Memoirs of the Faculty of Engineering, Okayama University, Vol.30, No. 1, pp.121-131, December 1995

Stability and Sensitivity Analysis in Convex Vector Optimization

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(Received October 2, 1995)

Abstract. In this paper we provide some theoretical results on stability and sensitivity analysis in convex vector optimization. Given a family of parametrized vector optimization problems, the perturbation maps are defined as the set-valued map which associates to each parameter value the set of minimal points (properly minimal points, weakly minimal points) of the perturbed feasible set with respect to an ordering convex cone. Sufficient conditions for the upper and lower semicontinuity of the perturbations map are obtained. We also provide quantitative properties of the perturbation maps under some convexity assumptions.

1. INTRODUCTION

In this paper we consider a family of parametrized vector optimization problems:

$$(\text{VOP})\qquad \begin{array}{ll} \text{minimize} & f(x,u) = (f_1(x,u),\cdots,f_p(x,u)) \\ \text{subject to} & g_j(x,u) \leqq 0,\quad j=1,\cdots,q, \\ & x \in R^n, \end{array}$$

where x is an n-dimensional decision variable, u is a perturbation parameter vector in R^m, f is a p-dimensional objective function and g is a q-dimensional constraint function. K is a nonempty pointed closed convex cone in R^p which serves us the domination cone in the objective space, where K is said to be pointed if $l(K) = K \cap (-K) = \{0\}$. Namely, K induces a partial order in R^p. We use the following notations. For $y,\ y' \in R^p$,

$$\begin{array}{ll} y \leqq_K y' & \text{if and only if}\ \ y'-y \in K, \\ y \leq_K y' & \text{if and only if}\ \ y'-y \in K \setminus l(K) = K\setminus\{0\}, \\ y <_K y' & \text{if and only if}\ \ y'-y \in \text{int}K. \end{array}$$

Let X be a set-valued map from R^m to R^n defined by

$$X(u) = \{x \in R^n : g_j(x,u) \leqq 0,\ j=1,\cdots,q\}.$$

We can define another set-valued map Y from R^m to R^p by

$$\begin{aligned} Y(u) &= \{y \in R^p : y = f(x,u),\ \text{for some}\ x \in X(u)\} \\ &= f(X(u),u). \end{aligned}$$

$Y(u)$ is the parametrized feasible set in the objective space.

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Definition 1.1 Let A be a nonempty subset of R^p.

(1) A point $\hat{y} \in A$ is a K-minimal point of A with respect to K if there exists no $y \in A$ such that $y \leq_K \hat{y}$. We denote the set of all K-minimal points of A by $\text{Min}_K A$, i.e.,

$$\begin{aligned} \text{Min}_K A &= \{\hat{y} \in A : \text{there exists no } y \in A \text{ such that } y \leq_K \hat{y}\} \\ &= \{\hat{y} \in A : (A - \hat{y}) \cap (-K) = \{0\}\}. \end{aligned}$$

(2) A point $\hat{y} \in A$ is a properly K-minimal point of A with respect to K if there exists a cone C such that $\hat{y} \in \text{Min}_C A$, where C is a convex cone with $C \neq R^p$ and $K \setminus \{0\} \subset \text{int}C$. We denote the set of all properly K-minimal points of A by $\text{PrMin}_K A$.

(3) A point $\hat{y} \in A$ is a weakly K-minimal point of A with respect to K if there exists no $y \in A$ such that $y <_K \hat{y}$. We denote the set of all weakly K-minimal points of A by $\text{WMin}_K A$, i.e.,

$$\begin{aligned} \text{WMin}_K A &= \{\hat{y} \in A : \text{there exists no } y \in A \text{ such that } y <_K \hat{y}\} \\ &= \{\hat{y} \in A : (A - \hat{y}) \cap (-\text{int}K) = \phi\}. \end{aligned}$$

Of course, every properly K-minimal point of A is K-minimal point and every K-minimal point is weakly K-minimal.

According to these three solution concepts we can define the following three set-valued maps W, G and S from R^m to R^p by $W(u) = \text{Min}_K Y(u)$, $G(u) = \text{PrMin}_K Y(u)$ and $S(u) = \text{WMin}_K Y(u)$, for any $u \in R^m$, respectively. These set-valued maps W, G and S are called the perturbation map, the proper perturbation map and the weak perturbation map, respectively.

Some quantitative results concerning the behavior of the perturbation map W were analyzed by Tanino [9,10], and Shi [8] improved some results of Tanino. Moreover, the authors (Kuk et al. [5]) established the behavior of the perturbation maps G and S in addition to W for general vector optimization problems.

In this paper the behavior of the perturbation maps are analyzed both qualitatively and quantitatively. First, some sufficient conditions which guarantee the upper and lower semicontinuity of the perturbation maps are provided. Next we investigate the relationships between the contingent derivative DY of Y and the contingent derivatives DG, DW and DS of G, W and S, respectively under some convexity assumptions. In virtue of convexity assumptions, we obtain the finer results than general case (see Kuk et al. [5]).

