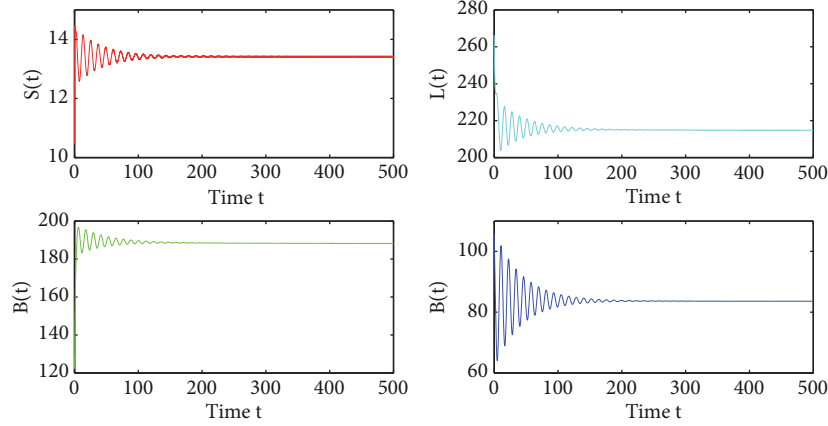


FIGURE 1: Time plots of S , L , B , and R with $\tau = 3.2575 < \tau_0 = 3.4685$.

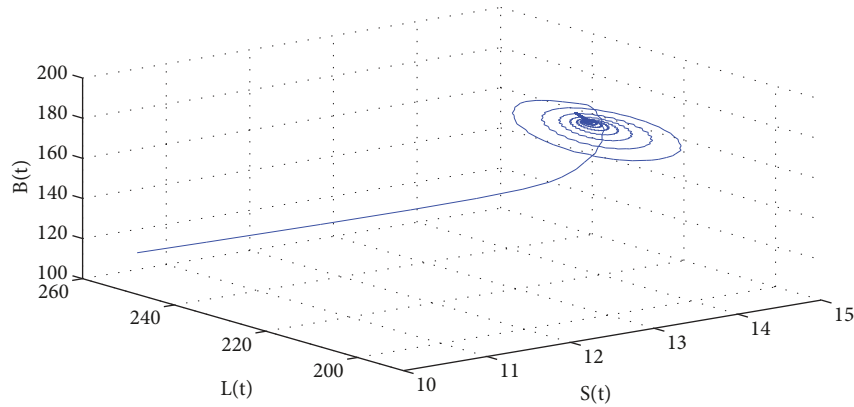


FIGURE 2: Dynamic behavior of system (61): projection on S-L-B with $\tau = 3.2575 < \tau_0 = 3.4685$.

Theorem 3. For system (2), if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical); if $\beta_2 < 0$ ($\beta_2 > 0$), then the bifurcating periodic solutions are stable (unstable); if $T_2 > 0$ ($T_2 < 0$), then the period of the bifurcating periodic solutions increases (decrease).

5. Numerical Simulations

In this section, we develop some numerical simulations in order to support the obtained results in our paper. A set of parameters of system (2) are chosen as follows: $b_1 = 1, b_2 = 1, b_3 = 2, b_4 = 1, \alpha = 0.35, \beta = 0.03, \gamma_1 = 0.1, \gamma_2 = 0.3, \eta = 0.5$, and $\mu_0 = 0.01$. Then, we obtain the following specific case of system (2):

$$\begin{aligned} \frac{dS(t)}{dt} &= 1 + 0.3L(t) + 0.3B(t) + 0.5R(t - \tau) \\ &\quad - 0.11S(t) - 0.03(L(t) + B(t))S(t), \end{aligned}$$

$$\frac{dL(t)}{dt} = 1 + 0.03(L(t) + B(t))S(t) - 0.76L(t),$$

$$\frac{dB(t)}{dt} = 2 + 0.35L(t) - 0.41B(t),$$

$$\begin{aligned} \frac{dR(t)}{dt} &= 1 + 0.1L(t) + 0.1B(t) + 0.1S(t) - 0.01R(t) \\ &\quad - 0.5R(t - \tau), \end{aligned} \tag{61}$$

Then, (4) becomes the following form:

$$-0.0113L^2 + 2.4034L + 6.7325 = 0, \tag{62}$$

from which we can obtain the unique positive root $L_* = 215.4556$. Further, we can verify that system (61) has a unique endemic equilibrium $E_*(13.4193, 215.4556, 188.8036, 83.6066)$ and all the conditions given in Theorem 1 are satisfied.

