

Stability and Stabilization of Piecewise Affine and Hybrid Systems: An LMI Approach

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Abstract

In this paper we present various algorithms both for stability analysis and state-feedback design for discrete-time piecewise affine systems. As in [13], our approach hinges on the use of piecewise quadratic Lyapunov functions that can be computed as the solution of a set of linear matrix inequalities. We show that the continuity of the Lyapunov function is not required in the discrete-time case. Moreover, the basic algorithms are made less conservative by exploiting the switching structure of piecewise affine systems and by using relaxation procedures.

1 Introduction

Hybrid systems have recently attracted the interest of the control and computer science communities, due to their ability to model the interaction between continuous dynamics and logic components. The analysis and control methodologies for hybrid systems strongly depend on the modeling framework adopted. It is known that a rich class of hybrid models can be described via Piecewise Affine (PWA) systems that are defined by partitioning the state-space in a finite number of polyhedral regions and associating to each region a different linear dynamic model. Sontag [17] showed that PWA systems and interconnections of linear systems and finite automata are equivalent. In [1] it is proved, by using a constructive argument, that the class of PWA and discrete-time hybrid systems in the Mixed Logic Dynamical (MLD) form [3] coincide. The expressivity of the MLD form is wide enough to represent a large class of hybrid systems, for instance linear hybrid dynamical systems, hybrid automata, some classes of discrete-event systems, and systems with qualitative inputs/outputs [3, 1]. The equivalence theorem (that is recalled in Section 2) allows to explicitly describe all these models in the PWA form.

The MLD/PWA framework is also useful both in investigating structural properties of hybrid systems such as observability, controllability [1] and in designing Model Predictive Control schemes [3], state-estimation algorithms [9, 2] and verification procedures [4, 5].

The first aim of this paper is to develop tests for checking the stability of the origin of a PWA system. The main motivation for this research is that there is no easy way to check the stability even of an autonomous PWA systems with two polyhedral regions. Indeed, Blondel and Tsitsiklis [6] showed that in general this problem is either \mathcal{NP} -complete or undecidable. Moreover, it is also hopeless to deduce the stability/instability of a PWA system from the stability/instability of its components subsystems [8]. These results highlight that, in order to derive practical stability tests, one must either resort to a restricted subclass of PWA system or look for *sufficient* conditions. A first algorithm to test semiglobal stability for discrete-time PWA systems was proposed in [5] by exploiting verification and robust simulation ideas. However these conditions apply only to PWA systems where the origin is strictly contained in one region. Nevertheless the method in [5] can be viewed as complementary to the tests proposed in this paper.

In this work we were largely inspired by the research of Johansson and Rantzer [13] where the stability of a continuous-time PWA system was investigated by looking for piecewise quadratic Lyapunov functions computed by solving Linear Matrix Inequalities (LMIs). This approach obviously leads to sufficient conditions, i.e. if the LMIs have a solution, the origin of the system can be classified as asymptotically stable but nothing can be said if no solution is found. We show that similar LMI based procedures can be fruitfully exploited for discrete-time PWA systems both for stability analysis and state-feedback design. However there are two major differences between the continuous and the discrete-time case. First, in the former only continuous Lyapunov functions can be used whereas in the latter discontinuous Lyapunov functions are also allowed. This obviously increases the flexibility of the method. Second, in discrete-time, switching can also occur between non-adjacent regions and this fact must be properly handled in the analysis/synthesis algorithms.

We propose various stability tests and controller synthesis procedures that exhibit different degrees of flexibility. The more general procedures are also computationally more demanding. Contrary to the optimization based receding horizon control scheme for MLD systems proposed in [3], the controllers synthesized in this work have the structure of a state feedback in closed-form. In Section 3 we review the basic results in stability analysis for PWA systems by using a common quadratic Lyapunov function. In this case we also show how to design a piecewise linear state-feedback. We generalize these results in Section 4 by considering piecewise quadratic Lyapunov functions. Here, the switching between non-adjacent regions can increase the complexity of the algorithms considerably and we discuss how to partially overcome this problem by avoiding to consider switches the system cannot exhibit. In Section 5 we propose a relaxation strategy for the LMIs involved in the stability analysis that leads to the most powerful test we propose. This relaxation is inspired by the works [13, 11]. Finally, we discuss the degree of conservativeness of the analysis algorithms proposed (Section 6) and provide an applicative example (Section 7).

2 The PWA form of Hybrid Systems

The state-space equations describing an MLD system are

$$x_{k+1} = Fx_k + G_1u_k + G_2\delta_k + G_3z_k \quad (1a)$$

$$y_k = Hx_k + D_1u_k + D_2\delta_k + D_3z_k \quad (1b)$$

$$V_2\delta_k + V_3z_k \leq V_1u_k + V_4x_k + V_5 \quad (1c)$$

where $x \in \mathbb{R}^{n_c} \times \{0, 1\}^{n_\ell}$ are the continuous and binary states, $u \in \mathbb{R}^{m_c} \times \{0, 1\}^{m_\ell}$ are the inputs, $y \in \mathbb{R}^{p_c} \times \{0, 1\}^{p_\ell}$ the outputs, and $\delta \in \{0, 1\}^{r_\ell}$, $z \in \mathbb{R}^{r_c}$ represent auxiliary binary and continuous variables respectively. The auxiliary variables are introduced when translating propositional logic into linear inequalities. For a detailed description of the modeling capabilities of MLD systems, we defer to [3, 1]. All constraints on state, input, and auxiliary variables are summarized in the inequality (1c) and we assume that the pair $(x, u) = (0, 0)$ satisfies (1c) for some values of δ and z . Note that, despite the fact that equations (1a)-(1b) are linear, nonlinearity is hidden in the integrality constraints over binary variables. We consider MLD systems that are *completely well-posed* [3], i.e. that for given $x(t)$ and $u(t)$, the values of $\delta(t)$ and $z(t)$ are uniquely defined through the inequalities (1c). This assumption is not restrictive and is always satisfied when real plants are described in the MLD form [3].

