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STABILITY AND UNIQUENESS OF POSITIVE SOLUTIONS FOR A SEMI-LINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

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Abstract. Positive solutions of the semilinear elliptic boundary value problem $-\Delta u(x) = g(x)f(u(x))$ on D, $Bu = 0$ on ∂D are studied where D is a bounded region and f is concave. It is proved that every positive non-constant solution is linearly stable and using fixed point index arguments results on existence and uniqueness of positive solutions are deduced. The results obtained are well-known in the case where g is positive on D; the proof presented in this paper applies also to the case, arising in population genetics, when g changes sign on D.

In this paper, we consider the stability, existence and uniqueness of positive solutions of the semi-linear elliptic boundary value problem

$$-\Delta u(x) = g(x)f(u(x)) \quad \text{in } D; \qquad B_\alpha u(x) = 0 \quad \text{on } \partial D \tag{1}$$

where D is a bounded region in $\mathbb{R}^n$ with smooth boundary, $g : D \to \mathbb{R}$ is a smooth function, and $B_\alpha u(x) = \alpha h(x)u(x) + (1-\alpha)\frac{\partial u}{\partial n}$ where $\alpha \in [0,1]$ is a constant and $h : \partial D \to \mathbb{R}^+$ is a smooth function with $h \equiv 1$ when $\alpha = 1$, i.e., the boundary condition may be of Dirichlet, Neumann or mixed type. We shall assume throughout that f satisfies

(f1) $f : I \to \mathbb{R}^+$ is a smooth function where $I = [0,r]$ or $[0,\infty)$, $f(0) = 0$ and $f''(u) < 0$ for all $u \in I$.

We say that u is a positive solution of (1) if u is a classical solution with $u(x) \in I$ for all $x \in \overline{D}$ and $u(x) > 0$ for all $x \in D$.

Our study of (1) is motivated by the fact that the equation arises in population genetics (see [5]) in which case the function g attains both positive and negative values on D. In the case when $g \equiv 1$ it is well known that (1) has at most one non-constant positive solution when f satisfies (f1) (see e.g., Cohen and Laetsch [4]) but may have multiple solutions when f is convex (see e.g. Amann [1]). At first sight it seems unlikely that a uniqueness theorem should hold for (1) under the assumption (f1) when g changes sign on D as $u \to g(x)f(u)$ is convex whenever

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$g(x) < 0$. However Tertikas has shown in [9] that when $n = 1$ or (1) possesses radial symmetry then (f1) ensures uniqueness of positive solutions even when g may change sign. His proof which is developed initially for problems on all of $\mathbb{R}$ or $\mathbb{R}^n$ uses a geometric shooting-type argument based on an integral identity satisfied by solutions of (1).

We shall give a very short and easy proof that every non-constant positive solution of (1) is stable whether or not g changes sign and then deduce various other results on the existence and uniqueness of solutions. Our stability result strengthens a result in Henry [6] where it is shown by applying the maximum principle to a certain parabolic equation that every non-constant positive solution of (1) with Neumann boundary conditions is non-degenerate.

Our results also complement the existence and uniqueness results obtained in [3] for (1) in the case on Neumann boundary conditions.

Our main theorem depends on the same algebraic strategy as that used in [6] and [9], viz, making use of an identity which does not contain the troublesome function g.

Theorem 1. *Suppose f satisfies $(f1)$. If u is a positive non-constant solution of* (1) *then the smallest eigenvalue of the linearized problem associated with* (1), *viz,*

$$-\Delta\psi - g(x)f'(u(x))\psi = \mu\psi \quad \textit{in } D; \qquad B_\alpha\psi = 0 \quad \textit{on } \partial D \tag{2}$$

is positive.

Proof: We prove the result only in the case of non-Dirichlet boundary conditions, i.e., $0 \le \alpha < 1$; the $\alpha = 1$ case is simpler.

It is well-konwn that a non-negative eigenfunction ϕ is associated with the smallest (principal) eigenvalue μ_1 of (2). Thus

$$-\Delta\phi - g(x)f'(u)\phi = \mu_1\phi \quad \text{in } D; \qquad B_\alpha\phi = 0 \quad \text{on } \partial D. \tag{3}$$

Multiplying (1) by $f'(u)\phi$ and (3) by $f(u)$, subtracting and integrating gives

$$\int_D [\Delta\phi f(u) - \Delta u f'(u)\phi]\, dx = -\mu_1 \int f(u)\phi\, dx.$$

Then using the boundary conditions and applying Green's theorem, we obtain

$$\frac{\alpha}{1-\alpha}\int_{\partial D} h\phi[uf'(u) - f(u)]\, dS + \int_D \phi|\nabla u|^2 f''(u)\, dx = -\mu_1 \int_D f(u)\phi\, dx. \tag{4}$$