2. CONTINUITY OF THE PERTURBATION MAPS

In this section, we investigate sufficient conditions for the upper semicontinuity of the weak perturbation map S and lower semicontinuity of the proper perturbation map G. First we introduce concepts of semicontinuity and cone-convexity of set-valued map.

Definition 2.1. Let F be a set-valued map from R^m to R^p.

(1) F is said to be upper semicontinuous at $u \in R^m$ if $u^k \to u$, $y^k \in F(u^k)$ and $y^k \to y$ all imply that $y \in F(u)$.

(2) F is said to be lower semicontinuous at $u \in R^m$ if $u^k \to u$ and $y \in F(u)$ imply the existence of an integer m and a sequence $\{y^k\} \subset R^p$ such that $y^k \in F(u^k)$ for $k \geqq m$ and $y^k \to y$.

(3) F is said to be continuous at $u \in R^m$ if it is both upper and lower semicontinuous at u.

Remark 2.1. F is upper semicontinuous on R^m if and only if the graph of F is a closed set in $R^m \times R^p$.

Definition 2.2. The set-valued map F from R^m to R^p is said to be K-convex if, for any u^1, $u^2 \in R^m$ and α, $0 \leqq \alpha \leqq 1$,

$$\alpha F(u^1) + (1-\alpha)F(u^2) \subset F(\alpha u^1 + (1-\alpha)u^2) + K,$$

or, equivalently, if the graph of the set-valued map $F+K$ is convex. Here $F+K$ is defined by

$$(F+K)(u) = F(u)+K, \quad \text{for each } u \in R^m.$$

F is said to be locally K-convex at u if there exists a neighborhood N_u of u such that $N_u \subset \text{dom}F = \{u \in R^m : F(u) \neq \phi\}$ and F is K-convex on N_u.

Lemma 2.1. ([9]) If F is convex and $\hat{u} \in \text{int}(\text{dom}F)$, then F is lower semicontinuous at $\hat{u}$.

The following theorem provides sufficient conditions for the upper semicontinuity of the map S at $\hat{u}$.

Theorem 2.1. Let Y be K-convex. If Y is upper semicontinuous at $\hat{u}$ and $\hat{u} \in \text{int}(\text{dom}Y)$, then the weak perturbation map S is upper semicontinuous at $\hat{u} \in R^m$.

Proof. Let $u^k \to \hat{u}$, $y^k \in S(u^k)$ and $y^k \to \hat{y}$. Since Y is upper semicontinuous at $\hat{u}$, $\hat{y} \in Y(\hat{u})$. Hence, if we suppose that $\hat{y} \notin S(\hat{u})$, then there exists $\bar{y} \in Y(\hat{u})$ such that $\hat{y} - \bar{y} \in \text{int}K$. Since $\hat{u} \in \text{int}(\text{dom}Y) = \text{int}(\text{dom}(Y+K))$ and $Y+K$ is convex, $Y+K$ is lower semicontinuous at $\hat{u}$ by Lemma 2.1, and hence there exist a sequence $\{\bar{y}^k\} \subset R^p$ and a positive integer m such that

$$\bar{y}^k \to \bar{y} \quad \text{and} \quad \bar{y}^k \in Y(u^k)+K \quad \text{for } k \geqq m.$$

Since $y^k - \bar{y}^k \to \hat{y} - \bar{y} \in \text{int}K$, $y^k - \bar{y}^k \in \text{int}K$ for all k sufficiently large. Hence $y^k \notin \text{WMin}_K(Y(u^k)+K)$ and so $y^k \notin \text{WMin}_K Y(u^k)$. This contradicts that $y^k \in S(u^k)$. Therefore $\hat{y} \in S(\hat{u})$. □

Next we consider sufficient conditions for the lower semicontinuity of the map G.

Definition 2.3. A set A in R^p is said to be K-dominated by $\text{Min}_K A$ if

$$A \subset \text{Min}_K A + K.$$

Remark 2.2. Since $\text{Min}_K A \subset A$, if A is K-dominated by $\text{Min}_K A$,

$$A+K = \text{Min}_K A + K.$$

Definition 2.4. For a nonempty set A in R^p, its recession cone A^+ is defined by

$$A^+ = \{y \in R^p : \text{there exist sequences } \{\lambda_k\} \subset \text{ and } \{y^k\} \subset R^p \text{ such that } \lambda_k > 0,\ \lambda_k \to 0,\ \lambda_k y^k \to y \text{ and } y^k \in A \text{ for all } k\}.$$

Remark 2.3. A^+ is a closed cone which contains the orgin. Moreover, if A is a nonempty closed convex set, A^+ coincides with the set 0^+A which is defined by

$$\begin{aligned} 0^+A &= \{y \in R^p : \bar{y} + \lambda y \in A \text{ for all } \lambda \geqq 0,\ \bar{y} \in A\} \\ &= \{y \in R^p : A + y \subset A\} \end{aligned}$$

and therefore it is a closed convex cone.