Recently, in [1], it was shown that MLD system can be represented in the PWA form and vice-versa. PWA systems are described by the state-space equations:

$$\begin{aligned} x_{k+1} &= A_i x_k + B_i u_k + a_i \\ y_k &= C_i x_k + c_i \\ (x, u) &\in \mathbb{X} \end{aligned}, \text{ for } \begin{bmatrix} x_k \\ u_k \end{bmatrix} \in \mathcal{X}_i \quad (2)$$

where the state+input set $\mathbb{X} \subset \mathbb{R}^{n_c} \times \{0, 1\}^{n_\ell} \times \mathbb{R}^{m_c} \times \{0, 1\}^{m_\ell}$ is either \mathbb{R}^n or a polyhedron containing the origin, $\{\mathcal{X}_i\}_{i=1}^s$ is a polyhedral partition¹ of \mathbb{X} and a_i, c_i are constant vectors of suitable dimension. We refer to each \mathcal{X}_i as a *cell*. In [3] it is shown that PWA systems can be represented in the MLD form. The converse is stated in the next proposition.

Proposition 1 ([1]). *Consider generic trajectories x_k, u_k, y_k of the MLD system (1). Then, there exist a polyhedral partition $\{\mathcal{X}_i\}_{i=1}^s$ of the state+input space \mathbb{X} and 5-tuples $(A_i, B_i, C_i, f_i, g_i)$, $i = 1, \dots, s$, such that x_k, u_k, y_k satisfy (2).*

¹Each set \mathcal{X}_i is a (not necessarily closed) convex polyhedron s.t. $\mathcal{X}_i \cap \mathcal{X}_j = \emptyset, \forall i \neq j, \bigcup_{i=1}^s \mathcal{X}_i = \mathbb{X}$.

Proposition 1 was proved in [1] by using a constructive argument that allows the explicit computations of the sets \mathcal{X}_i and the matrices A_i, B_i, C_i, a_i and c_i defining the PWA system.

According to the notation in [12], we call $\mathcal{I} = \{1, \dots, m\}$ the set of indices of the state space cells. \mathcal{I} is partitioned as $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1$, where \mathcal{I}_0 are the indices of the cells whose boundary contains the origin $x = 0$, and \mathcal{I}_1 are the indices of the cells, whose boundary that does not contain the origin.

In this paper we will focus on the stability of the origin. Then, we assume that $x = 0$ is an equilibrium of system (1). By using Proposition 1, this implies that $a_i = 0, \forall i \in \mathcal{I}_0$ in the PWA form (2). Moreover, to analyze stability, we restrict our attention to *autonomous* PWA systems, and, for the regulator design, we will consider a *piecewise linear* state feedback

$$u_k = K_i x_k \quad \forall x_k \in \mathcal{X}_i. \quad (3)$$

Definition 1. *The equilibrium state $x = 0$ of a system $x_{k+1} = f(x_k)$ is **stable** if, for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$, such that $\|x_0\| < \delta(\epsilon) \Rightarrow \|x_k\| < \epsilon, \forall k > 0$.*

*If, in addition, $\lim_{k \rightarrow +\infty} \|x_k\| = 0$ the origin is **asymptotically stable**.*

*Finally, let $\mathbb{X}_0 \subseteq \mathbb{X}$ such that $0 \in \mathbb{X}_0$. The origin $x = 0$ is **asymptotically stable on \mathbb{X}_0** if it is asymptotically stable for any initial state $x_0 \in \mathbb{X}_0$.*

3 Quadratic Lyapunov Function

3.1 Stability Analysis

Based on Lyapunov theory arguments, the asymptotic stability of a PWA discrete-time system can be checked with a quadratic Lyapunov function

$$V(x) = x^T P x \quad (4)$$

In [15] it is recalled that sufficient conditions on P for asymptotic stability are given by the LMIs:

$$P > 0 \quad (5)$$

$$A_i^T P A_i - P < 0 \quad \forall i \in \mathcal{I} \quad (6)$$

If such a P exists, the function (4) is called a common Lyapunov function for the matrices $\{A_1, \dots, A_s\}$ and system (2) is termed *quadratically stable* (Q-stable).

In [16] it is shown, that a sufficient condition for the existence of a quadratic Lyapunov function satisfying (5) and (6), is that all the matrices $A_i, i \in \mathcal{I}$ commute pairwise.

3.2 State Feedback Synthesis

We consider the synthesis of a piecewise linear state feedback (3) for system (2) that stabilizes the origin by means of a quadratic Lyapunov function (4). In other words we look for the matrices P and $K_i, \forall i \in \mathcal{I}$ that satisfy the conditions

$$P > 0 \quad (7)$$

$$(A_i + B_i K_i)^T P (A_i + B_i K_i) - P < 0 \quad \forall i \in \mathcal{I} \quad (8)$$

Since both variables P and K_i are unknown, the matrix inequality (8) is apparently nonlinear. However, as for LTI discrete-time systems, it can be rewritten as an LMI, as outlined in the following. The next Lemma is a standard application of Schur complements [7].