By means of Matlab software, we get $\omega_0 = 2.0684, \tau_0 = 3.4685$, and $\lambda'(\tau_0) = 0.0081 + 1.0307i$. Thus, we can obtain $C_1(0) = -0.0560 + 0.0092i, \mu_2 = 6.9136, \beta_2 = -0.1120$, and $T_2 = -0.9945$. It follows that $\mu_2 > 0, \beta_2 < 0$, and $T_2 > 0$. Fix $\tau = 3.2575 < \tau_0$, then we can see that the solution of system (61) would tend to the endemic equilibrium $E_*(13.4193, 215.4556, 188.8036, 83.6066)$. In other words, $E_*(13.4193, 215.4556, 188.8036, 83.6066)$ is locally asymptotically stable, which can be illustrated by Figures 1–3. However, when τ passes through the critical

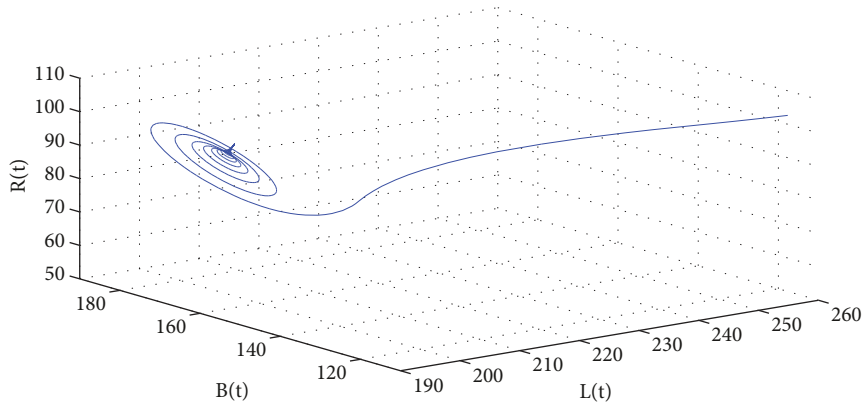


FIGURE 3: Dynamic behavior of system (61): projection on L-B-R with $\tau = 3.2575 < \tau_0 = 3.4685$.

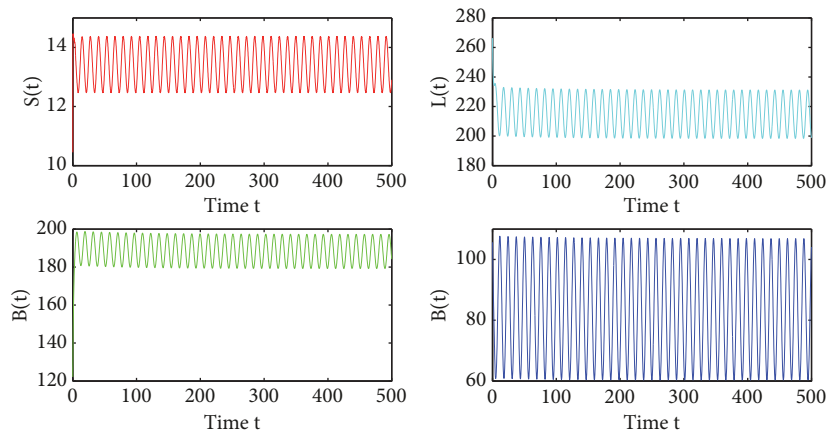


FIGURE 4: Time plots of $S, L, B,$ and R with $\tau = 3.6755 > \tau_0 = 3.4685$.

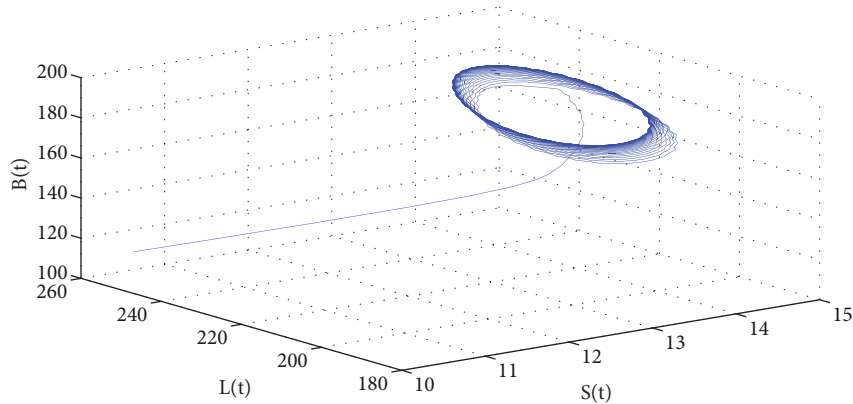


FIGURE 5: Dynamic behavior of system (61): projection on S-L-B with $\tau = 3.6755 > \tau_0 = 3.4685$.

value τ_0 , $E_*(13.4193, 215.4556, 188.8036, 83.6066)$ loses its stability, and a Hopf bifurcation occurs and a family of periodic solutions bifurcate from $E_*(13.4193, 215.4556, 188.8036, 83.6066)$. This property can be shown as in Figures 4–6. Since $\mu_2 > 0$, $\beta_2 < 0$, and $T_2 < 0$, we can conclude that the Hopf bifurcation occurring at $\tau_0 = 3.4685$ is supercritical and the bifurcating periodic solutions are stable and decrease. Next, we are interested to study the effect of some other parameters on the dynamics of system (62).

(i) Effect of the recovered rate (γ_1): in Figures 7(a)–7(d), we can see that the numbers of susceptible and recovered computers increase; nevertheless, the numbers of latent and breaking computers decrease, when the number of γ_1 increases. And the system changes its behavior from limit cycle to stable focus as we increase the value of γ_1 , from 0.1 to 0.3, which can be shown as in Figure 8.