Lemma 2.2. ([7]) A nonempty set A is bounded if and only if $A^+ = \{0\}$.

Definition 2.5. A nonempty set A in R^p is said to be K-bounded if $A^+ \cap (-K) = \{0\}$.

Lemma 2.3. ([7]) If $A \subset R^p$ is a nonempty closed convex set, the following statements are equivalent:

(1) A is K-bounded

(2) $\text{Min}_K A \neq \phi$
(3) A is K-dominated by $\text{Min}_K A$.

Lemma 2.4. ([7]) If F is K-convex, $\hat{u} \in \text{int}(\text{dom}F)$, and $F(\hat{u})$ is K-bounded, then there exists a neighborhood $N_{\hat{u}}$ of $\hat{u}$ such that $F(u)$ is K-bounded for all $u \in N_{\hat{u}}$.

Now we can obtain sufficient conditions for the lower semicontinuity of G.

Theorem 2.2. Let Y be K-convex. If $Y + K$ is upper semicontinuous in a neighborhood of $\hat{u}$ and $\hat{u} \in \text{int}(\text{dom}Y)$, then the proper perturbation map G is lower semicontinuous at $\hat{u}$.

Proof. Let $u^k \to \hat{u}$ and $\hat{y} \in G(\hat{u})$. Since $\hat{u} \in \text{int}(\text{dom}Y)=\text{int}(\text{dom}(Y + K))$ and $Y + K$ is convex, $Y + K$ is lower semicontinuous at $\hat{u}$ by Lemma 2.1, and hence there exist a sequence $\{y^k\}$ and a number m_1 such that

$$y^k \to \hat{y} \text{ and } y^k \in Y(u^k) + K \text{ for all } k \geqq m_1.$$

Since $Y(\hat{u}) + K$ is a nonempty closed convex set and $G(\hat{u}) \neq \phi$, $\text{Min}_K Y(\hat{u}) = \text{Min}_K(Y(\hat{u}) + K) \neq \phi$ and hence $Y(\hat{u}) + K$ is K-bounded by Lemma 2.3. Therefore, in view of Lemma 2.4, $Y(u) + K$ is K-bounded for all u in a certain neighborhood $N_{\hat{u}}$ of $\hat{u}$. From Lemma 2.3 and Remark 2.2, this implies that

$$G(u) + K = (Y(u) + K) + K = Y(u) + K$$

in a neighborhood of $\hat{u}$. Hence there exist a sequence $\{\hat{u}^k\}$ and a number $m_2 \geqq m_1$ such that

$$y^k - \hat{y}^k \in K \text{ and } \hat{y}^k \in G(u^k) \text{ for } k \geqq m_2.$$

First we will show that $\{\hat{y}^k\}$ is bounded. If this were not the case, from Lemma 2.2, we can take a subsequence of $\{\hat{y}^k\}$, for which there exist a sequence $\{\lambda_k\}$ of positive numbers and a nonzero vector $\tilde{y}$ such that $\lambda_k \to 0$ and $\lambda_k \hat{y}^k \to \tilde{y}$. Since $\lambda_k(y^k - \hat{y}^k) \in K$ and $y^k \to \hat{y}$, the limit $-\tilde{y}$ of $\{\lambda_k(y^k - \hat{y}^k)\}$ is contained in K. Take an arbitrary $\bar{y} \in Y(\hat{u}) + K$. Then there exist a sequence $\{\bar{y}^k\}$ and a number $m_3 \geqq m_2$ such that

$$\bar{y}^k \to \bar{y} \text{ and } \bar{y}^k \in Y(2\hat{u} - u^k) + K \text{ for } k \geqq m_3,$$

since $Y + K$ is lower semicontinuous at $\hat{u}$. Then, from the convexity of $Y + K$,

$$\frac{1}{2}(\hat{y}^k + \bar{y}^k) \in Y(\hat{u}) + K \text{ for } k \geqq m_3.$$

Moreover, $\lambda_k(\hat{y}^k + \bar{y}^k) \to \tilde{y}$. This implies that $\tilde{y} \in (Y(\hat{u}) + K)^+$ and hence leads to a contradiction to the K-boundedness of $Y(\hat{u}) + K$. Therefore $\{\hat{y}^k\}$ must be bounded. Hence $\{\hat{y}^k\}$ has a cluster point, which is denoted by y'. Since $y^k - \hat{y}^k \in K$ and $y^k \to \hat{y}$, $\hat{y} - y' \in K$. Since $Y + K$ is upper semicontinuous at $\hat{u}$, $y' \in Y(\hat{u}) + K$. Recalling that $\hat{y} \in G(\hat{u})$, we can conclude that $y' = \hat{y}$. In other words, $\hat{y}$ is the unique cluster point for the bounded sequence $\{\hat{y}^k\} \to \hat{y}$. Therefore $\hat{y}^k \to \hat{y}$, which indicates that G is lower semicontinuous at $\hat{u}$. □