Lemma 1. *The following LMIs are equivalent:*

•

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} > 0 \quad (9)$$

•

$$R > 0 \quad (10)$$

$$Q - S R^{-1} S^T > 0 \quad (11)$$

$$Q > 0 \quad (12)$$

$$R - S^T Q^{-1} S > 0 \quad (13)$$

$$\begin{pmatrix} R & S^T \\ S & Q \end{pmatrix} > 0 \quad (14)$$

By using the equivalence between (11) and (13), inequality (6) can be rewritten as:

$$P - A_i^T P A_i > 0 \Leftrightarrow P^{-1} - A_i P^{-1} A_i^T > 0 \quad (15)$$

Instead of considering inequalities (7) and (8) for asymptotic stability we can consider the conditions

$$Q - (A_i + B_i K_i) Q (A_i + B_i K_i)^T > 0 \quad (16)$$

$$Q > 0 \quad (17)$$

with $Q = P^{-1}$. Since Q is symmetric and positive definite, these (nonlinear) matrix inequalities are equivalent to

$$\begin{pmatrix} Q & (A_i Q + B_i K_i Q) \\ (A_i Q + B_i K_i Q)^T & Q \end{pmatrix} > 0 \quad (18)$$

where we used the equivalence between (11) and (9). Using again the fact that Q is positive definite, we can introduce new variables Y_i as

$$Y_i = K_i Q \quad (19)$$

and solve for Y_i instead of K_i . The resulting LMI is

$$\begin{pmatrix} Q & (A_i Q + B_i Y_i) \\ (A_i Q + B_i Y_i)^T & Q \end{pmatrix} > 0, \quad (20)$$

and the state feedback vector can be recovered as

$$K_i = Y_i Q^{-1} \quad (21)$$

To what concerns the region of attraction of the state-feedback derived so far, let $\bar{V}_1(x) = x^T P x$ and choose \mathbb{X}_0 as the largest level set $\{x \in \mathbb{X} : \bar{V}_1(x) \leq \xi, \xi > 0\}$ contained in \mathbb{X} . Then every state-trajectory of the controlled system starting from \mathbb{X}_0 , besides converging to the origin, does not leave the set \mathbb{X} . In summary, we have proved the following result

Lemma 2. *Let Q and Y_i be the solutions of the LMIs (17), (20) and K_i be computed as in (21). Then, the piecewise linear state feedback $u_k = K_i x_k, \forall x_k \in \mathcal{X}_i$ stabilizes asymptotically the origin of (2) on \mathbb{X}_0 .*

3.3 Bounded Inputs

The synthesis of the state feedback controller did not take into account any constraints on the control action so far. If u is constrained, as in (2), the values of K_i can possibly result in input values violating the bounds present in (2). According to [7, p. 103] we can add further LMI's that take into account these constraints. Assuming that the initial condition $x(0)$ satisfies $x(0)^T Q^{-1} x(0) \leq 1$, where Q is the same unknown appearing in Lemma 2, we can enforce the constraint

$$\max_{k \geq 0} \|u_k\| \leq \mu \quad (22)$$

with the LMIs

$$\begin{pmatrix} 1 & x(0)^T \\ x(0) & Q \end{pmatrix} \geq 0 \quad (23)$$

$$\begin{pmatrix} Q & Y_i^T \\ Y_i & \mu^2 I \end{pmatrix} \geq 0 \quad \forall i \in \mathcal{I} \quad (24)$$

Note that LMI (24) is more restrictive than condition (22), since it also holds for those states that are outside the region, where the i -th controller is active. The matrix Q is chosen such that the set $\{x : x^T Q^{-1} x \leq 1\}$ is invariant.

4 Piecewise Quadratic Lyapunov Function

4.1 Stability analysis

A Piecewise Quadratic (PWQ) Lyapunov function for a PWA system can be defined as

$$V(x) = x^T P_i x \quad \forall x \in \mathcal{X}_i \quad (25)$$

As in [13] we consider PWQ Lyapunov functions, but differently from the continuous-time case the function $V(x)$ can be discontinuous across cell boundaries. Indeed, we do not have to require continuity of $V(x)$ to prove stability, as long as the number of cells is finite, see appendix A.1. For stability, it has to hold that $V(x)$ is positive-definite in a neighborhood of the origin and that

$$\Delta V(x_k, x_{k-1}) = V(x_k) - V(x_{k-1}) < 0 \quad (26)$$

Assuming that $x_k \in X_i$ and $x_{k-1} \in X_j$, we have

$$\begin{aligned} \Delta V(x_k, x_{k-1}) &= x_k^T P_i x_k - x_{k-1}^T P_j x_{k-1} \\ &= x_{k-1}^T (A_j^T P_i A_j - P_j) x_{k-1} \end{aligned} \quad (27)$$

The LMIs to satisfy in this case are

$$\begin{aligned} A_j^T P_i A_j - P_j &< 0 \quad \forall (j, i) \in \mathcal{S}_{all} \\ P_i &> 0, \quad \forall i \in \mathcal{I} \end{aligned} \quad (28)$$

$$(29)$$

where $\mathcal{S}_{all} = \mathcal{I} \times \mathcal{I}$. In appendix A.2 we show, that these conditions are also sufficient for asymptotic stability on \mathbb{X} . When the LMIs (28)-(29) are feasible, we term the PWA systems *piecewise quadratically (PWQ) stable*. Now, the main difficulty, compared to the continuous-time case, is that we have to satisfy the LMIs (28) for all the pairs (i, j) because in principle the state may switch in one step between non adjacent cells. Without further analysis of the system we have to take into account all possible switches from each state space region to each other region. Therefore the number of possible switches grows quadratically with the number of cells. However, this condition can be relaxed because usually not all the transitions between cells \mathcal{X}_i and \mathcal{X}_j are allowed. Let \mathcal{S} be the set of all ordered pairs (j, i) of indices, denoting the possible switches from cell j to cell i :

$$\mathcal{S} = \{(j, i) : j, i \in \mathcal{I} \text{ and } \exists k \in \mathbb{N}_0, \text{ such that } x_k \in \mathcal{X}_i \text{ and } x_{k-1} \in \mathcal{X}_j\} \quad (30)$$

The set \mathcal{S} can be determined via reachability analysis for MLD systems [1, 4]. Since for asymptotic stability it is enough that (28)-(29) holds for $(j, i) \in \mathcal{S}$, we have the following result.