(ii) Effect of the rate of latent and breaking computers reinstall the operating system (γ_2): in the same manner,

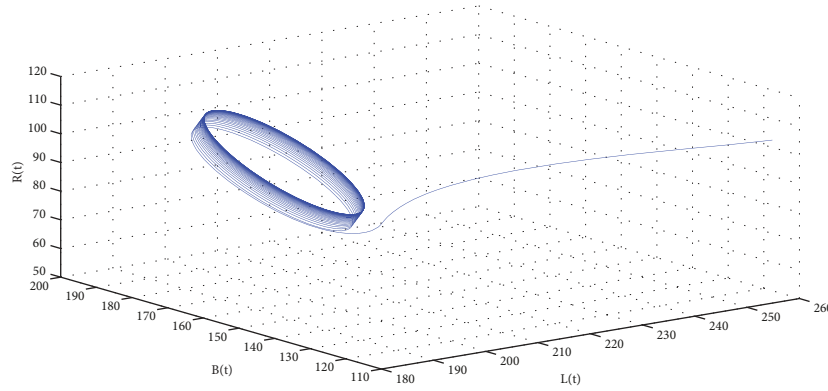


FIGURE 6: Dynamic behavior of system (61): projection on L-B-R with $\tau = 3.6755 > \tau_0 = 3.4685$.

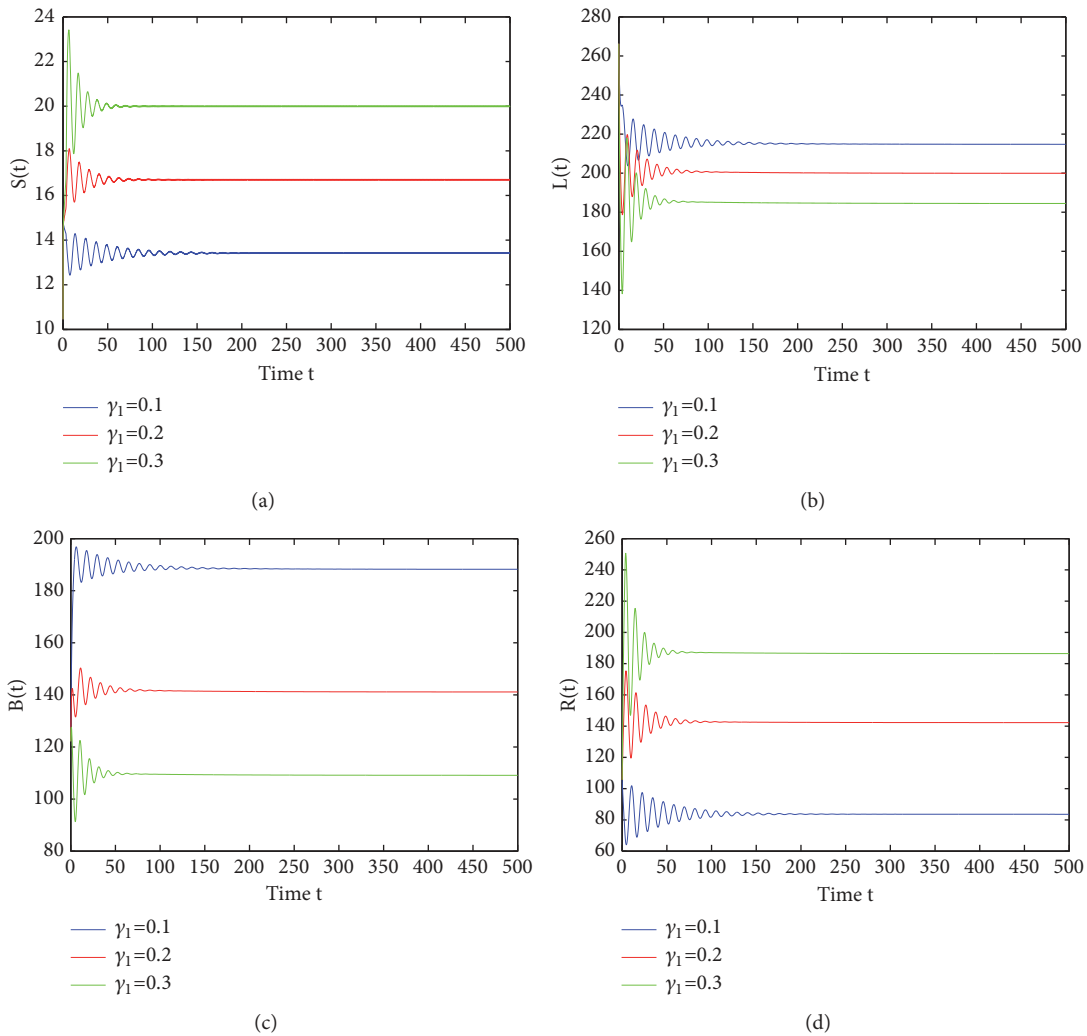


FIGURE 7: Time plots of $S, L, B,$ and R for different γ_1 at $\tau = 3.2575$. Rest of the parameters are taken as given in the text.

we can see from Figures 9(a)–9(d) that the numbers of susceptible and latent computers increase and the number of breaking computers decreases, when the number of γ_2 increases. But it does not affect the number of recovered computers, which can be also seen from the expression of R_* in Section 2. Also, we observe that γ_2 does not affect

the dynamics of the system; it remains at limit cycle when we choose $\tau = 3.6755$. This property can be illustrated by Figure 10.

