# Stability, causality, and Lorentz and *CPT* violation

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Stability and causality are investigated for quantum field theories incorporating Lorentz and *CPT* violation. Explicit calculations in the quadratic sector of a general renormalizable Lagrangian for a massive fermion reveal that no difficulty arises for low energies if the parameters controlling the breaking are small, but for high energies either energy positivity or microcausality is violated in some observer frame. However, this can be avoided if the Lagrangian is the sub-Planck limit of a nonlocal theory with spontaneous Lorentz and *CPT* violation. Our analysis supports the stability and causality of the Lorentz- and *CPT*-violating standard-model extension that would emerge at low energies from spontaneous breaking in a realistic string theory.

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## I. INTRODUCTION

Common folklore holds that the low-energy limit of any fundamental theory at the Planck scale is necessarily a local relativistic quantum field theory. If so, this would make it difficult to identify experiments showing directly any structural deviations from usual field theory occurring at the Planck scale, such as might perhaps be expected in string theories. However, this folklore is invalid if the fundamental theory violates one or more of the basic tenets of relativistic field theories. Remnant effects from the Planck scale might then be detectable at low energies, thereby providing valuable experimental information about nature at the smallest scales.

Lorentz symmetry, stability, and causality are examples of features normally expected to hold in physical quantum field theories. In relativistic field theories, stability and causality are closely intertwined with Lorentz invariance. For example, stability includes the need for energy positivity of Fock states of arbitrary momenta, while causality is implemented microscopically by the requirement that observables commute at spacelike separations [1]. Moreover, both energy positivity and microcausality are expected to hold in all observer inertial frames.

Although Lorentz symmetry is well established experimentally, it lacks the essential status of stability and causality. It would be difficult to make meaningful experimental predictions in a theory without either stability or causality, but a stable and causal theory without Lorentz symmetry could in principle still be acceptable. It is therefore worthwhile to consider the possibility that Lorentz symmetry might be violated and to examine the extent to which this violation conflicts with other fundamental properties of field theory. In particular, it would be of interest to establish the existence of a class of theories that incorporate Lorentz violation but that nonetheless maintain both stability and causality.

Lorentz symmetry is also one of the key ingredients in the *CPT* theorem [2]. This states under certain technical conditions that *CPT* is an exact symmetry of local relativistic quantum field theories. It is therefore to be expected that investigations of theories with Lorentz violation include a subset of cases in which *CPT* is also broken.

The present work is motivated by the development over the past decade of a framework allowing for Lorentz and *CPT* violation within realistic models. The basic idea is that spontaneous Lorentz violation could occur in an underlying Lorentz-covariant theory at the Planck scale [3]. Under certain circumstances, this would be accompanied by *CPT* violation. This mechanism appears theoretically viable and is motivated in part by the demonstration that spontaneous Lorentz and *CPT* violation can occur in the context of string theories with otherwise Lorentz-covariant dynamics. Lorentz- and *CPT*-violating effects could therefore provide a unique low-energy signature for qualitatively new physics from the Planck scale.

At presently accessible energy scales, these ideas lead to a phenomenology for Lorentz and *CPT* violation at the level of the standard model and quantum electrodynamics (QED) [4]. A general standard-model extension has been developed that provides a quantitative microscopic framework for Lorentz and *CPT* violation [5]. It preserves the usual $SU(3)\times SU(2)\times U(1)$ gauge structure and is power-counting renormalizable. Energy and momentum are conserved, and conventional canonical methods for quantization apply. The origin of the Lorentz violation in spontaneous symmetry breaking implies that the standard-model extension is covariant under observer Lorentz transformations: rotations or boosts of an observer's inertial frame leave the physics unaffected. The apparent Lorentz violations in the theory are associated with particle Lorentz transformations, which are rotations or boosts of the localized fields in a fixed observer inertial frame.

Since the standard-model extension is formulated at the level of the known elementary particles, it provides a quantitative basis on which to analyze a wide variety of Lorentz and *CPT* tests. In the QED context, investigations to date include tests in Penning traps [6–9], studies of photon birefringence and radiative effects [5,10,11], clock-comparison tests [12–16], experiments with spin-polarized matter [17,18], hydrogen and antihydrogen spectroscopy [19,20], and studies of muons [21,22]. In the broader context of the standard-model extension, studies of neutral-meson systems [23–25], baryogenesis [26], cosmic rays [27,28], and neutrinos [5,27,29] have been performed. Present experimental sensitivities are sufficient to detect Planck-suppressed effects. Moreover, the next generation of tests is expected to improve these results, in some cases by one or more orders of magnitude.

Given the substantial progress on the experimental front, it is of interest to study the regime of validity within which the standard-model extension can be applied directly and to develop a methodology for handling the corrections that are expected at high energies. Initiating this program is one of the goals of the present work. The point is that the standard-model extension contains the low-energy limit of any realistic fundamental theory incorporating spontaneous Lorentz and $CPT$ violation, and on general grounds it is expected to have a range of validity comparable to that of the standard model at sub-Planck energies. However, as Planck energies are approached, nonrenormalizable operators negligible at low energies should acquire importance. Since stability and causality are deeply related to Lorentz symmetry at the level of renormalizable quantum field theory, imposing them as requirements at high scales in the context of the standard-model extension might be expected to yield interesting insights into the structure of the nonrenormalizable terms.

The present work contains an investigation of the role of stability and causality in Lorentz- and $CPT$-violating theories, with particular emphasis on notions relevant to the fermion sector of the standard-model extension. We approach the subject by studying the quadratic fermion part of a general renormalizable Lagrangian with explicit Lorentz- and $CPT$-breaking terms. It is the single-fermion limit of the free-matter sector in the general standard-model extension. As a necessary part of the analysis, we develop further the results of Ref. [5] on the relativistic quantum mechanics of this theory and perform the corresponding free-field quantization. These results provide a complete quantization of the free-fermion sector of the Lorentz- and $CPT$-violating QED extension, including details such as the explicit general form of the one-particle dispersion relation. Interactions can be handled in the usual perturbative manner [5].

One of our goals is to establish the nature of the difficulties facing theories with explicit Lorentz violation, however small. We find violations of stability or causality occur for momenta outside a scale determined by the size of the explicit breaking terms. Although the scale in question may be large, consistency problems are typically present for any conventional quantum field theory of fermions with explicit Lorentz violation [30].

Another goal is to understand the mechanism by which spontaneous Lorentz breaking in string theory could overcome these difficulties. By itself, spontaneous Lorentz violation is an important ingredient. However, avoiding the problems with stability and causality seems to require in addition its transcendental suppression at high energies in the one-particle dispersion relations, through the appearance of nonrenormalizable terms that are unimportant at low energies. Interestingly, this requirement naturally leads to field interactions of a type related to those found in string field theory.

The analysis in this work leaves unaddressed several interesting theoretical issues associated with the transition from a fundamental theory with spontaneous Lorentz and $CPT$ violation at the Planck scale to the standard-model extension. These include the development of the observed hierarchy of scales in nature, the role of fluctuations about the tensor expectation values generating the extra terms in the standard-model extension, the explicit incorporation of gravity, and implications of nonminimality in the usual standard model such as supersymmetry and gauge-group unification. Although important in the development of a complete understanding, these issues lie beyond the present scope.

The results developed in this work provide both a guide to the regime of validity of theories with explicit Lorentz violation and insight into the nature of the expected nonrenormalizable corrections to the standard-model extension emerging as the Planck scale is approached. The twin demands of stability and causality lead from a renormalizable field theory to a nonlocal theory incorporating spontaneous Lorentz breaking. This supports the idea that the experimental observation of Lorentz violation would provide unique evidence for the nonlocality of nature at the Planck scale.

The outline of this paper is as follows. Some basics are provided in Sec. II. Section III studies relativistic quantum mechanics in a class of convenient inertial frames. Section IV performs the canonical quantization of the field theory and investigates stability and causality in arbitrary frames. The issue of how the associated problems are resolved in the context of spontaneous Lorentz and $CPT$ breaking in a fundamental theory is discussed in Sec. V. Finally, a summary is provided in Sec. VI. Throughout, we adopt the notations and conventions of Ref. [5].

## II. SOME BASICS

In this section, we provide background material and introduce some basic information used in later sections of this work. Some of this material is discussed in more detail in Ref. [5].

A general form for the quadratic sector of a renormalizable Lorentz- and $CPT$-violating Lagrangian describing a single massive spin-$\frac{1}{2}$ Dirac fermion is [5]

$$\mathcal{L}=\frac{1}{2}i\overline{\psi}\Gamma^{\nu}\overleftrightarrow{\partial}_{\nu}\psi-\overline{\psi}M\psi, \tag{1}$$

where

$$\Gamma^{\nu}:=\gamma^{\nu}+c^{\mu\nu}\gamma_{\mu}+d^{\mu\nu}\gamma_5\gamma_{\mu}+e^{\nu}+if^{\nu}\gamma_5+\frac{1}{2}g^{\lambda\mu\nu}\sigma_{\lambda\mu} \tag{2}$$

and

$$M:=m+a_{\mu}\gamma^{\mu}+b_{\mu}\gamma_5\gamma^{\mu}+\frac{1}{2}H^{\mu\nu}\sigma_{\mu\nu}. \tag{3}$$

In the above equations, the gamma matrices $1, \gamma_5, \gamma^{\mu}, \gamma_5\gamma^{\mu}, \sigma^{\mu\nu}$ have conventional properties. In the context of the standard-model and QED extensions, the parameters $a_{\mu}, b_{\mu}, c_{\mu\nu}, \ldots, H_{\mu\nu}$ are determined by expectation values of Lorentz tensors arising from spontaneous Lorentz breaking in a more fundamental theory.

For definiteness, it is assumed throughout this work that the mass $m$ of the fermion is nonzero. Our methods can in many cases be directly extended to the massless situation, although the distinctions between finite- and zero-mass representations of the Lorentz group introduce some additional complications that lie beyond our present scope. In any case, for most applications in the context of the fermionic sector of the standard-model extension, a nonzero mass is appropriate. One possible exception is the study of neutrinos, including neutrino oscillations. If neutrinos have mass then the results below can be applied, with minor modifications for Majorana fermions as necessary. If one or more neutrinos are massless, then more care may be required.

Hermiticity of the Lagrangian (1) implies that the coefficients for Lorentz violation are all real. Moreover, $c_{\mu\nu}$ and $d_{\mu\nu}$ can be taken as traceless, $g_{\lambda\mu\nu}$ antisymmetric in its first two indices, and $H_{\mu\nu}$ antisymmetric. All the parameters violate particle Lorentz invariance, while $a_\mu$, $b_\mu$, $e_\mu$, $f_\mu$, $g_{\lambda\mu\nu}$ also break $CPT$. The coefficients in Eq. (2) are dimensionless, while those in Eq. (3) have dimensions of mass. The reader is warned that field redefinitions may eliminate some of these coefficients without altering the physics [5]. For example, introducing a nonzero coefficient $a_\mu$ in a single-fermion theory such as Eq. (1) has no observable consequences. However, $a_\mu$-type coefficients can lead to physical effects in more general multifermion theories, including the standard-model extension. For completeness, we explicitly keep all terms in Eq. (1) in the present work.

The Lagrangian (1) is independent of the coordinate system. Observations made by any two inertial observers can be related by coordinate transformations, called observer Lorentz transformations. Since Eq. (1) is a scalar under these transformations, the theory exhibits observer Lorentz symmetry. However, in Eq. (1) observer coordinate transformations differ profoundly from boosts and rotations of particles or localized fields within a fixed inertial frame. The latter transformations, called particle Lorentz transformations, leave invariant the coefficients $a_\mu, b_\mu, \ldots, H_{\mu\nu}$ and so can modify the physics [31]. The particle Lorentz symmetry is therefore broken.

At the level of the present discussion, the observer Lorentz symmetry of the theory (1) is a consequence of choosing a Lagrangian invariant under Lorentz coordinate transformations. More general classes of theories with explicit Lorentz violation could in principle be considered. For example, the Lagrangian might be taken to transform nontrivially under the observer Lorentz group, or perhaps as a scalar under some non-Lorentz coordinate transformation. However, these possibilities represent radical departures from conventional physics and lack motivation. In contrast, the explicit Lorentz-violating terms in the Lagrangian (1) could arise from a more fundamental theory with a Lagrangian invariant under both observer and particle Lorentz symmetry, provided the interactions in the theory are such as to cause spontaneous Lorentz breaking. If so, then the coefficients $a_\mu, b_\mu, \ldots, H_{\mu\nu}$ for Lorentz and $CPT$ violation are related to vacuum expectation values of Lorentz tensor fields in the underlying theory, and Eq. (1) becomes a low-energy approximation to this theory in the Lorentz-breaking vacuum. The Lagrangian (1) therefore serves as a single-fermion model for the potentially realistic situation in which the standard-model extension emerges as the low-energy limit of spontaneous Lorentz violation in a fundamental theory at the Planck scale.