Theorem 1. *The origin of the PWA (2) is asymptotically stable on \mathbb{X} if there exist s matrices P_i , such that the following LMIs are satisfied:*

$$P_i > 0 \quad \forall i \in \mathcal{I} \quad (31)$$

$$A_j^T P_i A_j - P_j < 0 \quad \forall (j, i) \in \mathcal{S} \quad (32)$$

In this case the PWA system is termed \mathcal{S} -PWQ stable. Note that in \mathcal{S} there are in general also pairs of the form (i, i) for each cell that is not left in one step. For these cells Equation (32) states, that A_i must be stable. This means, that we cannot show stability of a stable piecewise linear system, whose components are unstable, with a piecewise quadratic Lyapunov function (25), if the system stays in the same unstable cell for more than one time step.

There are also PWA systems where the set \mathcal{S} contains both pairs of indices (i, j) and (j, i) , meaning for instance that the system trajectories point to the boundary of a state space cell, allowing the system to change from cell i to the cell j as well as from cell j to cell i . We define \mathcal{S}_2 as the set of such indices:

$$\mathcal{S}_2 = \{(i, j) \in \mathcal{S} : (j, i) \in \mathcal{S} \text{ and } i \neq j\} \quad (33)$$

We can state a necessary condition for the existence of a PWQ Lyapunov function (25) for systems having a nonempty set \mathcal{S}_2 .

Lemma 3. *Assume that for a PWA system (2), the set \mathcal{S}_2 is nonempty and contains $2N$ elements. If a PWQ Lyapunov function (25) satisfying (31)-(32) exists, then the $2N$ matrices $A_{i_1} A_{j_1}, A_{j_1} A_{i_1}, \dots, A_{i_N} A_{j_N}, A_{j_N} A_{i_N}$ must have all eigenvalues inside the unit circle.*

Proof. Given a system where boundary crossings can occur in both directions for cell i and j , the existence of a PWQ Lyapunov function implies the fulfillment of the LMI's

$$\begin{aligned} A_j^T P_i A_j - P_j &< 0 \\ A_i^T P_j A_i - P_i &< 0 \end{aligned}$$

Therefore,

$$P_j > A_j^T P_i A_j \geq A_j^T A_i^T P_j A_i A_j$$

where we made use of the fact, that for matrices of appropriate dimensions, it holds that

$$C > D \Rightarrow B^T C B \geq B^T D B$$

Analogously,

$$P_i > A_i^T A_j^T P_i A_j A_i$$

This means that both $A_i A_j$ and $A_j A_i$ must be stable matrices. These considerations apply to each pair $(i, j) \in S_2$ separately. Therefore the statement of the Lemma follows. \square

Note that for square matrices A, B of equal dimensions it holds that AB and BA have the same eigenvalues, see eg. [14]. To verify the nonexistence of a piecewise quadratic Lyapunov function with Lemma 3, we therefore only have to check the eigenvalues of at most N matrices.

We can generalize the result of Lemma 3 to systems that can cycle across an arbitrary number of cells:

$$j_1 \rightarrow j_2 \rightarrow j_3 \rightarrow \dots \rightarrow j_n \rightarrow j_1 \quad (34)$$

In this case the existence of a PWQ Lyapunov function implies the stability of all products of matrices denoting the cells, the system can cycle through. Define the set of ordered n -tuples of indices

$$\mathcal{S}_n = \{(j_1, \dots, j_n) \in \mathcal{I}^n : \{(j_1, j_2), \dots, (j_{n-1}, j_n), (j_n, j_1)\} \subset \mathcal{S} \text{ and } j_i \text{ all different}\} \quad (35)$$

and define the set of products of matrices

$$\tilde{A} = \cup_{\ell=2}^{\infty} \{A : A = \prod_{\tau=1}^{\ell} A_{i_{\tau}}, (i_1, \dots, i_{\ell}) \in \mathcal{S}_{\ell}\} \quad (36)$$

We have the following corollary:

Corollary 1. *Assume that for a PWA system (2), the sets \mathcal{S}_n ($n = 2 \dots \infty$) are not all empty. If a PWQ Lyapunov function (25) exists, then all matrices in \tilde{A} must have all eigenvalues inside the unit circle.*

