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STABILITY AND VIBRATIONS OF ELASTIC THICK-WALLED CYLINDRICAL AND SPHERICAL SHELLS SUBJECTED TO PRESSURE

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Abstract—Theoretical and experimental studies of the free vibrations and the loss of stability of thick-walled cylindrical and spherical shells subjected to external pressure are presented. General equations for the oscillations of the shell under pressure are formulated on the basis of a rigorous theory of finite elasticity. Loss of stability is determined when the fundamental natural frequency of the prestressed shell ceases to be real-valued. Numerical solutions are obtained by solving the specialized equations that are applicable to neo-Hookean materials. Experiments using specimens made of silicone rubber closely verify the theoretical results.

INTRODUCTION

STABILITY of thin elastic shells subjected to external pressure has been an important engineering problem of long standing. Several theories are now available for predicting the critical pressure and stresses [1]. It is known, however, that values of the critical pressure measured experimentally are usually lower than those found from theory. Continuous efforts have been made to explain this discrepancy since the first works of von Kármán and Tsien [2, 3].

Energy arguments and the concept of geometrical imperfection have usually been used to study the problem within the context of either linear or non-linear thin shell theories (see Nash [4]). In recent years various numerical techniques have been developed for shells of revolution [5]. Since in all thin shell theories, linear or non-linear, idealization is made that the thickness of the shell is small compared to its radius of curvature, it has been conjectured that the phenomenon of instability for shells of finite thickness could not be adequately described within the thin shell assumptions. This has led to numerous investigations on the stability of shells of arbitrary thickness. In 1955, Wilkes [6] studied the axial symmetric buckling of an elastic thick-walled tube under uniform end thrust. Thick-walled cylindrical and spherical shells subjected to external pressure were investigated by Lubkin [7] in 1957. Sensenig [8] re-considered these two problems again in 1964. In all these studies, a minimization of energy procedure was used in conjunction with an extended form of Hooke's law for the material. But such an energy approach, as applied to non-linear problems may be considered only approximate in nature [9].

As far as could be traced, only a few problems which deal with the instability of thick-walled shells are solved on the basis of a rigorous theory of elasticity. Most notable, however, are the recent works of Wesolowski [10] and Nowinski and Shahinpoor [11]. In the former, a thick-walled spherical shell subjected to an external pressure was analyzed by means of a theory advanced by Green, Rivlin and Shield [12]. Specifically, the shell undergoes first a finite initial static deformation caused by an external pressure, and is afterwards exposed to a

secondary static deformation. Instability of the initially deformed configuration is determined when uniqueness of equilibrium ceases. Such a stability criterion is based on Kirchhoff's uniqueness theorem and is generally referred to as adjacent equilibria or the equilibrium bifurcation. For conservative systems, investigations made by Ziegler [13], Guo and Urbanowski [14] and others indicate that the equilibrium criterion is synonymous with the kinematical criterion.

In Wesolowski's paper, the secondary deformation of the shell was expressed in a series of spherical harmonics, and the complex equations of equilibrium governing the secondary state of deformation were reduced to a single fourth-order ordinary differential equation. By requiring the vanishing of the surface tractions in the secondary state, the entire problem reduces to one of a boundary-value type. With the help of a finite-difference procedure, explicit results for shells made of rubber-like materials were obtained. These numerical solutions however, are incomplete, in that only a single spherical harmonic ($n = 2$) was considered. As a result, a singularity was encountered for relatively thin shells, where the critical pressure of the shell is indeterminate. An experiment was also carried out by Wesolowski, but the findings were quite remote from his analytical predictions.

Following a similar approach, Nowinski and Shahinpoor [11] studied the buckling of a long thick-walled tube subjected to a uniform radial pressure. In this case, the final boundary-value equations were solved using a Frobenius series. Probably because only two terms of the series were kept in the calculations, their numerical results appeared to over estimate a true solution. In particular, a singularity similar to that encountered in [10] was also found in the numerical solutions. Despite the unfavorable numerical results reported in these two papers, it is believed that the basic theory employed is essentially rigorous and should yield good solutions.

The present paper is, to some extent, a continuation of the works initiated in [10] and [11]. However, the formulation of the problems and the scope of the investigation have been more generalized. Specifically, the present work employs the same theory as that presented by Green *et al.* [12], but the superposed secondary displacements are now dynamical. We seek the frequencies of free vibrations of the initially deformed shells. The oscillatory characteristics, whether the shell softens or hardens under load, are explicitly determined by the prestressed state of the shell. Instability occurs when the superposed secondary motions cease to be periodic. This approach not only provides a stability criterion, but also the stress state and the vibration behavior of the shell prior to the occurence of instability.

Numerical solutions are obtained for cylindrical and spherical shells made of a rubber-like material which is assumed incompressible (the theory and the method of solution however do not preclude elastic compressible materials). The respective boundary value equations are solved numerically by the method of finite-differences.

An experimental study has also been carried out. The experiments are limited only to specimens having cylindrical geometry, since thick-walled spherical shells were already tested by Wesolowski [10]. Tubes of various thicknesses were made using a type of silicone rubber, which is quite elastic and virtually incompressible. The uniform radial pressure is applied by subjecting the inner cavity of the tube to a vacuum; but for thicker shells for which the atmospheric pressure is not sufficient to cause instability, the specimen is then placed in a pressurized, transparent chamber, together with the vacuuming accessories. Buckling modes and the corresponding pressure are then compared with those obtained analytically. Good agreement has been reached almost uniformly.

FORMULATION OF THE PROBLEMS

1. *The cylindrical shell*

Consider a long, circular cylindrical shell made of perfectly elastic and isotropic material. The inner and outer radii of the shell are denoted respectively by A_1 and A_2 (we follow the notation adopted in [11]). The shell is subjected to a uniform pressure q_1 on the interior surface and q_2 on the exterior surface. The pressure differences $q = q_2 - q_1$ produces an axially symmetric deformation with the new inner and the outer radii given by

$$a_1 = \mu_1 A_2 \qquad a_2 = \mu_2 A_2 \tag{1}$$

where μ_1 and μ_2 are constants depending on q and the material property. For simplicity, we assume the material incompressible.† Hence,

$$\mu_2^2 = 1 - (1 - \mu_1^2)A_1^2/A_2^2. \tag{2}$$

Let R be the material radial coordinate of the undeformed shell, and r be that of the deformed shell; I, II be the first and the second strain invariants, and $W(\text{I, II})$ be the strain energy density function of the (incompressible) material. The pressure necessary to produce the prescribed deformation is given by

$$q = q_2 - q_1 = \int\limits_{a_1}^{a_2} (Q^2 - Q^{-2})(\phi + \psi)\frac{dr}{r} \tag{3}$$

where

$$\phi = 2\partial W/\partial \text{I}, \qquad \psi = 2\partial W/\partial \text{II}$$

$$Q = R/r = \frac{1}{r}[r^2 + A_1^2(1 - \mu_1^2)]^{\frac{1}{2}}.$$

If $W(\text{I, II})$ is a linear function of I and II, then ϕ and ψ become constant and equation (3) reduces to

$$q = (\phi + \psi)\left[\frac{1}{2\mu_1^2}\left(\frac{\mu_2^2 - \mu_1^2}{\mu_2^2}\right) - \text{Ln}\,(\mu_1/\mu_2)\right]. \tag{4}$$

From equations (2)–(4) we can determine μ_1 and μ_2 for any given q, if a stable axially symmetric deformed state of the shell exists.