3. CONTINGENT DERIVATIVES OF THE PERTURBATION MAPS UNDER CONVEXITY ASSUMPTIONS

In this section we provide relationship between the contingent derivative DY of Y and the contingent derivative DG, DW, and DS of G, W and S, respectively, under some convexity assumption. Throughout this section, a cone $\tilde{K}$ is assumed to be a closed convex cone contained in $(\text{int}K) \cup \{0\}$. We first introduce the concept of contingent derivative of set-valued maps. Throughout this section, let F be a set-valued map from R^m to R^p and we denote it by $F : R^m \rightrightarrows R^p$..

Definition 3.1. Let A be a nonempty subset of R^m, and let $\hat{v} \in R^m$. The subset $T_A(\hat{v})$ defined below is called the contingent cone to A at $\hat{v}$:
$v \in T_A(\hat{v})$ if and only if

$$\begin{cases} \text{there exist sequences } \{h_k\} \subset \text{int}R_+,\ \{v^k\} \subset R^m \text{ such that } h_k \to 0,\ v^k \to v \\ \text{and for all } k,\ \hat{v} + h_k v^k \in A, \end{cases}$$

where $\text{int}R_+$ is the set of all positive real numbers.

The graph of set-valued map F is defined and denoted by

$$\text{graph}F = \{(u, y) : y \in F(u)\} \subset R^m \times R^p.$$

Definition 3.2. Let $(\hat{u}, \hat{y})$ be a point in graph$F$. The set-valued map $DF(\hat{u}, \hat{y})$ from R^m to R^p defined by the following is called the contingent derivative of F at $(\hat{u}, \hat{y})$:
$y \in DF(\hat{u}, \hat{y})(u)$ if and only if $(u, y) \in T_{\text{graph}F}(\hat{u}, \hat{y})(u)$.
In other words, $y \in DF(\hat{u}, \hat{y})(u)$ if and only if

$$\begin{cases} \text{there exist sequences } \{h_k\} \subset \text{int}R_+,\ \{u^k\} \subset R^m\ \ \{y^k\} \subset R^p \text{ such that } h_k \to 0,\ u^k \to v \\ y^k \to y \text{ and for all } k,\ \hat{y} + h_k y^k \in F(\hat{u} + h_k u^k), \end{cases}$$

The following proposition is obtained immediately from Proposition 2.1 in Tanino [10] and Lemma 3.1 in Shi [8].

Proposition 3.1. Let Y be locally K-convex at $\hat{u}$ and let $\hat{y} \in G(\hat{u})$. Then, for any $u \in R^m$,

$$DY(\hat{u}, \hat{y})(u) + K = D(Y + K)(\hat{u}, \hat{y})(u).$$

Remark 3.1. If Y is locally K-convex at $\hat{u}$ and $\hat{y} \in G(\hat{u})$, then we obtain, from Lemma 3.2 in Shi [8], for any $u \in R^m$,

$$\text{Min}_K DY(\hat{u}, \hat{y})(u) = \text{Min}_K D(Y + K)(\hat{u}, \hat{y})(u).$$

As for properly K-minimum and weakly K-minimum, we also have similar results as following theorem.

Theorem 3.1. Let Y be locally K-convex at $\hat{u}$ and let $\hat{y} \in G(\hat{u})$. Then, for any $u \in R^m$,

(1) $\text{PrMin}_K DY(\hat{u}, \hat{y})(u) = \text{PrMin}_K D(Y + K)(\hat{u}, \hat{y})(u)$,
(2) $\text{WMin}_K DY(\hat{u}, \hat{y})(u) = \text{WMin}_K D(Y + \tilde{K})(\hat{u}, \hat{y})(u)$.

Proof. (1) Let $y \in \text{PrMin}_K DY(\hat{u}, \hat{y})(u)$, i.e.,

$$y \in \text{Min}_C DY(\hat{u}, \hat{y})(u),$$

where C is the cone in the definition of the proper K-minimum. Then

$$y \in DY(\hat{u}, \hat{y})(u) \subset D(Y + K)(\hat{u}, \hat{y})(u).$$

Suppose that

$$y \notin \text{PrMin}_K D(Y + K)(\hat{u}, \hat{y})(u).$$

Then there exists a $y' \in D(Y + K)(\hat{u}, \hat{y})(u)$ such that

$$y - y' = k' \in C \setminus l(C).$$

For such y', from Proposition 3.1, there exist a $y'' \in DY(\hat{u},\hat{y})(u)$ such that

