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Stability by fixed point theory of impulsive differential equations with delay

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Abstract. In this paper we ensure that for some class of impulsive differential equations with delay the zero solution is asymptotically stable by means of fixed point theory.

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1 Introduction

Differential equations with impulses were considered for the first time by Milman and Myshkis [12] and then followed by a period of active research which culminated with the monograph by Halanay and Wexler [9]. Many phenomena and evolution processes in physics, chemical technology, population dynamics, and natural sciences may change state abruptly or be subject to short-term perturbations. These perturbations may be seen as impulses. Impulsive problems arise also in various applications in communications, mechanics (jump discontinuities in velocity), electrical engineering, medicine and biology. A comprehensive introduction to the basic theory is well developed in the monographs by Benchohra et al. [2], Graef et al. [7,8], Laskshmikantham et al. [10], Samoilenko and Perestyuk [13].

Fixed point theory has been proven to be a powerful tool dealing with the stability of functional differential equations. See, for instance, references [3–6].

In this paper we consider the following impulsive delay equations

$$\begin{cases} x'(t) = -a(t)\,x(t-r), & t \in J = [0,\infty),\ t \neq t_k,\ k = 1,2,\dots, \\ \Delta x_{t=t_k} = I_k(x(t_k^-)), & k = 1,2,\dots, \\ x(t) = \psi(t), & t \in [-r,0]; \end{cases} \tag{1.1}$$

where $a\colon J = [0,\infty) \to \mathbb{R}$ is a bounded and continuous function, r is a positive constant, and

$$\Delta x_{t=t_k} = x(t_k^+) - x(t_k^-),$$

where

$$x(t_k^+) = \lim_{h\to 0^+} x(t_k + h) \quad \text{and} \quad x(t_k^-) = \lim_{h\to 0^+} x(t_k - h).$$

Moreover, we suppose that $0 = t_0$, $t_k < t_{k+1}$ for all $k = 1,2,\dots$ and

$$\lim_{k\to\infty} t_k = \infty.$$

For any function x defined on $[-r,+\infty)$ and any $t \in J$, we denote by x_t the element of $PC([-r,0],\mathbb{R})$ defined by

$$x_t(\theta) = x(t+\theta), \qquad \text{for each} \quad \theta \in [-r,0].$$

Hence, $x_t(\cdot)$ represents the history of the state from time $t-r$, up to the present time t.

The plan of this paper is as follows. In Section 2 we introduce the space of solutions and the integral equations associated to problem (1.1); we also prove our first result of this paper via Banach's fixed point theorem. In Section 3, using Krasnoselskii's fixed point theorem we studied the asymptotically stability of zero solution of problem (1.1). Finally, in Section 4, we investigate the stability of the zero solution for some class of impulsive perturbation problems with delay.

2 Stability via Banach fixed point theorem

Consider the Banach space defined as

$$PC_b = \{y \in PC([-r,\infty),\mathbb{R}) :\ y \text{ is bounded}\},$$

where

$$PC([-r,\infty),\mathbb{R}) = PC_*([0,\infty),\mathbb{R}) \cap PC([-r,0],\mathbb{R})$$

and

$$PC_*([0,\infty),\mathbb{R}) = \{y\colon [-r,\infty) \to \mathbb{R},\ \tilde{y}_k = y_{|(t_k,t_{k+1})} \in C((t_k,t_{k+1}],\mathbb{R})\ \forall k \geq 0,\ \tilde{y}_k(t_k^-) \text{ and } \tilde{y}_k(t_k^+) \text{ exist and satisfy } \tilde{y}_k(t_k) = \tilde{y}_k(t_k^-)\ \forall k \geq 1\}.$$

We consider over PC_b the norm defined by

$$\|x\|_b = \sup\{|x(t)| : t \in [-r,\infty)\} \text{ for each } x \in PC_b.$$

Now $(PC_b, \|\cdot\|_b)$ is a Banach space.

Next we define what we mean by a solution of problem (1.1).