The distinction between observer and particle Lorentz transformations implies a dual role for Lorentz symmetry in studying stability and causality of Eq. (1). Thus, if a theory is to be stable and causal, then in a specified observer frame the implications of energy positivity and microcausality should hold for fields of different momenta related through particle Lorentz transformations, while energy positivity and microcausality should hold in arbitrary inertial frames related by observer Lorentz transformations. In later sections, it emerges that these two roles can be distinct. For example, a theory with spacelike 4-momentum for some one-particle states may maintain energy positivity under particle Lorentz transformations in a fixed frame, but it will violate this requirement in certain other frames obtained by suitable observer Lorentz transformations.

Since the various coefficients for Lorentz violation in Eq. (1) carry Minkowski indices, they vary with the observer as appropriate representations of the noncompact Lorentz group SO(3,1) and are in this sense unbounded. For some purposes, it is useful to introduce a special class of inertial frames in which the coefficients for Lorentz and $CPT$ violation represent only a small perturbation relative to the ordinary Dirac case. We call a member of this class of frames a *concordant frame*. If Lorentz and $CPT$ violation does indeed occur in nature, then on experimental grounds it must be true that any inertial frame in which the Earth moves nonrelativistically can serve as a concordant frame. The point is that no departures from Lorentz and $CPT$ symmetry have been observed to date, so any Lorentz and $CPT$ violation in an Earth-based laboratory must be minuscule, with the coefficients appearing in Eq. (2) much smaller than 1 and those in Eq. (3) much smaller than $m$.

In the present scenario, the Lorentz- and $CPT$-violating effects are regarded as originating in a more fundamental theory at some large scale $M_P$. It is plausible that $M_P$ is the Planck scale, since this is the natural scale for an underlying theory including gravity, and in what follows we refer to it as such. In any case, it is expected that observable effects in a low-energy theory with scale $m$ that arise from a fundamental theory with scale $M_P$ would be suppressed by some power of the ratio $m/M_P$. It is therefore likely that the order of magnitude of the coefficients appearing in Eq. (2) is no greater than $m/M_P$, while that of the coefficients in Eq. (3) is no greater than $m^2/M_P$.

In conventional special relativity, all inertial frames are equivalent in the sense that high-energy physics in one frame is in one-to-one correspondence with high-energy physics in any other frame. However, this equivalence fails in the present context. The coefficients for Lorentz and $CPT$ violation experienced by a high-energy particle in one frame can differ substantially from those experienced by a high-energy particle in a second frame because the particle Lorentz symmetry is broken. In particular, this means that statements restricting attention to Lorentz- and *CPT*-violating effects at high energies may be observer dependent.

Given this ambiguity in the conventional notion of high energy, it is useful to introduce a more precise definition. For purposes of the present work, the terminology of high and low energies relative to the scale of the underlying theory is always taken to refer to a concordant frame as defined above. From an experimental point of view, this terminology is sensible because by observation a laboratory frame moves nonrelativistically with respect to a concordant frame. The physics of high energies is therefore similar in both frames.

## III. RELATIVISTIC QUANTUM MECHANICS

In this section, we study the Lagrangian (1) in the context of relativistic quantum mechanics. The corresponding Hermitian Hamiltonian is derived, and the associated dispersion relation is obtained. We discuss properties of the eigenspinors and determine the general solution of the equations of motion. Throughout this section, we work exclusively in a concordant frame as defined in Sec. II.

### A. Hamiltonian

The construction of the relativistic quantum Hamiltonian $H$ from the Lagrangian $\mathcal{L}$ of Eq. (1) requires care because $\mathcal{L}$ contains time-derivative terms in addition to the usual one. In the concordant frame and a large class of associated observer frames, this difficulty can be resolved by a spinor redefinition chosen to eliminate the time-derivative couplings [6]. Writing $\psi = A\chi$, we require the non-singular matrix $A$ to be spacetime independent and to satisfy

$$A^\dagger \gamma^0 \Gamma^0 A = I, \tag{4}$$

where $I$ is the $4\times 4$ unit matrix. With this choice, $\mathcal{L}[\chi]$ contains no time derivatives outside the usual term $\frac{1}{2} i \overline{\chi} \gamma^0 \overleftrightarrow{\partial}_0 \chi$. This spinor redefinition amounts to a change of basis in spinor space, and as such it leaves unchanged the physics. Note that its explicit form depends on the choice of inertial frame.

It can be shown that $A$ exists if and only if all the eigenvalues of $\gamma^0\Gamma^0$ are positive. First, recall that an equivalence relation of the form $A^\dagger X A = Y$ between Hermitian matrices $X,Y$ is called a congruence [32]. In the present case, since both $I$ and $\gamma^0\Gamma^0$ are Hermitian, $A$ exists if and only if $\gamma^0\Gamma^0$ is congruent to $I$. Next, recall Sylvester's law of inertia, which implies that under a congruence the number of positive eigenvalues of a Hermitian matrix is invariant. Since $I$ has all positive eigenvalues, the claimed result holds.

It follows that $A$ always exists in the concordant frame. Define a matrix $\epsilon^0$ such that the zero component of Eq. (2) can be written in the form $\Gamma^0 = \gamma^0(I+\epsilon^0)$. Since the components of $\epsilon^0$ are small compared to 1 in the concordant frame by definition, the eigenvalues of $\gamma^0\Gamma^0 = I + \epsilon^0$ are indeed positive and $A$ therefore exists.

In Appendix A, we obtain an upper bound on the size of the coefficients for Lorentz and *CPT* breaking such that $A$ can exist. The bound is expressed in terms of a quantity $\delta^0$, defined as the largest absolute value of certain coefficients for Lorentz and *CPT* violation:

$$\delta^0 = \max_{\mu\nu}\{|c_{\mu 0}|,|d_{\mu 0}|,|e_0|,|f_0|,|g_{\mu\nu 0}|\}. \tag{5}$$

We prove that $\delta^0 < 1/480$ suffices for the spinor redefinition to exist. The numerical value of this bound is far larger than the maximum size of $\delta^0$ likely to be allowed on experimental grounds, showing that the spinor redefinition indeed exists for the realistic situation. Although it is sufficient for our purposes, this bound is not sharp. A determination of the sharp bound would be of interest. We conjecture it is of order 1.

Once the spinor redefinition has been performed, the Euler-Lagrange equations generate a modified Dirac equation in terms of the new spinor $\chi$. It can be written as

$$(i\partial_0 - H)\chi = 0, \tag{6}$$

where the Hamiltonian

$$H = -A^\dagger \gamma^0 (i\Gamma^j \partial_j - M)A \tag{7}$$

is Hermitian, as desired. Explicit forms for this Hamiltonian can be found in Ref. [12].

### B. Dispersion relation

As usual, a solution to Eq. (6) is a superposition of plane waves of the form

$$\chi(x) = e^{-i\lambda_\mu x^\mu} w(\vec{\lambda}). \tag{8}$$

Here, the 4-spinor $w(\vec{\lambda})$ must obey

$$(\lambda_0 - H) w(\vec{\lambda}) = 0, \tag{9}$$

where $H$ is now understood to be in $\lambda$-momentum space, and $\lambda_\mu$ must satisfy the dispersion relation

$$\det(\lambda_0 - H) = 0. \tag{10}$$

An alternative equivalent form for the dispersion relation is

$$\det(\Gamma^\mu \lambda_\mu - M) = 0, \tag{11}$$

since the non-singular matrices $\gamma^0$, $A$, and $A^\dagger$ relating the two forms of the Dirac equations contribute only overall multiplicative factors to the determinant.

To obtain an explicit expression for the dispersion relation, we write the matrix $\Gamma^\mu\lambda_\mu - M$ as

$$\Gamma^\mu \lambda_\mu - M = S + iP\gamma_5 + V^\mu \gamma_\mu + A^\mu \gamma_5 \gamma_\mu + T^{\mu\nu}\sigma_{\mu\nu}, \tag{12}$$

where we have introduced

$$S = e^{\mu}\lambda_{\mu} - m, \quad P = f^{\mu}\lambda_{\mu},$$

$$V^{\mu} = \lambda^{\mu} + c^{\mu\nu}\lambda_{\nu} - a^{\mu}, \quad A^{\mu} = d^{\mu\nu}\lambda_{\nu} - b^{\mu},$$

$$T^{\mu\nu} = \frac{1}{2} g^{\mu\nu\rho}\lambda_{\rho} - \frac{1}{2} H^{\mu\nu}. \tag{13}$$

Expansion of the determinant of this matrix yields

$$0 = 4(V_{\mu}A_{\nu} - A_{\mu}V_{\nu} - V_{\mu}V_{\nu} + A_{\mu}A_{\nu} + PT_{\mu\nu} - S\tilde{T}_{\mu\nu} + T_{\mu\alpha}T^{\alpha}{}_{\nu} + \tilde{T}_{\mu\alpha}\tilde{T}^{\alpha}{}_{\nu})^2 + (V^2 - A^2 - S^2 - P^2)^2 - 4(V^2 - A^2)^2 + 6(\epsilon_{\mu\nu\alpha\beta}A^{\alpha}V^{\beta})^2, \tag{14}$$

where $\tilde{T}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}T_{\alpha\beta}$ denotes the dual tensor.

The dispersion relation (14) can be viewed as a quartic equation for $\lambda^0(\vec{\lambda})$. In principle, it permits the explicit determination of the exact eigenenergies of a particle with given 3-momentum in the presence of Lorentz and *CPT* violation. Various approximate solutions can also be obtained. For example, in certain applications only the leading-order corrections to the conventional eigenenergies are of interest. However, we caution the reader that these cannot necessarily be obtained by keeping only leading contributions to the coefficients of the momentum in the dispersion relation and solving for the energies, as is argued in some of the published literature [33].

Many of the relevant properties of the dispersion relation can be established without an explicit algebraic solution. For example, since $H$ is Hermitian all four roots of the dispersion relation must be real. It follows from Eq. (11) that the roots are independent of the spinor redefinition (4), as expected. This equation also implies that the dispersion relation is observer Lorentz invariant and hence that $\lambda_{\mu}$ must be an observer Lorentz 4-vector.

In general, the fourfold degeneracy of the magnitudes of the roots of Eq. (11) is lifted, a feature different from the conventional Dirac case. Since the Lorentz and *CPT* violation is small in the concordant frame, one still anticipates two positive roots $\lambda^0_{+(\alpha)}(\vec{\lambda})$, $\alpha = 1,2$, and two negative roots $\lambda^0_{-(\alpha)}(\vec{\lambda})$. In Appendix B, we obtain a bound on the size of the coefficients for Lorentz and *CPT* violation such that this anticipation is correct. The bound is in terms of a quantity $\delta$, defined as

$$\delta = \max_{\mu,\nu,j}\{|a_{\mu}|, |b_{\mu}|, m|c_{\mu j}|, m|d_{\mu j}|, m|e_j|, m|f_j|, m|g_{\mu\nu j}|, |H_{\mu\nu}|\}, \tag{15}$$

where the Greek indices range from 0 to 3 and the Latin index ranges from 1 to 3, as usual. We find that for $\delta < m/124$ the dispersion relation has two positive and two negative solutions, as usual. This bound is independent of the spinor redefinition. Its numerical value is much larger than experimental observations are likely to allow, showing that the presence of Lorentz and *CPT* violation in nature would indeed leave unaffected the counting of positive- and negative-energy solutions. Although more than adequate for our purposes, this bound is not sharp, and it would be of interest to determine the sharp bound. We anticipate it is of order 1.