$$y' - y'' = k'' \in K.$$

Thus

$$y - y'' = k' + k'' \subset C \setminus l(C),$$

since C is a convex cone, which leads to a contradiction. Hence,

$$y \in \mathrm{PrMin}_K D(Y+K)(\hat{u},\hat{y})(u).$$

Conversely, let $y \in \mathrm{PrMin}_K D(Y+K)(\hat{u},\hat{y})(u)$. Then

$$y \in D(Y+K)(\hat{u},\hat{y})(u).$$

From Proposition 3.1, there exists a $y' \in DY(\hat{u},\hat{y})(u) \subset D(Y+K)(\hat{u},\hat{y})(u)$ such that

$$y - y' = k' \in K.$$

We may confirm that $k' = 0$, since $k' \neq 0$ implies that

$$k' \in K \setminus \{0\} \subset \mathrm{int}C.$$

Since C is not the whole space, int C is included by $C \setminus l(C)$ and hence,

$$y \notin \mathrm{PrMin}_K D(Y+K)(\hat{u},\hat{y})(u),$$

which leads to a contradiction. Therefore,

$$y \in \mathrm{PrMin}_K DY(\hat{u},\hat{y})(u).$$

(2) Let $y \in \mathrm{WMin}_K DY(\hat{u},\hat{y})(u)$. Then

$$y \in DY(\hat{u},\hat{y})(u) \subset D(Y+\tilde{K})(\hat{u},\hat{y})(u).$$

Assume that

$$y \notin \mathrm{WMin}_K D(Y+\tilde{K})(\hat{u},\hat{y})(u).$$

Then, there exists $y' \in D(Y+\tilde{K})(\hat{u},\hat{y})(u)$ such that

$$y - y' = k \in \mathrm{int}K.$$

For that y', from Proposition 3.1, there exists $y'' \in DY(\hat{u},\hat{y})(u)$ such that

$$y' - y'' = k' \in \tilde{K}.$$

Thus,

$$y - y'' = k + k' \in \mathrm{int}K,$$

since K is a convex cone, which contradicts $y \in \mathrm{WMin}_K DY(\hat{u},\hat{y})(u)$. Hence

$$y \in \mathrm{WMin}_K D(Y+\tilde{K})(\hat{u},\hat{y})(u).$$

Next, let $y \in \mathrm{WMin}_K D(Y+\tilde{K})(\hat{u},\hat{y})(u)$. It suffices to prove that

$$y \in DY(\hat{u},\hat{y})(u).$$

Since $y \in D(Y+\tilde{K})(\hat{u},\hat{y})(u)$, from Proposition 3.1 there exists

$$y' \in DY(\hat{u},\hat{y})(u) \subset D(Y+\tilde{K})(\hat{u},\hat{y})(u)$$

such that $y - y' = k' \in \tilde{K}$. We may confirm that $k' = 0$, because $k' \neq 0$ implies that

$$y \notin \mathrm{WMin}_K D(Y+\tilde{K})(\hat{u},\hat{y})(u).$$

This leads to a contradiction. Therefore, $y \in DY(\hat{u},\hat{y})(u)$. □

Definition 3.3. Given a set A and a convex cone D in R^p, A is said to be D-convex if $A+D$ is a convex set. A is said to be D-closed if $A+$ cl$D$ is closed.

Lemma 3.1. Let $Y(u)$ be a K-closed, K-convex set near $\hat{u}$. Then, for any $u \in R^m$,

$$DG(\hat{u},\hat{y})(u) = DW(\hat{u},\hat{y})(u).$$

Proof. Since, in view of Corollary 3.2.2 in Sawaragi et al. [7], $W(u) \subset \mathrm{cl}\ G(u)$ for any $u \in R^m$, $T_{\mathrm{graph}W}(\hat{u},\hat{y}) \subset T_{\mathrm{graph}(\mathrm{cl}G)}(\hat{u},\hat{y}) = T_{\mathrm{graph}G}(\hat{u},\hat{y})$, and hence, for any $u \in R^m$, $DW(\hat{u},\hat{y})(u) \subset DG(\hat{u},\hat{y})(u)$. Since graph$G \subset$ graph$W$, the converse inclusion is obvious. □

Remark 3.2. In view of Theorem 3.2.12 in Sawaragi et al. [7], if $Y(u)$ is a K-closed, K-convex set near $\hat{u}$, then the following (C1)∼(C3) are equivalent: for any $u \in R^m$ near $\hat{u}$,

(C1) $W(u) \neq \phi$,
(C2) $G(u) \neq \phi$,
(C3) $Y(u)$ is K-dominated by $W(u)$.