Definition 2.1. *A function $x \in PC([0,\infty),\mathbb{R})$ is said to be a solution of (1.1) if it satisfies*

$$\begin{array}{ll} x'(t) = -a(t)x(t-r), & t \in \mathbb{R}_+ \text{ and } t \neq t_k,\ \ k = 1,2,\dots, \\ \Delta x_{t=t_k} = I_k(x(t_k^-)), & k = 1,2,\dots, \\ x(t) = \psi(t), & t \in [-r,0]. \end{array}$$

Lemma 2.1. *The solution of problem (1.1) satisfies*

$$\begin{aligned} x(t) = {} & x(0)\exp\left(-\int_0^t a(s+r)\,ds\right) + \int_{t-r}^t a(u+r)x(u)\,du \\ & - \exp\left(-\int_0^t a(u+r)\,du\right)\int_{-r}^0 a(u+r)x(u)\,du \\ & - \int_0^t a(s+r)\,\exp\left(-\int_s^t a(u+r)\,du\right)\int_{s-r}^s a(u+r)x(u)\,du\,ds \\ & + \sum_{0<t_k<t} I_k(x(t_k))\,\exp\left(-\int_{t_k}^t a(s+r)\,ds\right), \qquad \text{if} \quad t > 0, \end{aligned}$$

and $x(t) = \psi(t)$, if $t \in [-r,0]$.

Proof. Let $x \in PC([-r,\infty),\mathbb{R})$ be solution of problem (1.1); then for $t \in [0,t_1]$ we have

$$\begin{aligned} & \int_0^t \left[x(s)\exp\left(\int_0^{s+r} a(u)\,du\right)\right]' ds \\ & \quad = \int_0^t \exp\left(\int_0^{s+r} a(u)\,du\right)\left(\frac{d}{ds}\int_{s-r}^s a(u+r)x(u)\,du\right) ds. \end{aligned}$$

Then, integrating by parts on the right-hand side, we obtain

$$\begin{aligned}
&x(t)\,\exp\left(\int_0^{t+r} a(u)\,du\right) - x(0)\,\exp\left(\int_0^{r} a(u)\,du\right)\\
&\quad = \exp\left(\int_0^{t+r} a(u)\,du\right)\int_{t-r}^{t} a(u+r)x(u)\,du\\
&\qquad - \exp\left(\int_0^{r} a(u)\,du\right)\int_{-r}^{0} a(u+r)x(u)\,du\\
&\qquad - \int_0^t \frac{d}{ds}\left[\exp\left(\int_0^{s+r} a(u)\,du\right)\right]\int_{s-r}^{s} a(u+r)x(u)\,du\,ds.
\end{aligned}$$

Thus, we have that,

$$\begin{aligned}
x(t) &= x(0)\,\exp\left(-\int_0^t a(s+r)\,ds\right) + \int_{t-r}^{t} a(u+r)x(u)\,du\\
&\quad - \exp\left(-\int_0^t a(u+r)\,du\right)\int_{-r}^{0} a(u+r)x(u)\,du\\
&\quad - \int_0^t a(s+r)\exp\left(-\int_s^t a(u+r)\,du\right)\int_{s-r}^{s} a(u+r)x(u)\,du\,ds.
\end{aligned}\tag{2.1}$$

For $t \in (t_1, t_2]$, we have

$$\begin{aligned}
&\int_{t_1}^{t}\left[x(s)\,\exp\left(\int_{t_1}^{s+r} a(u)\,du\right)\right]' ds\\
&\qquad = \int_{t_1}^{t}\exp\left(\int_{t_1}^{s+r} a(u)\,du\right)\frac{d}{ds}\int_{s-r}^{s} a(u+r)x(u)\,du\,ds.
\end{aligned}$$

Hence, as in the previous case,

$$\begin{aligned}
&x(t)\,\exp(\int_{t_1}^{t+r} a(u)\,du\Big) - x(t_1^{\,+})\,\exp\left(\int_{t_1}^{t_1+r} a(u)\,du\right)\\
&\quad = \exp\left(\int_{t_1}^{t+r} a(u)\,du\right)\int_{t-r}^{t} a(u+r)x(u)\,du\\
&\qquad - \exp\left(\int_{t_1}^{t_1+r} a(u)\,du\right)\int_{t_1-r}^{t_1} a(u+r)x(u)\,du\\
&\qquad - \int_{t_1}^{t} a(s+r)\exp\left(\int_{t_1}^{s+r} a(u)\,du\right)\int_{s-r}^{s} a(u+r)x(u)\,du\,ds.
\end{aligned}$$