(iii) Effect of the entering rates (b_1, b_2, b_3, b_4): as is shown in Figures 11–14, the numbers of all computers increase when the numbers of b_1 and b_4 increase. However, the

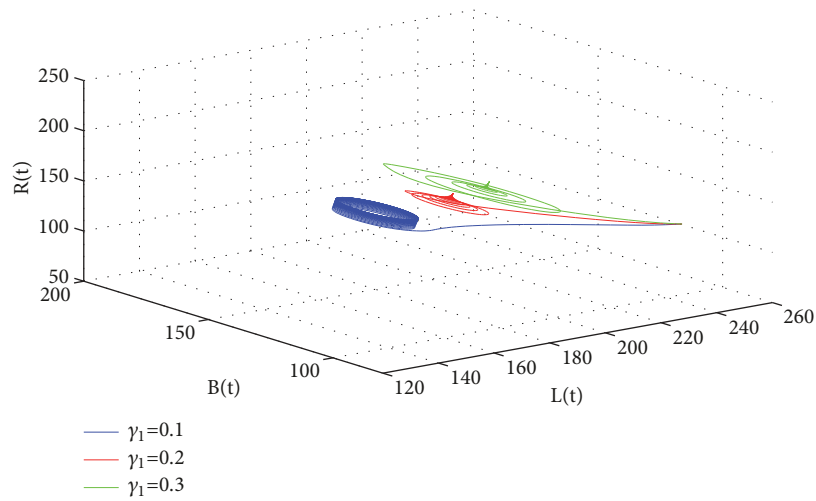


FIGURE 8: Dynamic behavior of system (61): projection on L-B-R with $\tau = 3.6755 > \tau_0 = 3.4685$ for different γ_1 . Rest of the parameters are taken as given in the text.

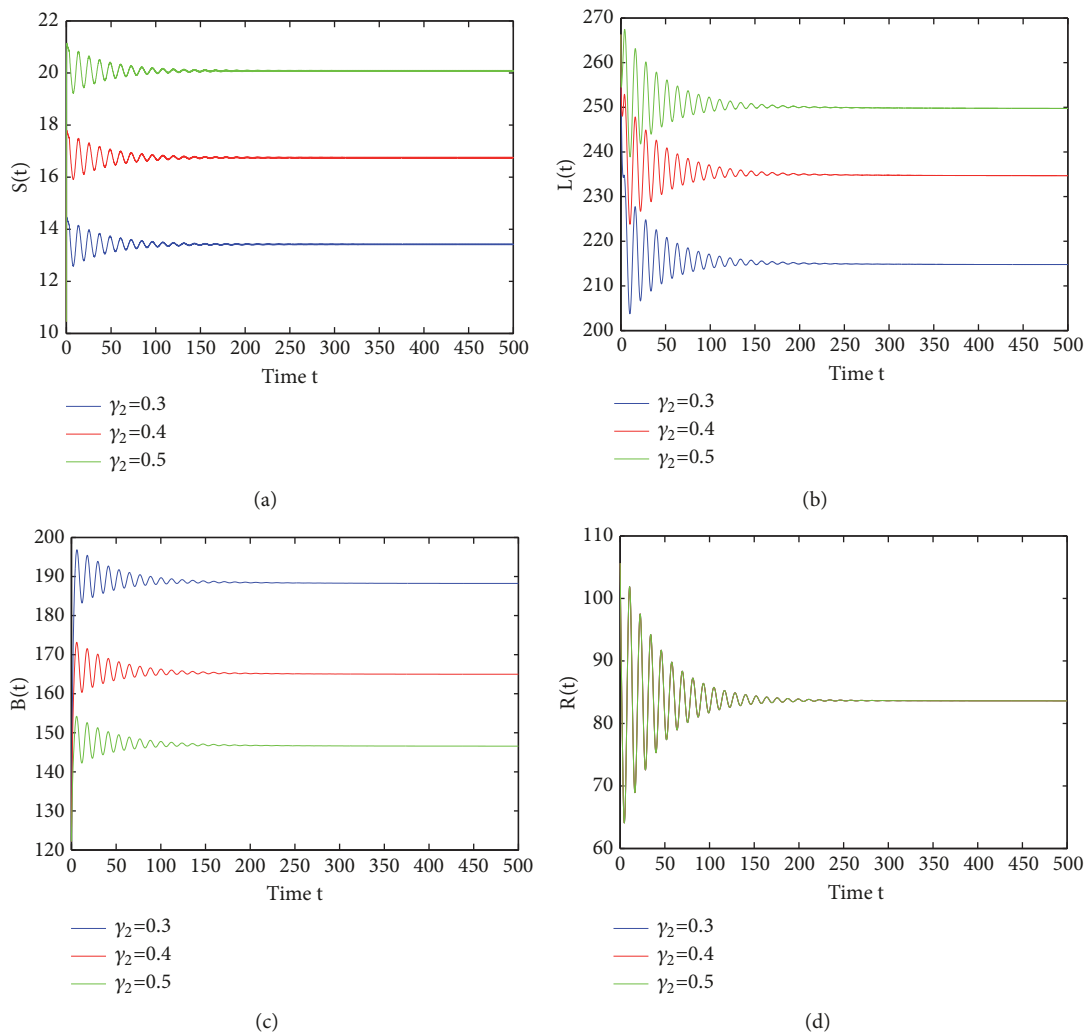


FIGURE 9: Time plots of $S, L, B,$ and R for different γ_2 at $\tau = 3.2575$. Rest of the parameters are taken as given in the text.