Theorem 3.2. Let Y be a locally K-convex at $\hat{u}$ and let $\hat{y} \in G(\hat{u})$. If Y is K-dominated by W near $\hat{u}$, then, for any $u \in R^m$,

$$\mathrm{PrMin}_K DY(\hat{u},\hat{y})(u) \subset DW(\hat{u},\hat{y})(u).$$

Proof. Since Y is a locally K-convex at $\hat{u}$ and $Y(u)$ is K-dominated by $W(u)$ near $\hat{u}$, W is also locally K-convex at $\hat{u}$. Hence, from Theorem 3.1 and K-dominatedness by $W(u)$ of $Y(u)$ near $\hat{u}$, for any $u \in R^m$,

$$\begin{aligned}\mathrm{PrMin}_K DY(\hat{u},\hat{y})(u) &= \mathrm{PrMin}_K D(Y+K)(\hat{u},\hat{y})(u)\\ &= \mathrm{PrMin}_K D(W+K)(\hat{u},\hat{y})(u)\\ &= \mathrm{PrMin}_K DW(\hat{u},\hat{y})(u)\\ &\subset DW(\hat{u},\hat{y})(u).\end{aligned}$$ □

The following example illustrates that K-dominatedness by $W(u)$ of $Y(u)$ near $\hat{u}$ is essential for Theorem 3.2.

Example 3.1. Let $Y : R \rightrightarrows R^2$, $K = R^2_+$ and Y be defined by

$$Y(u) = \begin{cases} \{y \in R^2 : y_2 \geqq (y_1)^2, y_1 \geqq 0, y_2 \geqq 0\}, & (u \geqq 0),\\ \{y \in R^2 : y_2 \geqq (y_1)^2, y_1 > 0, y_2 \geqq 0\}, & (u < 0).\end{cases}$$

Then Y is locally K-convex at $\hat{u} = 0$ and

$$W(u) = \begin{cases} \{(0,0)\}, & (u \geqq 0),\\ \phi, & (u < 0).\end{cases}$$

Hence, $Y(u)$ is not K-dominated by $W(u)$ near $\hat{u} = 0$. Let $\hat{y} = (0,0)$. Then

$$DY(\hat{u},\hat{y})(u) = \{y \in R^2 : y_1 \geqq 0, y_2 \geqq 0\}, \text{ for } u \in R,$$

$$\mathrm{PrMin}_K DY(\hat{u},\hat{y})(u) = \{(0,0)\}, \text{ for } u \in R.$$

On the other hand,

$$DW(\hat{u},\hat{y})(u) = W(u), \text{ for } u \in R.$$

Hence,

$$\text{PrMin}_K DY(\hat{u},\hat{y})(u) \not\subset DW(\hat{u},\hat{y})(u), \text{ for } u < 0.$$

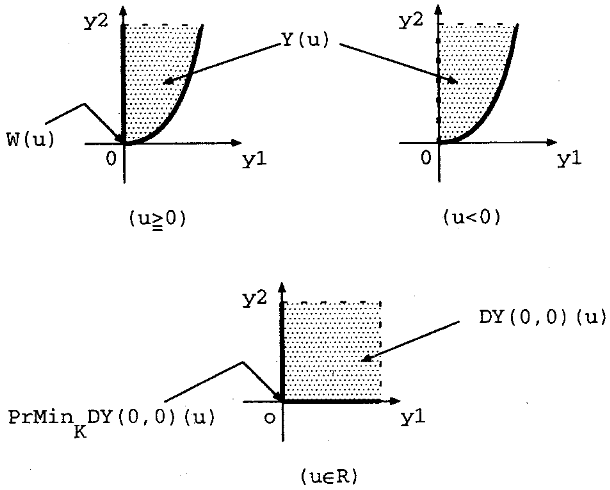

Figure 1: Example 3.1.

If Y is a locally K-convex at $\hat{u}$, then $Y(u)$ is K-convex set near $\hat{u}$. Hence, from Lemma 3.1 and Theorem 3.2, we obtain the following corollary.

Corollary 3.1. Let $Y(u)$ be a K-closed set near $\hat{u}$ and Y be a locally K-convex at $\hat{u}$ and let $\hat{y} \in G(\hat{u})$. If $Y(u)$ is K-dominated by $W(u)$ near $\hat{u}$, then for any $u \in R^m$,

$$\text{PrMin}_K DY(\hat{u},\hat{y})(u) \subset DG(\hat{u},\hat{y})(u).$$

Theorem 3.3. (1) Let Y be a locally K-convex at $\hat{u}$ and let $\hat{y} \in S(\hat{u})$. Then, for any $u \in R^m$,

$$DS(\hat{u},\hat{y})(u) \subset \text{WMin}_K DY(\hat{u},\hat{y})(u).$$

(2) Moreover, if $Y(u)$ is $\tilde{K}$-dominated by $S(u)$ near $\hat{u}$ and $\hat{y} \in G(\hat{u})$, then, for any $u \in R^m$,