Thus, we conclude that,

$$\begin{aligned} x(t) = &x(t_1^+) \exp\left(-\int_{t_1}^{t} a(s+r)\, ds\right) + \int_{t-r}^{t} a(u+r)x(u)\, du \\ &- \exp\left(-\int_{t_1}^{t} a(u+r)\, du\right) \int_{t_1-r}^{t_1} a(u+r)x(u)\, du \\ &- \int_{t_1}^{t} a(s+r) \exp\left(-\int_{s}^{t} a(u+r)\, du\right) \int_{s-r}^{s} a(u+r)x(u)\, du\, ds. \end{aligned} \tag{2.2}$$

From (2.1) and (2.2), we deduce

$$\begin{aligned} x(t) = &x(0) \exp\left(-\int_{0}^{t} a(s+r)\, ds\right) + \int_{t-r}^{t} a(u+r)x(u)\, du \\ &- \exp\left(-\int_{0}^{t} a(u+r)\, du\right) \int_{-r}^{0} a(u+r)x(u)\, du \\ &- \int_{0}^{t} a(s+r) \exp\left(-\int_{s}^{t} a(u+r)\, du\right) \int_{s-r}^{s} a(u+r)x(u)\, du\, ds \\ &+ I_1(x(t_1)) \exp\left(-\int_{t_1}^{t} a(s+r)\, ds\right). \end{aligned}$$

If we continue this process, we obtain for $t \in [0, b]$ that

$$\begin{aligned} x(t) = &x(0) \exp\left(-\int_{0}^{t} a(s+r)\, ds\right) + \int_{t-r}^{t} a(u+r)x(u)\, du \\ &- \exp\left(-\int_{0}^{t} a(u+r)\, du\right) \int_{-r}^{0} a(u+r)x(u)\, du \\ &- \int_{0}^{t} a(s+r) \exp\left(-\int_{s}^{t} a(u+r)\, du\right) \int_{s-r}^{s} a(u+r)x(u)\, du\, ds \\ &+ \sum_{0<t_k<t} I_k(x(t_k)) \exp\left(-\int_{t_k}^{t} a(s+r)\, ds\right) \qquad \text{if} \quad t > 0. \end{aligned}$$

□

Next we will use Banach fixed point theorem to prove that under some Lipschitz conditions on the jump functions I_k, $k = 1, 2, \ldots$, and for each small initial condition ψ there exists a bounded solution of problem (1.1) which tends to zero as t goes to ∞.

Theorem 2.2. *Assume that the following conditions hold:*

(H1) $\int_{t-r}^{t} |a(u+r)|du + \int_0^t |a(s+r)| \exp\left(-\int_s^t a(u+r)du\right) \int_{s-r}^{s} |a(u+r)|du\, ds \leq \alpha$;

(H2) $\int_0^t a(s+r)\, ds \to \infty$ *as* $t \to \infty$;

(H3) *for each* $k = 1, 2, \dots$ *there exist* $c_k \geq 0$, *such that*

$$|I_k(x) - I_k(y)| \leq c_k |x-y|, \qquad \textit{for all } x, y \in \mathbb{R} \textit{ and } I_k(0) = 0.$$

Then, problem (1.1) has a unique bounded solution in PC_b *which tends to zero as* t *goes to* ∞ *if*

$$\alpha + \sum_{k=1}^{\infty} c_k < 1.$$

Proof. Consider operator $N : PC \to PC$ given by

$$\begin{aligned}
[Nx]\,(t) = {} & \psi(0) \exp\left(-\int_0^t a(s+r)\, ds\right) + \int_{t-r}^{t} a(u+r)x(u)\, du \\
& - \exp\left(-\int_0^t a(u+r)\, du\right) \int_{-r}^{0} a(u+r)x(u)\, du \\
& - \int_0^t a(s+r) \exp\left(-\int_s^t a(u+r)\, du\right) \int_{s-r}^{s} a(u+r)x(u)\, du\, ds \\
& + \sum_{0<t_k<t} I_k(x(t_k)) \exp\left(-\int_{t_k}^{t} a(s+r)\, ds\right), \text{ if } t \in [0, \infty); \\
[Nx]\,(t) = {} & \psi(t), \text{ if } t \in [-r, 0].
\end{aligned}$$