We now consider a state of free, infinitesimal vibrations by superposing onto the deformed shell displacements $w_1 = u(r, \theta, t)$, $w_2 = rv(r, \theta, t)$, $w_3 = 0$. The equations of motion governing this secondary dynamic displacements are given by

$$Q^2 u_{,rr} + 2\left(\frac{Q^{-2}}{r} - \frac{p_{,r}}{2C_1}\right)u_{,r} + \frac{u_{,\theta\theta}}{r^2 Q^2} - \frac{v_{,\theta}}{r^2 Q^2} + \frac{p'_{,r}}{2C_1} = \frac{\rho u_{,tt}}{2C_1} \tag{5}$$

$$\left(\frac{1}{r^2 Q^2} - \frac{p_{,r}}{2rC_1}\right)(u_{,\theta} - v) + \left(\frac{1}{rQ^2} - \frac{p_{,r}}{2C_1}\right)v_{,r} - \frac{u_{,r\theta}}{rQ^2} + Q^2 v_{,rr} + \frac{p'_{,\theta}}{2rC_1} = \frac{\rho v_{,tt}}{2C_1} \tag{6}$$

where

$$p = C_1 \cdot [2\,\text{Ln}\,(\mu_1 Q) - Q^2 - \mu_1^{-2}] - q_1$$

† The theory does not preclude compressible materials. Formulations for the present problem using Hooke's material are contained in Ref. [15].

and $p'(r, \theta, t)$ is an unknown pressure associated with the secondary deformations. In obtaining the above, we have assumed for definiteness that the material is of the neo-Hookean type[†] and its strain energy density function has the form

$$W = C_1 (I - 3)$$

where C_1 is a material constant.

The incompressibility condition associated with the secondary state (see equation 1.31, Ref. [11]) yields the relation,

$$u_{,r} + \frac{1}{r} v_{,\theta} + \frac{u}{r} = 0. \tag{7}$$

Equations (5), (6) and (7) govern the unknown functions u, v and p' of the secondary (dynamic) state of the shell. These functions must satisfy the boundary conditions that the secondary surface tractions vanish (see equation 1.33, [11]),

$$\begin{aligned} 4C_1 Q^2 u_{,r} + p' &= 0 \\ u_{,\theta} + r v_{,r} - v &= 0 \end{aligned} \qquad \text{on } r = a_1, a_2. \tag{8}$$

Equations (5)–(8) are linear. We seek their steady-state solutions by proposing

$$u(r, \theta, t) = \sum_{n=0}^{\infty} R_{1n}(r) \cos n\theta \exp(i\omega_n t)$$

$$v(r, \theta, t) = \sum_{n=0}^{\infty} R_{2n}(r) \sin n\theta \exp(i\omega_n t) \tag{9}$$

$$p'(r, \theta, t) = \sum_{n=0}^{\infty} R_{3n}(r) \cos n\theta \exp(i\omega_n t).$$

Substitution of equation (9) into equations (5)–(7) and the subsequent elimination of the variables R_{2n} and R_{3n} yield, for each n, the single ordinary differential equation,

$$\frac{r^2 + k}{n^2} R_{1n}'''' + 2 \frac{3r^2 + k}{rn^2} R_{1n}''' + \left[\frac{\rho \omega_n^2 r^2}{2C_1 n^2} - \frac{r^2}{r^2 + k} - \frac{3k - 5r^2}{n^2 r^2} - \frac{r^2 + k}{r^2} \right] R_{1n}''$$

$$+ \left[3 \frac{\rho \omega_n^2 r}{2C_1 n^2} - 2 \frac{r^2 - k}{r^3} - \frac{r}{r^2 + k} - \frac{r^2 - 3k}{n^2 r^3} - \frac{2kr}{(r^2 + k)^2} - \frac{k - r^2}{r^3} \right] R_{1n}'$$

$$+ \left[\frac{\rho \omega_n^2}{2C_1 n^2} - \frac{\rho \omega_n^2}{2C_1} + \frac{n^2}{r^2 + k} - \frac{1}{r^2 + k} - \frac{1}{r^4} \left(r^2 - 3k + \frac{3k - r^2}{n^2} \right) \right] R_{1n} = 0 \tag{10}$$

where

$$k = A_1^2 (1 - \mu_1^2).$$

The corresponding boundary conditions (8) reduce to,

$$\frac{r^2}{n} R_{1n}'' + \frac{r}{n} R_{1n}' + \left(n - \frac{1}{n} \right) R_{1n} = 0 \qquad r = a_1, a_2 \tag{11}$$

$$\left(\frac{r^2 + k}{n^2} \right) R_{1n}''' + \frac{4r^2 + 2k}{rn^2} R_{1n}'' - \left[2 \frac{r^2 + k}{r^2} + \frac{r^2}{r^2 + k} - 2 \frac{r^2 - k}{n^2 r^2} - \frac{r^2}{n} \left(\frac{\rho \omega_n^2}{2C_1} - \frac{r^2 - k}{r^4} \right) \right]$$

[†] We used in the experiments a neo-Hookean material.

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$$\times R'_{1n} + \left[\frac{r^2 - k}{r^3} + \frac{r}{n^2}\left(\frac{\rho\omega_n^2}{2C_1} - \frac{r^2 - k}{r^4}\right)\right]R_{1n} = 0, \qquad r = a_1, a_2. \tag{12}$$

2. The spherical shell

In this section, we consider a spherical shell whose undeformed inner and outer radii are again denoted, respectively, by A_1 and A_2. An internal pressure q_1 and an external pressure q_2 are then applied to produce a spherically symmetric deformation with the new inner and outer radii $a_1 = \mu_1 A_1$, and $a_2 = \mu_2 A_2$.