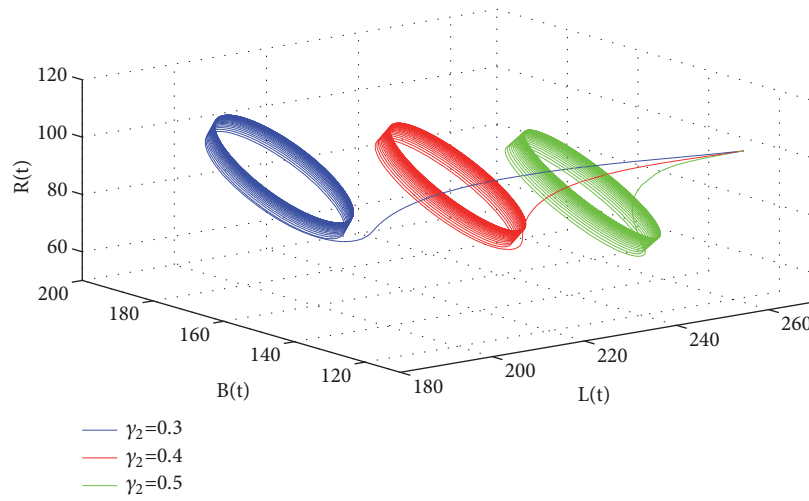


FIGURE 10: Dynamic behavior of system (61): projection on L-B-R with $\tau = 3.6755 > \tau_0 = 3.4685$ for different γ_2 . Rest of the parameters are taken as given in the text.

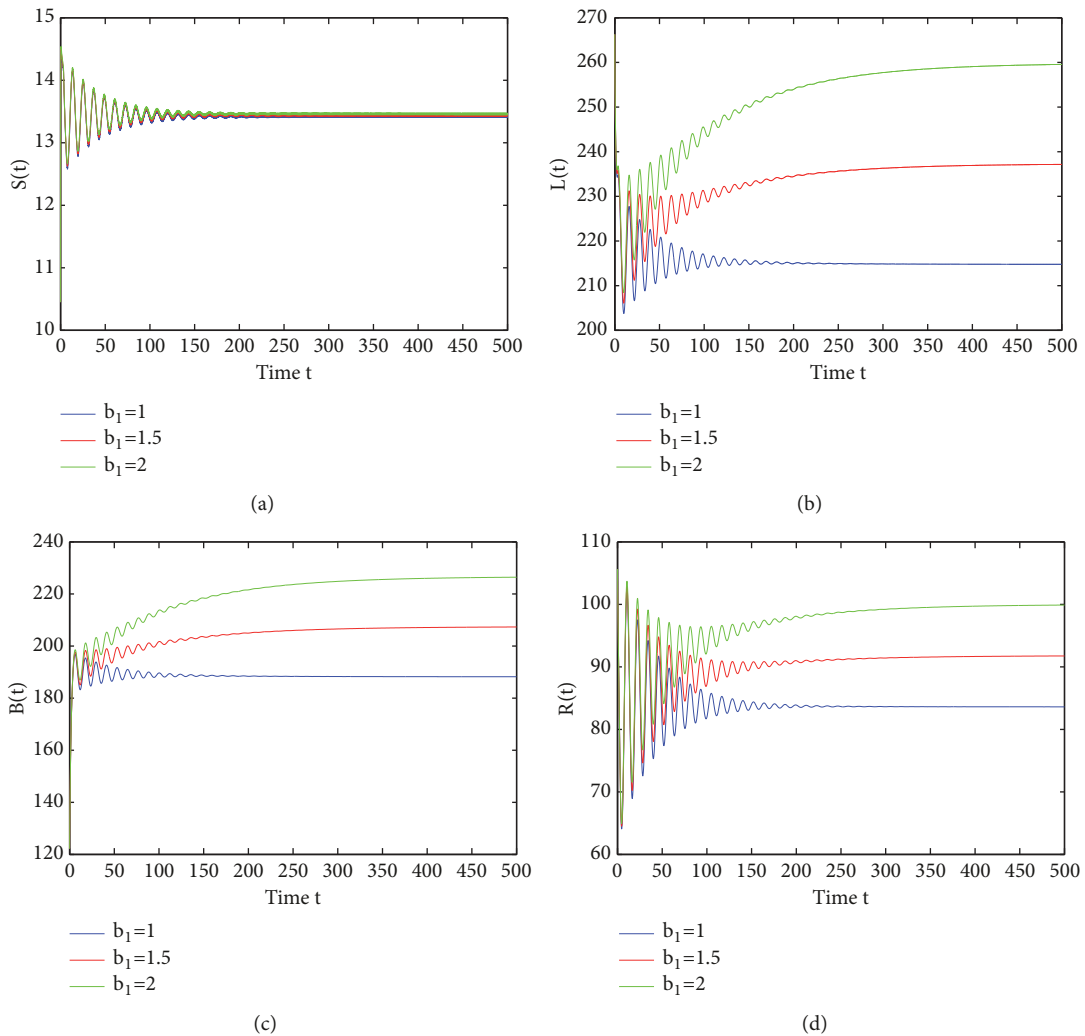


FIGURE 11: Time plots of $S, L, B,$ and R for different b_1 at $\tau = 3.2575$. Rest of the parameters are taken as given in the text.