From hypothesis $(H_1) - (H_3)$, we can easily prove that $N(PC_b) \subset PC_b$ and that N is contraction operator. Then, by Banach fixed point theorem, there exists a unique $x \in PC_b$ such that $x = N(x)$, which is solution of problem (1.1). Now we show that x is bounded and tends to zero as t goes to ∞.

Let $t \in [0, \infty)$, then we get,

$$\begin{aligned}
|x(t)| \leq {} & |\psi(0)| \exp\left(-\int_0^t a(s+r)\, ds\right) + \int_{t-r}^{t} |a(u+r)||x(u)|\, du \\
& + \exp\left(-\int_0^t a(u+r)\, du\right) \int_{-r}^{0} |a(u+r)||x(u)|\, du \\
& + \int_0^t |a(s+r)| \exp\left(-\int_s^t a(u+r)\, du\right) \int_{s-r}^{s} |a(u+r)||x(u)|\, du\, ds \\
& + \sum_{0<t_k<t} |I_k(x(t_k))| \exp\left(-\int_{t_k}^{t} a(s+r)\, ds\right).
\end{aligned}$$

Thus,

$$|x(t)| \leq |\psi(0)| \exp\left(-\int_0^t a(s+r)\,ds\right) + \int_{t-r}^t |a(u+r)| \sup_{u\in[0,t]} |x(u)|\,du$$
$$+ \exp\left(-\int_0^t a(u+r)\,du\right) \int_{-r}^0 |a(u+r)| \sup_{u\in[0,t]} |x(u)|\,du$$
$$+ \int_0^t |a(s+r)| \exp\left(-\int_s^t a(u+r)du\right) \int_{s-r}^s |a(u+r)| \sup_{u\in[0,t]} |x(u)| du\,ds$$
$$+ \sum_{0<t_k<t} c_k \sup_{u\in[0,t]} |x(u)|.$$

Hence,

$$|x(t)| \leq |\psi(0)| \exp\left(-\int_0^t a(s+r)\,ds\right)$$
$$+ \alpha \sup_{u\in[0,t]} |x(u)| + \exp\left(-\int_0^t a(u+r)\,du\right) \int_{-r}^0 |a(u+r)| \sup_{u\in[0,t]} |x(u)|$$
$$+ \sum_{0\leq t_k\leq t} c_k \sup_{u\in[0,t]} |x(u)|.$$

This implies that

$$\sup_{t\in[-r,\infty)} |x(t)| \leq \frac{\varepsilon}{1-\sum_{k=1}^{\infty} c_k}.$$

□

3 Stability via Krasnoselskii fixed point theorem

In this section we present a stability result for problem (1.1) via the following theorem.

Theorem 3.1. *([11]) Let $(X, \|\cdot\|)$ be a Banach space. Suppose that A and B are operators mapping X into X such that:*

(i) A is completely continuous;

(ii) B is a contraction operator;

(iii) the set defined as

$$\mathcal{M} = \left\{ x \in X : \; x = \lambda B\left(\frac{x}{\lambda}\right) + \lambda A(x), \;\; \lambda \in (0,1) \right\},$$

is a bounded set.

Then, there exists $x \in X$ with $Ax + Bx = x$.

The following compactness criterion on unbounded domains is a simple extension of a compactness criterion in $PC_b(\mathbb{R}^+, \mathbb{R})$ (see [1]).

Lemma 3.2. *Let $M \subset PC_b$. Then M is relatively compact if it satisfies the following conditions:*

(a) M is uniformly bounded in $PC_b(\mathbb{R}^+, \mathbb{R})$.