The constants μ_1 and μ_2 are related to the incompressibility of the material by

$$\mu_2^3 = 1 - (1 - \mu_1^3)(A_1/A_2)^3. \tag{13}$$

For a neo-Hookean material, we have

$$q = q_2 - q_1 = 4C_1\left[\frac{\mu_2 - \mu_1}{\mu_1\mu_2} + \frac{\mu_2^4 - \mu_1^4}{4\mu_1^4\mu_2^4}\right]. \tag{14}$$

Thus, for every given q we determine from equations (13) and (14) the values of μ_1 and μ_2, if a state of stable radial deformation as prescribed exists.

A secondary dynamic displacement field, $w_i(r, \theta, \phi, t)$, $i = 1, 2, 3$, is now superposed onto the initially deformed shell. The equations governing the motions are given by:

$$Q^4 w_{1,rr} + \frac{4}{r}(2Q - Q^4 - \tfrac{1}{2}Q^{-2})w_{1,r} - \frac{2}{r^2}Q^{-2}w_1 + \frac{1}{r^2}Q^{-2}(w_{1,\theta\theta} + w_{1,\theta}\cot\theta$$

$$+ w_{1,\phi\phi}\csc^2\theta) - \frac{2}{r^3}Q^{-2}(w_{2,\theta} + w_2\cot\theta + w_{3,\phi}\csc^2\theta) + \frac{p'_{,r}}{2C_1} = \frac{\rho}{2C_1}w_{1,tt} \tag{15}$$

$$-Q^{-2}w_{1,r\theta} + \frac{2}{r}(2Q - Q^4 - Q^{-2})w_{1,\theta} + Q^4 w_{2,rr} + \frac{4}{r}(Q - Q^4)w_{2,r} + \frac{2}{r^2}$$

$$\times (3Q^4 + Q^{-2} - 4Q)w_2 + Q^{-2}\csc^2\theta(w_{2,\phi\phi} - w_{3,\theta\phi}) + \frac{p'_{,\theta}}{2C_1} = \frac{\rho}{2C_1}\cdot w_{2,tt} \tag{16}$$

$$\frac{2}{r}(2Q - Q^4 - Q^{-2})w_{1,\phi} - Q^{-2}w_{1,r\phi} - \frac{1}{r^2}Q^{-2}(w_{2,\theta\phi} - w_{2,\phi}\cot\theta - w_{3,\theta\theta} + w_{3,\theta}\cot\theta)$$

$$+ Q^4 w_{3,rr} + \frac{4}{r}(Q - Q^4)w_{3,r} + \frac{2}{r^2}(3Q^4 - 4Q - Q^{-2})w_3 + \frac{p'_{,\phi}}{2C_1} = \frac{\rho}{2C_1}w_{3,tt} \tag{17}$$

where

$$Q = R/r = \frac{1}{r}[r^3 + A_1^3(1 - \mu_1^3)]^{\frac{1}{3}}$$

$$p = 2C_1(2Q - \tfrac{1}{2}Q^4 - 2\mu_1^{-1} - \tfrac{1}{2}\mu_1^{-4}) - q_1$$

and $p'(r, \theta, \phi, t)$ is an unknown secondary pressure. The associated incompressibility condition yields (cf. equation 2.12, Ref. [10]),

$$w_{1,r} + \frac{1}{r^2}(w_{2,\theta} + \csc^2\theta\, w_{3,\phi} + 2rw_1 + \cot\theta w_2) = 0. \tag{18}$$

The four equations (15)–(18) govern the four unknown functions w_i and p'. These functions

must satisfy the boundary conditions that the secondary surface tractions vanish (cf. equation 3.15, Ref. [10]),

$$4C_1 Q^4 w_{1,r} + p' = 0$$

$$w_{1,\theta} + w_{2,r} - \frac{2}{r} w_2 = 0 \qquad r = a_1, a_2 \tag{19}$$

$$w_{1,\phi} + w_{3,r} - \frac{2}{r} w_3 = 0.$$

Again, we seek the steady-state solutions to the system of equations (15)–(19). The solutions may be expressed in terms of the spherical harmonic functions

$$w_1 = \sum_{n=v}^{\infty} \sum_{v=1}^{\infty} R_{1nv}(r) Y_{nv}(\theta, \phi) \exp(i\omega t)$$

$$w_2 = \sum_{n=v}^{\infty} \sum_{v=1}^{\infty} R_{2nv}(r) \frac{\partial}{\partial \theta} Y_{nv}(\theta, \phi) \exp(i\omega t)$$

$$w_3 = \sum_{n=v}^{\infty} \sum_{v=1}^{\infty} R_{3nv}(r) \frac{\partial}{\partial \phi} Y_{nv}(\theta, \phi) \exp(i\omega t) \tag{20}$$

$$p' = \sum_{n=v}^{\infty} \sum_{v=1}^{\infty} R_{4nv}(r) Y_{nv}(\theta, \phi) \exp(i\omega t)$$

where the spherical function Y_{nv} satisfies the identity [16]

$$\left[\frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \csc^2 \theta \frac{\partial^2}{\partial \phi^2} + n(n+1) \right] Y_{nv} = 0 \tag{21}$$

for every $n, v = 1, 2, 3, \ldots . \infty$.

Since the spherical functions constitute a complete set of orthogonal functions in the domain $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$, and the system governed by equations (15)–(19) is homogeneous, each term in (20) may therefore be considered separately. To simplify the notation, the indices n and v will be hereafter omitted in all expressions.

By substituting the solutions in (20) into the system of equations (15)–(19), and by making use of the identity (21), we obtain a set of three simultaneous ordinary differential equations:

$$Q^4 R_{1,rr} + \frac{2}{r}(4Q - 2Q^4 - Q^{-2}) R_{1,r} + \left[\frac{\rho \omega_n^2}{2C_1} - \frac{2}{r^2} Q^{-2} - \frac{n(n+1)}{r^2 Q^2} \right] R_1$$

$$+ \frac{2}{r^3} Q^{-2} n(n+1) R_2 + R_{4,r} = 0 \tag{22}$$

$$-Q^{-2} R_{1,r} + \frac{2}{r}(2Q - Q^4 - Q^{-2})R_1 + Q^4 R_{2,rr} + \frac{4}{r}(Q - Q^4)R_{2,r}$$

$$+ \left[\frac{\rho \omega_n^2}{2C_1} + \frac{2}{r^2}(3Q^4 + Q^{-2} - 4Q) \right] R_2 + R_4 = 0 \tag{23}$$

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$$R_{1,r} + \frac{2}{r} R_1 - \frac{n(n+1)}{r^2} R_2 = 0 \tag{24}$$

with the boundary conditions

$$4C_1 Q^4 R_{1,r} + R_4 = 0$$

$$r = a_1, a_2. \tag{25}$$

$$R_1 + R_{2,r} - \frac{2}{r} R_2 = 0$$

It is noted that in deriving the above, the identity that $R_{2nv} = R_{3nv}$ was used, the proof of which is contained in Ref. [10].

