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ORIGINAL ARTICLE

Stability conditions for impulsive dynamical systems

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Abstract

In this work, we consider impulsive dynamical systems evolving on an infinite-dimensional space and subjected to external perturbations. We look for stability conditions that guarantee the input-to-state stability for such systems. Our new dwell-time conditions allow the situation, where both continuous and discrete dynamics can be unstable simultaneously. Lyapunov like methods are developed for this purpose. Illustrative finite and infinite dimensional examples are provided to demonstrate the application of the main results. These examples cannot be treated by any other published approach and demonstrate the effectiveness of our results.

Keywords Stability · Robustness · Impulsive systems · Infinite-dimensional systems · Nonlinear systems · Lyapunov methods · Input-to-state stability

Mathematics Subject Classification 34G20 · 35R12 · 37L15 · 34K34 · 93D05

1 Introduction

Impulsive dynamical systems provide a mathematical modeling framework for practical processes where a combination of continuous and discrete dynamics takes place. Such a hybrid dynamics appears in many applications, for example, in case of mechanical collisions or in control systems involving a combination of analog and digital controllers. As well in pandemic systems, a mass vaccination can be modeled as an impulsive action meaning a (nearly instantaneous) transition of a large amount of susceptible individuals to become immune.

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A combination of discrete and continuous dynamics leads to a higher complexity in the behavior of solutions compared with a purely discrete or purely continuous system. Such unexpected effects as instability [27] or chaos [14] can arise. In particular, such properties as stability and robustness are more difficult to investigate, especially in case of nonlinear systems.

Stability in the sense of Lyapunov of nonlinear impulsive systems has a long history of investigations, see [17,23]. Later, more general stability notions were developed for hybrid systems, which include impulsive ones as a particular case, see [12,28]. These notions use the generalized (hybrid) time concept, which allows to develop rather general results for a wide class of hybrid systems, including impulsive, switched and sampled data systems.

In case of systems having input signals, the notion of input-to-state stability (ISS) was introduced in [26] and was found very fruitful in many applications [16]. This framework was also successfully used for studying robust stability of impulsive systems, see [4,6,13,18,22]. In particular, [6] derives dwell-time conditions to establish the ISS property for nonlinear impulsive systems on infinite dimensional state spaces. This result is based on certain stability assumptions imposed either on the continuous or on the discrete dynamics. The ISS is assured by the stability property of either of both dynamics using a suitable dwell-time condition. Stability of interconnected impulsive systems is then studied in the case, when the ISS-Lyapunov functions are known for the subsystems, which leads to a combination of the dwell-time and small-gain conditions. The ISS property of impulsive systems where impulsive actions depend on time was studied in [4], where new and rather general dwell-time conditions were developed.

A small-gain theorem for $n \geq 2$ interconnected hybrid systems was established in [22] for the case where not all subsystems are assumed to be ISS, which extends the results of [6]. Similar results for nonlinear interconnected impulsive systems were developed in [8] for the case of absent external perturbations. This work derives sufficient stability conditions for interconnected systems by means of vector Lyapunov functions, which leads to conditions similar to the small-gain ones.

In most of works (e.g., in those mentioned above) studying stability of impulsive systems by means of Lyapunov methods, it is assumed that the discrete and continuous dynamics share a common Lyapunov function V which decays either on jumps or along the continuous flow. A dwell-time condition allows to compensate the destabilizing effect of one type of dynamics by the stabilizing property of the other one. Certainly, if V increases in both cases, then the system is unstable. However, in general, it can happen that the whole system is asymptotically stable even in the case when both discrete and continuous dynamics are unstable. Identification of such systems needs a more refined consideration of the interaction between both dynamics types. It is expected that stability conditions become more involved in this case. It should be noted that there are not many stability results in the literature that cover the case of simultaneous instability of discrete and continuous dynamics: [1,9–11,19,24], see also [3] and [2] for the linear case. However, these results cannot be extended directly to the case of nonlinear infinite-dimensional systems with inputs, also we note that only local stability was studied in the first group of these papers.

Vector Lyapunov functions were used in [19] to establish stability results, where second-order derivatives of Lyapunov functions along solutions enter to the stability conditions. These results were generalized in [9,10], where higher-order derivatives of Lyapunov functions are employed. This approach cannot be used in case of systems with inputs as one would need to require infinitely smooth disturbances, which is very restrictive in real applications.

Averaged dwell-time conditions were considered in [1,24] to establish stability of a linear impulsive system on a Banach space. Based on the identities from the commutator calculus new comparison theorems, constructions of Lyapunov functions and conditions for the local asymptotic stability were developed there. In this work, we are interested in global stability properties for nonlinear systems with inputs.

With an exception of [22], in the most of works devoted to investigations of the ISS-like properties of hybrid systems, it is assumed that either the continuous or discrete dynamics satisfies the ISS property. Hence, it is interesting to further develop the direct Lyapunov method for the case, when both types of dynamics fail to be ISS. Our paper contributes into this direction, providing a new approach and new stability conditions.

In this work, we improve the results of [9,10] and extend them to the case of ISS for nonlinear impulsive systems. We derive stability conditions by means of a series of Lyapunov-like functions. Instead of higher-order derivatives of Lyapunov functions employed in [9,10], we use an infinite sequence of auxiliary functions to provide estimates of the dwell-time in order to guarantee the ISS property. The obtained results are then applied to the studying of the global asymptotic stability of linear impulsive systems with continuous dynamics governed by a parabolic PDE. The ISS property is also studied for this type of systems. Moreover, we derive conditions for the ISS property of nonlinear locally homogeneous finite-dimensional impulsive systems.

The paper consists of six sections. Section 2 introduces the notation and several auxiliary inequalities used in the paper. The problem statement is described in Sect. 3. Section 4 contains the main results with their proofs. Application of the results to the investigation of GAS and ISS properties of linear impulsive systems in infinite-dimensional spaces and of nonlinear finite-dimensional systems is provided in Sect. 5. A brief discussion and conclusions are collected in Sect. 6. Proofs of several technical results are placed in Appendix.

2 Notation and preliminaries

We use the following classes of comparison functions:
$\mathcal{K} = \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ :$ continuous, strictly increasing and $\gamma(0) = 0\}$,
$\mathcal{K}_\infty = \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ : \gamma \in \mathcal{K}, \quad \gamma(s) \to \infty \quad \text{for} \quad s \to \infty\}$,
$\mathcal{L} = \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ :$ continuous, strictly decreasing and $\lim_{t\to\infty} \gamma(t) = 0\}$,
$\mathcal{KL} = \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ :$ continuous $\beta(\cdot, t) \in \mathcal{K}, \ \beta(s, \cdot) \in \mathcal{L}\}$.

By $C[0,l]$ we denote the space of continuous functions, defined on $[0,l]$ with values in $\mathbb{R}$ with norm $\|f\|_{C[0,l]} = \max_{x\in[0,l]} |f(x)|$, $C^k[0,l]$ denotes of k-times continuously differentiable functions with the norm $\|f\|_{C^k[0,l]} = \max_{p=0,\ldots,k} \max_{x\in[0,l]} |f^{(p)}(x)|$. $H^0[0,l] = L^2[0,l]$ is the Hilbert space of measurable square integrable functions with scalar product $(f, g)_{L^2[0,l]} = \int_0^l f(z)g(z)\,dz$. Let $\mathfrak{L}(L^2[0, l])$ be the Banach algebra of linear bounded operators defined on $L^2[0, l]$.

For $M \subset \mathbb{R}$ and a Banach space X by $L^\infty(M, X)$, we denote the space of mappings $f : M \to X$ with the norm $\|f\|_{L^\infty} = ess \sup_{m\in M} \|f(m)\|_X$. For $M = \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, we write $L^\infty(\mathbb{Z}_+, X) = l^\infty(X)$. By $B_r(x_0)$, we denote the open ball of radius $r > 0$ in X centered at x_0.

$C^1(U, \mathbb{R}^n)$, $U \subset \mathbb{R}$ is the set of continuously differentiable mappings $f : U \to \mathbb{R}^n$. $C^1(\mathbb{R}^n)$ is the space of continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}^n$, and for $f \in C^1(\mathbb{R}^n)$ by $\partial_x f(x)$, we denote the corresponding Jacobi matrix.

For a linear bounded operator A acting on a Banach space, $\sigma(A)$ denotes the spectrum of A and $r_\sigma(A)$ denotes its spectral radius.

$\mathbb{R}^{n\times m}$ is the space of $n \times m$-matrices, for $m = n$ the set $\mathbb{R}^{n\times n}$ is then a Banach algebra. We use the norm on $\mathbb{R}^{n\times n}$ induced by the Euclidean norm in $\mathbb{R}^n$: $\|A\| = \sup_{\|x\|=1} \|Ax\| = \lambda_{\max}^{1/2}(A^T A)$.

We will use the following well-known inequalities. For any $a, b \in \mathbb{R}_+$ and $\varrho \in \mathcal{K}$ holds

$$\varrho(a + b) \le \varrho(2a) + \varrho(2b); \tag{1}$$

for $p_1, p_2 \in (1, \infty)$ with $\frac{1}{p_1} + \frac{1}{p_2} = 1$, the Young's inequality is

$$xy \le \frac{x^{p_1}}{p_1} + \frac{y^{p_2}}{p_2}, \quad x \ge 0, \quad y \ge 0. \tag{2}$$

3 Problem statement and related stability notions

We consider dynamical systems with inputs defined similarly to [5,21,26] as follows

Definition 1 Let X be the state space with the norm $\|\cdot\|_X$ and $\mathcal{U}_1 \subset \{f : \mathbb{R} \to U_1\}$ be the space of input signals normed by $\|\cdot\|_{\mathcal{U}_1}$ with values in a nonempty subset U_1 of some linear normed space and invariant under the time shifts, that is, if $d_1 \in \mathcal{U}_1$ and $\tau \in \mathbb{R}$, then $\mathcal{S}_\tau d_1 \in \mathcal{U}_1$, where $\mathcal{S}_s : \mathcal{U}_1 \to \mathcal{U}_1$, $s \in \mathbb{R}$ is the linear operator defined by $\mathcal{S}_s u(t) = u(t + s)$.

The triple $\Sigma_c = (X, \mathcal{U}_1, \phi_c)$ is called dynamical system with inputs if the mapping $\phi_c : (t, t_0, x, d_1) \mapsto \phi_c(t, t_0, x, d_1)$ defined for all $(t, t_0, x, d_1) \in [t_0, t_0 + \epsilon_{t_0,x,d_1}) \times \mathbb{R} \times X \times \mathcal{U}_1$ for some positive ϵ_{t_0,x,d_1} and satisfies the following axioms

(Σ_c1) for $t_0 \in \mathbb{R}$, $x \in X$, $d_1 \in \mathcal{U}_1$, $t \in [t_0, t_0 + \epsilon_{t_0,x,d_1})$, the value of $\phi_c(t, t_0, x, d_1)$ is well-defined and the mapping $t \mapsto \phi_c(t, t_0, x, d_1)$ is continuous on $(t_0, t_0 + \epsilon_{t_0,x,d_1})$ with $\lim_{t\to t_0+} \phi_c(t, t_0, x, d_1) = x$;

(Σ_c2) $\phi_c(t, t, x, d_1) = x$ for any $(x, d_1) \in X \times \mathcal{U}_1$, $t \in \mathbb{R}$;

(Σ_c3) for any $t_0 \in \mathbb{R}$, $(t, x, d_1) \in [t_0, t_0 + \epsilon_{t_0,x,d_1}) \times X \times \mathcal{U}_1$ and $\widetilde{d}_1 \in \mathcal{U}_1$ with $d_1(s) = \widetilde{d}_1(s)$ for $s \in [t_0, t]$, and it holds that $\phi_c(t, t_0, x, d_1) = \phi_c(t, t_0, x, \widetilde{d}_1)$;