(b) The functions belonging to M are almost equicontinuous on $\mathbb{R}^+$, i.e. equicontinuous on every compact interval of $\mathbb{R}^+$.

(c) The functions from M are equiconvergent, that is, given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that

$$|x(\tau_1) - x(\tau_2)| < \varepsilon \quad \textit{for any} \quad \tau_1, \tau_2 \geq T(\varepsilon) \quad \textit{and} \quad x \in M.$$

Theorem 3.3. *Assume that (H_1) and (H_2) hold. Let us also suppose that $(H4)$ there exist α_k, $\beta_k \geq 0$, $k = 1, 2, \ldots$ such that*

$$|I_k(x)| \leq \alpha_k |x| + \beta_k, \qquad \textit{for all } x \in \mathbb{R}.$$

Then, problem (1.1) has a unique solution which is bounded if

$$\alpha + \sum_{k=1}^{\infty} \alpha_k < 1 \qquad \textit{and} \qquad \sum_{k=1}^{\infty} \beta_k < \infty.$$

Moreover, if $\beta_k = 0$ for all $k = 1, 2, \ldots,$ then such solution tends to 0 as t goes to ∞.

Proof. Let $N : PC_b \to PC_b$ be the operator defined in the proof of Theorem 2.2. Consider $N = A + B$, where A, $B : PC_b \to PC_b$ are defined by

$$[Bx](t) = \psi(0) \exp\left(-\int_0^t a(s+r)\,ds\right) + \int_{t-r}^t a(u+r)x(u)\,du$$
$$- \exp\left(-\int_0^t a(u+r)\,du\right) \int_{-r}^0 a(u+r)x(u)\,du$$
$$- \int_0^t a(s+r) \exp\left(-\int_s^t a(u+r)\,du\right) \int_{s-r}^s a(u+r)x(u)\,du\,ds, \text{if } t \in [0,\infty);$$

$[Bx](t) = \psi(t), \text{ if } t \in [-r, 0];$

and

$$[Ax](t) = \sum_{0<t_k<t} I_k(x(t_k)) \exp\left(-\int_{t_k}^{t} a(s+r)\,ds\right), \text{ if } t\in[0,\infty);$$
$$[Ax](t) = 0, \text{ if } t\in[-r,0].$$

Step 1: *B* is a contraction.
Let $x,\, y \in PC_b$, then

$$\begin{aligned}
|[Bx](t) - [By](t)| &\le \left|\int_{t-r}^{t} a(u+r)(x(u)-y(u))\,du\right| \\
&+ \left|\exp\left(-\int_0^t a(u+r)\,du\right)\int_{-r}^{0} a(u+r)\,(x(u)-y(u))\,du\right| \\
&+ \left|\int_0^t a(s+r)\exp\left(-\int_s^t a(u+r)\,du\right)\int_{s-r}^{s} a(u+r)(x(u)-y(u))\,du\,ds\right| \\
&\le \int_{t-r}^{t} |a(u+r)||x(u)-y(u)|\,du \\
&+ \exp(-\int_0^t a(u+r)\,du)\int_{-r}^{0} |a(u+r)||x(u)-y(u)|\,du \\
&+ \int_0^t |a(s+r)|\exp\left(-\int_s^t a(u+r)\,du\right)\int_{s-r}^{s} |a(u+r)||x(u)-y(u)|\,du\,ds.
\end{aligned}$$

Hence, we conclude that,

$$\|Bx - By\|_b \le \alpha\,\|x-y\|_b, \qquad \text{for all} \quad x,\, y\in PC_b.$$

Step 2: *A* is continuous.
Given $(x_n)_{n\in\mathbb{N}}$ a sequence such that $x_n \to x$ in PC_b, there exists $M>0$ such that

$$\|x_n\|_b \le M, \qquad \text{for all} \quad n\in\mathbb{N},$$

and

$$|[Ax_n](t) - [A\phi](t)| \le \sum_{0<t_k<t} |I_k(x_n(t_k)) - I_k(x(t_k))|.$$

Since

$$\sum_{k=0}^{\infty}\alpha_k < \infty \qquad \text{and} \qquad \sum_{k=1}^{\infty}\beta_k < \infty, \tag{3.1}$$

we have that for every $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0}^{\infty} \alpha_k < \frac{\varepsilon}{4\,M} \qquad \text{and} \qquad \sum_{k=k_0}^{\infty} \beta_k < \frac{\varepsilon}{4}.$$