Since f satisfies (f1), $uf'(u) - f(u) \le 0$ for $u > 0$ and so it follows from (4) that $\mu_1 > 0$. ∎

Theorem 1 shows that all non-constant positive solutions of (1) are non-degenerate and stable. This information is sufficient to give non-existence, existence and uniqueness results in a number of contexts given more information on f and g. We shall now assume also that

(f2) $I = [0, \infty)$ and $\lim_{u\to\infty} \frac{f(u)}{u} = 0$, i.e., f is sublinear, or

(f3) $I = [0, 1]$ and $f(1) = 0$, e.g., $f(u) = u(1-u)(\gamma(1-u) + (1-\gamma)u)$ as studied in [5] which also satisfies (f1) provided $\frac{1}{3} < \gamma < \frac{2}{3}$.

We prove our results by using the fixed point index in the framework of positive operators in ordered Banach spaces as described by Amann in [2]. We initially discuss only the case of Dirichlet boundary conditions; the other cases are similar but simpler technically.

First we must express (1) as an abstract equation in an appropriate function space. Let e denote the solution of

$$-\Delta u = 1 \quad \text{in } D; \qquad u\big|_{\partial D} = 0.$$

Let $E = C_e(\overline{D}) = \{u \in C(\overline{D}) :$ there exists a positive constant γ such that $-\gamma e \leq u \leq \gamma e\}$. Then E is a Banach space with norm $\|u\|_e = \inf\{\gamma > 0 : -\gamma e \leq u \leq \gamma e\}$. The ordering in E is given by the natural cone $P = \{u \in E : u \geq 0\}$.

Let $C > 0$ be chosen sufficiently large so that $u \to g(x)f(u) + Cu$ is strictly increasing for each x in D for all u in I. Then (1) is equivalent to the operator equation $u = KNu$ where K is the inverse of $-\Delta + C$ with Dirichlet boundary conditions and $N : E \to C(\overline{D})$ is defined by $Nu(x) = g(x)f(u(x)) + Cu(x)$. It follows from the maximum principle that $KN : E \to E$ is a compact strongly order-preserving map.

Let $i(KN, U, X)$ denote the fixed point index of KN over an open set $U \subset X$ where X is a retract of E and let $i(KN, u)$ denote the local index of KN at u where $u \in U$ is an isolated fixed point of KN such that a sufficiently small open ball in E centred at u is contained in U (see [2]). The value of $i(KN, u)$ can be determined from the eigenvalues of $KN'(u)$, the Frechet derivative of KN with respect to u.

Lemma 2. *The greatest eigenvalue of $KN'(u)$ is greater than, equal to or less than 1 according as the principal eigenvalue of (2) is negative, zero or positive.*

Proof: Suppose μ is the largest eigenvalue of $KN'(u)$. Since $KN'(u)$ is a positive operator, by the Krein-Rutman theorem there exists a positive eigenfunction ϕ corresponding to μ. Then

$$-\mu\Delta\phi + (\mu - 1)C\phi - g(x)f'(u)\phi = 0 \quad \text{in } D; \qquad \phi\big|_{\partial D} = 0.$$

Thus 0 is the principal eigenvalue of $L(\mu) = -\mu\Delta + (\mu - 1)C - g(x)f'(u)$. It follows from the variational characterisation of eigenvalues that the value of the principal eigenvalue of $L(\mu)$ is an increasing function of μ. But the differential expression in (2) is precisely $L(1)$ and so the result follows. ∎

We can now investigate the multiplicity of solutions of

$$-\Delta u - g(x)f(u) = 0 \quad \text{in } D; \qquad u = 0 \quad \text{on } \partial D. \tag{5}$$

Theorem 3. *Suppose f satisfies $(f1)$ and either $(f2)$ or $(f3)$.*

(i) *Suppose that the principal eigenvalue of $-\Delta - g(x)f'(0)$ with Dirichlet boundary conditions is positive, i.e., 0 is a stable solution of (5). Then (5) has no positive solution.*

(ii) *Suppose that the principal eigenvalue of $-\Delta - g(x)f'(0)$ is negative. Then (5) has exactly one positive solution.*

Proof: Suppose that there exists a positive supersolution $\overline{u} \in E$ of (5). Then

$$-\Delta\overline{u} + C\overline{u} \geq g(x)f(\overline{u}) + C\overline{u} \quad \text{in } D; \qquad \overline{u} = 0 \quad \text{on } \partial D.$$

Thus $KN\overline{u} \leq \overline{u}$. As $KN0 = 0$, $KN : [0,\overline{u}] \to [0,\overline{u}]$ where $[0,\overline{u}]$ denote the usual order interval in E. Since $[0,\overline{u}]$ is convex (and so a retract of E), $i(KN,[0,\overline{u}],[0,\overline{u}]) = 1$.