$$DS(\hat{u},\hat{y})(u) = \text{WMin}_K DY(\hat{u},\hat{y})(u).$$

Proof. (1) Let $y \in DS(\hat{u},\hat{y})(u) \subset DY(\hat{u},\hat{y})(u)$. Suppose that $y \notin \text{WMin}_K DY(\hat{u},\hat{y})(u)$. Then there exists a $\bar{y} \in DY(\hat{u},\hat{y})(u)$ such that $y - \bar{y} \in \text{int}K$. Since $\bar{y} \in DY(\hat{u},\hat{y})(u)$, there exist sequences $\{\bar{h}_k\} \subset \text{int}R_+$, $\{\bar{u}^k\} \subset R^m$ and $\{\bar{y}^k\} \subset R^p$ such that $\bar{h}_k \to 0$, $\bar{u}^k \to u$, $\bar{y}^k \to \bar{y}$ and

$$\hat{y} + \bar{h}_k\bar{y}^k \in Y(\hat{u} + \bar{h}_k\bar{u}^k), \quad \text{for all } k. \tag{1}$$

On the other hand, since $y \in DS(\hat{u},\hat{y})(u)$, there exist sequences $\{h_k\} \subset \text{int}R_+$, $\{u^k\} \subset R^m$ and $\{y^k\} \subset R^p$, such that $h_k \to 0$, $u^k \to u$, $y^k \to y$ and

$$\hat{y} + h_k y^k \in S(\hat{u} + h_k u^k), \quad \text{for all } k. \tag{2}$$

Since $h_k \to 0$, we may assume that $h_k \leqq \bar{h}_k$ by taking a subsequence if necessary. Since $\hat{y} + h_k y^k \in S(\hat{u} + h_k u^k)$, $(\hat{u} + h_k u^k, \hat{y} + h_k y^k)$ is a boundary point of convex set graph$(Y + K)$. Hence there exist a nonzero vector $(\lambda^k, \mu^k) \in R^m \times R^p$ such that

$$< \lambda^k, \hat{u} + h_k u^k > + < \mu^k, \hat{y} + h_k y^k > \geqq < \lambda^k, u' > + < \mu^k, y' >, \tag{3}$$

for all $(u', y') \in$ graph$(Y+K)$, where $< \cdot, \cdot >$ denotes the inner product. Since we may normalize these vectors so that $\| (\lambda^k, \mu^k) \| = 1$, we may assume that $\{(\lambda^k, \mu^k)\}$ converges to a nonzero vector $(\lambda, \mu) \in R^m \times R^p$. By taking the limit of (3) as $k \to \infty$, we see that

$$< \lambda, \hat{u} > + < \mu, \hat{y} > \geqq < \lambda, u' > + < \mu, y' >, \text{ for all } (u', y') \in \text{graph}(Y + K). \tag{4}$$

Since $\hat{u} \in$ int(dom Y), $\mu \neq 0$. Hence $< \mu, y'' > \leqq 0$ for all $y'' \in K$ and so $\mu \in K^o$, where $K^o = \{\mu \in R^p : < \mu, d > \leqq 0, \text{ for all } d \in K\}$. Since $y - \bar{y} \in$ int$K$,

$$< \mu, y > < < \mu, \bar{y} > . \tag{5}$$

Since $(\hat{u} + h_k u^k, \hat{y} + h_k y^k) \in$ graph$Y$ and $(\hat{u} + \bar{h}_k \bar{u}^k, \hat{y} + \bar{h}_k \bar{y}^k) \in$ graph$Y$, from locally K-convexity of Y at $\hat{u}$,

$$(\hat{u} + \frac{(h_k)^2}{h_k + \bar{h}_k} u^k + \frac{(\bar{h}_k)^2}{h_k + \bar{h}_k} \bar{u}^k, \hat{y} + \frac{(h_k)^2}{h_k + \bar{h}_k} y^k + \frac{(\bar{h}_k)^2}{h_k + \bar{h}_k} \bar{y}^k) \in \text{graph}(Y + K).$$

From (3), we have

$$\begin{aligned} < \lambda^k, h_k u^k > + < \mu^k, h_k y^k > \geqq & < \lambda^k, \frac{(h_k)^2}{h_k + \bar{h}_k} u^k + \frac{(\bar{h}_k)^2}{h_k + \bar{h}_k} \bar{u}^k > + \\ & < \mu^k \frac{(h_k)^2}{h_k + \bar{h}_k} y^k + \frac{(\bar{h}_k)^2}{h_k + \bar{h}_k} \bar{y}^k > . \end{aligned}$$

Hence

$$< \lambda^k, h_k \bar{h}_k u^k > + < \mu^k, h_k \bar{h}_k y^k > \geqq < \lambda^k, \bar{h}_k^2 \bar{u}^k > + < \mu^k, \bar{h}_k^2 \bar{y}^k >$$

and so

$$h_k(< \lambda^k, u^k > + < \mu^k, y^k >) \geqq \bar{h}_k(< \lambda^k, \bar{u}^k > + < \mu^k, \bar{y}^k >). \tag{6}$$