Using the fact that

$$\lim_{k \to \infty} t_k = \infty,$$

we have that there exists $n_0 \in \mathbb{N}$ such that if $k \geq n_0$ then $t_k \geq t_{k_0}$. From (H_4), we get

$$\begin{aligned}\|Ax_n - Ax\|_b &\leq \sum_{0 < t_k \leq t_{n_0 - 1}} |I_k(x_n(t_k)) - I_k(x(t_k))| + \sum_{k=n_0}^{\infty} (2\,M\alpha_k + 2\beta_k)\\ &\leq \sum_{k=1}^{n_0-1} |I_k(x_n(t_k)) - I_k(x(t_k))| + \frac{\varepsilon}{2\,M}\,2\,M + \frac{2\,\varepsilon}{4}.\end{aligned}$$

Since I_k are continuous functions, we have that

$$\sum_{k=1}^{n_0-1} |I_k(x_n(t_k)) - I_k(x(t_k))| \to 0 \qquad \text{as} \quad n \to \infty.$$

Hence,

$$\|Ax_n - Ax\|_b \to 0 \qquad \text{as} \quad n \to \infty.$$

Step 3: From (H_4), we can easily prove that A maps bounded sets into bounded sets in PC_b.

We will now show that $A(M)$ is contained in a compact set.

Step 4: *A* maps bounded sets in *PC_b* into almos, equicontinuous sets in *PC_b*.

Let $r > 0$ and $B_r = \{x \in PC_b : \|y\|_b \leq r\}$ be a bounded set in PC_b. If τ_1, $\tau_2 \in [0, \infty)$, $\tau_1 < \tau_2$, and $x \in B_r$, then we have that

$$[Ax]\,(\tau_1) = \sum_{0 \leq t_k \leq \tau_1} I_k(x(t_k))\, \exp\left(- \int_{t_k}^{\tau_1} a(s+r)\, ds \right)$$

and

$$\begin{aligned}[Ax]\,(\tau_2) &= \sum_{0 \leq t_k \leq \tau_2} I_k(x(t_k))\, \exp\left(- \int_{t_k}^{\tau_2} a(s+r)\, ds \right)\\ &= \sum_{0 \leq t_k \leq \tau_2} I_k(x(t_k))\, \exp\left(- \int_{t_k}^{\tau_1} a(s+r)\, ds - \int_{\tau_1}^{\tau_2} a(s+r)\, ds \right).\end{aligned}$$

Therefore, we deduce,

$$|[Ax](\tau_1) - [Ax](\tau_2)| \le \left[1 - \exp\left(-\int_{\tau_1}^{\tau_2} a(s+r)\,ds\right)\right] \sum_{k=1}^{\infty}(\alpha_k r + \beta_k).$$

Hence, we conclude that

$$|[Ax](\tau_1) - [Ax](\tau_2)| \to 0 \qquad \text{as} \qquad \tau_1 \to \tau_2.$$

Step 5: The set $A(\overline{B_r})$ is equiconvergent.
Let be $x \in B_r$ and $s < t$. Then

$$|[Ax](t) - [Ax](s)| \le \sum_{s \le t_k < t} |I_k(x(t_k))|.$$

Since (3.1) holds, there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0}^{\infty}(\alpha_k r + \beta_k) \le \varepsilon.$$

Hence, we can conclude that

$$|[Ax](t) - [Ax](s)| \le \varepsilon, \qquad \text{for all} \qquad t \ge k_0.$$

Then $A(B_r)$ is equiconvergent. From Lemma 3.2 and Steps 2-4, we conclude that A is completely continuous.