(Σ_c4) for any $(x, d_1) \in X \times \mathcal{U}_1$ and $t \geq \tau \geq t_0$ with $\tau \in [t_0, t_0 + \epsilon_{t_0,x,d_1})$, $t \in [\tau, \tau + \epsilon_{\tau,\phi_c(\tau,t_0,x,d_1),d_1}) \cap [t_0, t_0 + \epsilon_{t_0,x,d_1})$ it holds that

$$\phi_c(t, t_0, x, d_1) = \phi_c(t, \tau, \phi_c(\tau, t_0, x, d_1), d_1),$$

(Σ_c5) for any $(x, d_1) \in X \times \mathcal{U}_1$ and $t \in [t_0, t_0 + \epsilon_{t_0,x,d_1})$, it holds that

$$\epsilon_{t_0+\tau,x,d_1} = \epsilon_{t_0,x,\mathcal{S}_\tau d_1},$$
$$\phi_c(t + \tau, t_0 + \tau, x, d_1) = \phi_c(t, \tau, x, \mathcal{S}_\tau d_1).$$

Note that (Σ_c5) implies that for all $t \in [\tau, \tau + \epsilon_{\tau,x_0,d_1})$, $\tau \leq t$

$$\phi_c(t, \tau, x, d_1) = \phi_c(t - \tau, 0, x, \mathcal{S}_\tau d_1). \tag{3}$$

Systems with impulsive actions are defined as follows:

Definition 2 Let $\mathcal{E} = \{\tau_k\}_{k=0}^{\infty}$, $\tau_k \in \mathbb{R}$ be a strictly increasing time sequence of impulsive actions with $\lim\limits_{k\to\infty} \tau_k = \infty$. Let $\mathcal{U}_2 \subset \{f : \mathbb{Z}_+ \to U_2\}$ be the space of input signals normed by $\|\cdot\|_{\mathcal{U}_2}$ and taking values in a nonempty subset U_2 of some linear normed space. Let $g : X \times U_2 \to X$ be a mapping defining impulsive actions and the mapping ϕ be defined for all $(t, t_0, x, d_1, d_2) \in \mathbb{R} \times \mathbb{R} \times X \times \mathcal{U}_1 \times \mathcal{U}_2$, $t \geq t_0$.

The following data $\Sigma = (X, \Sigma_c, \mathcal{U}_2, g, \phi, \mathcal{E})$ defines an impulsive system if

(Σ_1) for all $(k, x, d_1) \in \mathbb{Z}_+ \times X \times \mathcal{U}_1$ the system Σ_c satisfies

$$\tau_{p(t_0)} - t_0 < \epsilon_{t_0,x,d_1}, \quad T_k := \tau_{k+1} - \tau_k < \epsilon_{\tau_k,x,d_1}$$

where we denote $p(t_0) := \min\{k \in \mathbb{Z}_+ : \tau_k \in \mathcal{E}_{t_0}\}$ with $\mathcal{E}_{t_0} = [t_0, \infty) \cap \mathcal{E}$; and

(Σ_2) the mapping ϕ satisfies

$$\begin{aligned} \phi(t, t_0, x, d_1, d_2) &= \phi_c(t, t_0, x, d_1), \quad \text{for all} \quad t \in [t_0, \tau_{p(t_0)}],\\ \phi(t, t_0, x, d_1, d_2) &= \phi_c(t, \tau_k, g(\phi(\tau_k, t_0, x, d_1, d_2), d_2(k)), d_1)\\ &\qquad \text{for all} \quad t \in (\tau_k, \tau_{k+1}], \quad k \in \mathbb{Z}_+, k \geq p(t_0). \end{aligned}$$

We will denote for short

$$\phi(\tau_k^+, t_0, x, d_1, d_2) = g(\phi(\tau_k, t_0, x, d_1, d_2), d_2(k)), \quad k \geq p(t_0), \ \tau_k \geq t_0$$

The conditions (Σ_c1) and (Σ_2) imply

$$\lim_{t\to\tau_k+} \phi(t, t_0, x, d_1, d_2) = \phi(\tau_k^+, t_0, x, d_1, d_2),$$
$$\lim_{t\to\tau_k-} \phi(t, t_0, x, d_1, d_2) = \phi(\tau_k, t_0, x, d_1, d_2);$$

and $(\Sigma_c 4)$, $(\Sigma_c 5)$, (Σ_2) imply that for $t \geq \tau \geq t_0$, $(x, d_1, d_2) \in X \times \mathcal{U}_1 \times \mathcal{U}_2$, the following holds:

$$\phi(t, t_0, x, d_1, d_2) = \phi(t, \tau, \phi(\tau, t_0, d_1, d_2), d_1, d_2). \tag{4}$$

The system Σ_c describes the continuous dynamics of the impulsive system Σ. One can also consider its discrete dynamics separately as a system Σ_d defined next

Definition 3 A discrete dynamical system with input $\Sigma_d = (X, g, \phi_d, \mathcal{U}_2)$ is given by a normed state space $(X, \|\cdot\|_X)$; a space of input signals $\mathcal{U}_2 \subset \{f : \mathbb{Z}_+ \to U_2\}$ with norm $\|\cdot\|_{\mathcal{U}_2}$ and values in a nonempty subset U_2 of a linear normed space; a mapping $g : X \times U_2 \to X$; and a mapping $\phi_d : (k, l, x, d_2) \mapsto \phi_d(k, l, x, d_2)$, for $(k, l, x, d_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \times X \times \mathcal{U}_2$, $k \geq l$ such that

$(\Sigma_d 1)$ $\phi_d(k, k, x, d_2) = x$, $\phi_d(k+1, l, x, d_2) = g(\phi_d(k, l, x, d_2), d_2(k))$ for all $k \geq l$.

Assumption For any $\tau \geq 0$, there exist $\xi,\ \xi_\tau \in \mathcal{K}_\infty$ and $\chi_\tau, \chi \in \mathcal{K}_\infty$ such that for all $(x, d_1, d_2) \in X \times \mathcal{U}_1 \times \mathcal{U}_2$, it holds that

$$\|\phi_c(t, 0, x, d_1)\| \leq \xi_\tau(\|x\|) + \chi_\tau(\|d_1\|_{\mathcal{U}_1}) \quad \text{for all} \quad t \in [0, \tau] \tag{5}$$

and

$$\|g(x, d_2)\| \leq \xi(\|x\|) + \chi(\|d_2\|_{U_2}). \tag{6}$$

This assumption is not restrictive; for example, if g is continuous, then (6) is satisfied.

We are interested in the stability properties of the system Σ and its robustness with respect to the input signals d_1 and d_2. To this end, we use the notion of input-to-state stability (ISS). It was originally introduced in [25] for time invariant finite-dimensional systems. In our case, we adapt it as follows for Σ

Definition 4 For a fixed time sequence $\mathcal{E}$ of impulsive actions, the system Σ is called ISS if there exist $\beta_{t_0} \in \mathcal{KL}$, $\gamma_{t_0} \in \mathcal{K}_\infty$, such that for any initial state $x \in X$, any $t \geq t_0$ and any $(d_1, d_2) \in \mathcal{U}_1 \times \mathcal{U}_2$, it holds that

$$\|\phi(t, t_0, x, d_1, d_2)\| \leq \beta_{t_0}(\|x\|, t) + \gamma_{t_0}(d), \qquad d := \|d_1\|_{\mathcal{U}_1} + \|d_2\|_{\mathcal{U}_2}. \tag{7}$$

Next, we introduce a class of functions that we will use as Lyapunov functions to study the ISS property.

Definition 5 A function $V : [t_0, \infty) \times X \to \mathbb{R}$ is said to be of class $\mathcal{V}(\mathcal{T}_0)$, where $\mathcal{T}_0 := [t_0, \infty) \setminus \mathcal{E}$, if it satisfies the following properties:

(1) V is continuous at any point $(t, x) \in \mathcal{T}_0 \times X$, left continuous in t for $t \in \mathcal{E}$, that is $\lim_{h \to 0-} V(t+h, x) = V(t, x)$ for all $(t, x) \in \mathcal{E} \times X$ and such that there exists $\lim_{h \to 0+} V(t+h, x) := V(t+0, x)$ for all $(t, x) \in \mathcal{E} \times X$;

(2) The Lie derivative $\dot{V}(t, x, \zeta)$ exists for all $(t, x, \zeta) \in \mathcal{T}_0 \times X \times U_1$ which is defined by

$$\dot{V}(t, x, \zeta) := \lim_{h \to 0+} \frac{1}{h}(V(t+h, \phi_c(t+h, t, x, \zeta)) - V(t, x)), \tag{8}$$

(3) For $(t, x, \zeta) \in \mathcal{E} \times X \times U_1$ the limits $\dot{V}(t \pm 0, x, \zeta) = \lim_{h \to 0+} \dot{V}(t \pm h, x, \zeta)$ exist and $\dot{V}(t - 0, x, \zeta) = \dot{V}(t, x, \zeta)$

4 Main results

In this section, we provide sufficient conditions to guarantee the ISS property for Σ. Complementary stability conditions are given in Theorems 1 and 2, respectively. Application of these results are illustrated in Sect. 5.

Theorem 1 *Assume that for Σ there are functions $V_i \in \mathcal{V}(\mathcal{T}_0)$, $i \in \mathbb{Z}_+$ with*

(1) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, it holds that

$$\alpha_1(\|x\|) \leq V_0(t, x) \leq \alpha_2(\|x\|), \quad \textit{for all} \quad (t, x) \in [t_0, \infty) \times X; \tag{9}$$

(2) there is a sequence $\eta_p \in \mathcal{K}_\infty$, $p \in \mathbb{Z}_+$ such that for all $(t, x, \zeta) \in \mathcal{T}_0 \times X \times U_1$ holds

$$\begin{aligned} \dot{V}_0(t, x, \zeta) &\leq V_1(t, x) + \eta_0(\|\zeta\|_{U_1}), \\ -\dot{V}_p(t, x, \zeta) &\leq V_{p+1}(t, x) + \eta_p(\|\zeta\|_{U_1}), \quad p \in \mathbb{N}; \end{aligned} \tag{10}$$

(3) there are $\eta \in \mathcal{K}_\infty$ and $W_k : X \to \mathbb{R}$, $k \in \mathbb{Z}_+$ so that $\forall\, (k, x, \zeta) \in \mathbb{Z}_+ \times X \times U_2$

$$V_0(\tau_k + 0, g(x, \zeta)) - V_0(\tau_k, x) \leq W_k(x) + \eta(\|\zeta\|_{U_2}); \tag{11}$$

holds;

(4) there exists $\delta \in \mathcal{K}_\infty$ such that for all $(k, x) \in \mathbb{Z}_+ \times X$, it holds that

$$G_{k+1}(x) := W_{k+1}(x) + \sum_{p=1}^{\infty} V_p(\tau_{k+1}, x) \frac{(\tau_{k+1} - \tau_k)^p}{p!} \leq -\delta(\|x\|); \tag{12}$$

(5) for any $\rho > 0$ exists $q_\rho \in [0, 1)$ such that $\lim_{p \to \infty} \frac{\eta_p(s)}{(p+1)\eta_{p-1}(s)} = q_\rho$ uniformly for $s \in [0, \rho]$ and for any $k \in \mathbb{Z}_+$, there is $\omega_k \in \mathcal{K}_\infty$ such that $|V_p(s, x)| \leq \omega_k(\|x\|)$ for all $(p, s) \in \mathbb{Z}_+ \times (\tau_k, \tau_{k+1}]$.

Then, Σ satisfies the ISS property.

Remark 1 The inequality (12) in condition (4) of the theorem restricts the time intervals between jumps, which is a dwell-time condition. If both continuous and discrete dynamics are stable, the existence of W_k and V_p so that $W_{k+1}(x)$ and $\sum_{p=1}^{\infty} V_p(\tau_{k+1}, x)\frac{(\tau_{k+1}-\tau_k)^p}{p!}$ are negative is guaranteed, so that (12) implies no restrictions on $\mathcal{E}$. However if one of the dynamics is unstable, the dwell-time is restricted. For example, if the discrete dynamics is stable but the continuous one is not, we have $W_{k+1}(x) < 0$, but the second summand in (12) can be positive, and hence it needs to be small enough in order to satisfy (12). This implies that the time distances $\tau_{k+1} - \tau_k$ need to be small enough. Moreover, (12) allows the situation, where both dynamics types are not stable. This will be illustrated in our examples.