Another important feature of the dispersion relation is the correspondence

$$\lambda^0_{-(1,2)}(\vec{\lambda}, a_{\mu}, d_{\mu\nu}, e_{\mu}, f_{\mu}, H_{\mu\nu}) = -\lambda^0_{+(2,1)}(-\vec{\lambda}, -a_{\mu}, -d_{\mu\nu}, -e_{\mu}, -f_{\mu}, -H_{\mu\nu}) \tag{16}$$

between the positive and negative solutions. In this equation, we have displayed only the dependence on the coefficients for Lorentz and *CPT* violation that change sign, and it is understood that the other coefficients are held constant. The numbering of the roots is chosen to agree with the results in Ref. [5]. Equation (16) can be regarded as a consequence of the identity $\det(\Gamma^{\mu}\lambda_{\mu} - M) = \det[C(\Gamma^{\mu}\lambda_{\mu} - M)C^{-1}]$, where $C$ is the usual charge-conjugation matrix. This implies the invariance of $\det(\Gamma^{\mu}\lambda_{\mu} - M)$ under the transformation

$$\{\vec{\lambda}, a_{\mu}, d_{\mu\nu}, e_{\mu}, f_{\mu}, H_{\mu\nu}\} \to \{-\vec{\lambda}, -a_{\mu}, -d_{\mu\nu}, -e_{\mu}, -f_{\mu}, -H_{\mu\nu}\} \tag{17}$$

and leads to the correspondence (16).

### C. Eigenspinors

The eigenfunctions corresponding to the two negative roots $\lambda^0_{-(\alpha)}$ can be reinterpreted as positive-energy reversed-momentum wave functions in the usual way. We define

$$\chi_{+}^{(\alpha)} = \exp(-ip_u^{(\alpha)} \cdot x) u^{(\alpha)}(\vec{p}),$$

$$\chi_{-}^{(\alpha)} = \exp(+ip_v^{(\alpha)} \cdot x) v^{(\alpha)}(\vec{p}), \tag{18}$$

where $u^{(\alpha)}(\vec{p})$ and $v^{(\alpha)}(\vec{p})$ are momentum-space spinors and the 4-momenta are given by

$$p_u^{(\alpha)} = (E_u^{(\alpha)}, \vec{p}), \quad E_u^{(\alpha)}(\vec{p}) = \lambda^0_{+(\alpha)}(\vec{p}),$$

$$p_v^{(\alpha)} = (E_v^{(\alpha)}, \vec{p}), \quad E_v^{(\alpha)}(\vec{p}) = -\lambda^0_{-(\alpha)}(-\vec{p}). \tag{19}$$

The symmetry (16) of the dispersion relation determines a relationship between the two sets of energies. We find

$$E_v^{(1,2)}(\vec{p}, a_{\mu}, d_{\mu\nu}, e_{\mu}, f_{\mu}, H_{\mu\nu}) = E_u^{(2,1)}(\vec{p}, -a_{\mu}, -d_{\mu\nu}, -e_{\mu}, -f_{\mu}, -H_{\mu\nu}). \tag{20}$$

Similarly, the spinors are related by

$$v^{(1,2)}(\vec{p}, a_{\mu}, d_{\mu\nu}, e_{\mu}, f_{\mu}, H_{\mu\nu}) = u^{(2,1)c}(\vec{p}, -a_{\mu}, -d_{\mu\nu}, -e_{\mu}, -f_{\mu}, -H_{\mu\nu}), \tag{21}$$

where the superscript $c$ denotes a charge-conjugate spinor defined by $w^c = C\overline{w}^T$, as usual.

The spinors $u$ and $v$ are the eigenvectors of the Hermitian matrix $H$ and they therefore span the spinor space. Orthogonality of the eigenspinors is automatic for nondegenerate eigenenergies and in any case can be imposed by choice. The normalization of $u$ and $v$ is constrained by the requirement $(\chi^c)^c = \chi$ but is otherwise arbitrary. For definiteness, we choose the conditions

$$u^{(\alpha)\dagger}(\vec{p})u^{(\alpha')}(\vec{p}) = \delta^{\alpha\alpha'}\frac{E_u^{(\alpha)}}{m},$$

$$v^{(\alpha)\dagger}(\vec{p})v^{(\alpha')}(\vec{p}) = \delta^{\alpha\alpha'}\frac{E_v^{(\alpha)}}{m},$$

$$u^{(\alpha)\dagger}(\vec{p})v^{(\alpha')}(-\vec{p}) = 0. \tag{22}$$

Note, however, that the conventional generalization of the orthogonality relation involving the Dirac-conjugate spinors $\overline{u}$ and $\overline{v}$ fails in the present case. Equation (22) implies the completeness relation

$$\sum_{\alpha=1}^{2}\left(\frac{m}{E_u^{(\alpha)}(\vec{p})}u^{(\alpha)}(\vec{p})\otimes u^{(\alpha)\dagger}(\vec{p}) + \frac{m}{E_v^{(\alpha)}(-\vec{p})}v^{(\alpha)}(-\vec{p})\otimes v^{(\alpha)\dagger}(-\vec{p})\right) = I. \tag{23}$$

With the above definitions, the general solution to the modified Dirac equation (6) can be written as

$$\chi(x) = \int \frac{d^3p}{(2\pi)^3}\sum_{\alpha=1}^{2}\left(\frac{m}{E_u^{(\alpha)}}b_{(\alpha)}(\vec{p})\exp(-ip_u^{(\alpha)}\cdot x)u^{(\alpha)}(\vec{p}) + \frac{m}{E_v^{(\alpha)}}d^*_{(\alpha)}(\vec{p})\exp(+ip_v^{(\alpha)}\cdot x)v^{(\alpha)}(\vec{p})\right), \tag{24}$$

where $b_{(\alpha)}(\vec{p})$ and $d^*_{(\alpha)}(\vec{p})$ are Fourier coefficients, as usual. For simplicity, the dependence of the eigenenergies and eigenspinors on the coefficients for Lorentz and $CPT$ violation is suppressed in this equation.

## IV. QUANTUM FIELD THEORY

In this section, we perform canonical quantization in a concordant frame by demanding energy positivity, as usual. We then study the issues of stability and causality in arbitrary frames.

### A. Canonical quantization and energy positivity

In the usual case, straightforward canonical quantization of a Dirac fermion is inadequate because the theory is singular. Appropriate quantization conditions can be found either by requiring the positivity of the conserved energy or, more formally, by extending the Dirac-bracket procedure to anticommuting fields [34]. We adopt the former procedure here.

We promote the complex weights in the expansion (24) to operators on a Fock space. The spinor $\chi$ thereby becomes a quantum field, as does the spinor $\psi$. The two fields are related through the redefinition $\psi = A\chi$, where $A$ is the same matrix discussed in the previous subsection.

We impose the following nonvanishing anticommutation relations:

$$\{b_{(\alpha)}(\vec{p}), b^\dagger_{(\alpha')}(\vec{p}\,')\} = (2\pi)^3\frac{E_u^{(\alpha)}}{m}\delta_{\alpha\alpha'}\delta(\vec{p}-\vec{p}\,'),$$

$$\{d_{(\alpha)}(\vec{p}), d^\dagger_{(\alpha')}(\vec{p}\,')\} = (2\pi)^3\frac{E_v^{(\alpha)}}{m}\delta_{\alpha\alpha'}\delta(\vec{p}-\vec{p}\,'). \tag{25}$$

These can be used to reconstruct the equal-time anticommutators for the fields $\chi$:

$$\{\chi_j(t,\vec{x}), \overline{\chi}_l(t,\vec{x}\,')\gamma^0_{lk}\} = \delta_{jk}\delta^3(\vec{x}-\vec{x}\,'),$$

$$\{\chi_j(t,\vec{x}), \chi_k(t,\vec{x}\,')\} = \{\overline{\chi}_l(t,\vec{x})\gamma^0_{lj}, \overline{\chi}_m(t,\vec{x}\,')\gamma^0_{mk}\} = 0, \tag{26}$$

where the spinor indices $j,k,l,m$ are displayed for clarity.

The above expressions permit the derivation of the equal-time anticommutators for the original fields $\psi$ as

$$\{\psi_j(t,\vec{x}), \overline{\psi}_l(t,\vec{x}\,')\Gamma^0_{lk}\} = \delta_{jk}\delta^3(\vec{x}-\vec{x}\,'),$$

$$\{\psi_j(t,\vec{x}), \psi_k(t,\vec{x}\,')\} = \{\overline{\psi}_l(t,\vec{x})\Gamma^0_{lj}, \overline{\psi}_m(t,\vec{x}\,')\Gamma^0_{mk}\} = 0. \tag{27}$$

Note that $\pi_\psi = \overline{\psi}\Gamma^0$ is the canonical conjugate of $\psi$, paralleling the usual Dirac case.

The vacuum state $|0\rangle$ of the Hilbert space in the concordant frame is defined by

$$b_{(\alpha)}(\vec{p})|0\rangle = 0, \quad d_{(\alpha)}(\vec{p})|0\rangle = 0. \tag{28}$$

The action of the creation operators $b^\dagger_{(\alpha)}(\vec{p})$ and $d^\dagger_{(\alpha)}(\vec{p})$ on $|0\rangle$ produces states describing particles and antiparticles with 4-momenta $p_u^{(\alpha)}$ and $p_v^{(\alpha)}$, respectively. This can be verified using the normal-ordered conserved momentum

$$P_\mu = \int d^3x :\Theta_{\mu 0}:, \tag{29}$$

where

$$\Theta_{\mu\nu} = \frac{1}{2}i\overline{\psi}\Gamma_\mu \overleftrightarrow{\partial}_\nu\psi \tag{30}$$

is the conserved canonical energy-momentum tensor.

In the present context, the one-particle states carry the 4-momenta $p_u^{(\alpha)}$ and $p_v^{(\alpha)}$ introduced in the previous section. It follows from Eq. (19) that the zero components of these 4-vectors are positive definite. This validates the quantization ansatz (25) in the concordant frame.

The Lagrangian (1) is observer Lorentz invariant by construction. The observables resulting from quantization should therefore be invariant or depend covariantly on the observer. In the usual case, Lorentz transformations are unitarily implemented on the Hilbert space of states, and so covariance follows directly. In contrast, in the present case the coefficients for Lorentz and *CPT* violation carry spacetime indices, and their values therefore depend on the observer. This implies that the Fock spaces constructed by different observers are inequivalent. Nonetheless, the invariance of observables may be implemented by suitable mappings between the Fock spaces for any two observers. These mappings then form a representation of the Lorentz group with group multiplication being the mapping composition. Note that the existence of this group structure is assured if the Lorentz violation is spontaneous. In this case, although the observer Lorentz symmetry cannot be unitarily implemented on the Fock space, the freedom to select the physical vacuum among all Lorentz-equivalent choices means that all observers have Fock spaces in one-to-one correspondence.

The field quantization presented above can be performed provided the bounds on $\delta^0$ and $\delta$ in Sec. III are satisfied, so that the Lorentz-violating time-derivative terms can be removed and the usual eigenenergy-sign structure holds. These conditions involve the size of individual components of observer Lorentz tensors and are thus inherently noninvariant under observer Lorentz transformations. There is therefore a class of observers, strongly boosted relative to a concordant frame, for whom these bounds are violated and the present technique of field quantization fails. However, as discussed above, the observer Lorentz invariance guarantees a one-to-one correspondence of the Fock spaces among all observers, so some difficulties must also exist even for the quantization scheme in a concordant frame. It turns out these are associated with the stability and causality of the theory. The next two subsections discuss these issues in detail.

### B. Stability

In usual Lorentz-covariant free-field theories, energy positivity in a particular frame translates under certain assumptions to the statement that the vacuum is stable in any frame. One assumption is that the 4-momenta of all one-particle states in the particular frame are timelike or lightlike with nonnegative 0th components. This is satisfied in the usual Dirac theory. Since an observer Lorentz transformation cannot change the sign of these 0th components, energy positivity is in this case a Lorentz-invariant notion even though it is a statement about a 4-vector component.

In the present case with Lorentz and *CPT* violation, energy positivity in a concordant frame is assured if the bound on $\delta$ discussed in Sec. III B is satisfied. However, stability of the quantized theory in all observer frames requires more than just energy positivity in a concordant frame. In fact, one of the usual assumptions fails: some of the energy-momentum 4-vectors solving the dispersion relation (11) may under certain circumstances be spacelike in all observer frames.

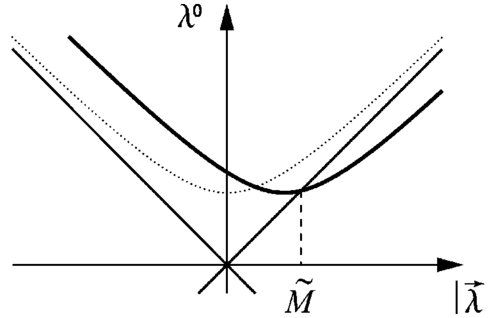


FIG. 1. Dispersion relation for a model with only a large nonzero $b_0$ in a concordant frame. One of the two positive roots is displayed. It intersects the light cone at a 3-momentum of magnitude $\tilde{M}$. The dotted line is the conventional dispersion relation for a massive particle.