When the variables R_2 and R_4 are eliminated from the expressions in (22)–(24), we obtain a single ordinary differential equation governing R_1;

$$R_1'''' + \frac{8}{r} Q^{-3} R_1''' + \left[\frac{\rho \omega_n^2}{2C_1 Q^4} - \frac{n^2 + n + 10}{r^2} + \frac{16}{r^2 Q^3} + \frac{6 - n^2 - n}{r^2 Q^6} \right] R_1''$$

$$+ \left[\frac{2\rho \omega_n^2}{C_1 r Q^4} + \frac{16 - 4n^2 - 4n}{r^3 Q^6} + \frac{20 + 2n^2 + 2n}{r^3} - \frac{4n^2 + 4n + 32}{r^3 Q^3} + \frac{2n^2 + 2n - 4}{r^3 Q^9} \right]$$

$$\times R_1' + \left[\frac{\rho \omega^2 (2 - n^2 - n)}{2C_1 r^2 Q^4} + \frac{10(n^2 + n - 2)}{r^4} + \frac{n^4 + 2n^3 + n^2 - n}{r^4 Q^6} \right.$$

$$\left. + \frac{16(2 - n^2 - n)}{r^4 Q^3} + \frac{4(n^2 + n - 2)}{r^4 Q^9} \right] R_1 = 0. \tag{26}$$

The corresponding boundary conditions become,

$$R_1'' + \frac{2}{r} R_1' + \frac{n^2 + n - 2}{r^2} R_1 = 0, \qquad r = a_1, a_2 \tag{27}$$

$$Q^4 R_1''' + \frac{2Q(2 + Q^3)}{r} R_1'' + \left[\frac{\rho \omega_n^2}{2C_1} - \frac{4Q^4}{r^2} + \frac{8Q}{r^2} + \frac{2 - n^2 - n}{r^2 Q^2} - \frac{2Q^4(n^2 + n)}{r^2} \right] R_1'$$

$$+ \left[\frac{\rho \omega_n^2}{C_1 r} + \frac{4Q^4}{r^3} + \frac{4Q(n^2 + n - 2)}{r^3} + \frac{2(2 - n^2 - n)}{r^3 Q^2} - \frac{2Q^4(n^2 + n)}{r^3} \right] R_1 = 0, r = a_1, a_2. \tag{28}$$

SOLUTIONS TO THE PROBLEMS

1. *Exact solution for $n = 0$; pure radial vibrations.*

Closed form solutions for the frequency of oscillations corresponding to $n = 0$ may be obtained by solving the systems (10)–(12) and (26)–(28). Thus, for the cylindrical shell, we have

$$\omega_0^2 = \frac{4C_1}{\rho} \left[\frac{2A_1^2 - k}{(A_1^2 - k)^2} - \frac{2A_2^2 - k}{(A_2^2 - k)^2} \right] \Big/ \mathrm{Ln}\left(\frac{A_2^2 - k}{A_1^2 - k} \right) \tag{29}$$

and for the spherical shell †

† The breathing mode of vibrations for inhomogeneous thick-walled spherical shells has recently been studied by Nowinski et al. [17]. Equation (30) here compares with Equation 42, Ref. [17] when the material is assumed to be neo-Hookean.

$$\omega_0^2 = \frac{4C_1}{\rho} \cdot \frac{2[a_2^7(a_1^3 + k)^{\frac{4}{3}} - a_1^7(a_2^3 + k)^{\frac{4}{3}}] - [a_2^7(a_1^3 + k)^{\frac{1}{3}} - a_1^7(a_2^3 + k)^{\frac{1}{3}}]k}{a_1^6 a_2^6 (a_2 - a_1)} \quad (30)$$

where $k = A_1^2(1 - \mu_1^2)$ in equation (29) and $k = A_1^3(1 - \mu_1^3)$ in equation (30).

Physically, $n = 0$ corresponds to pure radial oscillations about the finitely deformed shell. If the shells are subjected to a net inward pressure, i.e. $q > 0$, it can be shown from (29) and (30) that ω_0 remains positive and is monotonically increasing with q. Theoretically, if confined in a pure radial mode, the system hardens under a net inward pressure, and instability of the shell will not occur.

On the other hand, if the net pressure is outward, i.e. $q < 0$, the shells are softening with increasing pressure and ω_0 varies from being real to imaginary. The latter occurs in the case of the cylindrical shell,† when

$$q \to 2C_1 \operatorname{Ln}(A_1/A_2) \quad (31)$$

and in the case of the spherical shell when q attains a maximum value, q_{max}, which may be calculated from equation (14). In both cases, as $q \to q_{max}$ the shells fail without bound in a mode of radial expansion.

If the primary deformation of the shells is infinitesimal, i.e. $q \simeq 0$, then μ_1, $\mu_2 \to 1$ and $k \to 0$. From equations (29) and (30) we obtain, respectively.

$$\omega_0^2 = \frac{4C_1(A_2^2 - A_1^2)}{\rho A_1^2 A_2^2 \operatorname{Ln}(A_2/A_1)} \quad (32)$$

which is the classical result of small radial oscillations for cylindrical shells (see, [18], equation 37), and

$$\omega_0^2 = \frac{8C_1}{\rho} \frac{A_1^2 + A_1 A_2 + A_2^2}{A_1^2 + A_2^2} \quad (33)$$

which is the classical result of pure radial oscillations for spherical shells (see [19], equation 32).

2. Finite-difference solutions for $n \geq 1$: asymmetric vibrations

For $n \geq 1$, the system of equations (10)–(12) for the cylindrical shell, and (26)–(28) for the spherical shell are first non-dimensionalized and then written in finite-difference form. In both problems, a central-difference method is used. The thickness of the shells (the range of the variable r) is divided into N equal parts, and each of the two sets of equations is rewritten into a system of $(N + 1)$ homogeneous algebraic equations in the $(N + 1)$ unknown nodal functions of $R_1(r)$. For non-trivial solutions of these nodal functions we require the vanishing of the respective characteristic determinant. Hence, for each integer n we obtain a transcendental equation relating the frequency of the vibrations, ω_n, and the deformation parameter k. Note that k is associated with μ_1 and is convertible to the pressure q via equation (4) for the cylindrical shell, and (14) for the spherical shell.