Let us fix any $(t_0, x_0) \in \mathbb{R} \times X$ as the initial data and assume without loss of generality that $t_0 \le \tau_0$. For any disturbances $(d_1, d_2) \in \mathcal{U}_1 \times \mathcal{U}_2$, the corresponding solution to Σ will be denoted by $x(t) = \phi(t, t_0, x_0, d_1, d_2)$.

Lemma 1 *Under the conditions of Theorem 1 for any $k \in \mathbb{Z}_+$, $n \ge 2$ the following inequality holds true:*

$$\begin{aligned} V_0(\tau_k + 0, x(\tau_k + 0)) \ge V_0(\tau_{k+1}, x(\tau_{k+1})) - \sum_{p=1}^{n-1} V_p(\tau_{k+1}, x(\tau_{k+1}))\frac{(\tau_{k+1} - \tau_k)^p}{p!} \\ - \int_{\tau_k}^{\tau_{k+1}}\int_{s_1}^{\tau_{k+1}} \dots \int_{s_{n-2}}^{\tau_{k+1}} V_n(s_{n-1}, x(s_{n-1}))\, ds_0 \dots ds_{n-1} \\ - \sum_{p=1}^{n} \frac{\eta_{p-1}(\|d_1\|_{\mathcal{U}_1})\theta^p}{p!}. \end{aligned} \tag{13}$$

The proof can be found in Appendix.

Corollary 1 *Under the conditions of Theorem 1, the following estimate is true:*

$$\begin{aligned} &V_0(\tau_{k+1}, x(\tau_{k+1})) - V_0(\tau_k + 0, x(\tau_k + 0)) \\ &\quad \le \sum_{p=1}^{\infty} V_p(\tau_{k+1}, x(\tau_{k+1}))\frac{(\tau_{k+1} - \tau_k)^p}{p!} + \sum_{p=1}^{\infty} \frac{\eta_{p-1}(\|d_1\|_{\mathcal{U}_1})\theta^p}{p!}. \end{aligned} \tag{14}$$

Proof The estimate follows after taking the limit for $n \to \infty$ in the inequality (13), which is possible under the conditions of Theorem 1. □

Proof (of Theorem 1) The combination of (11) and (14) implies that

$$\begin{aligned} &V_0(\tau_{k+1} + 0, x(\tau_{k+1} + 0)) - V_0(\tau_k + 0, x(\tau_k + 0)) \\ &\quad \le W_{k+1}(x(\tau_{k+1})) + \sum_{p=1}^{\infty} V_p(\tau_{k+1}, x(\tau_{k+1}))\frac{(\tau_{k+1} - \tau_k)^p}{p!} + \sum_{p=1}^{\infty} \frac{\eta_{p-1}(\|d_1\|_{\mathcal{U}_1})\theta^p}{p!} \\ &\qquad + \eta(\|d_2\|_{\mathcal{U}_2}). \end{aligned}$$

Let $\widehat{\eta}(s) := \sum_{p=1}^{\infty} \frac{\eta_{p-1}(s)\theta^p}{p!} + \eta(s)$, then $\widehat{\eta} \in \mathcal{K}_\infty$. Recall that $d = \|d_1\|_{\mathcal{U}_1} + \|d_2\|_{\mathcal{U}_2}$, then for all $k \in \mathbb{Z}_+$, we have

$$V_0(\tau_{k+1} + 0, x(\tau_{k+1} + 0)) - V_0(\tau_k + 0, x(\tau_k + 0)) \le G_{k+1}(x(\tau_{k+1})) + \widehat{\eta}(d).$$

From this inequality together with condition (4) of Theorem 1, it follows that for all $k \in \mathbb{Z}_+$

$$V_0(\tau_{k+1} + 0, x(\tau_{k+1} + 0)) - V_0(\tau_k + 0, x(\tau_k + 0)) \le -\delta(\|x(\tau_{k+1})\|) + \widehat{\eta}(d). \quad (15)$$

Let $\epsilon \in (0, 1)$, $r = \delta^{-1}(\frac{\widehat{\eta}(d)}{1-\epsilon})$. Let us show by contradiction that for some $k^* \in \mathbb{Z}_+$, it holds that $x(\tau_{k^*}) \in B_r(0)$. Assume that this is not true, that is for all $k \in \mathbb{Z}_+$, we have $\|x(\tau_k)\| \ge r$, then from (15) follows:

$$\begin{aligned} &V_0(\tau_{k+1} + 0, x(\tau_{k+1} + 0)) - V_0(\tau_k + 0, x(\tau_k + 0)) \\ &\quad \le -\delta(r) + \widehat{\eta}(d) = -\frac{\widehat{\eta}(d)}{1-\epsilon} + \widehat{\eta}(d) < 0. \end{aligned}$$

This means that the bounded from below sequence $\{V_0(\tau_k + 0, x(\tau_k + 0))\}_{k \in \mathbb{Z}_+}$ is decreasing, hence there exists the limit $m := \lim_{k\to\infty} V_0(\tau_k + 0, x(\tau_k + 0))$. From (15) follows then $\limsup_{k\to\infty} \delta(\|x(\tau_{k+1})\|) \le \widehat{\eta}(d)$. This implies that

$$\frac{\widehat{\eta}(d)}{1-\epsilon} = \delta(r) \le \limsup_{k\to\infty} \delta(\|x(\tau_{k+1})\|) \le \widehat{\eta}(d),$$

which leads to a contradiction. Hence, for some k^*, it holds that $\|x(\tau_{k^*})\| < r$. Let $R := \max\{(\alpha_1^{-1} \circ \alpha_2)(\xi(r) + \chi(d)), r\}$. We show that for $k \ge k^*$, the inequality $\|x(\tau_k + 0)\| \le R$ is true. Indeed, if for some $m \ge k^*$

$$\|x(\tau_m)\| < r, \quad \|x(\tau_{m+i})\| \ge r \quad \text{for all} \quad i = 1, \dots, j(m),$$

where $1 \le j(m) \le \infty$, then from (15) follows that

$$\begin{aligned} &V_0(\tau_{m+i} + 0, x(\tau_{m+i} + 0)) - V_0(\tau_{m+i-1} + 0, x(\tau_{m+i-1} + 0)) \\ &\quad \le -\delta(\|x(\tau_{m+i})\|) + \widehat{\eta}(d) < 0; \end{aligned}$$

hence, by condition (1) of Theorem 1, we obtain

$$\|x(\tau_{m+i} + 0)\| \le (\alpha_1^{-1} \circ \alpha_2)(\|x(\tau_m + 0)\|).$$

Taking (6) into account, we obtain $\|x(\tau_m + 0)\| \le \xi(\|x(\tau_m)\|) + \chi(d)$. Hence,

$$\|x(\tau_{m+i} + 0)\| \le (\alpha_1^{-1} \circ \alpha_2)(\xi(r) + \chi(d)) \le R.$$

Let $S_r := \{k \in \mathbb{Z} \ : \ \forall l, \ 0 \leq l \leq k \ \|x(\tau_l)\| \geq r\}$ and

$$N := \begin{cases} \max S_r, & \text{for} \quad S_r \neq \emptyset, \\ 0 \quad \text{for} \quad S_r = \emptyset \end{cases}$$

We need to consider the case $N \geq 1$. Let $k \in \mathbb{N}$ be such that $1 \leq k \leq N$, then $\widehat{\eta}(d) \leq (1-\epsilon)\delta(\|x(\tau_k)\|)$ and from (15), it follows that

$$\begin{aligned} &V_0(\tau_k + 0, x(\tau_k + 0)) - V_0(\tau_{k-1} + 0, x(\tau_{k-1} + 0)) \\ &\quad \leq -\delta(\|x(\tau_k)\|) + \widehat{\eta}(d) \leq -\epsilon\delta(\|x(\tau_k)\|). \end{aligned} \tag{16}$$

The inequality (6) implies that

$$\begin{aligned} \|x(\tau_k + 0)\| &\leq \xi(\|x(\tau_k)\|) + \chi(d) \\ &\leq \xi(\|x(\tau_k)\|) + \chi(\widehat{\eta}^{-1}((1-\epsilon)\delta(\|x(\tau_k)\|))) := \varphi(\|x(\tau_k)\|). \end{aligned}$$

It is easily seen that $\varphi \in \mathcal{K}_\infty$, hence $\|x(\tau_k)\| \geq \varphi^{-1}(\|x(\tau_k + 0)\|)$ and from (16) follows

$$\begin{aligned} &V_0(\tau_k + 0, x(\tau_k + 0)) - V_0(\tau_{k-1} + 0, x(\tau_{k-1} + 0)) \\ &\quad \leq -\epsilon(\delta \circ \varphi^{-1})(\|x(\tau_k + 0)\|) \\ &\quad \leq -\epsilon(\delta \circ \varphi^{-1} \circ \alpha_2^{-1})(V_0(\tau_k + 0, x(\tau_k + 0))). \end{aligned} \tag{17}$$

We denote $\delta_1 := \delta \circ \varphi^{-1} \circ \alpha_2^{-1} \in \mathcal{K}_\infty$, $v_k := V_0(\tau_k + 0, x(\tau_k + 0))$, and conclude that for all k, $1 \leq k \leq N$, the inequality (17) can be written as

$$v_k - v_{k-1} \leq -\epsilon\delta_1(v_k). \tag{18}$$

Let us define the sequence $\widehat{v}_k$ for $k \in \mathbb{N}$ by $\widehat{v}_k := v_k$ for $1 \leq k \leq N$ and so that for $k \geq N+1$ the $\widehat{v}_k$ satisfies the difference equation $\widehat{v}_k - \widehat{v}_{k-1} = -\epsilon\delta_1(\widehat{v}_k)$.

Hence for all $k \in \mathbb{N}$, the sequence $\{\widehat{v}_k\}_{k\in\mathbb{N}}$ satisfies the inequality

$$\widehat{v}_k - \widehat{v}_{k-1} \leq -\epsilon\delta_1(\widehat{v}_k). \tag{19}$$

Together with the inequality (19), we consider the comparison equation

$$w_k - w_{k-1} = -\epsilon\delta_1(w_k), \quad w_0 = \widehat{v}_0. \tag{20}$$

First, let us show that for all $k \in \mathbb{Z}_+$, the inequality $\widehat{v}_k \leq w_k$ is true. Indeed, if for some $k_1 \in \mathbb{N}$ $\widehat{v}_{k_1-1} \leq w_{k_1-1}$ and $\widehat{v}_{k_1} > w_{k_1}$, then from $\mathrm{id} + \epsilon\delta_1 \in \mathcal{K}_\infty$, it follows that

$$0 < \widehat{v}_{k_1} - w_{k_1} \leq (\mathrm{id} + \epsilon\delta_1)^{-1}(\widehat{v}_{k_1-1}) - (\mathrm{id} + \epsilon\delta_1)^{-1}(w_{k_1-1}) \leq 0,$$

which is a contradiction. From (20) follows $w_k = (\mathrm{id} + \epsilon\delta_1)^{-1}(w_{k-1})$, $k \in \mathbb{N}$. By the properties of comparison functions, there exists $\delta_2 \in \mathcal{K}_\infty$ such that $(\mathrm{id} + \epsilon\delta_1)^{-1} = \mathrm{id} - \delta_2$, hence (20) can be written as

$$w_k - w_{k-1} = -\delta_2(w_{k-1}), \quad w_0 = \widehat{v}_0. \tag{21}$$

Let $\widehat{\delta_2}(s) := \min\{s, \delta_2(s)\}$, then

$$w_k - w_{k-1} \le -\widehat{\delta_2}(w_{k-1}), \quad w_0 = \widehat{v}_0. \tag{22}$$