Step 6: The set defined as follows is bounded:

$$\mathcal{M} = \left\{x \in PC_b([0,\infty),\mathbb{R}) : \; x = \lambda B\left(\frac{x}{\lambda}\right) + \lambda A(x), \; \lambda \in (0,1)\right\}.$$

Let $x \in \mathcal{M}$ and $t \in [0,\infty)$. Then, we have

$$\begin{aligned}
|x(t)| \le |\psi(0)| \exp\left(-\int_0^t a(s+r)\,ds\right) + \int_{t-r}^{t} |a(u+r)||x(u)|\,du \\
+ \exp\left(-\int_0^t a(u+r)\,du\right) \int_{-r}^{0} |a(u+r)||x(u)|\,du \\
+ \int_0^t |a(s+r)| \exp\left(-\int_s^t a(u+r)\,du\right) \int_{s-r}^{s} |a(u+r)||x(u)|\,du\,ds \\
+ \sum_{0<t_k<t} |I_k(x(t_k))| \exp\left(-\int_{t_k}^{t} a(s+r)\,ds\right).
\end{aligned}$$

Therefore,

$$|x(t)| \leq \alpha \sup_{s\in[0,t]} |x(s)| + \exp\left(-\int_0^t a(u+r)\,du\right)\int_{-r}^0 |a(u+r)| \sup_{s\in[0,t])} |x(s)|$$
$$+ \sum_{0<t_k<t} \alpha_k \sup_{s\in[0,t]} |x(s)| + \sum_{0<t_k<t} \beta_k + |\psi(0)|.$$

Hence, finally we deduce that

$$|x(t)| \leq \frac{\sum_{k=1}^{\infty} \beta_k}{1-\alpha-\sum_{k=1}^{\infty}\alpha_k}.$$

Now, we can deduce by Theorem 3.1, that problem (1.1) has a bounded solution.

Let ψ be an initial condition and $\varepsilon > 0$ such that $\|\psi\|_\infty \leq \varepsilon$. Then,

$$|x(t)| \leq \varepsilon + \sum_{0\leq t_k\leq t} \alpha_k \sup_{s\in(0,t)} |x(s)|.$$

So,

$$|x(t)| \leq \frac{\epsilon}{1-\sum_{k=1}^{\infty}\alpha_k}.$$

□

4 Perturbation problem

In this section, we will prove the existence of a bounded solution and the asymptotically stability of the following problem:

$$\begin{cases} x'(t) = -a(t)\,x(t-r) + f(t,x_t), & t\in J=[0,\infty),\, t\neq t_k,\, k=1,2,\dots, \\ \Delta x_{t=t_k} = I_k(x(t_k^-)), & k=1,2,\dots, \\ x(t)=\psi(t), & t\in[-r,0]; \end{cases} \tag{4.1}$$

where $f:[0,\infty)\times P([-r,0],\mathbb{R})$ is continuous function.

Theorem 4.1. *Suppose that* (H_1), (H_2) *and the following condition hold: there exist a measurable function* $p: [0,\infty) \longrightarrow \mathbb{R}_+$ *and a positive number* $M \geq 0$ *such that*

$$|f(t,x)| \leq p(t)\|x\|_\infty, \qquad \textit{for all} \quad x \in PC([-r,0],\mathbb{R}),$$

satisfying

$$\int_0^t p(s) \exp\left(-\int_u^s a(l)\,dl\right) ds \leq M, \qquad \textit{for all} \quad t \in [0,\infty)$$

and for every $t \in [0,\infty)$ *the function*

$$t \to f(t,x_t)$$

is measurable. Then, problem (4.1) has at least one solution and all its solutions are bounded. Moreover, if in (H_4) *we have*

$$\sum_{k=1}^{\infty} \beta_k = 0,$$

then for every small initial condition, the solution of problem (4.1) tends to zero as t *goes to* ∞.