Computational procedure and the determination of instability

It is noted that, if instability is not considered, the value of μ_1 will vary from 1 to ∞ as q increases outwardly; from 1 to 0 as q increases inwardly. Thus, in the numerical calculation,

† This limit was first found by Knowles [18] for radial oscillations of a tube.

we determine from the (ω_n, k)-equation the fundamental frequency ω_n for any given k, or any μ_1 with $0 \leqslant \mu_1 \leqslant \infty$. We seek the range of μ_1 in which ω_n^2 is positive real. In this range of μ_1, the strained shell (characterized by μ_1) will be oscillatory in the mode corresponding to n. Outside this range ω_n^2 will be non-positive, and the system will not be oscillatory. In particular, if ω_n^2 is negative, the system will then deform without bound according to the solutions (9) or (20). Thus, we define the critical deformation, $(\mu_1)_{cr}$ as the one which corresponds to the vanishing of ω_n^2. The pressure q associated with $(\mu_1)_{cr}$ is defined as the critical pressure, under which the shell fails in the mode corresponding to n. Clearly, q_{cr} is dependent upon the value of n; the shell fails first, however, in the mode which yields the smallest q_{cr}.

Some remarks on the convergence of solutions

In examining the convergence of the solution scheme, we compute for each given shell, for each given n and for each given $\mu_1 < (\mu_1)_{cr}$ the frequency of oscillation ω_n as a function of the nodal number N. We then examine whether or not the solution approaches an asymptote as N increases in value.[†] Figure 1 shows the convergence nature of the critical deformation as plotted against N. Generally speaking, thicker shells require more nodal points
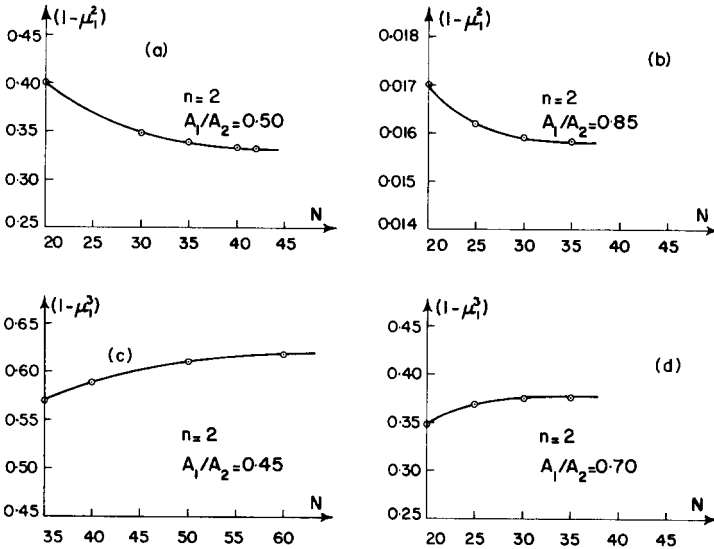

Fig. 1. Convergence of the finite-difference Solutions: (a, b) critical deformation vs. the pivotal number N for cylindrical shells; (c, d) critical deformation vs. the pivotal number N for spherical shells.

for the same degree of convergence as thinner shells and the solutions converge faster for higher harmonics than for lower harmonics. It is also interesting to note that the solution for the cylindrical shell approaches its asymptote from above while that for the spherical shell approaches from below.

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† The computations were found to be quite sensitive numerically. Scattering of solutions occurred when only eight significant digits were used; but the situation was corrected by doubling the precision to 16 digits. For details, see [15].

Vibrations about the undeformed state

When the applied pressure q is small, the shell is virtually unstrained and the free vibrations of such a shell reduces to a classical problem. In fact, for thin shells, classical results may be readily found in the texts of Rayleigh [20] and Kraus [21] both of whom used an energy approach to determine the natural frequencies of the shells. In the present study, we can calculate the frequencies for shells of arbitrary thickness by letting $\mu_1 \to 1$. The following Table illustrates the differences between the present solutions and the classical solutions:

TABLE 1

	Cylindrical		Spherical	
	Present	Rayleigh [20]	Present	Kraus [21]
$n = 2$	$\bar{\omega} = 0.235$	0.252	---	---
$= 3$	$= 0.610$	0.624	1.50	1.69
$= 4$	$= 1.15$	1.36	1.90	2.21

where $\bar{\omega}$ is the non-dimensionalized frequency (by a factor of $[3\rho A_1^2/E]^{\frac{1}{2}}$, E being the Young's modulus of the material). The thickness of both shells is $A_1/A_2 = 0.85$. It is seen that the present solutions are uniformly lower than the classical solutions. This is more so in the higher harmonics.

Vibrations about the finitely deformed state

When the applied pressure q is finite, then the shells will be finitely strained. But the behavior of the shell depends profoundly on the initial loading characteristics.

When the net pressure is outward, the lowest frequency of vibration is a function of the initial state of strain. Figure 2 shows for a cylindrical shell having $A_1/A_2 = 0.8$ the frequencies corresponding to $n = 0, 1$ and 2. It is seen that for small initial deformation the lowest (flexural) mode of vibration is $n = 2$; the behavior of the shell is of a hardening nature. But as q increases, the lowest frequency corresponds to the breathing mode. The shell fails in excessive inflation at a limiting pressure q_{max} calculated from equation (31). For the mode $n = 0$, the shell behaves as a softening system. A similar character is shown in Fig. 3 for spherical shells.

In both the cylindrical and spherical shells the mode corresponding to $n = 1$ is not the lowest. Physically, $n = 1$ represents a radial breathing coupled with a rigid body motion. The rigid body motion has a stiffening effect on the breathing mode since it is seen that the frequency corresponding to $n = 1$ under any pressure is always higher than that corresponding to $n = 0$.

When the net pressure is inward the behavior of the shells is reversed; the hardening effect is now associated with the breathing mode $(n = 0, 1)$ while the softening effect corresponds to the flexural mode, $n \geqslant 2$. It is seen however, throughout the compression regime and for any thickness of the shell, that $n = 2$ gives the lowest frequency for all cylindrical shells. Figure 4a illustrates this point for a shell having $A_1/A_2 = 0.85$, while Figure 4b shows the effect of the thickness on the frequency (for $n = 2$).