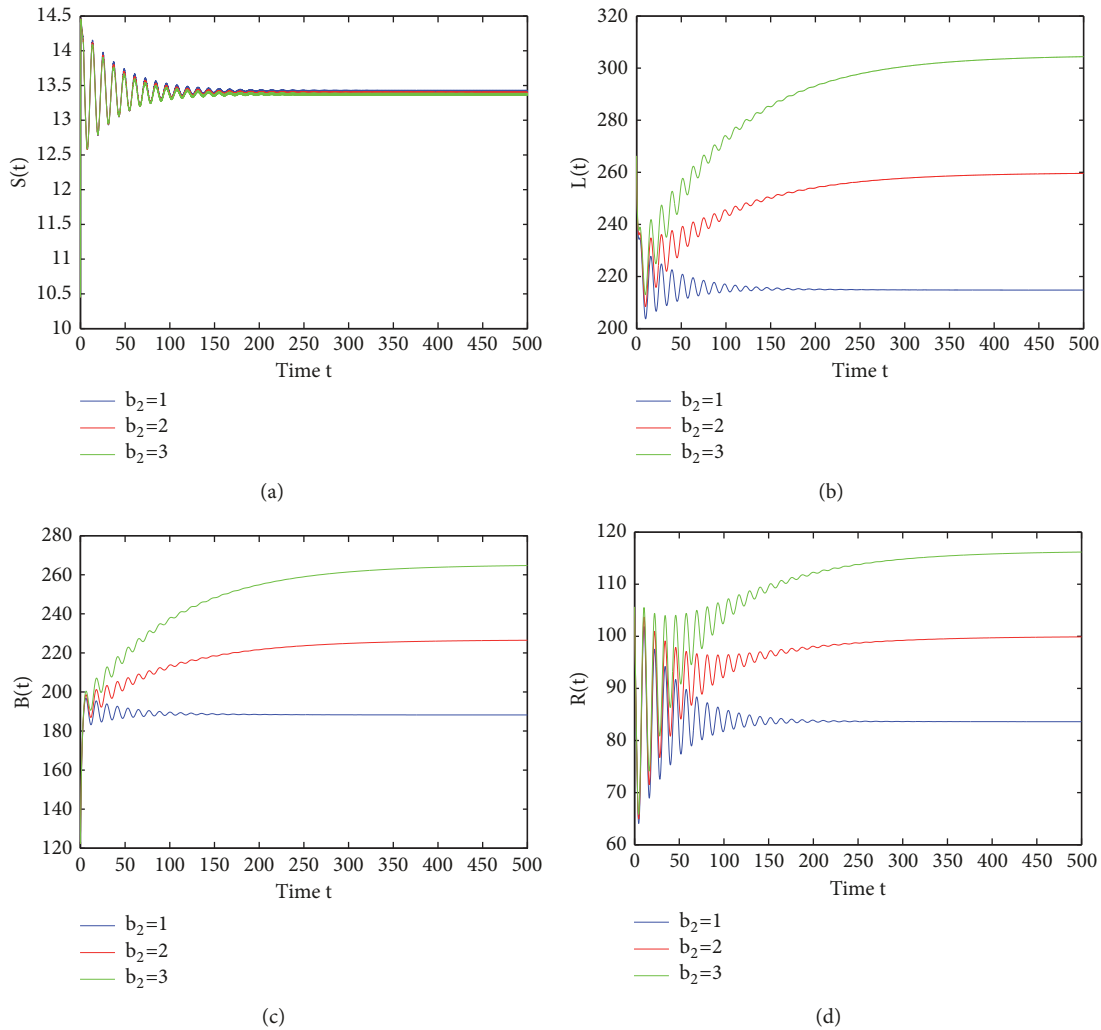


FIGURE 12: Time plots of $S, L, B,$ and R for different b_2 at $\tau = 3.2575$. Rest of the parameters are taken as given in the text.

number of susceptible computers decreases and the numbers of latent, breaking, and recovered computers increase, when the numbers of b_2 and b_3 increase. In addition, we find that the entering rates does not affect the dynamics of the systems.

6. Conclusions

In this paper, a delayed SLBRS computer virus model is presented by incorporating the time delay due to the temporary immunity period of the recovered computers based on the model proposed in [27]. Compared with the model in [27], we mainly consider the effect of the time delay on its dynamic behavior. Compared with other computer virus models, we assume that every computer can enter the Internet, which is consistent with the reality. Further, we also consider the effect of antivirus software on the susceptible computers in the presented model. Thus, the computer virus model proposed in our paper is more general.