4.2 State Feedback Synthesis

The same rationale used in section 3.2 applies also to the case of a piecewise quadratic function. We can formally exploit the equivalence between equation (13) and (14) as well as the equivalence between equation (9) and (11), yielding

$$P_j - (A_j + B_j K_j)^T P_i (A_j + B_j K_j) > 0 \quad (37)$$

$$\Leftrightarrow P_i^{-1} - (A_j + B_j K_j) P_j^{-1} (A_j + B_j K_j)^T > 0 \quad (38)$$

$$\Leftrightarrow P_i^{-1} - (A_j + B_j K_j) P_j^{-1} P_j P_j^{-1} (A_j + B_j K_j)^T > 0 \quad (39)$$

$$\Leftrightarrow P_i^{-1} - (A_j P_j^{-1} + B_j K_j P_j^{-1}) P_j (A_j P_j^{-1} + B_j K_j P_j^{-1})^T > 0 \quad (40)$$

$$\Leftrightarrow \begin{pmatrix} P_i^{-1} & (A_j P_j^{-1} + B_j K_j P_j^{-1}) \\ (A_j P_j^{-1} + B_j K_j P_j^{-1})^T & P_j^{-1} \end{pmatrix} > 0 \quad (41)$$

To obtain an LMI, we substitute

$$W_j = K_j P_j^{-1} \quad \text{or} \quad K_j = W_j P_j \quad (42)$$

which gives

$$\begin{pmatrix} P_i^{-1} & (A_j P_j^{-1} + B_j W_j) \\ (A_j P_j^{-1} + B_j W_j)^T & P_j^{-1} \end{pmatrix} > 0 \quad (\forall (j, i) \in \bar{\mathcal{S}}), \quad (43)$$

where the set $\bar{\mathcal{S}}$ will be defined next. We still have to impose the condition

$$Q_i = P_i^{-1} > 0, \quad \forall i \in \mathcal{I} \quad (44)$$

The LMIs (43), (44) provide the synthesis of a piecewise linear controller that stabilizes the origin, as can be shown with the PWQ Lyapunov function (25).

The LMI (43) must be fulfilled again for all possible pairs (j, i) corresponding to regions, the system can switch to in one step. However, the switching must be predicted in *closed loop*. Lacking of better knowledge we can always choose $\bar{\mathcal{S}} = \mathcal{S}_{all} = \mathcal{I} \times \mathcal{I}$. This takes into account all conceivable pairs of cells in a system with s cells. Such a brute force approach could however introduce an unnecessarily high level of conservativeness. For a particular system at hand, there are some considerations that allow to take $\bar{\mathcal{S}}$ as a smaller set than \mathcal{S}_{all} . Take for instance a system controlled on the high level by a finite state machine. If the finite state machine does not allow arbitrary state transitions, the corresponding PWA system will have a set $\bar{\mathcal{S}}$, which is smaller than \mathcal{S}_{all} . In some cases one can also know, from practical insights, that $\bar{\mathcal{S}} = \mathcal{S}$.

To what concerns the region of attraction of the controlled system, let $\bar{V}_2(x) = x^T P_i x$, $\forall x \in \mathcal{X}_i$ and choose \mathbb{X}_0 as the largest level set $\{x \in \mathbb{X} : \bar{V}_2(x) \leq \xi, \xi > 0\}$ contained in \mathbb{X} . Then, even if \bar{V}_2 is only piecewise continuous, it is easy to prove that every state-trajectory of the controlled system starting from \mathbb{X}_0 , besides converging to the origin, does not leave the set \mathbb{X} . This is due to the fact that the number of regions \mathcal{X}_i is finite. The results derived so far are summarized in the following Lemma:

Lemma 4. *If $\bar{\mathcal{S}}$ contains all possible pairs of indices corresponding to the cells the closed-loop PWA system can switch in one step, a stable piecewise linear state feedback that stabilizes asymptotically the origin on \mathbb{X}_0 can be found solving the LMIs (43) and (44) for Q and Y_i . K_i is then given by equation (42).*

5 Piecewise Quadratic Lyapunov Function with Relaxations

5.1 Stability Analysis

In this Section we present another method to relax the conservativeness of the stability analysis via PWQ Lyapunov functions. These ideas have been proposed for continuous time systems in [13] and [11]. Here, we consider some relaxations on the LMIs denoting the negative or positive definiteness of the matrices of interest. Since the state space is partitioned into polyhedral cells, we can find matrices E_i and e_i such that

$$\mathcal{X}_i = \{x : \bar{E}_i \begin{pmatrix} x \\ 1 \end{pmatrix} = [E_i \ e_i] \begin{pmatrix} x \\ 1 \end{pmatrix} \geq 0\} \quad (45)$$

The inequality sign in (45) means that each entry of the vector on the left hand side has to be positive. Following the rationale of [12] we note that the LMIs (31) and (32) are actually valid on the whole state space, even though they would only be required to hold in the regions i and j, i respectively. We can remove some conservativeness if we can find matrices f_i and $g_{j,i}$ such that

$$x^T f_i x \geq 0 \quad \text{if } x \in \mathcal{X}_i \quad (46)$$

$$x^T f_i x \leq 0 \quad \text{if } x \notin \mathcal{X}_i \quad (47)$$

$$x^T g_{i,j} x \geq 0 \quad \text{if } x \in \mathcal{X}_j \cup \mathcal{X}_i \quad (48)$$

$$x^T g_{i,j} x \leq 0 \quad \text{if } x \notin \mathcal{X}_j \cup \mathcal{X}_i \quad (49)$$

Note that we do not require the matrices f and g to be positive or negative definite, we only require that the quadratic forms take on the signs mentioned above in the corresponding regions. With these matrices the following relaxed conditions for stability can be formulated,

$$P_i - f_i > 0 \quad \forall i \in \mathcal{I} \quad (50)$$

$$A_j^T P_i A_j - P_j + g_{i,j} < 0 \quad \forall (j, i) \in \mathcal{S}. \quad (51)$$

From the assumptions on f and g it follows that the fulfillment of the LMIs (50) and (51) implies the fulfillment of (31) and (32) in the regions of relevance. On the other hand, for $x \notin \mathcal{X}_i \cup \mathcal{X}_j$, the expressions (47) and (49), may render the LMIs (50) and (51) easier to satisfy.