In the case of the spherical shells, $n = 2$ does not always give the lowest frequency. Figure 5a shows the frequencies corresponding to $n = 2, 3$ and 4 for a spherical shell of $A_1/A_2 = 0.8$. It is seen that $n = 2$ is the lowest mode only when the deformation is small. As the pressure
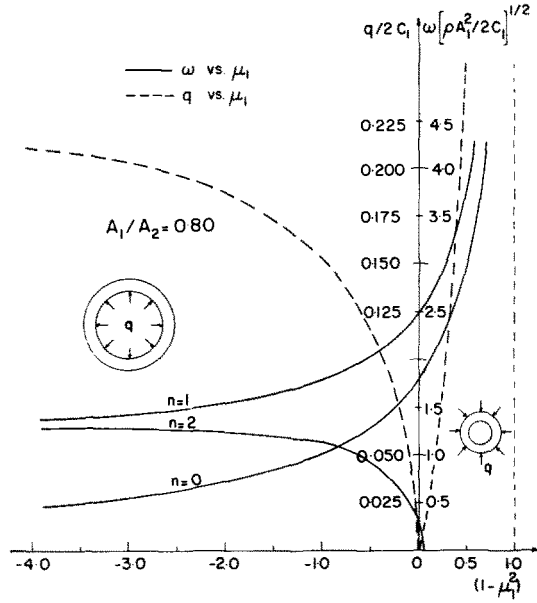
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FIG. 2. Frequencies and pressure vs. deformation for a cylindrical shell.
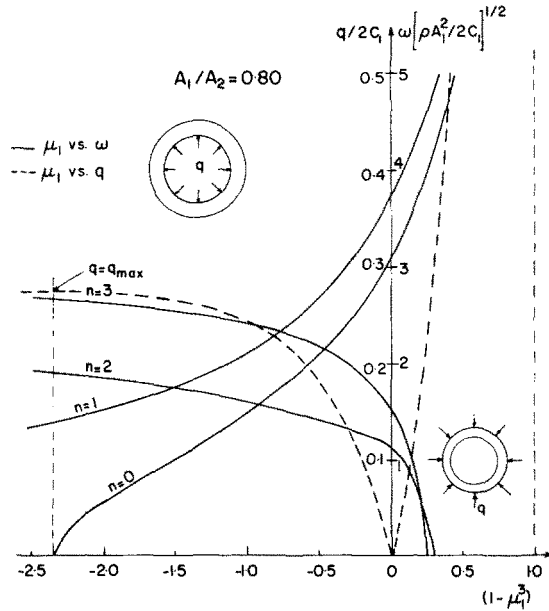

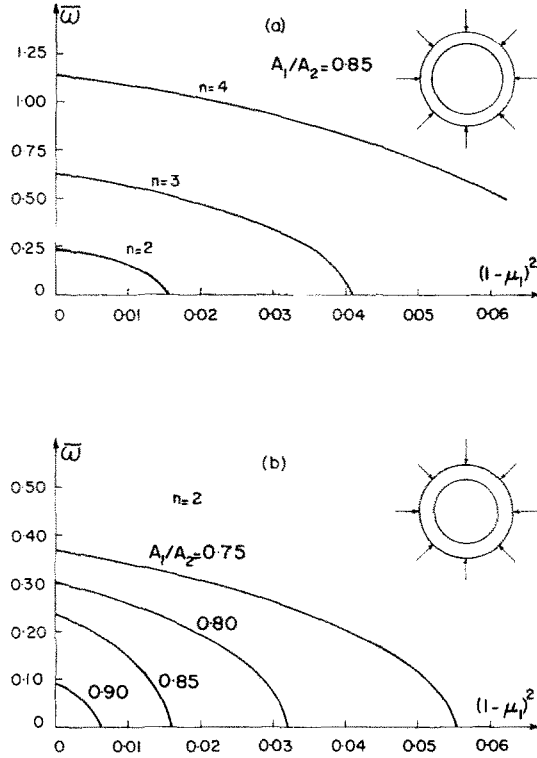FIG. 3. Frequencies and pressure vs. deformation for a spherical shell.

FIG. 4. Frequencies vs. deformation for cylindrical shells: (a) at different harmonic numbers; (b) at different thicknesses for $n = 2$.

increases $n = 3$ eventually takes over as the lowest mode until the shell fails by instability. Figure 5b is for a thinner shell, $A_1/A_2 = 0.85$. Again, $n = 2$ governs the fundamental mode when q is small; as q increases, $n = 3$ takes over and eventually $n = 4$ is the lowest until the shell fails by instability. Thus, thinner shells fail in a mode of higher harmonics. Calculations have been made for $A_1/A_2 = 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.85, 0.9, 0.95$. Approximately, for shells of A_1/A_2 up to 0.7, $n = 2$ governs the failure mode; between 0.7 and 0.82, $n = 3$ governs; between 0.82 and 0.86, $n = 4$ governs; between 0.86 and 0.89 $n = 5$ governs; at 0.90 $n = 6$ or 7 governs; at 0.95 $n \geqslant 11$ governs (see Fig. 6). Such wave number dependent character for thin spherical shells has been discussed by Flügge [22] and others. Recently, Long [23] has studied the stability of thin spherical shells under uniform external pressure for Hookean materials, and has found a similar behavior.

Instability due to inward pressure

Having determined the loss of stability of shells of any thickness under inward pressure, we now present some explicit results. For comparison, we first illustrate the results using the graphs shown in Refs. [10] and [11]. Figure 6a shows the deformation parameter $\mu_2 = a_2/A_2$ at instability as plotted against the thickness parameter for the cylindrical shells. It is seen that a shell of any thickness will collapse in an oval shape ($n = 2$). The dotted lines are results reported in [11].
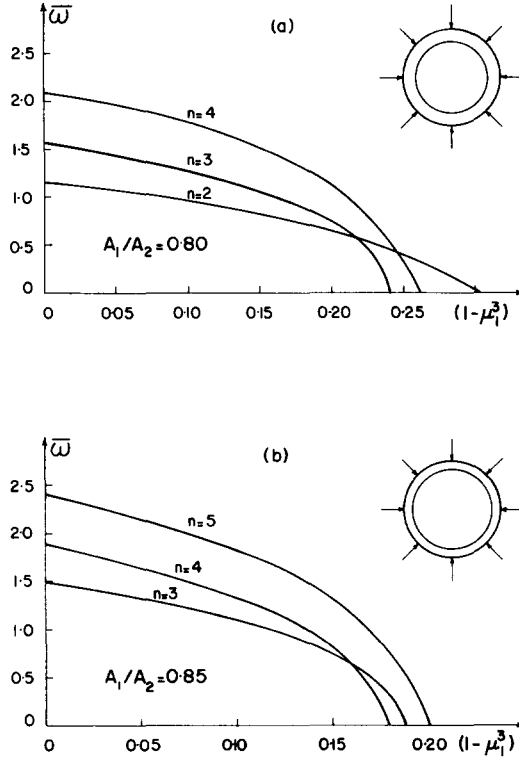
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FIG. 5. Frequencies vs. deformation for spherical shells: (a) $A_1/A_2 = 0.8$; (b) $A_1/A_2 = 0.85$.