As an example, consider the dispersion relation

$$(\lambda^2-b^2-m^2)^2+4b^2\lambda^2-4(b\cdot\lambda)^2=0 \tag{31}$$

for a model with a $b_\mu$ coefficient only. One can show that for any nonzero $b_\mu$, no matter how small, it is always possible to choose an observer frame in which $b_\mu=(b_0,0,0,b_3)$ and $b_3^2>m^2+|b^\mu b_\mu|$. Defining the real quantities $p_\pm$ by

$$p_\pm^2=(2b_3^2+b^2-m^2)\pm\sqrt{(2b_3^2+b^2-m^2)^2-(m^2+b^2)^2}, \tag{32}$$

the spacelike 4-vectors $\lambda^\mu{}_\pm=(0,0,0,p_\pm)$ can be shown to satisfy the dispersion relation (31), as the reader is invited to verify. Moreover, the existence of such spacelike solutions to the dispersion relation is unaffected by the inclusion of a nonzero $a_\mu$, for example.

Although the instabilities introduced by the existence of spacelike solutions exist in any frame, including a concordant frame as discussed below, they are most transparent by considering observer Lorentz boosts. An appropriate observer boost involving a velocity less than 1 can always convert a spacelike vector with a positive 0th component to one with a negative 0th component. In the present instance, this means that there exist otherwise acceptable observer frames in which a single root of the dispersion relation involves both positive and negative energies. In such frames, the canonical quantization procedure fails.

In Figs. 1 and 2, the appearance of negative energies in a strongly boosted frame is illustrated for a model with only a nonzero $b_0$ in a concordant frame. The dispersion relation as seen by an observer in a concordant frame is shown in Fig. 1. One of the two positive roots is displayed. The energy is manifestly positive for all 3-momenta. However, the dispersion relation crosses the light cone [36] at a finite value $\tilde{M}$ of the 3-momentum. Beyond this value, points lying on the

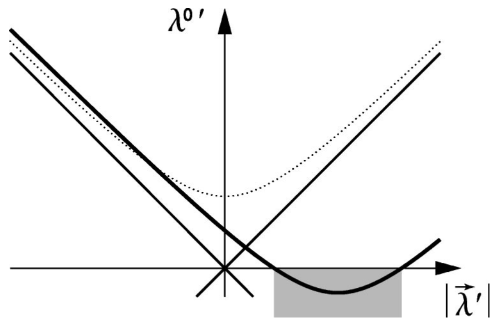


FIG. 2. Dispersion relation for the model of Fig. 1 as seen by an observer strongly boosted relative to the concordant frame. The occurrence of negative energies is apparent in the shaded region. The dotted line is the conventional dispersion relation for a massive particle.

curve can be regarded as represented by spacelike vectors relative to the origin. All these spacelike vectors have positive 0th components.

For a suitable boost, some of the spacelike vectors are converted to spacelike vectors with negative 0th components. Figure 2 shows the result of a large boost. A portion of the dispersion relation has dipped below the energy zero. The corresponding negative-energy states represent a stability problem for the theory when interactions are introduced. We remark in passing that under the same boost the other roots of the dispersion relation are positioned so as to preclude eliminating the negative energies by a simple shift of the energy zero.

The scale $\widetilde{M}$ of the 3-momentum at which the 4-momentum turns spacelike can be calculated explicitly in various models. For example, consider the case of a timelike $b_\mu$, as above. In an observer frame with $b_\mu=(b_0,\vec{0})$, we find

$$\widetilde{M}=\frac{m^2+b_0^2}{2|b_0|} \gtrsim \mathcal{O}(M_P). \tag{33}$$

The approximate equality in the last step is attained for the case of a single suppression factor from the Planck scale, $b_0 \sim \mathcal{O}(m^2/M_P)$, following the discussion in Sec. II.

This estimate reveals that the instabilities in the model emerge only for Planck-scale 4-momenta in a concordant frame. The corresponding negative energies appear only for observers undergoing a Planck-scale boost relative to this frame. It follows that the concordant-frame quantization we have presented above maintains stability for all experimentally attainable physical momenta and in all experimentally attainable observer frames.

Inspection of the dispersion relation for the $b_\mu$ model reveals that in all observer frames the asymptotes of the dispersion relation are parallel to the usual light-cone asymptotes. The behavior can also be seen in the example in Figs. 1 and 2. We see that, to avoid spacelike 4-momenta, the asymptotes of the dispersion relation must remain inside the usual light cone. In terms of the group velocity $\vec{v}_g$ of a wave packet in the theory, given as usual by

$$\vec{v}_g=\frac{\partial E}{\partial \vec{p}}, \tag{34}$$

this requirement on the asymptotes implies the following necessary condition for energy positivity:

$$|\vec{v}_g| \geq 1, \quad |\vec{p}| \to \infty. \tag{35}$$

The reader is reminded that the relation between momentum and group velocity is unconventional [5]. In particular, $\vec{p}$ and $\vec{v}_g$ need not be parallel.

Since the physics is invariant under observer boosts, the appearance of negative energies in a strongly boosted frame indicates that spacelike 4-momenta lead to a stability problem also in a concordant frame, albeit only for particles with energies exceeding the Planck scale. As an illustration, consider the following process in a concordant frame: a Planck-energy fermion emits a virtual photon, which then decays into a fermion-antifermion pair. We can write this as

$$f_{+1} \to f_{+1}+f_{+1}+\overline{f}_{-1}, \tag{36}$$

where $f$ and $\overline{f}$ denote fermions and antifermions, respectively, and the subscript labels the helicity state. In conventional QED, this decay is kinematically forbidden even though both the U(1) charge and angular momentum are conserved. However, for Planck energies it can occur in the context of the Lorentz- and $CPT$-violating QED extension with a nonzero $b_0$ coefficient. The dispersion relation for the 4-momentum $(E,\vec{p})$ of a fermion of helicity $+1$ or an antifermion of helicity $-1$ is given in Appendix B of the first paper in Ref. [5] as

$$E=\sqrt{m^2+(|\vec{p}|-b_0)^2}. \tag{37}$$

Taking for simplicity the 3-momentum $|\vec{q}|$ of the incoming fermion as

$$|\vec{q}|=\frac{2m^2+b_0^2}{b_0}+b_0 \gtrsim \mathcal{O}(M_P), \tag{38}$$

we find the process (36) is kinematically allowed with all final 3-momenta equal to $\vec{q}/3$. A single-particle state describing a fermion of sufficiently large 3-momentum (38) and helicity $+1$ is therefore unstable. The instability also occurs for other high-energy single-particle states, although the final 3-momenta are then unequal.

It can be shown that an initial spacelike 4-momentum is a necessary condition allowing the process (36), as expected. The decay process (36) could therefore occur repeatedly in a cascade until the energy of the decay products reaches the order of the Planck scale in a concordant frame. Although unusual, this behavior and related phenomena involving other decays might be phenomenologically admissible. However, in what follows we focus on the possibility of maintaining stability at the Planck scale despite the presence of Lorentz violation.

The conclusion that instabilities enter at $\mathcal{O}(M_P)$, as in Eq. (33), may fail for models with a nonzero coefficient $c_{\mu\nu}$. This coefficient is special because the associated quadratic field term has the same general spinorial and derivative structure as the usual Dirac kinetic term, and so it acts as a first-order correction to an existing zeroth-order term. No other Lorentz-violating term has this feature.

As an explicit example, consider a model with only the coefficient $c_{00}$ nonzero in a concordant frame [37]. The dispersion relation for this model in an arbitrary frame is

$$(\eta_{\alpha\mu}+c_{\alpha\mu})(\eta^{\alpha}{}_{\nu}+c^{\alpha}{}_{\nu})\lambda^{\mu}\lambda^{\nu}-m^2=0. \tag{39}$$

In the concordant frame, this takes the form

$$\zeta^2\lambda_0^2-\vec{\lambda}^2-m^2=0, \tag{40}$$

where we define $\zeta=1+c_{00}$. For the case $c_{00}>0$, we then find that spacelike 4-momenta occur at a scale $\tilde{M}$ given by

$$\begin{aligned}\tilde{M}&=\frac{m}{\sqrt{\zeta^2-1}}\approx\frac{1}{\sqrt{2c_{00}}}m+\mathcal{O}(c_{00})\\&\gtrsim\mathcal{O}(\sqrt{mM_P}),\end{aligned} \tag{41}$$

where in the last step the approximate equality is attained for a single suppression factor from the Planck scale, $c_{00}\sim\mathcal{O}(m/M_P)$.

The result (41) implies that instabilities occur at energies well below the scale $M_P$ of the underlying theory in the $c_{00}$ model with $c_{00}>0$. We show in the next section that if $c_{00}<0$ instead, then microcausality violations arise at the same scale. If these results continue to hold in the full underlying theory, they could have observable physical implications. As one example, Coleman and Glashow have suggested [27] the interesting possibility that high-energy effects from $c_{00}$-type terms might be responsible for the apparent excess of cosmic rays in the region of $10^{19}$ GeV. This scale is potentially comparable to $\sqrt{mM_P}$. However, if stability and causality are imposed on the theory, then the $c_{00}$ dispersion relation (40) must be modified. This in turn is likely to modify the physical implications at high energies. In Sec. V, we discuss some possible high-energy corrections to Eq. (40) that would preserve stability and causality. It would be of interest to revisit the cosmic-ray analysis in light of these requirements.

In any case, given the impracticality of achieving Planck-scale energies or boosts in the laboratory, the issues with spacelike 4-momenta are largely unimportant at the level of the standard-model extension. However, they do confirm the expectation that corrections to the theory at high energies are needed for complete stability. Requiring stability therefore has the potential to provide insight into the nature of the corrections. This situation is qualitatively different from that occurring in conventional special relativity, where Planck-scale boosts are admissible without generating instabilities internal to the theory. Since the standard-model extension contains all relevant renormalizable operators, the resolution of the stability issue must involve nonrenormalizable operators that are irrelevant at low energies. We return to this topic in Sec. V.

### C. Microcausality

A quantum field theory is microcausal if any two local observables with spacelike separation commute. In the Lorentz- and $CPT$-violating Dirac theory (1), the local quantum observables are fermion bilinears as usual, and microcausality holds if

$$iS(x-x')=\{\psi(x),\overline{\psi}(x')\}=0,\quad (x-x')^2<0. \tag{42}$$

We work directly with the original field $\psi$ rather than $\chi$ because the observer Lorentz symmetry holds for the Lagrangian (1) written in terms of $\psi$, whereas the conversion to $\chi$ is frame dependent. Note that the anticommutator function $S(x-x')$ depends only on coordinate differences, due to the translational invariance of the theory.

To investigate the conditions under which Eq. (42) holds, it is useful to obtain an integral representation for $S(x-x')$. The latter can be found in terms of Green functions for the modified Dirac equation. In the conventional case, one usually starts with the Fourier decomposition of the field operators and proceeds by identifying spinor projection operators. The latter are then expressed in terms of gamma matrices, the momentum, and the mass. However, in the present case a straightforward generalization of this last step is obstructed by the complexity of the modified Dirac equation. Instead, a more general argument can be adopted.

We proceed in a concordant frame. First, define the function

$$iG_R(x,x')=\Theta(t-t')\{\psi(x),\overline{\psi}(x')\}, \tag{43}$$

where $\Theta$ denotes the usual Heaviside step function. With the help of the canonical anticommutators (27), it can explicitly be checked that $G_R$ satisfies

$$(i\Gamma^{\mu}\partial_{\mu}-M)G_R(x,x')=\delta^{(4)}(x-x'). \tag{44}$$

It follows that $G_R(x,x')$ is a Green function of the modified Dirac equation, and therefore it can be written as

$$G_R(x,x')=\int_{C_R}\frac{d^4\lambda}{(2\pi)^4}\frac{e^{-i\lambda\cdot(x-x')}}{\Gamma^{\mu}\lambda_{\mu}-M}. \tag{45}$$

Inspection shows that $C_R$ is the contour of the retarded Green function passing above all poles in the complex $\lambda^0$ plane. Similarly, it can be shown that the function defined by

$$iG_A(x,x')=-\Theta(t'-t)\{\psi(x),\overline{\psi}(x')\} \tag{46}$$

is the advanced Green function, with the same representation as Eq. (45) except that the contour $C_R$ is replaced with a contour $C_A$ passing below all the poles.