Suppose that u is a positive solution of (5) lying in $[0,\overline{u}]$. By Theorem 1 all eigenvalues of (2) are positive and so by Lemma 2 all eigenvalues of $KN'(u) < 1$. Hence $u - KN'(u)$ is invertible and so u is an isolated solution of $u = KNu$. Suppose that $\{u_k\}$ is a sequence of positive solutions of (5) lying in $[0,\overline{u}]$ such that $\lim_{k\to\infty} u_k = 0$ in E. Thus $KNu_k = u_k$ for all k and so $\lambda = 1$, $u = 0$ is a bifurcation point for $KNu = \lambda u$. A standard bifurcation argument shows that 1 is an eigenvalue of the linear operator $KN'(0)$ and that there is a corresponding positive eigenfunction v. Thus v is a positive eigenfunction of $-\Delta - g(x)f'(0)$ corresponding to the eigenvalue $\lambda = 0$, i.e., the principal eigenvalue of $-\Delta - g(x)f'(0)$ is zero and this is impossible from the hypotheses of the theorem. Hence $u = 0$ is an isolated solution. Therefore, since KN is compact, there can exist only finitely many solutions of $u = KNu$ in $[0,\overline{u}]$, say $0, u_1, u_2, \ldots, u_k$. Thus

$$1 = i(KN,[0,\overline{u}],[0,\overline{u}]) = i(KN, P_\rho, 0) + \sum_{j=1}^{k} i(KN, u_j) \tag{6}$$

where $P_\rho = \{v \in [0,\overline{u}] : \|v\|_e < \rho\}$ and ρ is chosen sufficiently small so that 0 is the only fixed point of KN in P_ρ.

It has been already shown that all the eigenvalues of $KN'(u_j) < 1$ and so $i(KN, u_j) = 1$ for $j = 1, \ldots, k$ (see [2] Theorem 11.4). Hence (6) becomes

$$i(KN, P_\rho, 0) + k = 1.$$

The hypotheses in (i) [respectively (ii)] ensure that $i(KN, P_\rho, 0) = 1$ [respectively 0] (see [2] Lemma 13.1). Hence (5) has no positive solution in $[0,\overline{u}]$ in case (i) and exactly one positive solution in case (ii).

We complete the proof by constructing appropriate supersolutions $\overline{u}$. First suppose f satisfies (f2). Choose $K_1 <$ first eigenvalue of $-\Delta$ with Dirichlet boundary conditions. Then there exists a constant K_2 such that $\|g\|_\infty f(u) \leq K_1 u + K_2$ for all $u > 0$. If $\overline{u}$ is the unique solution of

$$-\Delta u - K_1 u = K_2 \quad \text{in } D; \qquad u = 0 \quad \text{on } \partial D$$

then $\overline{u} \in E$ and is a supersolution of (5). Since K_2 may be chosen arbitrarily large in the construction, $\overline{u}$ may be chosen arbitrarily large and the proof is complete for this case.

Suppose f satisfies (f3). Then $u \equiv 1$ is a supersolution of (5) and so the iteration defined by $u_1 \equiv 1$ and

$$-\Delta u_{n+1} + Cu_{n+1} = g(x)f(u_n) + Cu_n \quad \text{in } D; \qquad u_{n+1} = 0 \quad \text{on } \partial D$$

gives a monotonic decreasing sequence of supersolutions of (5) converging to the maximal solution of (5). Thus, if $\overline{u} = u_2$, i.e., the unique solution of

$$-\Delta u + Cu = g(x)f(1) + C \quad \text{in } D; \qquad u = 0 \quad \text{on } \partial D$$

then $\overline{u} \in E$ and $\overline{u}$ is a supersolution of (5) which is greater than or equal to the maximal solution of (5). Hence all positive solutions of (5) lie in $[0, \overline{u}]$ and the proof is complete. ∎

If $g(x) < 0$ for all x in D, it follows from Theorem 3(i) and is in any case easy to prove directly that (5) has no positive solutions. If $g(x) > 0$ for all x in D or g changes sign on D, then either (i) or (ii) of Theorem 3 may apply and Theorem 3 does not specify what happens when the principal eigenvalue of $-\Delta - g(x)f'(0)$ equals zero. This latter case can be understood by considering the nonlinear eigenvalue problem

$$-\Delta u = \lambda g(x)f(u) \quad \text{in } D; \qquad u = 0 \quad \text{on } \partial D. \tag{7_λ}$$

We consider first the case where g changes sign on D. It is well known that the corresponding linearized problem

$$-\Delta u = \lambda g(x)f'(0)u \quad \text{in } D; \qquad u = 0 \quad \text{on } \partial D$$

possesses sequences of eigenvalues $0 < \lambda_1 < \lambda_2 \leq \ldots$ and $0 > \lambda_{-1} > \lambda_{-2} \geq \ldots$.