Figure 6b shows a similar graph for the spherical shells. Here, it is seen that for A_1/A_2 up to 0·7, $n = 2$ governs the failure mode. As the shell becomes thinner, higher harmonics govern. An envelope of failure is thus formed. The earlier results reported by Wesolowski correspond to the curve labelled by $n = 2$. When A_1/A_2 increases beyond 0·85, Wesolowski encountered what he termed the singularity of the solution. Actually, the shell will never fail in the mode of $n = 2$ if the shell has its A_1/A_2 ratio larger than 0·7. The isolated points in Fig. 6b are the experimental data reported in [10]. Excepting for one point, the present results agree well with the experiments.

Figures 7a and 7b show the critical (lowest) pressure as plotted against the thickness of the shell. Figure 7a shows the pressure-thickness relation for cylindrical shells; the isolated points are obtained by experiments carried out in this study. In Fig. 7b, the same is shown for spherical shells. The dotted lines in both figures represent the classical thin shell solution, which may be calculated by the simple formula,

$$(q_{cr})_{cy} = 16C_1 \left[\frac{A_2 - A_1}{A_2 + A_1} \right]^3, \qquad (q_{cr})_{sp} = 32C_1 \left[\frac{A_2 - A_1}{A_2 + A_1} \right]^2. \qquad (34)$$

EXPERIMENTAL STUDY

Since in our analysis no simplifying assumption was made with regard to the magnitude
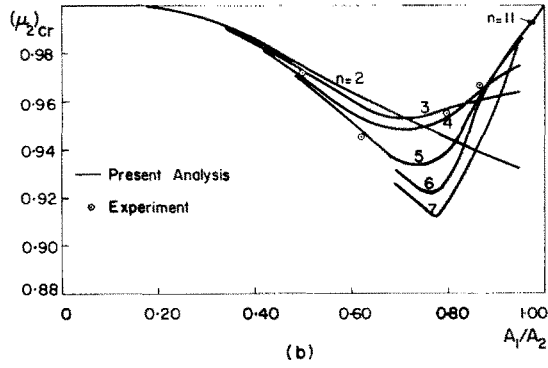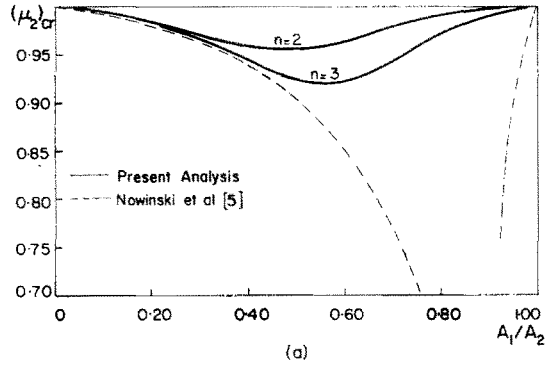
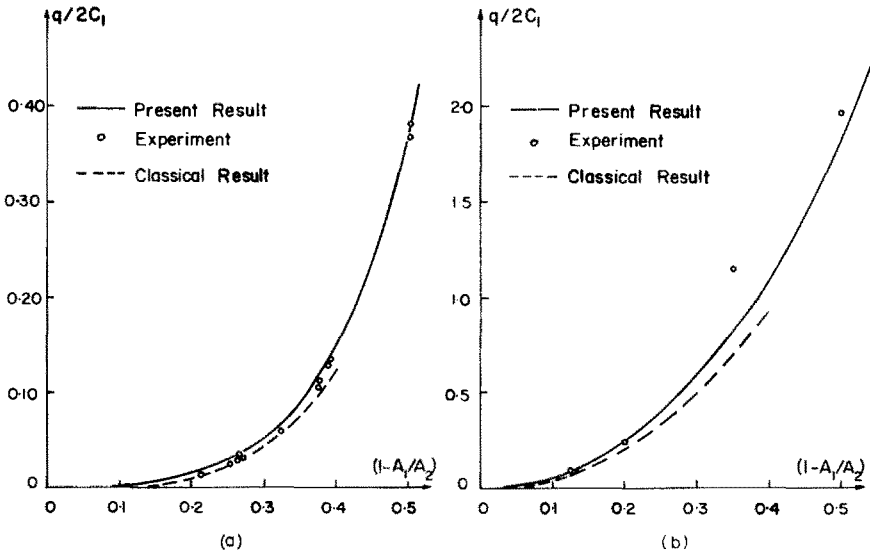FIG. 6. Critical deformation vs. thickness of the shells: (a) cylindrical shells; (b) spherical shells.

FIG. 7. Critical pressure vs. thickness of the shells: (a) cylindrical shells; (b) spherical shells.

of deformation, the thickness of the shells or the linearity of the material (the only assumptions are isotropy, homogeneity and incompressibility), we used in the experiments a rubber-like material which is capable of sustaining large elastic deformations and is materially non-linear.† To establish the constitutive equation of such a material, we examined first the incompressible nature of the rubber by performing uniaxial tension tests on four coupons, each of which was subjected to more than 350 per cent elongation without failure. All specimens were observed to return to their original length immediately after the load was removed. We assumed next that the material is of the neo-Hookean type. This assumption was confirmed by the tension tests, that the neo-Hookean model holds good within 200 per cent elongation. In addition, we manufactured cylindrical tubes of various thicknesses and tested them under triaxial stress conditions. Again, the neo-Hookean model represented the material behavior well in all stress levels before instability occurred in each case.

In the buckling tests of the cylindrical shells twelve specimens were examined. Each of these specimens had a length of at least 10 times the inner diameter of the tube, so that the end effects will be minimum at the center of the specimen. Uniform external pressure was supplied by subjecting the inner cavity of the tube to a vacuum, and the pressure difference between the inside and the outside was monitored by pressure gages. For thick shells with $A_1/A_2 \leqslant 0.5$, it was necessary to place the specimen in a transparent, high-pressure chamber and to apply additional pressure on the outside of the shell.

The specimens, under slowly increasing pressure, were first observed to undergo considerable concentric deformation. Then suddenly, the crosssection of the specimen became eccentric to resemble an oval shape ($n = 2$). Beyond this point, the pressure reading remained essentially constant while the specimen continued to register additional deformation: increasing on one diameter and decreasing on the other, the two diametral axes being approximately orthogonal. The final failure pressure for the shell was determined by the well-known Southwell plots (see Timoshenko and Gere [24]). Details of the experiments and other aspects of the present work are contained in Ref. [15].