The anticommutator function $S(x-x')$ can be written as $S=G_R-G_A$. The integral represention for $S$ has the same form as Eq. (45) except that $C_R$ is replaced by a contour $C$ encircling all poles in the clockwise direction. If the matrix in the integrand of Eq. (45) is explicitly inverted, we can replace $\lambda^\mu \to i\partial^\mu$ in the matrix of cofactors $\mathrm{cof}(\Gamma^\mu \lambda_\mu - M)$ to obtain

$$S(z)=\mathrm{cof}(\Gamma^\mu i\partial_\mu - M)\int_C \frac{d^4\lambda}{(2\pi)^4}\frac{e^{-i\lambda\cdot z}}{\det(\Gamma^\mu\lambda_\mu - M)}. \tag{47}$$

The interchange of differentiation and integration is justified because the contour can be deformed so that the integrand is analytic in the neighborhood of $C$ [35].

Next, we take advantage of observer Lorentz invariance and boost to a frame such that $z^\mu = (0,\vec{z})$. The evaluation of $S(z)$ outside the light cone is simplified when the spinor redefinition discussed in Sec. III A can be performed in *all* observer frames. A sufficient condition for this is

$$c_{\mu\nu}=d_{\mu\nu}=e_\mu=f_\mu=g_{\lambda\mu\nu}=0, \tag{48}$$

so that the derivative couplings take the standard form with $\Gamma^\mu=\gamma^\mu$. In this case, a Hermitian Hamiltonian always exists, and the four poles of the integrand in Eq. (47) remain on the real axis in the complex $\lambda^0$ plane.

Under the condition (48), we can directly perform the contour integration in Eq. (47). For simplicity, we assume here that all four roots $E_{(j)}(\vec{p})$, $j=1,\ldots,4$, of the dispersion relation are nondegenerate. Cases with degenerate roots can be treated similarly with slight algebraic changes. Explicit calculation yields

$$\int_C \frac{d\lambda^0}{2\pi}\frac{1}{(\lambda^0-E_{(1)})(\lambda^0-E_{(2)})(\lambda^0-E_{(3)})(\lambda^0-E_{(4)})}$$
$$=\frac{i}{(E_{(1)}-E_{(2)})(E_{(1)}-E_{(3)})(E_{(1)}-E_{(4)})}$$
$$+\frac{i}{(E_{(2)}-E_{(1)})(E_{(2)}-E_{(3)})(E_{(2)}-E_{(4)})}$$
$$+\frac{i}{(E_{(3)}-E_{(1)})(E_{(3)}-E_{(2)})(E_{(3)}-E_{(4)})}$$
$$+\frac{i}{(E_{(4)}-E_{(1)})(E_{(4)}-E_{(2)})(E_{(4)}-E_{(3)})}=0, \tag{49}$$

where the dependence of the $E_{(j)}$ on $\vec{p}$ has been suppressed.

This calculation shows that $S(z)$ vanishes outside the light cone if Eq. (48) is satisfied. Thus, microscopic causality is ensured for the Dirac quantum field theory in the presence of Lorentz and $CPT$ violation controlled by the coefficients $a_\mu$, $b_\mu$, and $H_{\mu\nu}$.

The above argument can fail when Eq. (48) is invalid. For this more general case, the poles of the integrand in Eq. (47) may no longer lie on the real $\lambda^0$ axis in an arbitrary observer frame, and the contour $C$ may therefore fail to encircle them all. This corresponds to the case where the bound on $\delta^0$ discussed in Sec. III A is violated, so that the Hamiltonian cannot be made Hermitian and the roots of the dispersion relation can therefore be complex.

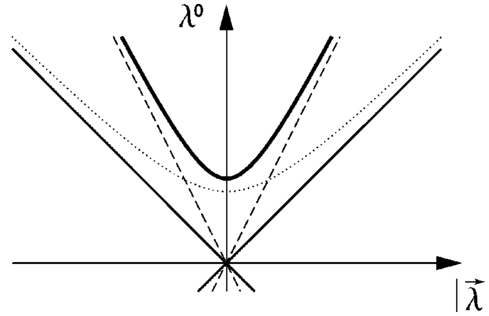


FIG. 3. Dispersion relation for a model with only a large negative nonzero $c_{00}$ in a concordant frame. The degenerate positive roots are displayed. The dashed lines show their asymptotes. The dotted line is the conventional dispersion for a massive particle.

As an explicit example, let us return to the $c_{00}$ model with dispersion relation (40) discussed in the previous subsection, but without imposing $c_{00}>0$. For this model, the integration in Eq. (47) can be performed analytically to yield

$$S(z)=(i\zeta\gamma^0\partial^0 - i\gamma^j\partial^j + m)\frac{1}{4\pi\zeta r}\frac{\partial}{\partial r}[\Theta(w^2)J_0(m\sqrt{w^2})], \tag{50}$$

where $r=|\vec{z}|$, $w^2=(z^0/\zeta)^2-\vec{z}^2$, and $J_0(y)$ is the zeroth-order Bessel function. Thus, the anticommutator function $S(z)$ vanishes only in the region defined by $z^0 < (1+c_{00})|\vec{z}|$. Outside this region, $S(z)$ could be nonzero. Signal propagation therefore could occur with maximal speed $1/(1+c_{00})$. When $c_{00}$ is negative, this exceeds 1 and hence violates microcausality.

To make further progress, it is useful to introduce a definition of the velocity of a particle valid for an arbitrary 3-momentum. Even in the usual case without Lorentz and $CPT$ violation, the notion of a quantum velocity operator is nontrivial. The presence of Lorentz and $CPT$ violation further complicates the issue [5]. For definiteness, we consider here the group velocity defined for a monochromatic wave in terms of the dispersion relation by Eq. (34). This choice is appropriate for several reasons. For one-particle states in the theory, the flow velocities of the conserved momentum $P_\mu$ and the U(1) charge can be calculated from the corresponding conserved currents, and they agree with the group velocity (34). Also, we have checked explicitly that $\langle d\vec{x}/dt\rangle = \vec{v}_g$ in the relativistic quantum mechanics of the $c_{00}$ model. Moreover, for the explicit examples considered above, involving either no derivative couplings or a $c_{00}$ coupling only, the magnitude of the maximal attainable group velocity is equal to the maximal speed of signal propagation determined from the anticommutator function.

Figures 3 and 4 illustrate the situation for the $c_{00}$ model. The dispersion relation in a concordant frame is displayed in Fig. 3. This figure shows that the maximal speed is attained asymptotically for large 3-momenta. Figure 4 shows the group velocity as determined from the dispersion relation in

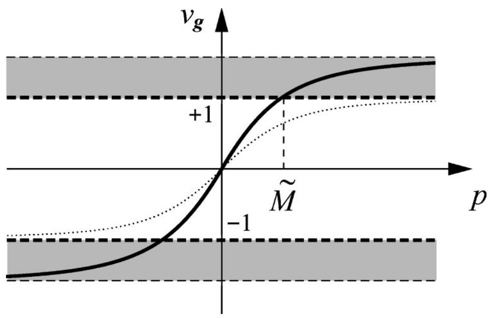


FIG. 4. Group velocity for the dispersion relation of the model in Fig. 3 as a function of the 3-momentum in a fixed direction. The asymptotic development of velocities exceeding 1 is apparent in the shaded region, which lies above a momentum scale $\tilde{M}$. The heavy dashed lines correspond to the usual limiting velocities $\pm 1$. The dotted line is the usual result for a massive particle.

the same frame. Above a certain value $\tilde{M}$ of the 3-momentum magnitude, all the group-velocity magnitudes exceed 1.

It follows from the above considerations that a necessary condition to avoid microcausality violations is that the asymptotic behavior of the energy must have a slope less than or equal to that of the usual light cone:

$$|\vec{v}_g| \leqslant 1, \quad |\vec{p}| \to \infty. \tag{51}$$

Combined with Eq. (35), we see that a necessary condition for a positive root to avoid both negative energies in some observer frame and microcausality violations is that the asymptotic behavior of the dispersion relation must lie inside the forward light cone and satisfy

$$|\vec{v}_g| = 1, \quad |\vec{p}| \to \infty. \tag{52}$$

Although this is only an asymptotic condition, it nonetheless provides an interesting constraint on possible stable and causal models for Lorentz and *CPT* violation.

Insight about the scale $\tilde{M}$ of microcausality breakdown can be obtained by determining the value of the 3-momentum at which the group velocity reaches 1: $|\vec{v}_g|(|\vec{p}| = \tilde{M}) = 1$. For the $c_{00}$ model, the dispersion relation (40) gives

$$\tilde{M} = \frac{\zeta}{\sqrt{1-\zeta^2}} m \approx \frac{1}{\sqrt{-2c_{00}}} m + \mathcal{O}(c_{00}) \\ \gtrsim \mathcal{O}(\sqrt{m M_P}). \tag{53}$$

In the last step, the approximate equality holds for a single suppression factor $c_{00} \sim \mathcal{O}(m/M_P)$.

The result (53) is a special feature of models with a nonzero $c_{\mu\nu}$ parameter. It is the same as that for the case with $c_{00} > 0$, given in Eq. (41). We see that group velocities exceeding 1 occur in the $c_{00}$ model at energies well below the scale $M_P$ of the underlying theory. This may have physical implications, as mentioned in the previous subsection.

To see what happens for other Lorentz- and *CPT*-violating terms with derivative couplings, consider a model with only a nonzero $e_\mu$ term. Its dispersion relation is

$$\lambda^2 - (m - \lambda \cdot e)^2 = 0. \tag{54}$$

For simplicity, we take $e_\mu$ to be timelike and choose the concordant frame to have $\vec{e} = 0$. The scale $\tilde{M}$ of microcausality violation is then found to be

$$\tilde{M} = \frac{1}{e_0} m \\ \gtrsim \mathcal{O}(M_P), \tag{55}$$

where in the last step the approximate equality is attained for a single Planck-scale suppression factor, $e_0 \sim \mathcal{O}(m/M_P)$, as before. This confirms that microcausality is violated in the $e_\mu$ model at the scale of the underlying theory, as expected.

The $e_\mu$ model can also be used to illustrate the relation between microcausality and Hermiticity of the Hamiltonian *H*. In the $e_\mu$ model, the matrix $\gamma^0 \Gamma^0$ takes the explicit form

$$\gamma^0 \Gamma^0 = \begin{pmatrix} 1+e_0 & 0 & 0 & 0 \\ 0 & 1+e_0 & 0 & 0 \\ 0 & 0 & 1-e_0 & 0 \\ 0 & 0 & 0 & 1-e_0 \end{pmatrix} \tag{56}$$

in the Pauli-Dirac representation. Provided $|e_0| < 1$, the spectrum of $\gamma^0 \Gamma^0$ containes positive numbers only, a matrix *A* satisfying Eq. (4) can be found, and a Hermitian Hamiltonian *H* exists. However, if $|e_0| > 1$, two eigenvalues become negative, $\gamma^0 \Gamma^0$ is no longer congruent to the identity, the spinor-redefinition matrix *A* cannot exist, and a Hermitian *H* cannot be found.

The same problem is reflected at the level of the dispersion relation (54). Its solutions

$$\lambda^0_{\pm} = \frac{e_0(m + \vec{\lambda} \cdot \vec{e}) \pm \sqrt{(m + \vec{\lambda} \cdot \vec{e})^2 + (1 - e_0^2)\vec{\lambda}^2}}{e_0^2 - 1} \tag{57}$$

can become complex for $|e_0| > 1$. Since it is always possible to find an observer frame in which this condition is satisfied, the model is inconsistent with observer invariance of the Hermiticity of *H*. This again indicates that the argument for microcausality can fail when the condition (48) is invalid.

Figures 5 and 6 illustrate in the context of the $e_\mu$ model how eigenenergies can be real in one observer frame and complex in another, despite the observer invariance of the dispersion relation. Figure 5 shows the dispersion relation for a model with a nonzero $e_0$ only, in a concordant frame. One of the two positive roots and its negative partner are displayed. The eigenenergies are real for all 3-momenta. However, the slope of the dispersion relation exceeds 1 for a sufficiently large 3-momentum. The effect of this on a positive root and its negative partner as seen by an observer in a strongly boosted frame is displayed in Fig. 6. These two roots admit no real value of the energy for 3-momenta in the

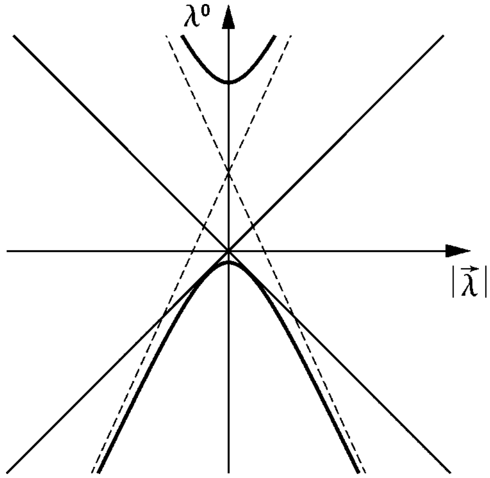


FIG. 5. Dispersion relation for a model with only a large nonzero $e_0$ in a concordant frame. One positive root and its negative partner are displayed. The dashed lines show the asymptotes.

shaded region. Moreover, there is a range of 3-momenta for which the dispersion relation has multiple-valued roots.