We define $\Delta_{v_0}(s) := \int\limits_s^{v_0} \frac{d\tau}{\widehat{\delta_2}(\tau)}$, then $\Delta_{v_0}(s) \to +\infty$ for $s \to 0+$. By the mean value theorem for some $s^* \in (w_k, w_{k-1})$ from (22), we obtain

$$\Delta_{v_0}(w_k) - \Delta_{v_0}(w_{k-1}) = \int\limits_{w_k}^{w_{k-1}} \frac{ds}{\widehat{\delta_2}(s)} = \frac{w_{k-1} - w_k}{\widehat{\delta_2}(s^*)} \ge \frac{\widehat{\delta_2}(w_{k-1})}{\widehat{\delta_2}(s^*)} \ge 1.$$

Hence, $\Delta_{v_0}(w_k) \ge \Delta_{v_0}(v_0) + k = k$, which implies the estimate

$$V(\tau_k + 0, x(\tau_k + 0)) \le \Delta_{v_0}^{-1}(k) \quad \text{for all} \quad k \in \mathbb{Z}_+.$$

This means that for all k, $0 \le k \le N$, we can estimate $\|x(\tau_k + 0)\| \le (\alpha_1^{-1} \circ \Delta_{v_0}^{-1})(k)$. From (5), we have that for all $k \in \mathbb{Z}_+$ follows

$$\|x(t)\| \le \xi_\theta(\|x(\tau_k + 0)\|) + \chi_\theta(d) \quad \text{for all} \quad t \in (\tau_k, \tau_{k+1}]$$

That is for all $t \in (\tau_k, \tau_{k+1}]$, $0 \le k \le N - 1$, the following inequality holds:

$$\|x(t)\| \le (\xi_\theta \circ \alpha_1^{-1} \circ \Delta_{v_0}^{-1})(k) + \chi_\theta(d). \tag{23}$$

Let $\vartheta_k := (\xi_2 \circ \alpha_1^{-1} \circ \Delta_{v_0}^{-1})(k)$, and we define the function

$$\widetilde{\beta}(v_0, t) = \vartheta_{k-1} + \frac{t - \tau_{k-1}}{\tau_k - \tau_{k-1}}(\vartheta_k - \vartheta_{k-1}) \quad \text{for} \quad t \in (\tau_{k-1}, \tau_k], \quad k \in \mathbb{N}.$$

It is easily seen that $\widetilde{\beta} \in \mathcal{KL}$. From (23), we get for all $t \in (\tau_0, \tau_N]$, the inequality

$$\|x(t)\| \le \widetilde{\beta}(V(\tau_0 + 0, x(\tau_0 + 0)), t) + \chi_\theta(d) \le \widetilde{\beta}(\alpha_2(\|x(\tau_0 + 0)\|), t) + \chi_\theta(d). \tag{24}$$

Applying the estimates

$$\|x(\tau_0 + 0)\| \le \xi(\|x(\tau_0)\|) + \chi(d) \quad \text{and} \quad \|x(\tau_0)\| \le \xi_{\tau_0 - t_0}(\|x_0\|) + \chi_{\tau_0 - t_0}(d)$$

and the inequality (1), we obtain that for some functions $\widetilde{\xi}_{\tau_0-t_0}, \widetilde{\chi}_{\tau_0-t_0} \in \mathcal{K}_\infty$

$$\|x(\tau_0+0)\| \leq \widetilde{\xi}_{\tau_0-t_0}(\|x_0\|) + \widetilde{\chi}_{\tau_0-t_0}(d)$$

Applying again (1), we derive

$$\alpha_2(\|x(\tau_0+0)\|) \leq (\alpha_2 \circ 2\widetilde{\xi}_{\tau_0-t_0})(\|x_0\|) + (\alpha_2 \circ 2\widetilde{\chi}_{\tau_0-t_0})(d).$$

From (24) using again (1), we get that for all $t \in (\tau_0, \tau_N]$, the inequality

$$\begin{aligned} \|x(t)\| &\leq \widetilde{\beta}(2(\alpha_2 \circ 2\widetilde{\xi}_{\tau_0-t_0})(\|x_0\|), t) + \widetilde{\beta}(2(\alpha_2 \circ 2\widetilde{\chi}_{\tau_0-t_0})(d), t) + \chi_\theta(d) \\ &\leq \widehat{\beta}_{t_0}(\|x_0\|, t) + \widehat{\chi}_{t_0}(d). \end{aligned} \tag{25}$$

holds, where we have denoted

$$\begin{aligned} \widehat{\beta}_{t_0}(s,t) &:= \widetilde{\beta}(2(\alpha_2 \circ 2\widetilde{\xi}_{\tau_0-t_0})(s), t), \\ \widehat{\chi}_{t_0}(d) &= \widetilde{\beta}(2(\alpha_2 \circ 2\widetilde{\chi}_{\tau_0-t_0})(d), \tau_0+0) + \chi_\theta(d) \end{aligned}$$

It easy to check by definition that $\widehat{\beta}_{t_0} \in \mathcal{KL}$, $\widehat{\chi}_{t_0} \in \mathcal{K}_\infty$. There exists a function $\beta_{t_0} \in \mathcal{KL}$ such that $\beta_{t_0}(s,t) \geq \widehat{\beta}_{t_0}(s,t)$ for $t \in (\tau_0, \tau_N]$ and $\beta_{t_0}(s,t) \geq \widetilde{\xi}_{\tau_0-t_0}(s)$ for $t \in [t_0, \tau_0]$. Hence, from (25), it follows that for all $t \in [t_0, \tau_N]$

$$\|x(t)\| \leq \beta_{t_0}(\|x_0\|, t) + \widehat{\chi}_{t_0}(d) + \chi_{\tau_0-t_0}(d). \tag{26}$$

Recall that for $k \geq N$, we have $\|x(\tau_k+0)\| \leq R$. Hence from the estimate $\|x(t)\| \leq \xi_\theta(\|x(\tau_k+0)\|) + \chi_\theta(d)$, $t \in (\tau_k, \tau_{k+1}]$ it follows that

$$\|x(t)\| \leq \xi_\theta(R) + \chi_\theta(d) := \widehat{\chi}(d), \quad t > \tau_N, \tag{27}$$

where $\hat{\chi} \in \mathcal{K}_\infty$ by definition. Combining the inequalities (26) and (27), we see that for some $\gamma_{t_0} \in \mathcal{K}_\infty$, the following estimate holds

$$\|x(t)\| \leq \beta_{t_0}(\|x_0\|, t) + \gamma_{t_0}(d), \quad t \geq t_0,$$

which proves the theorem. □

Remark 2 Condition (5) of Theorem 1 assures the convergence of three last terms in (14) for $n \to \infty$. However, if $V_n(t,x) \leq 0$ for some $n \geq 1$, then we can set $V_p(t,x) \equiv 0$ for all $p > n$ and this condition (5) can be dropped. Stability investigation in this case is essentially easier because we deal with a finite number of auxiliary functions instead of an infinite sequence. The class of systems, where this simplification is possible becomes wider due to the next result (see the difference in the sign before $\dot{V}_p$ in conditions (2) of the previous and the next theorem). Such simplification will be used in some of our examples later.

Theorem 2 *Assume that for the system Σ, there are $V_i \in \mathcal{V}(\mathcal{T}_0)$ such that*

(1) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ it holds that

$$\alpha_1(\|x\|) \le V_0(t,x) \le \alpha_2(\|x\|), \quad \textit{for all} \quad (t,x) \in [t_0, +\infty) \times X, \tag{28}$$

(2) there is a sequence $\eta_p \in \mathcal{K}_\infty$, $p \in \mathbb{Z}_+$ such that $\forall\,(t,x,\zeta) \in \mathcal{T}_0 \times X \times U_1$

$$\dot{V}_p(t,x,\zeta) \le V_{p+1}(t,x) + \eta_p(\|\zeta\|_{U_1}), \quad p \in \mathbb{Z}_+, \tag{29}$$

(3) there are $\eta \in \mathcal{K}_\infty$ and $W_k : X \to \mathbb{R}$, $k \in \mathbb{Z}_+$ so that $\forall\,(k,x,\zeta) \in \mathbb{Z}_+ \times X \times U_2$

$$V_0(\tau_k + 0, g(x,\zeta)) - V_0(\tau_k, x) \le W_k(x) + \eta(\|\zeta\|_{U_2}), \tag{30}$$

(4) there are $Q_{kp} : X \to \mathbb{R}$ and $\pi_p \in \mathcal{K}_\infty$, $(k,p) \in \mathbb{Z}_+ \times \mathbb{N}$, $(x,\zeta) \in X \times U_2$ with

$$V_p(\tau_k + 0, g(x,\zeta)) - V_p(\tau_k, x) \le Q_{kp}(x) + \pi_p(\|\zeta\|_{U_2}), \tag{31}$$

(5) there is $\delta \in \mathcal{K}_\infty$ such that for all $(k,x) \in \mathbb{Z}_+ \times X$, the next inequality holds

$$G_k(x) := W_k(x) + \sum_{p=1}^{\infty} (V_p(\tau_k, x) + Q_{kp}(x)) \frac{(\tau_{k+1} - \tau_k)^p}{p!} \le -\delta(\|x\|), \tag{32}$$

(6) For any $\rho > 0$, there exists $q_\rho \in [0,1)$ such that $\lim_{p\to\infty} \frac{\eta_p(s)}{(p+1)\eta_{p-1}(s)} \le q_\rho$ and $\lim_{p\to\infty} \frac{\pi_p(s)}{(p+1)\pi_{p-1}(s)} \le q_\rho$ exist uniformly for $s \in [0,\rho]$, and for each $k \in \mathbb{Z}_+$ exists $\omega_k \in \mathcal{K}_\infty$ such that

$$|V_p(s,x)| \le \omega_k(\|x\|), \quad |Q_{kp}(x)| \le \omega_k(\|x\|)$$

for all $(p,s,x) \in \mathbb{Z}_+ \times (\tau_k, \tau_{k+1}] \times X$. Then, system Σ is ISS.

Lemma 2 *Under the conditions of Theorem 2, we have for all $k \in \mathbb{Z}_+$, $n \ge 2$*

$$\begin{aligned} V_0(\tau_{k+1}, x(\tau_{k+1})) \le\ & V_0(\tau_k + 0, x(\tau_k + 0)) \\ & + \sum_{p=1}^{n-1} V_p(\tau_k + 0, x(\tau_k + 0)) \frac{(\tau_{k+1} - \tau_k)^p}{p!} \\ & + \int_{\tau_k}^{\tau_{k+1}} \int_{\tau_k}^{s_1} \dots \int_{\tau_k}^{s_{n-2}} V_n(s_{n-1}, x(s_{n-1}))\, ds_0 \dots ds_{n-1} \\ & + \sum_{p=1}^{n} \frac{\eta_{p-1}(\|d\|_{\mathcal{U}_1})\theta^p}{p!}. \end{aligned} \tag{33}$$

Corollary 2 *Under the conditions of Theorem 2, the following holds:*

$$
\begin{aligned}
V_0(\tau_{k+1}, x(\tau_{k+1})) \le\ & V_0(\tau_k + 0, x(\tau_k + 0)) \\
&+ \sum_{p=1}^{\infty} V_p(\tau_k + 0, x(\tau_k + 0)) \frac{(\tau_{k+1} - \tau_k)^p}{p!} \\
&+ \sum_{p=1}^{\infty} \frac{\eta_{p-1}(\|d_1\|_{\mathcal{U}_1})\theta^p}{p!}.
\end{aligned} \tag{34}
$$

Proof Condition (6) of Theorem 2 assures the possibility to take the limit for $n \to \infty$ in (33) which implies the assertion. □

Proof (of Theorem 2) From (30) and (34) follows

$$
\begin{aligned}
V_0(\tau_{k+1}, x(\tau_{k+1})) \le\ & V_0(\tau_k, x(\tau_k)) + W_k(x(\tau_k)) \\
&+ \sum_{p=1}^{\infty} V_p(\tau_k + 0, x(\tau_k + 0)) \frac{(\tau_{k+1} - \tau_k)^p}{p!} \\
&+ \eta(\|d_2\|_{\mathcal{U}_2}) + \sum_{p=1}^{\infty} \frac{\eta_{p-1}(\|d_1\|_{\mathcal{U}_1})\theta^p}{p!}.
\end{aligned} \tag{35}
$$