It has been shown that the endemic equilibrium $E_*(S_*, L_*, B_*, R_*)$ is locally asymptotically stable when

$\tau \in [0, \tau_0)$ under some certain conditions. In this case, the propagation of the computer virus in system (2) can be controlled easily. Once the value of the time delay passes through τ_0 , $E_*(S_*, L_*, B_*, R_*)$ loses its stability and a Hopf bifurcation occurs and a family of periodic solutions bifurcate from $E_*(S_*, L_*, B_*, R_*)$. In this case, the numbers of the four classes computers in system (2) will oscillate in the vicinity of $E_*(S_*, L_*, B_*, R_*)$. Namely, the propagation of the computer virus will be out of control. Therefore, the results obtained in the present paper can help us to gain insight into the spreading process of computer viruses. Also, sufficient conditions for global stability of the endemic equilibrium are derived by constructing a suitable Lyapunov function. Furthermore, properties of the Hopf bifurcation are investigated by using the normal form theory and center manifold theorem. Numerical simulations are presented to verify the analytical predictions. In addition, it has been observed in our simulations that the recovered rate γ_1 can change the dynamics of the system from limit cycle to stable focus as its value increases. Thus, it is strongly

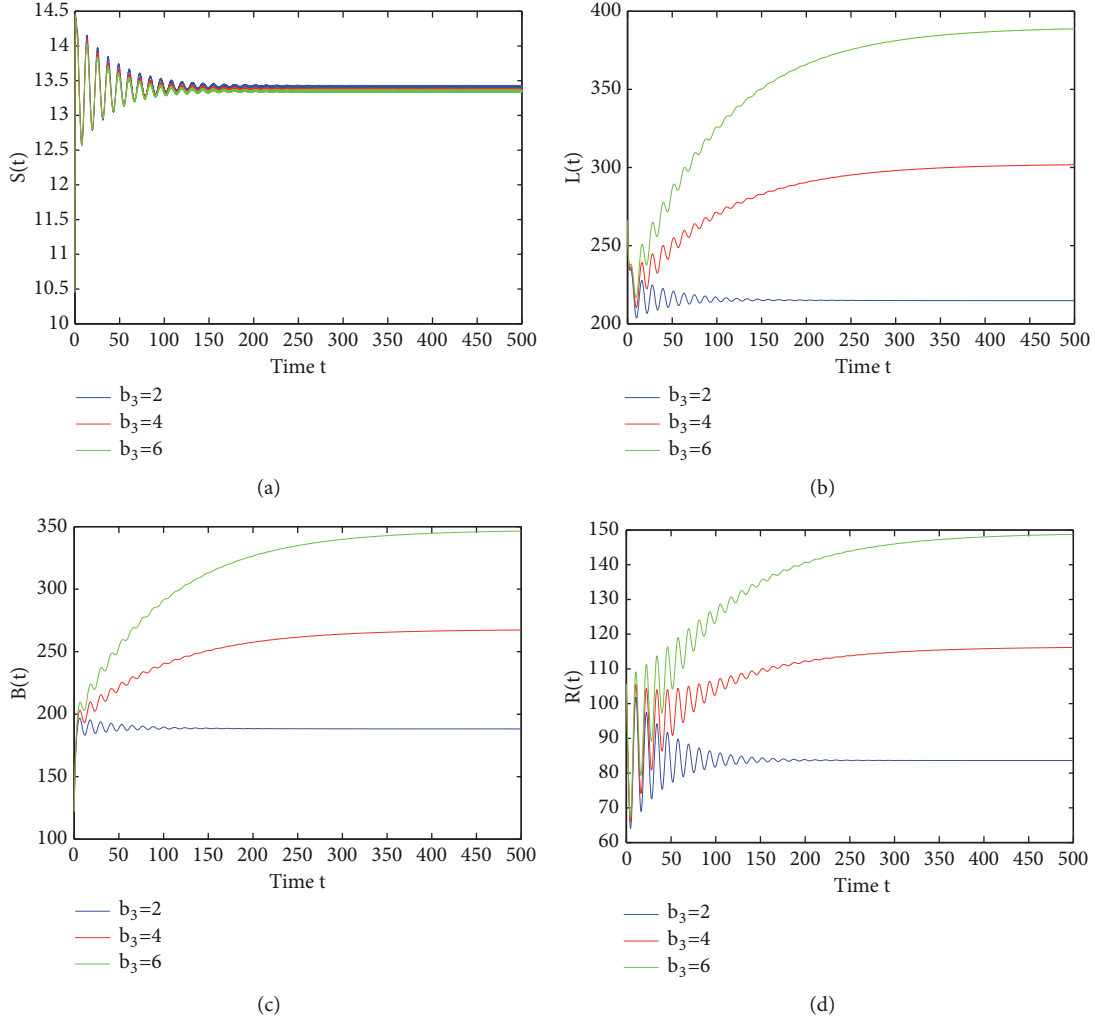


FIGURE 13: Time plots of S , L , B , and R for different b_3 at $\tau = 3.2575$. Rest of the parameters are taken as given in the text.

recommended that users of computers connected to Internet should periodically run antivirus software of the newest version. From the point of this view, we can conclude that the results of the proposed model in our paper can be used to evaluate the effectiveness of antivirus software. In addition, the numbers of latent and breaking computers decrease, when the reinstalling of the operating system rate increases. Thus, it can be concluded that users should reinstall operating system if necessary. Finally, the numbers of latent and breaking computer will also increase, when the values of entering rates of all computers b_1 , b_2 , b_3 , and b_4 increase. Therefore, the manager of a network should control the number of computers connected to the network properly.