This feature can be expected in the general case, whenever the magnitude $|\vec{v}_g|$ of the slope of the dispersion relation in a concordant frame exceeds 1. More generally, the individual branches of the dispersion relation should remain one-to-one mappings under observer Lorentz transformations, so that each 3-momentum has exactly one image point. The number of real solutions to the dispersion relation is then invariant under observer boosts. In terms of the asymptotic behavior of the dispersion relation in the general case, we see that the existence requirements for the spinor redefinition (4) and for a Hermitian Hamiltonian $H$ also lead to the condition (51).

The above analysis reveals that difficulties with causality in the Lorentz- and $CPT$-violating Dirac theory arise primarily for Planck-scale 4-momenta in a concordant frame or for observers undergoing a Planck boost relative to this frame. Nonetheless, it would be theoretically interesting to have a framework for Lorentz and $CPT$ violation in which microcausality is exactly preserved. Moreover, constraints from the requirement of causality may offer insight into the nature of an underlying theory with Lorentz and $CPT$ violation. This is the subject of the following section.

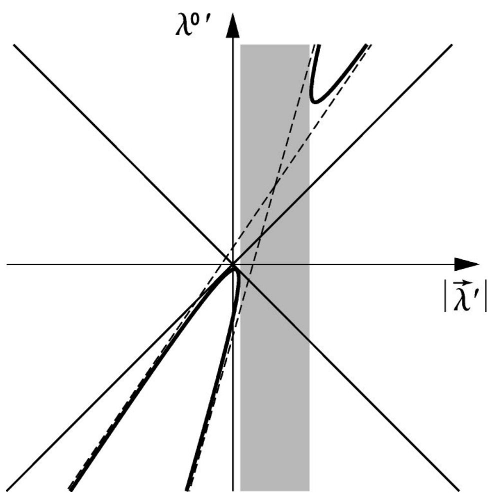


FIG. 6. Dispersion relation for the model of Fig. 5 as seen by an observer strongly boosted relative to the concordant frame. The occurrence of multiple-valued energies for a given root is apparent. The positive root and its negative partner have no real values of the energy for 3-momenta in the shaded region. The dashed lines show the asymptotes.

## V. PLANCK-SCALE EFFECTS

The results of the previous section indicate that a quantum field theory of massive fermions with terms containing explicit Lorentz and $CPT$ violation generically develops difficulties with stability or causality. However, if the coefficients controlling the violation are Planck-suppressed, as in the standard-model extension, the difficulties arise only at high energies or high boosts determined by the Planck scale.

Many possible sets of values of the coefficients $a_\mu, b_\mu, \ldots, H_{\mu\nu}$ for Lorentz and $CPT$ violation in Eq. (1) eliminate one of the two difficulties. However, we are unaware of any combination of the coefficients that simultaneously maintains both stability and causality. Although it is conceivable that a satisfactory combination would be naturally selected by a mechanism for Lorentz and $CPT$ breaking in an underlying theory, we conjecture that no such combination exists. A definitive argument to settle this issue would be of interest but appears hampered by the complexity of the dispersion relation (14).

We have previously advocated spontaneous Lorentz and $CPT$ breaking in a Lorentz-covariant theory at the Planck scale as a possible mechanism that could generate the apparent Lorentz and $CPT$ violations at low energies [3,4]. Indeed, the standard-model extension includes by construction all possible renormalizable terms maintaining the usual gauge structure while potentially originating in spontaneous Lorentz breaking. This reasoning is a top-down approach, with theoretical considerations at the Planck scale suggesting that spontaneous Lorentz violation might emerge as the apparent violation in the standard-model extension. However, the requirements of stability and causality appear strong enough to adopt the inverse line of reasoning. Thus, as the Planck scale is approached, higher-order nonrenormalizable operators coming from the fundamental theory should play an increasing role. The structure of the standard-model extension as a conventional quantum field theory should therefore undergo a corresponding modification, which could provide insight into the nature of the fundamental theory at the Planck scale. In the remainder of the present section, we fill in some details for this set of ideas.

### A. Spontaneous Lorentz and $CPT$ breaking

Since a theory with spontaneous Lorentz and $CPT$ violation starts from a Lorentz-invariant Lagrangian and hence has Lorentz-covariant dynamics, it is unsurprising that it avoids at least some of the difficulties plaguing more general models involving Lorentz and $CPT$ violation. For example, one consequence of spontaneous violation is the natural maintenance of observer Lorentz invariance, which the previous sections have shown to be an important advantage. Thus, given a Lagrangian invariant under both observer and particle Lorentz transformations, spontaneous symmetry breaking violates only the latter. The point is that observer Lorentz invariance is a statement about physical behavior under certain coordinate changes made by an independent external observer, and once this property is built into a theory it cannot be removed by the behavior of fields internal to the theory. In contrast, imposing observer Lorentz invariance in a theory with explicit Lorentz breaking requires an additional *ad hoc* choice.

Spontaneous violation manifests itself physically because the Fock-space states are constructed on a noninvariant vacuum. Any difficulties with spontaneous Lorentz and $CPT$ violation must therefore be a consequence of Lorentz- and $CPT$-violating properties of the ground state. However, the link between stability, causality, and Lorentz symmetry does indeed depend in part on the notion of an invariant vacuum. The difficulties uncovered in the previous section can be regarded as a consequence of vacuum noninvariance. For example, the vacuum state in one frame is not necessarily the lowest-energy state in all frames. Despite its advantages, one therefore might expect that spontaneous Lorentz and $CPT$ violation alone may be insufficient to guarantee stability and causality at all scales in a generic quantum field theory.

To gain insight into this issue, it is useful to consider a toy quantum field theory describing a Dirac fermion $\psi$ interacting with a vector field $B_\mu$, with a potential for the vector that induces spontaneous Lorentz and $CPT$ violation [38]. The Lagrangian is

$$\mathcal{L}=\overline{\psi}\left(\frac{1}{2}i\gamma^\mu \overleftrightarrow{\partial}_\mu - m - \xi\gamma_5\gamma^\mu B_\mu\right)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}\lambda(B^\mu B_\mu - \beta^2)^2. \qquad (58)$$

The fermion $\psi$ has mass $m$ and is chirally coupled to the vector $B_\mu$ with dimensionless strength $\xi$. The field strength $F_{\mu\nu}$ for $B_\mu$ is defined as $F_{\mu\nu}=\partial_\mu B_\nu - \partial_\nu B_\mu$, as usual, while the potential term for $B_\mu$ is controlled by a dimensionless constant $\lambda$ and by a constant $\beta$ with dimensions of mass satisfying $\beta^2>0$.

The Lagrangian (58) is a scalar under both observer and particle Lorentz transformations and contains no explicit Lorentz- and $CPT$-violating terms. However, the last term triggers a Lorentz- and $CPT$-violating vacuum expectation value $\langle B_\mu\rangle=\beta_\mu$, where $\beta_\mu$ is a constant 4-vector satisfying $\beta_\mu\beta^\mu=\beta^2$. Note the close analogy to spontaneous symmetry breaking in the standard O($N$) model with $N=4$. The Lorentz invariance of the Lagrangian (58) means that the constant vector $\beta_\mu$ can be arbitrarily chosen, but a definite choice must be specified to establish the quantum physics. This choice forces the particle Lorentz symmetry to be spontaneously broken on the Fock space.

The physics of interest is described by fluctuations about the vacuum. Redefining $B_\mu \to \beta_\mu + B_\mu$ in parallel with the usual case yields

$$\begin{aligned}\mathcal{L}&=\overline{\psi}\left[\frac{1}{2}i\gamma^\mu \overleftrightarrow{\partial}_\mu - m - \xi\gamma_5\gamma^\mu(\beta_\mu + B_\mu)\right]\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\\ &\quad - \frac{1}{4}\lambda(B^\mu B_\mu - 2B\cdot\beta)^2\\ &=\overline{\psi}\left(\frac{1}{2}i\gamma^\mu \overleftrightarrow{\partial}_\mu - m - \gamma_5\gamma^\mu b_\mu\right)\psi + \mathcal{L}',\end{aligned} \qquad (59)$$

where in the last step we have identified $\xi\beta_\mu$ with $b_\mu$ and explicitly displayed all the quadratic fermion terms in $\mathcal{L}$. The remaining piece $\mathcal{L}'$ of the Lagrangian contains only bosonic quadratic terms and interactions. We see that the spontaneous Lorentz and $CPT$ violation in the Lagrangian (58) has generated the $b_\mu$ model discussed in previous sections.

The free-field Fock space of the quantum theory associated with $\mathcal{L}$ contains one-fermion states determined by the quadratic terms in Eq. (59). These states have dispersion relations given by Eq. (31), as before. They therefore suffer from the same problems of instability as the $b_\mu$ model discussed in Sec. IV B. This leads to difficulties within the standard framework of perturbative quantum field theory, since the interacting fields are normally constructed iteratively from the free fields under the assumption that the effects of interactions are small. The toy model therefore still has interpretational difficulties, despite the spontaneous nature of the Lorentz and $CPT$ violation.

A similar argument applies to more general models. Since the theory described by Eq. (1) contains the most general terms quadratic in the fermion fields and arising in a renormalizable theory, any conventional fermion field theory with spontaneous Lorentz and $CPT$ violation analogous to Eq. (58) must generate free-fermion Fock-space states with dispersion relations contained as a subset of Eq. (14). If all such dispersion relations indeed lead to either stability or causality violations at some large scale, as expected from the discussion in Sec. IV, then it follows that no conventional Lagrangian of fermions with spontaneous Lorentz and $CPT$ violation has a completely satisfactory perturbative quantum field theory. Although it is conceivable that a nonperturbative analysis taking the full structure of the theory into account would reveal a consistent theory satisfying stability and causality, this appears unlikely. Even this possibility is excluded if the quantum field theory is *defined* in terms of its perturbative expansion, as is sometimes done in the literature.

The above discussion shows that spontaneous symmetry breaking in a conventional quantum field theory can naturally generate Lorentz- and $CPT$-violating terms of the form in Eq. (1) and ensures various desirable features such as observer Lorentz symmetry. Provided the coefficients for Lorentz and $CPT$ violation are small, as in the standard-model extension, difficulties arise only at large scales. However, by itself spontaneous Lorentz violation is insufficient to ensure stability and causality at energies determined by the Planck scale. Maintaining stability and causality requires an additional ingredient that goes beyond conventional quantum field theory. This is consistent with the idea that the observation of Lorentz and $CPT$ violation would provide a unique signal of Planck-scale physics.

### B. Nonlocality

If indeed the requirements of stability and causality are to be satisfied by free-field terms, then it is of interest to identify a class of theories for which no difficulties arise in the quadratic Lagrangian. Such theories would need to include terms beyond the ones in Eq. (1). The new terms must be nonrenormalizable, and in a realistic scenario with spontaneous Lorentz violation they would correspond to higher-dimensional nonrenormalizable operators correcting the standard-model extension at energies determined by the Planck scale.

The first step is to determine whether any type of dispersion relation can satisfy all the requirements for consistency. In a concordant frame, a satisfactory dispersion relation describing Lorentz and $CPT$ violation would reproduce the physics of Eq. (14) for small 3-momenta but would avoid spacelike 4-momenta and group velocities exceeding 1 for large 3-momenta. Moreover, its asymptotic behavior would need to obey Eq. (52). These requirements could be implemented by combining the coefficients for Lorentz and $CPT$ violation with a suitable factor suppressing them only at large 3-momenta. A factor of this type must be essentially constant at small 3-momenta and must overwhelm polynomial powers at large 3-momenta. Since the distinction between small and large 3-momenta is a frame-dependent concept, it is to be expected that a suitable factor would also be frame-dependent and hence involve Lorentz- and $CPT$-violating coefficients.