Taking (31) into account, we obtain

$$
\begin{aligned}
V_0(\tau_{k+1}, x(\tau_{k+1})) \le\ & V_0(\tau_k, x(\tau_k)) + W_k(x(\tau_k)) \\
&+ \sum_{p=1}^{\infty} (V_p(\tau_k, x(\tau_k)) + Q_{kp}(x(\tau_k))) \frac{(\tau_{k+1} - \tau_k)^p}{p!} \\
&+ \eta(\|d_2\|_{\mathcal{U}_2}) + \sum_{p=1}^{\infty} \frac{(\pi_p(\|d_2\|_{\mathcal{U}_2}) + \eta_{p-1}(\|d_1\|_{\mathcal{U}_1}))\theta^p}{p!}.
\end{aligned} \tag{36}
$$

Let $\widehat{\eta}(s) := \eta(s) + \sum_{p=1}^{\infty} \frac{(\pi_p(s)+\eta_{p-1}(s))\theta^p}{p!}$. Obviously $\widehat{\eta} \in \mathcal{K}_\infty$ and from (32) and (36) follows (recall that $d = \|d_1\|_{\mathcal{U}_1} + \|d_2\|_{\mathcal{U}_2}$)

$$
V_0(\tau_{k+1}, x(\tau_{k+1})) \le V_0(\tau_k, x(\tau_k)) - \delta(\|x(\tau_k)\|) + \widehat{\eta}(d). \tag{37}
$$

Let $\epsilon \in (0, 1)$, $r = \delta^{-1}(\frac{\widehat{\eta}(d)}{1-\epsilon})$. First we show by contradiction that there exists $k^* \in \mathbb{Z}_+$ such that $\|x(\tau_{k^*})\| < r$. Indeed, otherwise, for all $k \in \mathbb{Z}_+$ $\|x(\tau_k)\| \ge r$ and hence from (37), it follows that

$$
V_0(\tau_{k+1}, x(\tau_{k+1})) - V_0(\tau_k, x(\tau_k)) \le -\delta(r) + \widehat{\eta}(d) = -\frac{\widehat{\eta}(d)}{1-\epsilon} + \widehat{\eta}(d) < 0.
$$

This means that the sequence $\{V_0(\tau_k, x(\tau_k))\}_{k \in \mathbb{Z}_+}$ is strictly decreasing and is bounded from below, hence it possesses a nonnegative limit $m^* = \lim_{k\to\infty} V_0(\tau_k, x(\tau_k))$. From

(37) follows

$$\frac{\widehat{\eta}(d)}{1-\epsilon} \leq \delta(r) \leq \lim \sup_{k\to\infty} \delta(\|x(\tau_k)\|) \leq \widehat{\eta}(d),$$

which leads to a contradiction.

We denote $R := \max\{(\alpha_1^{-1} \circ \alpha_2)(\xi_\theta(\xi(r) + \chi(d)) + \chi_\theta(d)), r\}$. Let us show that for $k \geq k^*$ the inequality $\|x(\tau_k)\| \leq R$ is true. Indeed, if for some $m \geq k^*$

$$\|x(\tau_m)\| < r, \quad \|x(\tau_{m+i})\| \geq r \quad \text{for all} \quad i = 1, \ldots, j(m),$$

where $1 \leq j(m) \leq \infty$, then from (37), we have that for $i \geq 2$

$$V_0(\tau_{m+i}, x(\tau_{m+i})) - V_0(\tau_{m+i-1}, x(\tau_{m+i-1})) \leq -\delta(\|x(\tau_{m+i-1})\|) + \widehat{\eta}(d) < 0,$$

and hence from (5) and by the condition (1) of Theorem 2, it follows that

$$\|x(\tau_{m+i})\| \leq (\alpha_1^{-1} \circ \alpha_2)(\|x(\tau_{m+1})\|) < (\alpha_1^{-1} \circ \alpha_2)(\xi_\theta(\|x(\tau_m + 0)\|) + \chi_\theta(d)).$$

Taking (6) into account, we obtain

$$\|x(\tau_{m+i})\| \leq (\alpha_1^{-1} \circ \alpha_2)(\xi_\theta(\xi(r) + \chi(d)) + \chi_\theta(d)) \leq R.$$

Similarly, for $i = 1$ we get

$$\begin{aligned}\|x(\tau_{m+1})\| &\leq \xi_\theta(\|x(\tau_m + 0)\|) + \chi_\theta(d) \leq \xi_\theta(\xi(\|x(\tau_m)\|) + \chi(d)) + \chi_\theta(d) \\ &\leq \xi_\theta(\xi(r) + \chi(d)) + \chi_\theta(d) \leq R.\end{aligned}$$

Let $S_r := \{k \in \mathbb{Z}_+ \ : \ \forall l,\ 0 \leq l \leq k\ \|x(\tau_l)\| \geq r\}$ and

$$N = \begin{cases}\max S_r, & \text{for} \quad S_r \neq \emptyset, \\ 0 \quad \text{for} \quad S_r = \emptyset\end{cases}$$

It is enough to consider the case $N \geq 1$. Let k be such that $0 \leq k \leq N$, then $\widehat{\eta}(d) \leq (1-\epsilon)\delta(\|x(\tau_k)\|)$, and from (37), the next inequality follows:

$$V_0(\tau_{k+1}, x(\tau_{k+1})) \leq V_0(\tau_k, x(\tau_k)) - \epsilon\delta(\|x(\tau_k)\|). \tag{38}$$

By condition (1) of Theorem 2, we obtain $\|x(\tau_k)\| \geq \alpha_2^{-1}(V_0(\tau_k, x(\tau_k)))$ and hence (38) can be written as

$$V_0(\tau_{k+1}, x(\tau_{k+1})) \leq V_0(\tau_k, x(\tau_k)) - \epsilon\delta_1(V_0(\tau_k, x(\tau_k))),$$

where $\delta_1 = \delta \circ \alpha_2^{-1} \in \mathcal{K}_\infty$. Let us denote $v_k := V_0(\tau_k, x(\tau_k))$, then for all $k \in \mathbb{Z}_+$, we have the inequality $v_{k+1} - v_k \le -\epsilon \delta_1(v_k)$. Let $\widehat{\delta}_1(s) := \min\{s, \delta_1(s)\} \in \mathcal{K}_\infty$, then

$$v_{k+1} - v_k \le -\epsilon \widehat{\delta}_1(v_k).$$

We define the $\{\widehat{v}_k\}_{k\in\mathbb{Z}_+}$ sequence recurrently setting $\widehat{v}_k := v_k$ for $0 \le k \le N$ and

$$\widehat{v}_{k+1} - \widehat{v}_k = -\epsilon \widehat{\delta}_1(\widehat{v}_k)$$

for $k \ge N$. From this definition follows that for all $k \in \mathbb{Z}_+$

$$\widehat{v}_{k+1} - \widehat{v}_k \le -\epsilon \widehat{\delta}_1(\widehat{v}_k) \tag{39}$$

and $0 \le \widehat{v}_k \le v_0$.

Let $\Delta_{v_0}(s) := \int\limits_s^{v_0} \frac{d\tau}{\widehat{\delta}_1(\tau)}$. By the mean value theorem $\exists\, s^* \in (\widehat{v}_{k+1}, \widehat{v}_k)$ with

$$\Delta_{v_0}(\widehat{v}_{k+1}) - \Delta_{v_0}(\widehat{v}_k) = \int\limits_{\widehat{v}_{k+1}}^{\widehat{v}_k} \frac{ds}{\widehat{\delta}_1(s)} = \frac{\widehat{v}_k - \widehat{v}_{k+1}}{\widehat{\delta}_1(s^*)} \ge \epsilon \frac{\widehat{\delta}_1(\widehat{v}_k)}{\widehat{\delta}_1(s^*)} \ge \epsilon.$$

Hence, $\Delta_{v_0}(\widehat{v}_k) \ge k\epsilon$ for all $k \in \mathbb{Z}_+$. From its definition, it follows that $\Delta_{v_0}(s)$ is strictly growing with respect to v_0 for a fixed s and strictly decreasing in s for a fixed v_0 with $\Delta_{v_0}(s) \to \infty$ for $s \to 0+$. This implies that

$$V(\tau_k, x(\tau_k)) \le \Delta_{v_0}^{-1}(k\epsilon) \quad \text{for all} \quad k \in \mathbb{Z}_+.$$

The desired ISS estimate follows similarly as in the proof of Theorem 1, which proves Theorem 2. □

Remark 3 If for some $n \ge 1$ we have $V_n(t, x) \le 0$, then we can take $V_p(t, x) \equiv 0$ for all $p > n$, and in this case, condition (5) of Theorem 2 can be dropped.

5 Examples

5.1 Heat equation with variable coefficients

Let X be the normed linear vector space defined by

$$X = \{f \ : \ [0, l] \to \mathbb{R} \mid f \in C^1(0, l),\ f|_{(0,2\delta)} \in C^2(0, 2\delta), \quad f|_{(2\delta,l)} \in C^2(2\delta, l)\},$$

with $l > 0$, $0 < 2\delta < l$ and the norm $\|\cdot\|_X := \|\cdot\|_{L^2[0,l]}$.

Consider the following linear system with impulsive actions

$$\begin{aligned} u_t(z, t) &= a^2 u_{zz}(z, t) + b(z)u(z, t), \quad t \notin \mathcal{E} \\ u(z, t^+) &= u(z, t) + c(z)u(z, t), \quad t \in \mathcal{E} \end{aligned} \tag{40}$$

with the following initial and boundary conditions

$$\begin{aligned} u(0,t) &= u(l,t) = 0, \\ u(z,0+0) &= \varphi(z), \quad \varphi \in X, \quad \varphi(0) = \varphi(l) = 0, \end{aligned} \tag{41}$$

where $(z,t) \in [0,l] \times [0,\infty)$, $u : [0,l] \times [0,\infty) \to \mathbb{R}$. We assume that $a > 0$, $b : [0,l] \to \mathbb{R}$ is piecewise continuous bounded function which can be discontinuous only at $z = 2\delta$, and $c : [0,l] \to \mathbb{R}$, $c \in C^2[0,l]$.

We are interested in the properties of classical solutions to (40)–(41) defined next. Let us denote $\mathcal{T}_T = [0,T] \cap \mathcal{T}_0$, $\mathcal{E}_T = [0,T] \cap \mathcal{E}$ for $T > 0$.

Definition 6 A function $u : [0,l] \times [0,T] \to \mathbb{R}$ is called classical solution to (40)-(41), if

$$u_t(z,t) = a^2 u_{zz}(z,t) + b(z)u(z,t), \quad (z,t) \in ((0,2\delta) \cup (2\delta,l)) \times \mathcal{T}_T;$$

For $z = 2\delta$, it holds that

$$\begin{aligned} u(2\delta+0,t) &= u(2\delta-0,t), \quad u_z(2\delta+0,t) = u_z(2\delta-0,t), \quad t \in \mathbb{R}_+; \\ u(z,t^+) &= u(z,t) + c(z)u(z,t), \quad (z,t) \in [0,l] \times \mathcal{E}_T; \end{aligned}$$

and u satisfies the initial and boundary conditions (41).

Since $c \in C^2[0,l]$, it follows that the state space X is invariant under jumps, that is, $f \in X$ implies $(1+c)f \in X$. It is also clear that the jump operator is consistent with the boundary conditions. Theorem 1 from [15] implies the existence and uniqueness of solutions to (40)—(41) so that $u(\cdot,t) \in X$ for $t \geq 0$.

We consider the question of asymptotic stability of solutions to (40)–(41) with respect to the norm in $L^2[0,l]$. The next proposition verifies the estimates (5)–(6).