Proof. Consider operators N_*, $A_*: PC_b \longrightarrow PC_b$ defined as

$$\begin{aligned}
[N_*x](t) = {}& \psi(0) \exp\left(-\int_0^t a(s+r)\,ds\right) + \int_{t-r}^t a(u+r)x(u)\,du \\
& - \exp\left(-\int_0^t a(u+r)\,du\right) \int_{-r}^0 a(u+r)x(u)\,du \\
& - \int_0^t a(s+r) \exp\left(-\int_s^t a(u+r)\,du\right) \int_{s-r}^s a(u+r)x(u)\,du\,ds \\
& + \sum_{0<t_k<t} I_k(x(t_k)) \exp\left(-\int_{t_k}^t a(s+r)\,ds\right) \\
& + \int_0^t f(s,x_s) \exp\left(-\int_s^t a(u+r)\,du\right) ds, \text{ if } t \in [0,\infty); \\
[N_*x](t) = {}& \psi(t), \text{ if } t \in [-r,0];
\end{aligned}$$

and

$$\begin{aligned}
[A_*x](t) = {}& \sum_{0<t_k<t} I_k(x(t_k)) \exp\left(-\int_{t_k}^t a(s+r)\,ds\right) \\
& + \int_0^t f(s,x_s) \exp\left(-\int_s^t a(u+r)\,du\right) ds, \text{ if } t \in [0,\infty); \\
[A_*x](t) = {}& 0, \text{ if } t \in [-r,0].
\end{aligned}$$

Then, we have that

$$N_*(x) = A_*(x) + B(x), \text{ for each } x \in PC_b,$$

where B was defined in the proof of Theorem 3.3. As in theorem 3.3, we can prove that A_* is completely continuous and from (H_1), we deduce that the operator A is contractive. Also we can easily prove that the set

$$\mathcal{M} = \left\{ x \in PC_b(J, \mathbb{R}) : \ x = \lambda A_* \left(\frac{x}{\lambda} \right) + \lambda B(x), \ \ \lambda \in (0,1) \right\}$$

is bounded. Hence, by Theorem 3.1, problem (4.1) has at least one solution and all its solutions are bounded.

Now we show that for every small initial condition the solution x of problem (4.1) tends to zero as $t \to \infty$.

Let $t \in [0, \infty)$, then, we get

$$\begin{aligned}
|x(t)| \leq \ & |\psi(0)| \exp\left(- \int_0^t a(s+r)\, ds \right) \\
& + \int_{t-r}^t |a(u+r)||x(u)|\, du + \exp\left(- \int_0^t a(u+r)\, du \right) \int_{-r}^0 |a(u+r)||x(u)|\, du \\
& + \int_0^t |a(s+r)| \exp\left(- \int_s^t a(u+r)\, du \right) \int_{s-r}^s |a(u+r)||x(u)|\, du\, ds \\
& + \sum_{0<t_k<t} |I_k(x(t_k))| \exp\left(- \int_{t_k}^t a(s+r)\, ds \right) \\
& + \int_0^t f(s, x_s) \exp\left(- \int_s^t a(u+r)\, du \right) dr.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|x(t)| \leq & \int_{t-r}^t |a(u+r)| \sup_{u \in [0,t]} |(u)|\, du \\
& + \exp\left(- \int_0^t a(u+r)\, du \right) \int_{-r}^0 |a(u+r)| \sup_{u \in [0,t]} |x(u)|\, du \\
& + \int_0^t |a(s+r)| \exp\left(- \int_s^t a(u+r)\, du \right) \int_{s-r}^s |a(u+r)| \sup_{u \in [0,t]} |x(u)|\, du\, ds \\
& + \sum_{0<t_k<t} \alpha_k \sup_{u \in [0,t]} |x(u)| + \int_0^t \|x_s\|_\infty p(s) \exp\left(- \int_s^t a(u+r)\, du \right) ds + |\psi(0)|.
\end{aligned}$$

Hence,

$$\mu(t) \leq \frac{1}{1-\sum_{k=1}^{\infty} \alpha_k} \left(\varepsilon + \int_0^t \mu(s)\, p(s)\, \exp\left(-\int_s^t a(u+r)\, du \right) ds \right),$$

where

$$\mu(t) = \sup\{|x(s)| : \ s \in [-r, t]\}.$$

Finally, by Gronwall's lemma, we obtain

$$|x(t)| \leq \frac{\epsilon}{1-\sum_{k=1}^{\infty} \alpha_k} \left(1 + \frac{M\ \exp(M)\, \varepsilon}{1-\sum_{k=1}^{\infty} \alpha_k} \right).$$

□

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