In conclusion, it may be remarked that, in view of the close agreement between the theory and the experiments, the fundamental theory [12] is applicable to the present problem and the criterion for the loss of stability is accurate. In addition, the numerical solution method adapted in the present work also provided reliable information. It should be mentioned, however, that since none of the specimens tested had a thickness-to-radius ratio smaller than 10^{-1}, they must all be classified as being moderately thick or thick-walled shells. The pressure necessary to buckle a shell is approximately proportional to the thickness squared in the case of spherical shells, and to the thickness cubed in the case of cylindrical shells. Thus, for relatively thin shells (say t/R is of the order of 10^{-2}), the buckling pressure is extremely small; the effects of geometrical or material imperfections may, therefore, become significant in comparison with the buckling pressure. These effects are negligible in the case of moderately thick or thick-walled shells. This may also help to explain why the present experimental results are so close to those predicted by theory.

† Silastic RTV Rubber, a product of the Dow Corning Corp., Midland, Michigan.

REFERENCES

[1] Y. C. Fung and E. E. Sechler, Instability of thin elastic shells, *Proc. 1st Symp. Naval Struct. Mech.* pp. 115–168 (Ed. J. N. Goodier and N. J. Hoff), Pergamon Press (1960).

[2] T. von Kármán and H. S. Tsien, The buckling of spherical shells by external pressure, *J. Aero Sci.* 7, 43–50 (1939).

[3] H. S. Tsien, A theory for the buckling of thin shells, *J. Aero Sci.* 9, 373–384 (1942).

[4] W. A. Nash, Instability of thin shells, *Appl. Mech. Survey*, pp. 339–348 (Ed. H. N. Abramson *et al.*), Spartan Books (1966).

[5] R. E. Fulton, Numerical analysis of shells of revolution, *IUTAM Symp. High-Speed Computer Elastic Struct.*, Liege, Belgium (1970).

[6] E. W. Wilkes, On the stability of a circular tube under end thrust, *Q. J. Mech. appl. Math.*, 8, 88–100 (1955).

[7] S. Lubkin, Determination of buckling criteria by minimization of total energy, *Inst. Math. Sci. N.Y.U.*, Report IMM-NYU 241 (1957).

[8] C. B. Sensenig, Instability of thick elastic shells, *Communs. pure appl. Math.* 17, 451–491 (1964).

[9] C. Truesdell and W. Noll, The non-linear field theories of mechanics, *Encycl. Phys.* Vol. III/3, Springer Verlag, p. 225 (1965).

[10] Z. Wesolowski, Stability of an elastic thick-wall spherical shell loaded by an external pressure, *Arch. Mech. Stosow.*, Vol. 19, pp. 3–44 (1967).

[11] J. L. Nowinski and M. Shahinpoor, Stability of an elastic circular tube of arbitrary wall thickness subjected to an external pressure, *Int. J. Non-linear Mech.* 4, 143–158 (1969).

[12] A. E. Green, R. S. Rivlin and R. T. Shield, General theory of small elastic deformation superposed on finite elastic deformations, *Proc. Roy. Soc.*, A, Vol. 211 (1952).

[13] H. Ziegler, On the concept of elastic stability, *Adv. Appl. Mech.* Vol. 4 (1956); H. Ziegler, ZAMP, Vol. 4 (1953).

[14] Z. H. Guo and W. Urbanowski, Stability of non-conservative systems in the theory of elasticity of finite deformations, *Arch. Mech. Stosow.* 15 (1963).

[15] A. Ertepinar, *Theoretical and experimental studies on shells of arbitrary wall thickness subjected to an external or internal pressure*, Dissertation, Drexel University (1972).

[16] J. W. Dettman, *Mathematical Methods in Physics and Engineering*, McGraw-Hill, New York, pp. 172–182 (1962).

[17] J. L. Nowinski and M. Shahinpoor, Radial oscillations of a thick-walled spherical highly stressed non-homogeneous shell, *J. Franklin Inst.* 291, 293–304 (1971).

[18] J. K. Knowles, On a class of oscillations in the finite deformation theory of elasticity, *J. Appl. Mech.* 29, 283–286 (1962).

[19] A. S. D. Wang and A. Ertepinar, Large amplitude oscillations of laminated thick-walled spherical shells using Galerkin's method, *Solid Mech. Study Series*, Vol. 5, pp. 603–614 (Ed. G. M. L. Gladwell), University of Waterloo (1971).

[20] J. W. S. Rayleigh, *The Theory of Sound*, Vol. 1, Dover (1945).

[21] H. Kraus, *Thin Elastic Shells*, pp. 340–345, Wiley, New York (1967).

[22] W. Flugge, *Stresses in Shells*, pp. 472–478, Springer-Verlag (1967).

[23] C. F. Long, Vibration of a radially loaded spherical shell, *J. Eng. Mech., ASCE* 92, 235–250 (1966).

[24] S. P. Timoshenko and J. M. Gere, *Theory of Elastic Stability*, McGraw-Hill, New York p. 191 (1961).

Résumé—On étudie théoriquement et expérimentalement les vibrations libres et la perte de stabilité de coques épaisses, cylindriques ou sphériques, soumises à une pression externe. Nous formulons les équations générales pour les oscillations de la coque sous pression à partir d'une théorie rigoureuse de l'élasticité finie. La perte de stabilité est déterminée lorsque la valeur de la fréquence fondamentale naturelle de la coque cesse d'être réelle. On obtient des solutions numériques en résolvant les équations particulières qui s'appliquent aux matériaux néo-élastiques. Des expériences utilisant des échantillons faits de caoutchouc au silicone vérifient de près les résultats théoriques.

Zusammenfassung—Theoretische und experimentelle Untersuchungen über die freien Schwingungen und den Stabilitätsverlust dickwandiger zylindrischer und kugelförmiger Schalen unter äusserem Druck werden widergegeben. Allgemeine Gleichungen für die Schwingungen der Schale unter Druck werden mit Hilfe einer strengen Theorie der endlichen Elastizität formuliert. Stabilitätsverlust tritt ein, wenn die Grundeigenfrequenz der vorgespannten Schale aufhört, eine reale Zahl zu sein. Numerische Lösungen werden aufgestellt indem spezielle

Gleichungen gelöst werden, die für neo-Hookesche Materialien anwendbar sind. Experimente an Gummimodellen bestätigen die theoretischen Ergebnisse.

Аннотация—Излагаются теоретические и экспериментальные исследования свободных колебаний и потери устойчивости толстостенных цилиндрических и сферических оболочек под действием внешнего давления. Составляются общие уравнения колебаний оболочки на основе строгой теории упругости для конечных перемещений. Потеря устойчивости определяется когда основная частота собственных колебаний предварительно нагруженной оболочки становится нереальной. Численные решения полнчены путём решения специализированных управнений применимых для материалов Треллоара.

Теоретические результаты точно подтверждаются экспериментами на образцах из силиконовой резины.