Proposition 1 *Solutions of (40)—(41) satisfy the estimates (5)–(6) with the following choice of functions*

$$\xi_\tau(s) = e^{\tau b_{\max}} s, \quad \xi(s) = (1 + c_{\max})s, \quad \chi_\tau(s) = 0 \quad \chi(s) = 0,$$

where $b_{\max} = \sup_{z\in[0,l]} |b(z)|$, $c_{\max} = \sup_{z\in[0,l]} |c(z)|$.

The proof can be found in Appendix.

To study stability properties, we introduce the following Lyapunov function:

$$V_0(u(\cdot,t)) = \int_0^l u^2(z,t)\,dz,$$

for which we have

$$\begin{aligned}\dot V_0(u(\cdot,t)) &= 2\int_0^l u(z,t)u_t(z,t)\,dz = 2\int_0^l u(z,t)(a^2u_{zz}(z,t)+b(z)u(z,t))\,dz\\ &= -2a^2\int_0^l u_z^2(z,t)\,dz + 2\int_0^l b(z)u^2(z,t)\,dz := V_1(u(\cdot,t)).\end{aligned} \tag{42}$$

Let us calculate the full time derivative of $V_1(u(\cdot,t))$ along solutions to (40)–(41):

$$\begin{aligned}\dot V_1(u(\cdot,t)) &= -4a^2\int_0^l u_z(z,t)u_{zt}(z,t)\,dz + 4\int_0^l b(z)u(z,t)u_t(z,t)\,dz\\ &= 4a^2\int_0^l u_{zz}(z,t)(a^2u_{zz}(z,t)+b(z)u(z,t))\,dz\\ &\quad +4\int_0^l b(z)u(z,t)(a^2u_{zz}(z,t)+b(z)u(z,t))\,dz\\ &= 4\int_0^l (a^2u_{zz}(z,t)+b(z)u(z,t))^2\,dz \geq 0.\end{aligned}$$

Hence, we can apply Theorem 1 taking $V_2 = 0$. From (40) for $t \in \mathcal{E}$, we obtain

$$\begin{aligned}&V_0((1+c(z))u(\cdot,t)) - V_0(u(\cdot,t))\\ &\quad \leq \int_0^l (2c(z)+c^2(z))u^2(z,t)\,dz = W(u(\cdot,t)).\end{aligned} \tag{43}$$

With help of Theorem 1, we arrive to the following conditions for the global asymptotic stability

$$G(u(\cdot,z)) = V_1(u(\cdot,z))T_k + W(u(\cdot,t)) < -\varepsilon_0\|u(\cdot,t)\|^2_{L^2[0,l]} \tag{44}$$

for some $\varepsilon_0 > 0$, where $T_k = \tau_{k+1} - \tau_k$ denotes the dwell-time between two impulsive actions. By means of the Friedrich's inequality, we arrive to the following condition guaranteeing the GAS property of the system (40)–(41)

$$\sup_{k\in\mathbb{Z}_+}\max_{z\in[0,l]}\Big(T_k\big(b(z)-\frac{\pi^2a^2}{l^2}\big)+c(z)+\frac{1}{2}c^2(z)\Big) < 0. \tag{45}$$

For illustration, we apply this condition to the following particular case of (40)-(41): $a = 1, l = \pi$ and

$$b(z) = \begin{cases} 1 - 2\varepsilon, z \in [0, 2\delta], \\ 1 + \varepsilon, z \in (2\delta, \pi], \end{cases}$$

$$c(z) = \begin{cases} \varepsilon, \quad z \in [0, \delta], \\ \varepsilon(-18(\frac{z}{\delta})^5 + 135(\frac{z}{\delta})^4 - 390(\frac{z}{\delta})^3 + 540(\frac{z}{\delta})^2 - 360\frac{z}{\delta} + 94), z \in (\delta, 2\delta], \\ -2\varepsilon, \quad z \in (2\delta, \pi], \end{cases}$$

where $\varepsilon > 0$, $\delta < \pi/2$, $c \in C^2$, $-2\varepsilon \leq c(z) \leq \varepsilon$. We will see that for some choice of parameters ε, δ both continuous and discrete dynamics have unstable behavior.

In this case, the dwell-time condition (45) reduced to the following two inequalities:

$$-2\varepsilon T_k + \varepsilon + \frac{1}{2}\varepsilon^2 < 0, \quad T_k\varepsilon - 2\varepsilon + 2\varepsilon^2 < 0$$

or equivalently to the following explicit condition applied on the dwell-times

$$\varepsilon \in (0, 2/3), \quad 0.5 + 0.25\varepsilon < \inf_{k\in\mathbb{Z}_+} T_k \leq \sup_{k\in\mathbb{Z}_+} T_k < 2 - 2\varepsilon. \tag{46}$$

Let us consider continuous and discrete dynamics of (40)–(41) separately.

First, we consider the stability properties of the differential equation

$$u_t(z, t) = u_{zz}(z, t) + b(z)u(z, t), \tag{47}$$

with the following boundary and initial conditions

$$u(0, t) = u(\pi, t) = 0, \quad t \in \mathbb{R}_+, \quad u(z, 0 + 0) = \varphi(z), \quad \varphi \in X. \tag{48}$$

The corresponding self-adjoint spectral problem is

$$\psi''(z) + b(z)\psi(z) = \lambda\psi(z), \quad \psi(0) = \psi(\pi) = 0. \tag{49}$$

We can show that at least for ε small enough there exists some critical δ^*, ($\delta^* \approx 0.651331$) such that for all $\delta < \delta^*$ the linear system (47)–(48) is not stable, and for $\delta > \delta^*$, this system is asymptotically stable.

This follows immediately from the next proposition proved in Appendix:

Proposition 2 *The largest eigenvalue $\lambda_{\max}(\varepsilon)$ of the spectral equation (49) can be represented asymptotically as*

$$\lambda_{\max}(\varepsilon) = \frac{2}{\pi}\Big(\frac{3}{4}\sin(4\delta) + \frac{\pi}{2} - 3\delta\Big)\varepsilon + O(\varepsilon^2), \quad \varepsilon \to 0+.$$

Indeed, the function $f(\delta) = \frac{3}{4}\sin(4\delta) + \frac{\pi}{2} - 3\delta$ is decreasing on $[0, \pi/2]$, and we have $f(0)f(\pi/2) < 0$, hence the equation $f(\delta) = 0$ has a unique solution in $[0, \pi/2]$, which we denote by δ^*. Then for $\delta < \delta^*$ for sufficiently small ε, we have $\lambda_{\max}(\varepsilon) > 0$, and for $\delta > \delta^*$, we have $\lambda_{\max}(\varepsilon) < 0$.

This mens that for sufficiently small ε, the parameter δ can be chosen so that the partial differential equation (47)–(48), which describes the continuous dynamics of the original impulsive system (40)–(41), is unstable.

We consider the difference equation in the state space X, that describes the discrete dynamics of the original impulsive system (40)—(41), and demonstrate its unstable behavior. Let $C\psi := (1 + c(z))\psi$ for $\psi \in X$, then $C \in \mathfrak{L}(X)$ and $\|C^n\| = \sup_{\|\psi\|_{L^2[0,\pi]}=1} \|(1 + c(z))^n\psi\|_{L^2[0,l]} \geq \sup_{\|\psi\|_{L^2[0,\pi]}=1, \text{supp } \psi\subset[0,\delta]} \|(1 + \varepsilon)^n\psi\|_{L^2[0,l]} = (1 + \varepsilon)^n$, which implies that $r_\sigma(C) \geq 1 + \varepsilon > 1$, showing the instability property.

Remark 4 Let us note that in case of an unstable scalar ODE of the first order subjected to destabilizing impulsive actions, the overall dynamics of the whole impulsive system is always unstable. In contrary to this, our example shows that for the considered unstable PDE with destabilizing impulses the overall dynamics is stable under certain condition on dwell times.

5.2 Nonlinear impulsive ODE

Consider the following nonlinear ODE system with impulsive actions

$$\begin{aligned} \dot{x}(t) &= \frac{f(x(t)) + d_1(t)}{\sqrt{1 + \|x(t)\|^{2m}}}, \quad t \neq k\theta, \\ x(t^+) &= x(t) + \frac{g(x(t)) + d_2(k)}{\sqrt{1 + \|x(t)\|^{2m}}}, \quad t = k\theta, \end{aligned} \tag{50}$$

where $x \in \mathbb{R}^n$, $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are homogeneous functions of the order $m > 1$ with odd m, that is, $f(\lambda x) = \lambda^m f(x), g(\lambda x) = \lambda^m g(x)$ for all $(\lambda, x) \in \mathbb{R}^{n+1}$, $f \in C^1(\mathbb{R}^n)$. Let $U_1 = U_2 = \mathbb{R}^n, \mathcal{U}_1 = L^\infty(\mathbb{R}_+, \mathbb{R}^n), \mathcal{U}_2 = L^\infty(\mathbb{Z}_+, \mathbb{R}^n) = l^\infty(\mathbb{R}^n)$.

Let there exist a positive definite matrix P such that for all $x \in \mathbb{R}^n$ such that the function f, g satisfies the inequalities

$$\begin{aligned} &\|g(x)\| \leq b_0\|x\|^m, \quad x^{\mathrm{T}} P(\theta f(x) + g(x)) \leq -c_0\|x\|^{m+1}, \\ &\quad \times (Pf(x) + \partial_x f^{\mathrm{T}}(x)Px)^{\mathrm{T}} f(x) \geq \nu_0\|x\|^{2m}, \\ &\quad \times \|x\|^2(Pf(x) + \partial_x f^{\mathrm{T}}(x)Px)^{\mathrm{T}} f(x) - mx^{\mathrm{T}} Pf(x)x^{\mathrm{T}} f(x) \geq \nu_1\|x\|^{2m+2} \end{aligned} \tag{51}$$

for some positive constants b_0, c_0, ν_0 and ν_1. For short, we denote

$$a_0 := \sup_{\|x\|=1} \|f(x)\|, \quad a_1 := \sup_{\|x\|=1} \|\partial_x f(x)\|.$$

Proposition 3 *Solutions of the system (50) satisfy the estimates (5)—(6) with functions*

$$\xi_\tau(s) = \omega_1(\tau)s, \quad \chi_\tau(s) = \omega_2(\tau)s^{(m+1)/2m}, \quad \xi(s) = (1+b_0)s, \quad \chi(s) = s,$$

where ω_i, $i = 1, 2$ are certain positive functions (provided explicitly in the proof).

The proof can be found in Appendix.