A possible choice for f is:

$$f_i = \bar{E}_i^T U_i \bar{E}_i, \quad (52)$$

where U_i has non-negative entries. The choice of g is more difficult however, because in general it cannot be defined in terms of the matrices \bar{E}_i , due to the switching between different regions. One possibility to circumvent all problems is:

$$g_{j,i} = \begin{cases} \bar{E}_i^T W_i \bar{E}_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad (53)$$

where W_i has non-negative entries. This avoids relaxing the LMIs (51), where two different indices for the regions i and j appear. By summarizing the previous discussion we have the following result.

Lemma 5. *If the LMIs*

$$P_i - \bar{E}_i^T U_i \bar{E}_i > 0 \quad \forall i \in \mathcal{I} \quad (54)$$

$$A_j^T P_i A_j - P_j < 0 \quad \forall (j, i) \in \mathcal{S}, i \neq j \quad (55)$$

$$A_i^T P_i A_i - P_i + \bar{E}_i^T W_i \bar{E}_i < 0 \quad \forall (i, i) \in \mathcal{S}. \quad (56)$$

admit a solution in P , U_i and W_i , the origin of (2) is asymptotically stable on \mathbb{X} and the PWA system is termed \mathcal{S} -PWQ stable with relaxations.

To prove this result it is enough to note that the LMIs denoting the decrease from region j to region i are not relaxed. Therefore, Lemma 5 is a consequence of Theorem 1.

Remark 1 As in [13] one can impose some structure to the matrices U_i and W_i , for instance by taking them diagonal or lower triangular. This decreases the flexibility of the relaxation procedure but also reduces the number of unknowns in the LMIs (54)-(56).

6 Conservativeness of the Various Stability Analysis Algorithms

The stability criteria presented so far exhibit different levels of conservativeness and complexity in their applications. This is summarized in figure 1. In general, the higher the degree of conservativeness of the Lyapunov function the lower are the computational requirements to find the corresponding Lyapunov function. The classification in figure 1 has been obtained by considering several examples of piecewise linear systems with two states and cell partitioning crossing the origin. For each field of the diagram we were able to find a system, therefore none of the categories in figure 1 is empty. Some of the relations denoted in figure 1 are quite obvious, for instance it is clear that quadratic stability implies piecewise quadratic stability of the system. The relations become more complex if we consider relaxations and minimal switching sets \mathcal{S} . For instance systems can be found, that are not piecewise quadratically stable, but are both piecewise quadratically stable with relaxations and \mathcal{S} -piecewise quadratically stable.

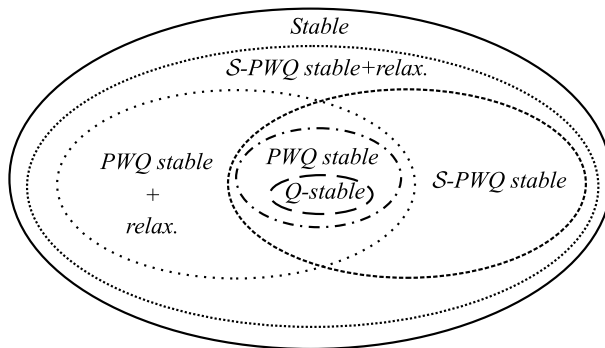


Figure 1: Conservativeness of the different stability analysis approaches.

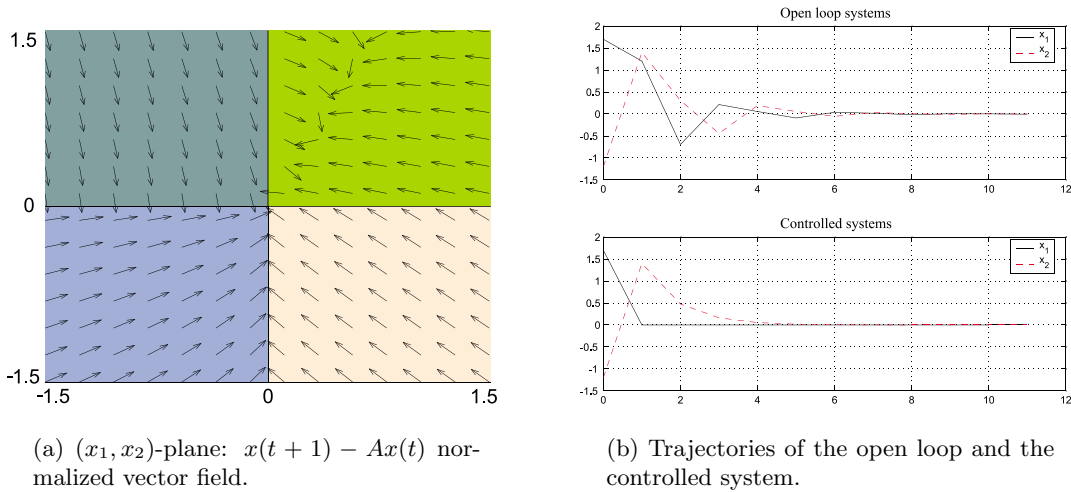


Figure 2: Open and closed loop behaviour of system (57).