Theorem 4. *Suppose f satisfies the hypotheses of Theorem 3 and g changes sign on D. Then $(7)_\lambda$ has no positive solution if $0 \leq \lambda \leq \lambda_1$ and exactly one positive solution when $\lambda > \lambda_1$.*

Proof: It is shown in [7] that the principal eigenvalue $\mu(\lambda)$ of

$$-\Delta\phi - \lambda g(x)f'(0)\phi = \mu(\lambda)\phi \quad \text{in} D; \qquad \phi = 0 \quad \text{on } \partial D$$

is positive, zero or negative according as $0 < \lambda < \lambda_1$, $\lambda = \lambda_1$ or $\lambda > \lambda_1$. Thus by Theorem 3, (7_λ) has no positive solution if $\lambda < \lambda_1$ and exactly one positive solution if $\lambda > \lambda_1$.

Suppose that (7_λ) has a positive solution u_1 when $\lambda = \lambda_1$. If $F : W^{2,p}_{B_1}(D) \times \mathbb{R} \to L^p(D)$ is defined by $F(u, \lambda) = -\Delta u - \lambda g(x)f(u)$ then F has Frechet derivative $F_u(\lambda_1, u_1)(\phi) = -\Delta\phi - \lambda_1 g(x)f'(u_1)\phi$. Hence by Theorem 1, $F_u(\lambda_1, u_1)$ is invertible. Therefore by the implicit function theorem there exists a continuous curve of solutions $(\lambda, u(\lambda))$ passing through (λ_1, u_1) and so there must exist a positive solution of (7_λ) for λ just less than λ_1 which is impossible. Hence (7_λ) has no positive solution when $\lambda = \lambda_1$ and the proof is complete. ∎

The results in the preceding theorem along with standard results on local and global bifurcation show that in the bifurcation diagram for (7_λ) in the (λ, u) plane all positive solutions lie on a curve emanating from and lying on the right of $(\lambda_1, 0)$. The same result can be obtained by almost exactly the same arguments when $g(x) > 0$ on D; this latter result is already well known but the corresponding result for the case where g changes sign appears to be new.

Exactly analogous results hold for mixed boundary conditions, i.e., $0 < \alpha < 1$. In fact it is easy to see that for each $\alpha \in (0, 1]$ analogues of Theorem 3 and 4 hold also when assumption (f2) is weakened to $\lim_{u\to\infty} \frac{f(u)}{u} < \lambda_1(\alpha)$ where $\lambda_1(\alpha)$ is the principal eigenvalue of $-\Delta$ together with the boundary condition $B_\alpha u = 0$.

The above method does not yield existence results for Neumann boundary conditions when f satisfies (f2); interestingly this is exactly the case dealt with in [3]. In the case of Neumann boundary conditions when f satisfies (f3) the situation is complicated by the fact that $u \equiv 1$ is also a solution of (1) and the stability of the solutions $u \equiv 0$ and $u \equiv 1$ is influenced by the sign of $\int_D g\,dx$. However, by considering the fixed point indices of the constant solutions in the various cases it can be shown that there exists at most one non-constant positive solution and precise results can be given about the existence of solutions. For example

Theorem 5. *Suppose f satisfies $(f1)$ and $(f3)$ and $\int_D g\,dx = 0$. Then*

$$-\Delta u = g(x)f(u) \quad in\ D; \qquad \frac{\partial u}{\partial n} = 0 \quad on\ \partial D \tag{8}$$

has a unique non-constant positive solution.

Proof: Equation (8) is equivalent to an operator equation of the form $u = KNu$ where $K : C(\bar{D}) \to C(\bar{D})$ and N are the natural analogues of the operators used in the Dirichlet case. Since $KN : [0,1] \to [0,1]$, $i(KN, [0,1], [0,1]) = 1$. It is shown in [8] that the linearized problem

$$-\Delta\phi - g(x)f'(0)\phi = \mu\phi \quad \text{in } D; \qquad \frac{\partial \phi}{\partial n} = 0 \quad \text{on } \partial D$$

has negative principal eigenvalue. Hence an argument similar to that used in the proof of Lemma 2 shows that the largest eigenvalue of $KN'(0)$ exceeds 1. Hence, $i(KN, P_\rho, [0,1]) = 0$. Similarly $i(KN, \hat{P}_\rho, [0,1]) = 0$ where $\hat{P}_\rho = \{v \in [0,1] : \|1 - v\| < \rho\}$. Therefore, by the argument used in the proof of Theorem 3, equation (8) has exactly one non-constant solution in [0, 1]. ∎

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