Of course, when we pursue a low level of infections, we should also consider the cost of the measures we carry out. In addition, it should be pointed out that the model investigated in the literature [27] and our present paper assumes that the latent computers and the breaking computers have the same infection rate β . In the near future, we will investigate the optimal control problem of the following general system (63)

so as to achieve a low level of infections at a low cost by using the method introduced in [32]:

$$\begin{aligned}
 \frac{dS(t)}{dt} &= b_1 + \gamma_2 L(t) + \gamma_2 B(t) + \eta R(t - \tau) - \gamma_1 S(t) \\
 &\quad - \mu_0 S(t) - \beta_1 S(t) L(t) - \beta_2 S(t) B(t), \\
 \frac{dL(t)}{dt} &= b_2 + \beta_1 S(t) L(t) + \beta_2 S(t) B(t) - \gamma_1 L(t) \\
 &\quad - \gamma_2 L(t) - \mu_0 L(t) - \alpha L(t), \\
 \frac{dB(t)}{dt} &= b_3 + \alpha L(t) - \gamma_1 B(t) - \gamma_2 B(t) - \mu_0 B(t), \\
 \frac{dR(t)}{dt} &= b_4 + \gamma_1 L(t) + \gamma_1 B(t) + \gamma_1 S(t) - \mu_0 R(t) \\
 &\quad - \eta R(t - \tau),
 \end{aligned} \tag{63}$$

where β_1 and β_2 are the infection rate of the latent computers and the breaking computers, respectively.

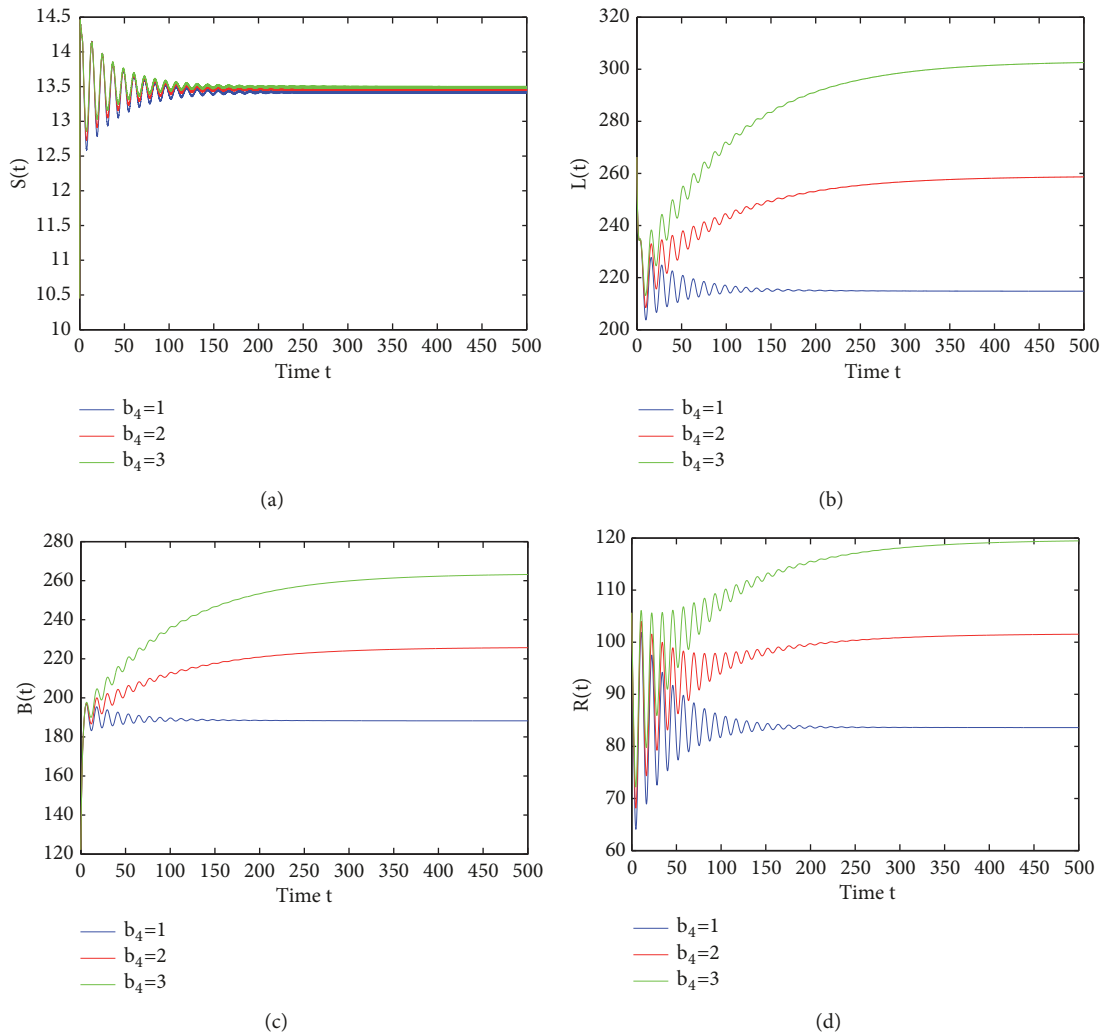


FIGURE 14: Time plots of S , L , B , and R for different b_3 at $\tau = 3.2575$. Rest of the parameters are taken as given in the text.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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