On the other hand, since $(\hat{u}, \hat{y}) \in$ graph$(Y + K)$, we obtain, from (3),

$$< \lambda^k, u^k > + < \mu^k, y^k > \geqq 0.$$

Since $\bar{h}_k - h_k \geqq 0$, we have

$$(\bar{h}_k - h_k)(< \lambda^k, u^k > + < \mu^k, y^k >) \geqq 0. \tag{7}$$

Hence, from (6) and (7),

$$< \lambda^k, u^k > + < \mu^k, y^k > \geqq < \lambda^k, \bar{u}^k > + < \mu^k, \bar{y}^k > .$$

By taking the limit as $k \to \infty$, we have

$$< \mu, y > \geqq < \mu, \bar{y} >,$$

which contradicts (5). Therefore, we obtain

$$y \in \text{WMin}_K DY(\hat{u}, \hat{y})(u).$$

(2) Since Y is a locally K-convex at $\hat{u}$ and $Y(u)$ is $\tilde{K}$-dominated by $S(u)$ near $\hat{u}$, S is also locally K-convex at $\hat{u}$. Hence, from the similar way of the proof of Theorem 3.2, we obtain

$$\text{WMin}_K DY(\hat{u}, \hat{y})(u) \subset DS(\hat{u}, \hat{y})(u).$$

Hence, we have

$$\mathrm{WMin}_{K} DY(\hat{u},\hat{y})(u) = DS(\hat{u},\hat{y})(u)$$

by (1) of this theorem. □

The following example shows that the $\tilde{K}$-dominated by $S(u)$ of $Y(u)$ near $\hat{u}$ is essential for Theorem 3.3 (2).

Example 3.2. Let $Y : R \rightrightarrows R^2$, $\tilde{K} = \mathrm{int}R^2_+ \cup \{0\}$ and Y be defined by

$$Y(u) = \{y \in R^2 : y_1 \geqq 0,\ y_2 \geqq 0\} \cup \{y \in R^2 : y_2 > (y_1)^2\}, \text{ for } u \in R.$$

Then

$$S(u) = \{y \in R^2 : y_1 \geq 0,\ y_2 = 0\}, \text{ for } u \in R.$$

Hence $Y(u)$ is not $\tilde{K}$-dominated by $S(u)$ near $\hat{u} = 0$. Let $\hat{u} = 0$ and $\hat{y} = (0,0)$. Then

$$DY(\hat{u},\hat{y})(u) = \{y \in R^2 : y_2 \geqq 0\}, \text{ for } u \in R,$$

$$\mathrm{WMin}_K DY(\hat{u},\hat{y})(u) = \{y \in R^2 : y_2 = 0\}, \text{ for } u \in R,$$

$$DS(\hat{u},\hat{y})(u) = \{y \in R^2 : y_1 \geqq 0,\ y_2 = 0\}, \text{ for } u \in R.$$

Thus,

$$\mathrm{WMin}_K DY(\hat{u},\hat{y})(u) \not\subset DS(\hat{u},\hat{y})(u), \text{ for } u \in R.$$

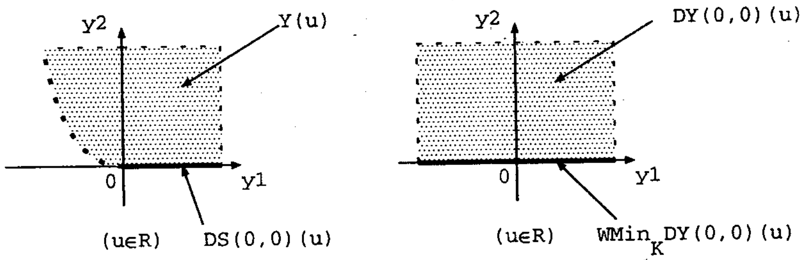

Figure 2: Example 3.2.

Finally, we obtain the following theorem.

Theorem 3.4. Let Y be a locally K-convex at $\hat{u}$ and let $\hat{y} \in G(\hat{u})$. If $Y(u)$ is $\tilde{K}$-dominated by $W(u)$ near $\hat{u}$, then for any $u \in R^m$,

$$DW(\hat{u},\hat{y})(u) = DS(\hat{u},\hat{y})(u) = \mathrm{WMin}_K DY(\hat{u},\hat{y})(u).$$

Proof. From Theorem 5.2 in Shi [8], we have, for any $u \in R^m$,

$$DW(\hat{u},\hat{y})(u) = \mathrm{WMin}_K DY(\hat{u},\hat{y})(u).$$

If $Y(u)$ is $\tilde{K}$-dominated by $W(u)$ near $\hat{u}$, then $Y(u)$ is also $\tilde{K}$-dominated by $S(u)$ near $\hat{u}$. Hence, from Theorem 3.3 (2), we obtain the result of theorem. □

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