Stability properties will be studied with help of the following Lyapunov function $V_0(x) = x^{\mathrm{T}} Px$, $P \succ 0$. Its time derivative along solutions to (50) for $t \neq k\theta$ is

$$\dot{V}_0(x, d_1) = \frac{2x^{\mathrm{T}} Pf(x) + 2x^{\mathrm{T}} Pd_1}{\sqrt{1+\|x\|^{2m}}}. \tag{52}$$

By means of the Young's inequality (2), we have

$$2|x^{\mathrm{T}} Pd_1| \leq 2\|P\|\|x\|\|d_1\| \leq 2\|P\| \left(\frac{\tau^{m+1}}{m+1}\|x\|^{m+1} + \frac{m\tau^{-(m+1)/m}}{m+1}\|d_1\|^{1+1/m} \right)$$

for any $\tau > 0$. Hence from (52), we obtain

$$\dot{V}_0(x, d_1) \leq \frac{2x^{\mathrm{T}} Pf(x) + 2\|P\|\frac{\tau^{m+1}}{m+1}\|x\|^{m+1}}{\sqrt{1+\|x\|^{2m}}} + 2\|P\|\frac{m\tau^{-(m+1)/m}}{m+1}\|d_1\|^{1+1/m}. \tag{53}$$

We chose

$$V_1(x) := \frac{2x^{\mathrm{T}} Pf(x) + 2\|P\|\frac{\tau^{m+1}}{m+1}\|x\|^{m+1}}{\sqrt{1+\|x\|^{2m}}}, \quad \eta_0(s) := 2\|P\|\frac{m\tau^{-(m+1)/m}}{m+1}s^{1+1/m}. \tag{54}$$

On the jumps due to impulsive actions from the definition of V_0, we calculate

$$\begin{aligned} V_0\Big(x + \frac{g(x)+d_2}{\sqrt{1+\|x\|^{2m}}}\Big) - V_0(x) &= \frac{2x^{\mathrm{T}} Pg(x)}{\sqrt{1+\|x\|^{2m}}} + \frac{g^{\mathrm{T}}(x)Pg(x)}{1+\|x\|^{2m}} \\ &+ \frac{2x^{\mathrm{T}} Pd_2}{\sqrt{1+\|x\|^{2m}}} + \frac{2g^{\mathrm{T}}(x)Pd_2 + d_2^{\mathrm{T}} Pd_2}{1+\|x\|^{2m}}. \end{aligned} \tag{55}$$

Again by the Young's inequality (2), we obtain

$$2|x^{\mathrm{T}} Pd_2| \leq 2\|P\| \left(\frac{\tau^{m+1}}{m+1}\|x\|^{m+1} + \frac{m\tau^{-(m+1)/m}}{m+1}\|d_2\|^{1+1/m} \right),$$

$$2|g^{\mathrm{T}}(x)Pd_2| \leq 2\|P\|b_0\|x\|^m\|d_2\| \leq \|P\|b_0\Big(\tau\|x\|^{2m} + \tau^{-1}\|d_2\|^2\Big).$$

From (51), it follows that $|g^{\mathrm{T}}(x)Pg(x)| \le \|P\|b_0^2\|x\|^{2m}$. Hence from (55), the following estimate follows:

$$
\begin{aligned}
&V_0\Big(x + \frac{g(x)+d_2}{\sqrt{1+\|x\|^{2m}}}\Big) - V_0(x) \le \frac{2x^{\mathrm{T}}Pg(x) + 2\|P\|\frac{\tau^{m+1}}{m+1}\|x\|^{m+1}}{\sqrt{1+\|x\|^{2m}}} \\
&\quad + (b_0+\tau)\|P\|b_0\frac{\|x\|^{2m}}{1+\|x\|^{2m}} + \frac{2\|P\|m\tau^{-(m+1)/m}\|d_2\|^{1+1/m}}{(m+1)\sqrt{1+\|x\|^{2m}}} \\
&\quad + \frac{b_0\tau^{-1}+1}{1+\|x\|^{2m}}\|P\|\|d_2\|^2 \\
&\quad \le W(x) + \eta(\|d_2\|),
\end{aligned}
$$

where

$$
\begin{aligned}
W(x) &= \frac{2x^{\mathrm{T}}Pg(x) + 2\|P\|\frac{\tau^{m+1}}{m+1}\|x\|^{m+1}}{\sqrt{1+\|x\|^{2m}}} + (b_0+\tau)\|P\|b_0\frac{\|x\|^{2m}}{1+\|x\|^{2m}}, \\
\eta(s) &= 2\|P\|\frac{m}{m+1}\tau^{-(m+1)/m}s^{1+1/m} + (b_0\tau^{-1}+1)\|P\|s^2.
\end{aligned} \tag{56}
$$

Now, consider $\dot{V}_1(x, d_1)$ for $t \ne k\theta$

$$
\begin{aligned}
\dot{V}_1(x,d_1) &= \frac{(2f^{\mathrm{T}}(x)P + 2x^{\mathrm{T}}P\partial_x f(x) + 2\tau^{m+1}\|P\|\|x\|^{m-1}x^{\mathrm{T}})(f(x)+d_1)}{1+\|x\|^{2m}} \\
&\quad - 2m\left(x^{\mathrm{T}}Pf(x) + \|P\|\frac{\tau^{m+1}}{m+1}\|x\|^{m+1}\right)\frac{\|x\|^{2m-2}x^{\mathrm{T}}(f(x)+d_1)}{(1+\|x\|^{2m})^2} \\
&= \frac{2(Pf(x) + \partial_x f^{\mathrm{T}}(x)Px)^{\mathrm{T}}f(x)}{1+\|x\|^{2m}} - \frac{2mx^{\mathrm{T}}Pf(x)\|x\|^{2m-2}x^{\mathrm{T}}f(x)}{(1+\|x\|^{2m})^2} \\
&\quad + \frac{2\tau^{m+1}\|P\|\|x\|^{m-1}x^{\mathrm{T}}f(x)}{1+\|x\|^{2m}} - 2m\|P\|\frac{\tau^{m+1}}{m+1}\|x\|^{3m-1}\frac{x^{\mathrm{T}}f(x)}{(1+\|x\|^{2m})^2} \\
&\quad + \frac{(2f^{\mathrm{T}}(x)P + 2x^{\mathrm{T}}P\partial_x f(x) + 2\tau^{m+1}\|P\|\|x\|^{m-1}x^{\mathrm{T}})d_1}{1+\|x\|^{2m}} \\
&\quad - 2m\left(x^{\mathrm{T}}Pf(x) + \|P\|\frac{\tau^{m+1}}{m+1}\|x\|^{m+1}\right)\frac{\|x\|^{2m-2}x^{\mathrm{T}}d_1}{(1+\|x\|^{2m})^2}.
\end{aligned}
$$

By means of the Cauchy inequality and from (51), it follows that

$$
\begin{aligned}
&|x^{\mathrm{T}}f(x)| \le a_0\|x\|^{m+1},\ \|f^{\mathrm{T}}(x)P + x^{\mathrm{T}}P\partial_x f(x) \\
&\qquad + \tau^{m+1}\|P\|\|x\|^{m-1}x^{\mathrm{T}}\| \le \|P\|(a_0+a_1+\tau^{m+1})\|x\|^m.
\end{aligned}
$$

Hence,

$$\begin{aligned}\dot{V}_1(x,d_1) \geq \; & \frac{2\nu_0\|x\|^{2m} + 2\nu_1\|x\|^{4m}}{(1+\|x\|^{2m})^2} \\ & - \frac{2\|P\|a_0\tau^{m+1}\|x\|^{2m}}{1+\|x\|^{2m}} - \frac{2ma_0\tau^{m+1}\|P\|\|x\|^{4m}}{(m+1)(1+\|x\|^{2m})^2} \\ & - \frac{2\|P\|(a_0+a_1+\tau^{m+1})\|x\|^m\|d_1\|}{1+\|x\|^{2m}} - \frac{2m\|P\|(a_0+\frac{\tau^{m+1}}{m+1})\|x\|^{3m}\|d_1\|}{(1+\|x\|^{2m})^2}.\end{aligned}$$

Now, we apply the Young's inequality with the parameter $\tau > 0$

$$\begin{aligned}\|x\|^m\|d_1\| &\leq \frac{\tau}{2}\|x\|^{2m} + \frac{\tau^{-1}}{2}\|d_1\|^2, \\ \|x\|^{3m}\|d_1\| &\leq \frac{3\tau^{4/3}}{4}\|x\|^{4m} + \frac{\tau^{-4}}{4}\|d_1\|^4\end{aligned}$$

and obtain

$$\begin{aligned}\dot{V}_1(x,d_1) \geq \; & \frac{(2\nu_0 - \tau\|P\|(a_0+a_1+\tau^{m+1}) - 2a_0\|P\|\tau^{m+1})\|x\|^{2m}}{(1+\|x\|^{2m})^2} \\ & + \frac{(2\nu_1 - \frac{3}{2}m\tau^{4/3}\|P\|(a_0+\frac{\tau^{m+1}}{m+1}) - \frac{4m+2}{m+1}a_0\tau^{m+1}\|P\| - \tau\|P\|(a_0+a_1+\tau^{m+1}))\|x\|^{4m}}{(1+\|x\|^{2m})^2} \\ & - \tau^{-1}\|P\|(a_0+a_1+\tau^{m+1})\|d_1\|^2 - \frac{m\tau^{-4}\|P\|(a_0+\frac{\tau^{m+1}}{m+1})}{2}\|d_1\|^4.\end{aligned}$$

Denoting

$$\begin{aligned}\vartheta_1(\tau) &= 2\nu_0 - \tau\|P\|(a_0+a_1+\tau^{m+1}) - 2a_0\|P\|\tau^{m+1}, \\ \vartheta_2(\tau) &= 2\nu_1 - 1.5m\tau^{4/3}\|P\|\left(a_0+\frac{\tau^{m+1}}{m+1}\right) \\ &\quad - \frac{2(2m+1)}{m+1}a_0\tau^{m+1}\|P\| - \tau\|P\|(a_0+a_1+\tau^{m+1}), \\ \eta_2(s) &= \tau^{-1}\|P\|(a_0+a_1+\tau^{m+1})s^2 + \frac{m\tau^{-4}\|P\|(a_0+\frac{\tau^{m+1}}{m+1})}{2}s^4,\end{aligned}$$

we can write

$$\dot{V}_1(x,d_1) \geq \frac{\vartheta_1(\tau)\|x\|^{2m} + \vartheta_2(\tau)\|x\|^{4m}}{(1+\|x\|^{2m})^2} - \eta_2(\|d_1\|) := -V_2(x) - \eta_2(\|d_1\|).$$

For sufficiently small $\tau > 0$, we have $\vartheta_1(\tau) > 0$, $\vartheta_2(\tau) > 0$. Since $V_2(x) \leq 0$, we have

$$-\dot{V}_1(x,d_1) \leq \eta_2(\|d_1\|) \tag{57}$$

and we can set $V_3(x) = 0$.

In order to apply Theorem 1 (see Remark 2), we need to estimate the function

$$G(x) = \theta V_1(x) + W(x).$$

From the estimates (54) and (56), we obtain

$$\begin{aligned} G(x) &\le \left(-2c_0 + 2\|P\|(\theta+1)\frac{\tau^{m+1}}{m+1}\right)\frac{\|x\|^{m+1}}{\sqrt{1+\|x\|^{2m}}} \\ &\quad + \|P\|(b_0^2 + b_0\tau)\frac{\|x\|^{2m}}{1+\|x\|^{2m}} \\ &= -\frac{\|x\|^{m+1}}{\sqrt{1+\|x\|^{2m}}}\left(\sigma_3(\tau) - \sigma_4(\tau)\frac{\|x\|^{m-1}}{\sqrt{1+\|x\|^{2m}}}\right), \end{aligned} \tag{58}$$

where $\sigma_3(\tau) = 2c_0 - 2\|P\|(\theta+1)\frac{\tau^{m+1}}{m+1}$, $\sigma_4(\tau) = \|P\|(b_0^2 + b_0\tau)$.

Proposition 4 *If*

$$\sigma_4(0)\frac{(m-1)^{(m-1)/2m}}{\sqrt{m}} < \sigma_3(0),$$

then for $\tau > 0$ small enough and for all $s \in \mathbb{R}_+$, the next inequality is true

$$\sigma_3(\tau) - \sigma_4(\tau)\frac{s^{m-1}}{\sqrt{1+s^{2m}}} > 0.$$

The proof can be found in Appendix.

Proposition 5 *Let system (50) satisfy the conditions of (51) as well as the inequality*

$$\|P\|b_0^2\frac{(m-1)^{(m-1)/2m}}{\sqrt{m}} < 2c_0.$$

Then, system (50) is ISS.

Proof Number $\tau > 0$ can be chosen small enough, so that from the inequality

$$\|P\|b_0^2\frac{(m-1)^{(m-1)/2m}}{\sqrt{m}} < 2c_0$$

the following estimation $G(x) \le -\delta(\|x\|)$ c $\delta(s) = \varepsilon_1 s^{m+1}/\sqrt{1+s^{2m}}$, follows where $\varepsilon_1 > 0$ is small enough. Hence, system (50) satisfies all conditions of Theorem 1 (see Remark 2) from which desired assertion follows. □

To illustrate this result, we take

$$f(x) = \begin{pmatrix} -2.5x_1^3 + 0.1x_1^2 x_2 \\ 0.1x_2^2 x_1 + 0.3x_2^3 \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0.5x_1^3 - 0.1x_1^2 x_2 \\ -0.1x_2^2 x_1 - x_2^3 \end{pmatrix},$$

$P = I$, $\theta = 1$, then $b_0^2 = 1.003329653$, $c_0 = 0.5185185185$, $\nu_0 = 0.2511999583$, $\nu_1 = 0.08780118723$.