A complete treatment of the possibilities lies outside the scope of the present work. Instead, we prove by example that suitable dispersion relations can in principle exist by providing explicit situations with the desired features. We present here two cases that are closely related to the $b_\mu$ and $c_{\mu\nu}$ models discussed in Sec. IV. To simplify the discussion, we disregard here issues associated with the size of the coefficients for Lorentz and $CPT$ violation and take all masses and Lorentz- and $CPT$-breaking coefficients to be of order 1 in appropriate units. This permits a focus on resolving the problems of stability and causality at Planck-scale energies in a concordant frame without the complications introduced by the hierarchy of scales.

Consider first a dispersion relation obtained from Eq. (31) for the $b_\mu$ model by combining all appearances of $b_\mu$ with an appropriate exponential factor. For simplicity, we take a model with only a nonzero $b_0$ in a concordant frame. Multiplication of each factor of $b_0$ by $\exp[-(b_0\lambda_0)^2]$ suppresses the effect of $b_0$ at high energies with minimal effect at low energies. In an arbitrary frame, observer Lorentz invariance implies the resulting modified dispersion relation takes the form

$$[\lambda^2 - b^2 \exp[-2(b\cdot\lambda)^2] - m^2]^2 + 4b^2\lambda^2 \exp[-2(b\cdot\lambda)^2] - 4(b\cdot\lambda)^2 \exp[-2(b\cdot\lambda)^2] = 0. \quad (60)$$

For $b_0$ of appropriate size, the positive roots of this modified dispersion relation remain positive in all frames. This provides a proof by example that a suitable modification of the dispersion relation can be found that removes the difficulty with stability in arbitrary frames.

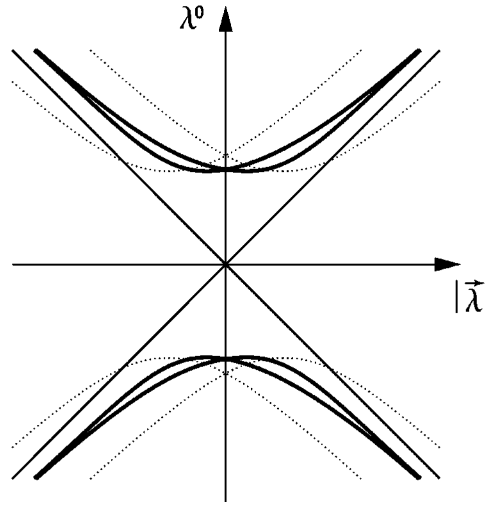


FIG. 7. Dispersion relation for a model with only a large nonzero $b_0$ in a concordant frame and exponential suppression at large energy. All four roots are displayed. None cross the light cone. The dotted lines are the four roots for the $b_0$ model *without* the exponential suppression.

Figure 7 shows the dispersion relation for the modified $b_\mu$ model in the special case where only $b_0$ is nonzero in a concordant frame. At small energies, the exponential factors are negligible and the behavior is essentially like that of the original $b_0$ model. However, at large energy the exponential factors dominate, causing the dispersion relation to remain within the light cone while asymptotically approaching it as required by condition (52). The modified $b_\mu$ dispersion relation (60) therefore has no difficulties with energy positivity in any frame.

To establish that microcausality is also preserved, the group velocity of the modified dispersion relation (60) can be examined. Figure 8 shows that the group velocity can indeed lie between the usual limiting values $\pm 1$ for all values of the 3-momentum despite the modification to the dispersion relation. Note that the asymmetry of this plot reflects the asymmetry of the corresponding curve in Fig. 7.

It is also possible to find examples where the difficulties with causality are absent. For example, consider the disper-

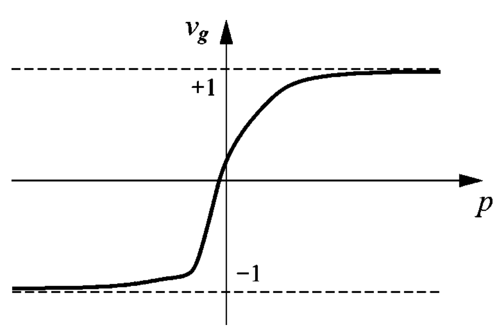


FIG. 8. Group velocity for the dispersion relation of the model in Fig. 7 as a function of the 3-momentum in a fixed direction. The modified dispersion has no group velocity exceeding 1. The dashed lines correspond to the usual limiting velocities $\pm 1$.

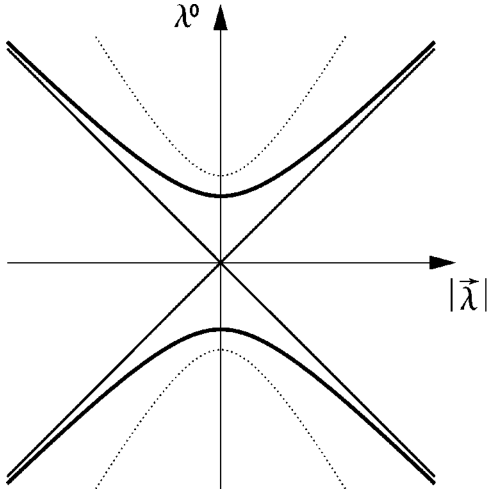


FIG. 9. Dispersion relation for a model with only a large nonzero $c_{00}$ in a concordant frame and exponential suppression at large energy. Only two curves appear because there is a two-fold degeneracy among the four roots. The dotted lines are the corresponding roots for the $c_{00}$ model *without* the exponential suppression.

sion relation obtained from Eq. (40) for the $c_{00}$ model with $c_{00}<0$ by multiplying each factor of $c_{00}$ with an exponential factor $\exp(c_{00}\lambda_0^2)$. In an arbitrary frame, the result is a modification of Eq. (39) given by

$$(\eta_{\alpha\mu}+c_{\alpha\mu}\exp(c_{\beta\gamma}\lambda^{\beta}\lambda^{\gamma}))(\eta^{\alpha}{}_{\nu}+c^{\alpha}{}_{\nu}\exp(c_{\beta\gamma}\lambda^{\beta}\lambda^{\gamma}))\lambda^{\mu}\lambda^{\nu} - m^2=0. \tag{61}$$

The exponential factors remove the microcausality violations that previously occurred at large $\lambda_\mu$. Indeed, it can be shown that the group velocity remains below 1 for all values of $\vec{\lambda}$. This proves by example that a suitable modification of the dispersion relation can eliminate difficulties with microcausality [39].

Figure 9 displays the dispersion relation for the special case of a modified model with only a nonzero $c_{00}$ in a concordant frame. At small energies, the exponential factors are negligible and the behavior is essentially like that of the original $c_{00}$ model. However, at large energy the exponential factors dominate, so the group velocities never exceed 1 and causality is maintained. The asymptotes of the dispersion relation coincide with the light cone, as required by Eq. (52). The group velocity of the modified dispersion relation (61) is shown as a function of the 3-momentum in Fig. 10. It remains within the usual limiting velocities everywhere, as desired.

The above demonstrations prove that dispersion relations violating Lorentz and *CPT* while maintaining stability and causality can exist. It would be of interest to identify theories from which these dispersion relations emerge naturally. The appearance of transcendental functions of the momenta corresponds to the occurrence of derivative couplings of arbitrary order in the Lagrangian. A satisfactory theory with Lorentz and *CPT* violation appears necessarily to be nonlocal in this sense. Although it is conceivable that a theory with explicit Lorentz breaking might satisfy the requirements of stability and causality, it would appear somewhat contrived to implement both the necessary observer Lorentz invariance and nonlocal couplings by hand. In contrast, we see that spontaneous Lorentz and *CPT* violation in a nonlocal theory can naturally yield the desired ingredients for stability and causality at all scales.

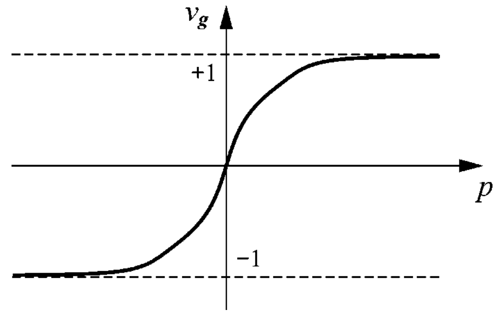


FIG. 10. Group velocity for the dispersion relation of the model in Fig. 9 as a function of the 3-momentum in a fixed direction. The modified dispersion has no group velocity exceeding 1. The dashed lines correspond to the usual limiting velocities $\pm 1$.

### C. String theory

Our field-theoretic considerations seeking the nature of Planck-scale corrections to a low-energy quantum field theory with Lorentz and *CPT* violation have thus led naturally to the case of a nonlocal theory with spontaneous symmetry breaking. String theories have nonlocal interactions, and it is of interest to determine whether they could be of the desired kind. Although a satisfactory realistic string theory has yet to be formulated, string field theories do exist for some simple string models and have already been used to investigate microcausality in the Lorentz-invariant case [40]. Moreover, studies of string field theory provided the original motivation for identifying spontaneous Lorentz and *CPT* violation as a serious candidate signal from the Planck scale [3] and for the construction of the standard-model extension as the appropriate low-energy limit.

In the remainder of this section, we examine the structure of the field theory for the open bosonic string to see whether it is compatible with dispersion relations of the desired type. Although this theory is unrealistic in detail, the structural features of interest are generic to string field theories and so provide insight into the possibility of generating a consistent theory with spontaneous Lorentz and *CPT* violation.

The open bosonic string has no fermion modes, so instead we focus on the dispersion relation for the scalar tachyon mode in the presence of Lorentz- and *CPT*-violating expectation values of tensor fields. In general, the analogue of Eq. (1) for a single real massive scalar field $\phi$ is [5]

$$\mathcal{L}=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}m^2\phi^2+\frac{1}{2}k_{\mu\nu}\partial^\mu\phi\partial^\nu\phi. \tag{62}$$

Here, $k_{\mu\nu}$ is a dimensionless coefficient for Lorentz violation that preserves *CPT*. It can be taken as real, symmetric, and traceless. The dispersion relation for this theory is closely related to that for the Lagrangian (1) with a nonzero coefficient $c_{\mu\nu}$ only. For the special case with only $k_{00}$ nonzero in a concordant frame, the dispersion relation of the theory (62) is just that in Eq. (40) with the identification $\zeta^2=1+k_{00}$. Studying the dispersion relation of the scalar tachyon mode in the presence of Lorentz violation is therefore more appropriate than might perhaps be expected *a priori*.

The action for the Witten string field theory [41] can be written in the Chern-Simons form

$$I(\Psi)=\frac{1}{2\alpha'}\int \Psi\star Q\Psi+\frac{g}{3}\int \Psi\star\Psi\star\Psi, \qquad (63)$$

where $\alpha'$ is the Regge slope and $g$ is the on-shell 3-tachyon coupling at zero momentum. The operator $Q$ acts as a quadratic kinetic operator. The interactions are controlled by the star operator $\star$, which joins the left half of one string to the right half of another. The integral joins the left half of a string onto its own right half.

The vibrational modes of the string are the particle states. The field $\Psi$ can be decomposed as a linear combination of ordinary particle fields with coefficients that are solutions of the first-quantized theory, expressed as creation operators $\alpha_{-1}, \ldots$ acting on a vacuum $|0\rangle$. Following the notation of Ref. [42], the fields in $\Psi$ are found to include among others a scalar $\phi$ (the tachyon) and a series of $2j$-tensors $B_{\mu\nu}, D_{\mu\nu\rho\sigma}, \ldots$:

$$\Psi=\left(\phi+\cdots+\frac{1}{\sqrt{2}}B_{\mu\nu}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1} + \frac{1}{2\sqrt{6}}D_{\mu\nu\rho\sigma}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\alpha^{\rho}_{-1}\alpha^{\sigma}_{-1}+\cdots\right)|0\rangle. \qquad (64)$$

The explicit Lagrangian for the theory in terms of particle fields to low orders has been obtained in Ref. [42]. Our interest here lies merely in determining whether the theory can in principle contain the types of term necessary for a stable and causal dispersion relation involving Lorentz violation. We therefore proceed under the assumption that spontaneous Lorentz violation has occurred, possibly along the lines discussed in Ref. [3], and has generated nonzero expectation values for the $2j$-tensors: $\langle B_{\mu\nu}\rangle, \langle D_{\mu\nu\rho\sigma}\rangle, \ldots$. Note that this assumption preserves $CPT$, as desired.