7 An Example

To illustrate the analysis and synthesis methods described in the previous sections we consider the following PWA system:

$$x_{k+1} = \begin{cases} \begin{pmatrix} -0.04 & -0.461 \\ -0.139 & 0.341 \end{pmatrix} x_k + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_k & E_1 x_k \geq 0 \\ \begin{pmatrix} 0.936 & 0.323 \\ 0.788 & -0.049 \end{pmatrix} x_k + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_k & E_2 x_k \geq 0 \\ \begin{pmatrix} -0.857 & 0.815 \\ 0.491 & 0.62 \end{pmatrix} x_k + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_k & E_3 x_k \geq 0 \\ \begin{pmatrix} -0.022 & 0.644 \\ 0.758 & 0.271 \end{pmatrix} x_k + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_k & E_4 x_k \geq 0 \end{cases} \quad (57)$$

The partitioning corresponds to the four quadrants of the two dimensional $x_1 - x_2$ plane, i.e.

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad E_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad E_4 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

For this system the stability analysis gave conclusive results only if we were looking for a piecewise quadratic Lyapunov function with minimal set \mathcal{S} and relaxations. All the more conservative tests did not give any results. The vector field of the system in the two-dimensional plane can be seen in figure 2-a. The system trajectories starting from some nonzero initial condition are in figure 2-b.

The controller synthesis succeeded in the search for a piecewise linear state feedback showing stability with a quadratic Lyapunov function. The closed loop matrices are given by the expressions:

$$A_{cli} = A_i + B_i K_i^T \quad (58)$$

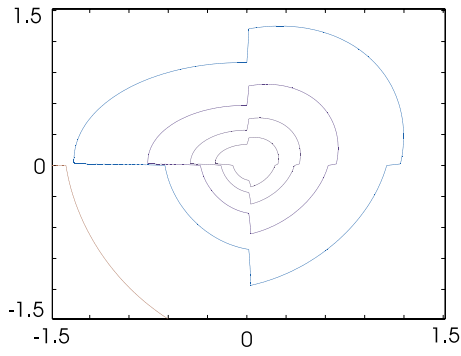
where the feedback gains are given by:

$$K_1 = \begin{pmatrix} 0.04 \\ 0.461 \end{pmatrix} \quad K_2 = \begin{pmatrix} -0.936 \\ -0.323 \end{pmatrix} \quad K_3 = \begin{pmatrix} 0.857 \\ -0.815 \end{pmatrix} \quad K_4 = \begin{pmatrix} 0.022 \\ -0.644 \end{pmatrix} \quad (59)$$

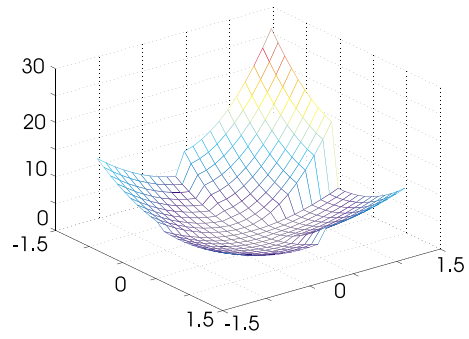
The contour-lines and a 3-D plot of the Lyapunov function can be seen in figure 3 whereas the state trajectories of the controlled system is depicted in figure 2-b.

8 Conclusions

In this paper we presented various algorithms for the stability analysis and the state-feedback design for discrete-time PWA systems. All these procedures hinge on the use of piecewise quadratic Lyapunov functions computed as the solution of a set of LMIs. Due to the LMI formulation, it is important to note that the synthesis criteria we presented can be combined with other design specifications (e.g. robustness requirements) opening the way to multi-objective control for hybrid systems.



(a) Contour Plot of the Lyapunov function.



(b) 3D plot of the Lyapunov Function.

Figure 3: Open loop Lyapunov function for system (57).

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A Discontinuous Lyapunov Functions

A.1 Stability

In this Appendix we show that the piecewise quadratic Lyapunov functions do not need to be continuous across state space cells in order to prove stability. We follow the reasoning presented in [10].

Definition 2. A scalar function $V(x)$ is **positive definite** in a neighborhood N of $x = 0$, if $V(0) = 0$, and if there exists a continuous, nondecreasing, scalar function w such that

$$w(0) = 0, \quad V(x) \geq w(\|x\|) > 0 \quad \forall x \in N \setminus \{0\} \quad (60)$$

Theorem 2. Assume that there exists a positive definite function $V(x_k)$ possessing a nonpositive forward difference $\Delta V(x_k, x_{k-1})$,

$$\Delta V(x_k, x_{k-1}) = V(x_k) - V(x_{k-1}), \quad \forall k \in \mathbb{N}$$

and such that $\forall \epsilon > 0$, exists $\delta(\epsilon) > 0$ that, for all $\|x_0\| < \delta(\epsilon)$, gives

$$\|x_0\| < \epsilon \quad (61)$$

$$V(x_0) < w(\epsilon). \quad (62)$$

Then, the equilibrium $x = 0$ is stable.

Proof. Since $\Delta V(x_k, x_{k-1}) \leq 0$, we have

$$V(x_0) \geq V(x_k) \geq w(\|x_k\|) \quad (63)$$

Inequality (63) holds because of (60). Therefore, $w(\epsilon) > V(x_0) \geq w(\|x_k\|)$, where the first inequality holds because of (62) and the second holds because of (63). But since w is nondecreasing, it follows that $\|x_k\| < \epsilon$ for all $k \geq 0$ and for all $\|x_0\| < \delta(\epsilon)$. \square

In [10] it was explicitly assumed, that $V(x)$ was continuous. This assumption entered the proof to guarantee the existence of a $\delta(\epsilon)$, given any ϵ , as in Theorem 2. We now show that we can find such a $\delta(\epsilon)$ even if we assume V to be piecewise quadratic, as in (25).

Theorem 3. Assume $|\mathcal{I}| < \infty$. The equilibrium $x = 0$ of (2) is stable if there exists a function $V(x)$ as in (25) possessing a nonpositive forward difference $\Delta V(x_k, x_{k-1})$.

$$\Delta V(x_k, x_{k-1}) = V(x_k) - V(x_{k-1}) = x_k^T P_i x_k - x_{k-1}^T P_j x_{k-1}, \quad x_{k-1} \in \mathcal{X}_j, x_k \in \mathcal{X}_i.$$

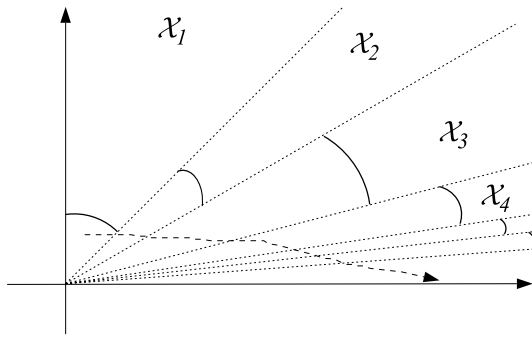


Figure 4: Infinite partition of the state space. The arcs are a level curve of the Lyapunov function. Even though the trajectory is moving to smaller and smaller levels of the Lyapunov function, the state diverges.

Proof. For the whole proof, the expressions min and max are taken over the set \mathcal{I} . Let $\lambda_{max}(P)$ and $\lambda_{min}(P)$ denote the largest and the smallest eigenvalue of P respectively. For each P_i choose $\bar{k}_i = \lambda_{max}(P_i)$ and $\underline{k}_i = \lambda_{min}(P_i)$. This implies $V(x_k) \leq \max_i \{\bar{k}_i\} \|x_k\|^2$.

Choosing $w(x) = \frac{\min_i \{\underline{k}_i\}}{2} \|x\|^2$, we have $V(x) \geq w(\|x\|)$ so showing that $V(x)$ is positive definite.

The next step is to prove that (61) and (62) hold. For each $\epsilon > 0$ we can choose $\delta = \sqrt{\frac{\min_i \{\underline{k}_i\}}{2 \max_i \{\bar{k}_i\}}} \epsilon$, so obtaining $\delta < \epsilon$.

Choosing further $\|x_0\| < \delta$ we have

$$V(x_0) \leq \max_i \{\bar{k}_i\} \delta^2 = \max_i \{\bar{k}_i\} \frac{\min_i \{\underline{k}_i\}}{2 \max_i \{\bar{k}_i\}} \epsilon^2 \leq \frac{\min_i \{\underline{k}_i\}}{2} \epsilon^2 = w(\epsilon)$$

Then, Theorem 2 can be applied to conclude the proof. \square

Note that we assumed that the state space partitioning is finite. Otherwise the expressions \bar{k}_i and \underline{k}_i in the proof might not be defined. This assumption allows us to exclude cases as depicted in figure 4.

A.2 Attractivity

In this Appendix we show that conditions (31) and (32) are actually sufficient for asymptotic stability on a suitable set \mathbb{X}_0 of the origin.

Theorem 4. *Assume that $|\mathcal{I}| < \infty$ and that there exists a function $V(x)$ as in (25) satisfying (31) and (32) (i.e. possessing a negative forward difference $\Delta V(x_k, x_{k-1})$). Let \mathbb{X}_0 be the largest level set $\{x \in \mathbb{X} : V(x) \leq \xi, \xi > 0\}$ contained in \mathbb{X} . Then, the equilibrium $x = 0$ of (2) is asymptotically stable on \mathbb{X}_0 .*

Proof. We first proof asymptotic stability of the origin. Since the assumptions of Theorem 3 are satisfied, from its proof we know that $\lim_{k \rightarrow +\infty} V(x_k) = V_L \geq 0$. We aim at showing that $V_L = 0$. We start looking for a continuous nondecreasing scalar function $r(\cdot)$ such that

$$\Delta V(x_k, x_{k-1}) \leq -r(\|x_{k-1}\|). \quad (64)$$

Let

$$\eta = \max_{(i,j) \in \mathcal{S}} \lambda_{max}(A'_j P_i A_j - P_j).$$

Then (64) is satisfied with $r(\|x_{k-1}\|) = \eta \|x_{k-1}\|^2$, because, from (32), all the matrices $A'_j P_i A_j - P_j$ are negative definite. Now, assume by contradiction that $V_L > 0$. This implies that there exists $\mu > 0$ such that $\|x_k\| > \mu, \forall k \in \mathbb{N}$. Then $\Delta V(x_k, x_{k-1}) \leq -r(\mu) < 0$. By writing $V(x_k)$ in terms of its forward difference, we obtain

$$V(x_k) = V(x_0) + \sum_{i=1}^k \Delta V(x_i, x_{i-1}) \leq V(x_0) - kr(\mu).$$

Since $V(x_k) \geq 0$, $V(x_0) - kr(\mu)$ must not become negative. The only way this can be satisfied for k arbitrarily large is to have $r(\mu) = 0$. Hence $\mu = 0$ and $\|x_k\| \rightarrow 0$ as $k \rightarrow +\infty$.

Finally, the asymptotic stability on \mathbb{X}_0 of the origin follows from standard arguments [10] relying on the fact that $V(\cdot)$ is radially unbounded (to see this it is enough to consider the lower bound $w(\cdot)$ as in the proof of Theorem 3). \square

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