Follow the approach of Sec. V A, we directly extract relevant quadratic terms in the Lagrangian involving the tachyon. This procedure yields the Lagrangian

$$\mathcal{L}\supset\frac{1}{2}\partial_\mu\phi\partial^\mu\phi+(\alpha'^{-1}+k_0)\phi^2+\cdots+k_1\langle B_{\mu\nu}\rangle\partial^\mu\phi\partial^\nu\phi + \cdots+k_2\langle D_{\mu\nu\rho\sigma}\rangle\partial^\mu\phi\partial^\nu\phi\partial^\rho\phi\partial^\sigma\phi+\cdots. \qquad (65)$$

Here, the scalar parameters $k_0, k_1, k_2, \ldots$ are fixed by the theory, but their specific values are irrelevant for the present considerations. Each ellipsis represents quadratic terms involving other tensor expectation values and terms with powers of $\partial^2$.

For a plane-wave tachyon solution, the dispersion relation resulting from this Lagrangian takes the form

$$\lambda^2+(\alpha'^{-1}+k_0)+\cdots+k_1\langle B_{\mu\nu}\rangle\lambda^\mu\lambda^\nu+\cdots +k_2\langle D_{\mu\nu\rho\sigma}\rangle\lambda^\mu\lambda^\nu\lambda^\rho\lambda^\sigma+\cdots=0. \qquad (66)$$

We see that the structure of this equation does indeed contain features similar to those needed for a dispersion relation satisfying criteria for stability and causality. Thus, for example, the type of term in the toy dispersion relation (61) is a subset of the terms displayed in Eq. (66), when only 0th components of the $2j$ tensors are nonzero and the $2j$th-tensor expectation value is proportional to $(k_{00})^j$.

We emphasize that the purpose of the above discussion is only to provide an outline indicating how an acceptable dispersion relation for Lorentz violation might emerge in the context of string theory. In particular, we make no claim that the tachyon itself must *necessarily* obey such a relation, although it is conceivable that it does [3]. Here, the tachyon dispersion relation is used merely as an example to display explicitly the appearance of nonlocal couplings in string theory that could be appropriate for a stable and causal theory with spontaneous Lorentz violation. Such couplings are generic both for other fields in the open bosonic string and for fields in other string theories, including ones with fermions.

It would be of interest to find an explicit analytical construction for a Lorentz-violating solution in some string field theory and demonstrate its stability and causality. The most accessible case is likely to be the open bosonic string, but other string field theories with fermions could be amenable to investigation. If such a solution exists, it may be possible to find it using the methods of Ref. [43]. These interesting issues lie beyond the scope of the present work.

## VI. SUMMARY

In this paper, we have investigated the issues of stability and causality in quantum field theories incorporating Lorentz and $CPT$ violation. No difficulties arise at low energies provided the coefficients for Lorentz violation are small. However, local quantum field theories of fermions involving Lorentz violation generically develop difficulties with either stability or causality at some scale in every inertial frame.

On experimental and theoretical grounds, it is to be expected that the parameters controlling the Lorentz and $CPT$ violation are Planck suppressed in any Earth-based laboratory frame. In this physical situation, except for a special case involving a scale intermediate between the low-energy and the Planck scales, the difficulties appear only for particles with Planck-scale energies or in inertial frames undergoing Planck-scale boosts. In particular, the detailed analysis can be applied to the fermion sector of the standard-model extension, which is thereby seen to have a regime of validity comparable in many respects to that expected for the usual standard model. The high-energy difficulties are characterized by one-particle dispersion relations with tails either crossing the light cone or developing group velocities exceeding 1. The former result in instabilities, while the latter produce microcausality violations.

As part of the analysis, we have presented the relativistic quantum mechanics and the quantum field theory of a massive fermion governed by the quadratic sector of a renormalizable Lagrangian with general Lorentz- and *CPT*-violating terms. Much of the discussion can be extended to quadratic terms in a quantum field theory for a massive scalar with Lorentz and *CPT* violation, by virtue of the generality of the dispersion relation (14) and the usual type of connection between the Dirac and Klein-Gordon equations. Some of the results should also apply to the case of massless particles, including any massless neutrinos and the photon or other gauge bosons. However, further effort is likely to be required to account correctly for the differences between massive and massless representations of the Lorentz group and for the effects of gauge symmetry. Our methodology and general results are also applicable to nonrenormalizable terms in an effective theory. The limitation to renormalizable terms in our analysis is largely a matter of convenience, chosen to minimize complications in the identification of the origin and resolution of the difficulties with Lorentz and *CPT* violation.

The issues with stability and causality can be resolved under suitable circumstances. An important ingredient in this is the requirement of observer Lorentz invariance, which is guaranteed if the Lorentz and *CPT* violation develops spontaneously in a Lorentz-covariant underlying theory. This provides a link between the Fock spaces constructed by different inertial observers. In contrast, in theories based on explicit Lorentz violation instead, this condition must either be imposed by hand or be replaced by some other *ad hoc* condition.

We have shown explicitly that spontaneous Lorentz and *CPT* violation in suitable nonlocal theories can generate dispersion relations avoiding the problems with stability and causality. In particular, the necessary structures appear in the context of string field theories. We find it noteworthy that imposing stability and causality on quantum field theories with Lorentz violation leads naturally both to insight about the nonrenormalizable terms emerging as the Planck scale is approached and to requirements compatible with string field theories. This reverses the usual chain of reasoning by which spontaneous Lorentz and *CPT* violation in some fundamental theory leads to the standard-model extension in the low-energy limit where nonrenormalizable terms become irrelevant.

The analysis in this work supports the idea that a stable and causal realistic fundamental theory involving spontaneous Lorentz and *CPT* violation exists. If so, it would lead to potentially observable effects at sub-Planck energies described by the Lorentz- and *CPT*-violating standard-model extension. This offers the promising possibility of providing a unique experimental signature of Planck-scale physics.

## ACKNOWLEDGMENTS

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## APPENDIX A: BOUND FOR $\delta^0$

The key to bounding $\delta^0$ is to obtain a bound on $\det(\gamma^0\Gamma^0)=\det(I+\epsilon^0)$ in terms of the components of the matrix $\epsilon^0$ controlling the Lorentz and *CPT* violation. Expanding the determinant yields $4!=24$ terms, each a product of 4 matrix elements of $I+\epsilon^0$. It can be written

$$\det(I+\epsilon^0)=(1+\epsilon^0_{11})(1+\epsilon^0_{22})(1+\epsilon^0_{33})(1+\epsilon^0_{44})+\cdots, \tag{A1}$$

where $\epsilon^0_{jk}$ denotes the $jk$ element of $\epsilon^0$ and the ellipsis represents the 23 remaining terms, none of which are at zeroth order in $\epsilon^0$.

Define $\epsilon=\max_{j,k}\{|\epsilon^0_{jk}|\}$, the matrix element with the largest absolute value. Then, a lower bound for the term displayed in the expansion (A1) is $(1-\epsilon)^4$. Provided $\epsilon<\frac{1}{2}$, the largest of the remaining terms is bounded above by $\epsilon(1+\epsilon)^3$. It follows that

$$\det(I+\epsilon^0)\geqslant(1-\epsilon)^4-23\epsilon(1+\epsilon)^3. \tag{A2}$$

Subtraction of suitable non-negative terms from the right-hand side of this inequality yields

$$\det(I+\epsilon^0)\geqslant(1-\epsilon)^3(1-30\epsilon). \tag{A3}$$

Explicitly, we have

$$\epsilon^0=\gamma^0\left(c^{\mu0}\gamma_\mu+d^{\mu0}\gamma_5\gamma_\mu+e^0+if^0\gamma_5+\frac{1}{2}g^{\lambda\mu0}\sigma_{\lambda\mu}\right). \tag{A4}$$

Noting the antisymmetry properties of $\sigma_{\lambda\mu}$ and $g^{\lambda\mu\nu}$, we see that $\epsilon^0$ is the sum of 16 terms, each being a product of one Lorentz- and *CPT*-violating parameter with one of the 16 gamma matrices. Since the absolute value of an arbitrary entry of any gamma matrix does not exceed 1, it follows from the definition (5) of $\delta^0$ that $\epsilon\leqslant16\delta^0$. Together with Eq. (A3), this implies

$$\det(\gamma^0\Gamma^0)>0,\quad 0\leqslant\delta^0<\frac{1}{480}. \tag{A5}$$

In the trivial case $\delta^0=0$, $\gamma^0\Gamma^0=I$ has four positive eigenvalues. The continuity of the determinant implies this must also hold true for all $\delta^0$ in the above range. An eigenvalue sign change would be accompanied by a vanishing determinant, contradicting Eq. (A5).

## APPENDIX B: BOUND FOR $\delta$

Equation (11) shows that the four roots of the dispersion relation can be interpreted as eigenvalues of $(\Gamma^0)^{-1}(\Gamma^j\lambda_j-M)$. Note that the matrix $\Gamma^0$ is invertible provided the spinor redefinition (4) exists, as we assume here. We proceed by obtaining an upper bound on the quantity $\delta$ in Eq. (15) such that

$$\det(\gamma^0\Gamma^j\lambda_j-\gamma^0M)\neq0, \tag{B1}$$

where the factor of $\gamma^0$ has been inserted for convenience. With the bound on $\delta$ in hand, the continuity of the determinant in Eq. (B1) as the coefficients for Lorentz and *CPT* violation vanish then implies the same eigenenergy-sign structure as occurs in the usual Dirac case.

To simplify the notation, define $\epsilon^j$ and $\epsilon(M)$ such that Eqs. (2) and (3) take the forms

$$\Gamma^j = \gamma^j + \gamma^0 \epsilon^j, \quad M = m + \gamma^0 \epsilon(M). \tag{B2}$$

An argument similar to that following Eq. (A3) shows the components $\epsilon^j_{kl}$ and $\epsilon_{kl}(M)$ of $\epsilon^j$ and $\epsilon(M)$ obey

$$m\epsilon^j_{kl} < 16\delta, \quad \epsilon_{kl}(M) < 14\delta. \tag{B3}$$

Using this notation, we can write

$$\gamma^0(\Gamma^j \lambda_j - M) = \gamma^0(\gamma^j \lambda_j - m) + (\epsilon^j \lambda_j - \epsilon(M)), \tag{B4}$$

where the first term on the right-hand side is just the usual free Dirac Hamiltonian $H_D$ and the second term controls the Lorentz and $CPT$ violation.

For Eq. (B1) to hold, the kernel of $\gamma^0(\Gamma^j \lambda_j - M)$ must be empty. Thus, $\gamma^0(\Gamma^j \lambda_j - M)v \neq 0$ must hold for all complex spinors $v$. The norm $|v|$ of $v$ can be set to 1 without loss of generality. A sufficient condition for the vanishing of the kernel is then

$$|H_D v|^2 > |(\epsilon^j \lambda_j - \epsilon(M))v|^2 \tag{B5}$$

for all $v$, where we have used Eq. (B4).

The left-hand side of this inequality is just $\vec{\lambda}^2 + m^2$, as can be seen by expanding $v$ in eigenspinors of $H_D$. An upper bound for the right-hand side is determined by $64(\sqrt{3} \cdot 8|\vec{\lambda}| + 7m)^2 \delta^2$, where we have used Eq. (B3) and the assumption $|v| = 1$. Some algebra then directly yields the bound on $\delta$ given in the text.


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cally contain quartic roots involving $\vec{\lambda}$ and the coefficients for Lorentz violation. A fourth-order term can therefore generate a first-order energy correction. Another difficulty is that the momentum is a variable. For example, even if the momentum component $p_z$ in the $z$ direction occurs only at second order, when large it may dominate other first-order terms.

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[36] Note that what we are calling the light cone may be unrelated to the properties of light in the present case. In the standard-model extension, the appropriate cone is defined by the behavior of observables under observer Lorentz transformations. This is determined by the spontaneous nature of the Lorentz violation, which preserves the Minkowski metric associated with the fundamental theory at the Planck scale. Nonetheless, for simplicity we keep the terminology light cone. Note also that throughout this paper the dispersion relations are in momentum space, so all references are to the energy-momentum light cone rather than the spacetime light cone.

[37] The coefficients $c_{\mu\nu}$ in the Lagrangian (1) are defined to be traceless, so a model with only a nonzero $c_{00}$ might appear inappropriate. However, the effect of a nonzero trace in $c_{\mu\nu}$ is merely to scale the normalization of the Dirac field [5]. For example, a model with a nonzero $c_{00}=c$ describes physics similar to a model with $c_{00}=3c/4$, $c_{jj}=c/4$, $j=1,2,3$. The former choice has been adopted in the text to simplify the presentation.

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