It is easy to see that conditions of the last proposition are satisfied. Hence, the nonlinear impulsive system (50) is ISS. Note that in this case, both continuous and discrete dynamics are unstable.

6 Discussion and conclusions

The main results of this paper are given by Theorems 1 and 2. They allow us to study the ISS property of nonlinear impulsive systems with different assumptions imposed on the discrete and continuous dynamics. These theorems can establish the ISS property even for the case when neither discrete nor continuous dynamics are ISS. Our approach enables usage of a wider class of ISS-Lyapunov functions to study the ISS property of nonlinear impulsive systems. One advantage of our approach, demonstrated in the examples, is that a rather simple (energetic) Lyapunov function equipped with a sequence of auxiliary functions allows to derive desired stability condition to assure the ISS or GAS property.

An interesting direction for future research would be to develop an approach of stability investigation of nonlinear impulsive systems by means of a combination of a Lyapunov and Chetaev functions, which was used for studying local stability of finite-dimensional systems without inputs [11]. This can provide stability conditions alternative to the ones developed in the current paper. First steps in this direction can be found in [7].

Another interesting direction of research is to explore an extension possibility of our results to the strong ISS notion in the context of time varying impulsive systems (see [20]).

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7 Appendix

7.1 Proof of Lemma 1

The proof is by induction as follows. First, we check that the statements is true for $n = 2$ from the obvious equality

$$V_0(\tau_k + 0, x(\tau_k + 0)) = V_0(\tau_{k+1}, x(\tau_{k+1})) - \int_{\tau_k}^{\tau_{k+1}} \dot{V}_0(s, x(s), d_1(s))\, ds.$$

and the first inequality of (10) for $s \in (\tau_k, \tau_{k+1}]$:

$$\dot{V}_0(s, x(s), d_1(s)) \leq V_1(s, x(s)) + \eta_0(\|d_1(s)\|_{U_1})$$

it follows that

$$\begin{aligned} V_0(\tau_k + 0, x(\tau_k + 0)) &= V_0(\tau_{k+1}, x(\tau_{k+1})) - \int_{\tau_k}^{\tau_{k+1}} \dot{V}_0(s, x(s), d_1(s))\, ds \\ &\geq V_0(\tau_{k+1}, x(\tau_{k+1})) - \int_{\tau_k}^{\tau_{k+1}} V_1(s, x(s))\, ds \\ &\quad - \int_{\tau_k}^{\tau_{k+1}} \eta_0(\|d_1(s)\|_{U_1})\, ds. \end{aligned}$$

From the next two obvious relations

$$V_1(s, x(s)) = V_1(\tau_{k+1}, x(\tau_{k+1})) - \int_{s}^{\tau_{k+1}} \dot{V}_1(s_1, x(s_1), d_1(s_1))\, ds_1,$$

$$\int_{\tau_k}^{\tau_{k+1}} \eta_0(\|d_1(s)\|_{U_1})\, ds \leq \eta_0(\|d_1\|_{\mathcal{U}_1})(\tau_{k+1} - \tau_k)$$

we obtain the desired inequality

$$\begin{aligned} V_0(\tau_k+0, x(\tau_k+0)) \geq\ & V_0(\tau_{k+1}, x(\tau_{k+1})) - V_1(\tau_{k+1}, x(\tau_{k+1}))(\tau_{k+1}-\tau_k) \\ & - \int_{\tau_k}^{\tau_{k+1}}\int_{s_0}^{\tau_{k+1}} V_2(s_1, x(s_1))ds_0ds_1 - \int_{\tau_k}^{\tau_{k+1}}\int_{s_0}^{\tau_{k+1}} \eta_1(\|d_1(s_1)\|_{U_1})ds_0\,ds_1 \\ & - \eta_0(\|d_1\|_{\mathcal{U}_1})(\tau_{k+1}-\tau_k) \geq V_0(\tau_{k+1}, x(\tau_{k+1})) \\ & - V_1(\tau_{k+1}, x(\tau_{k+1}))(\tau_{k+1}-\tau_k) \\ & - \int_{\tau_k}^{\tau_{k+1}}\int_{s_0}^{\tau_{k+1}} V_2(s_1, x(s_1))ds_0ds_1 - \frac{\eta_1(\|d_1\|_{\mathcal{U}_1})\theta^2}{2!} - \eta_0(\|d_1\|_{\mathcal{U}_1})\theta. \end{aligned}$$

Now, let the statement be true for $n = q$, that is

$$\begin{aligned} V_0(\tau_k+0, x(\tau_k+0)) \geq\ & V_0(\tau_{k+1}, x(\tau_{k+1})) - \sum_{p=1}^{q-1} V_p(\tau_{k+1}, x(\tau_{k+1}))\frac{(\tau_{k+1}-\tau_k)^p}{p!} \\ & - \int_{\tau_k}^{\tau_{k+1}}\int_{s_1}^{\tau_{k+1}}\cdots\int_{s_{q-2}}^{\tau_{k+1}} V_q(s_{q-1}, x(s_{q-1}))\,ds_0\ldots ds_{q-1} \\ & - \sum_{p=1}^{q}\frac{\eta_{p-1}(\|d_1\|_{\mathcal{U}_1})\theta^p}{p!}. \end{aligned} \tag{59}$$

Similarly to the previous step, we write

$$\begin{aligned} V_q(s_{q-1}, x(s_{q-1})) =\ & V_q(\tau_{k+1}, x(\tau_{k+1})) \\ & - \int_{s_{q-1}}^{\tau_{k+1}} \dot{V}_q(s_q, x(s_q), d_1(s_q))ds_q,\ s_{q-1} \in (\tau_k, \tau_{k+1}], \end{aligned}$$

and from the inequalities (10), we obtain

$$\begin{aligned} V_q(s_{q-1}, x(s_{q-1})) \leq\ & V_q(\tau_{k+1}, x(\tau_{k+1})) + \int_{s_{q-1}}^{\tau_{k+1}} V_{q+1}(s_q, x(s_q))\,ds_q \\ \leq\ & V_q(\tau_{k+1}, x(\tau_{k+1})) + \int_{s_{q-1}}^{\tau_{k+1}} V_{q+1}(s_q, x(s_q))\,ds_q \\ & + \eta_q(\|d_1\|_{\mathcal{U}_1})(\tau_{k+1}-s_{q-1}). \end{aligned}$$

We substitute this estimate into (59) and obtain the desired inequality

$$
\begin{aligned}
V_0(\tau_k+0, x(\tau_k+0)) &\geq V_0(\tau_{k+1}, x(\tau_{k+1})) - \sum_{p=1}^{q-1} V_p(\tau_{k+1}, x(\tau_{k+1}))\frac{(\tau_{k+1}-\tau_k)^p}{p!} \\
&\quad - \int_{\tau_k}^{\tau_{k+1}} \int_{s_1}^{\tau_{k+1}} \dots \int_{s_{q-2}}^{\tau_{k+1}} V_q(\tau_{k+1}, x(\tau_{k+1}))\, ds_0 \dots ds_{q-1} \\
&\quad - \int_{\tau_k}^{\tau_{k+1}} \int_{s_1}^{\tau_{k+1}} \dots \int_{s_{q-1}}^{\tau_{k+1}} V_{q+1}(s_q, x(s_q))\, ds_0 \dots ds_{q-1} ds_q \\
&\quad - \int_{\tau_k}^{\tau_{k+1}} \int_{s_1}^{\tau_{k+1}} \dots \int_{s_{q-2}}^{\tau_{k+1}} \eta_q(\|d_1\|_{\mathcal{U}_1})(\tau_{k+1}-s_{q-1})\, ds_0 \dots ds_{q-1} \\
&\quad - \sum_{p=1}^{q} \frac{\eta_{p-1}(\|d_1\|_{\mathcal{U}_1})\theta^p}{p!} \\
&\geq V_0(\tau_{k+1}, x(\tau_{k+1})) - \sum_{p=1}^{q} V_p(\tau_{k+1}, x(\tau_{k+1}))\frac{(\tau_{k+1}-\tau_k)^p}{p!} \\
&\quad - \int_{\tau_k}^{\tau_{k+1}} \int_{s_1}^{\tau_{k+1}} \dots \int_{s_{q-1}}^{\tau_{k+1}} V_{q+1}(s_q, x(s_q))\, ds_0 \dots ds_q \\
&\quad - \sum_{p=1}^{q+1} \frac{\eta_{p-1}(\|d_1\|_{\mathcal{U}_1})\theta^p}{p!},
\end{aligned}
$$

which proves the Lemma.

7.2 Proof of Lemma 2

The proof is again by the mathematical induction. Let $n = 2$, then from (29), it follows that

$$
\begin{aligned}
V_0(\tau_{k+1}, x(\tau_{k+1})) &= V_0(\tau_k + 0, x(\tau_k + 0)) + \int_{\tau_k}^{\tau_{k+1}} \dot{V}_0(s, x(s), d_1(s))\, ds \\
&\leq V_0(\tau_k + 0, x(\tau_k + 0)) + \int_{\tau_k}^{\tau_{k+1}} V_1(s, x(s))\, ds \\
&\quad + \int_{\tau_k}^{\tau_{k+1}} \eta_0(\|d_1(s)\|_{U_1})\, ds.
\end{aligned}
$$

Applying (29) for $s \in (\tau_k, \tau_{k+1}]$, we obtain

$$
\begin{aligned}
V_1(s, x(s)) &= V_1(\tau_k + 0, x(\tau_k + 0)) + \int_{\tau_k}^{s} \dot{V}_1(s, x(s), d_1(s))\, ds \\
&\leq V_1(\tau_k + 0, x(\tau_k + 0)) + \int_{\tau_k}^{s} V_2(s_1, x(s_1))\, ds_1 + \int_{\tau_k}^{s} \eta_1(\|d_1(s_1)\|_{U_1})\, ds_1
\end{aligned}
$$

which implies the desired inequality:

$$
\begin{aligned}
V_0(\tau_{k+1}, x(\tau_{k+1})) &\leq V_0(\tau_k + 0, x(\tau_k + 0)) + \int_{\tau_k}^{\tau_{k+1}} V_1(\tau_k + 0, x(\tau_k + 0))\, ds \\
&\quad + \int_{\tau_k}^{\tau_{k+1}} \int_{\tau_k}^{s_0} V_2(s_1, x(s_1))\, ds_0\, ds_1 \\
&\quad + \int_{\tau_k}^{\tau_{k+1}} \int_{\tau_k}^{s} \eta_1(\|d_1(s_1)\|)\, ds_0\, ds_1 + \int_{\tau_k}^{\tau_{k+1}} \eta_0(\|d_1(s)\|_{U_1})\, ds \\
&\leq V_0(\tau_k + 0, x(\tau_k + 0)) + V_1(\tau_k + 0, x(\tau_k + 0))(\tau_{k+1} - \tau_k) \\
&\quad + \int_{\tau_k}^{\tau_{k+1}} \int_{\tau_k}^{s_0} V_2(s_1, x(s_1))\, ds_0\, ds_1 + \frac{\eta_1(\|d_1\|_{\mathcal{U}_1})\theta^2}{2!} + \frac{\eta_0(\|d_1\|_{\mathcal{U}_1})\theta}{1!}.
\end{aligned}
$$

Now, we assume that the Lemma 2 is true for $n = q$; then from (29), it follows for $s_{q-1} \in (\tau_k, \tau_{k+1}]$ that

$$\begin{aligned} V_q(s_{q-1}, x(s_{q-1})) &= V_q(\tau_k + 0, x(\tau_k + 0)) + \int_{\tau_k}^{s_{q-1}} \dot{V}_q(s_q, x(s_q), d_1(s_q))\, ds_q \\ &\le V_q(\tau_k + 0, x(\tau_k + 0)) + \int_{\tau_k}^{s_{q-1}} V_{q+1}(s_q, x(s_q))\, ds_q \\ &\quad + \int_{\tau_k}^{s_{q-1}} \eta_q(\|d_1(s_q)\|_{U_1})\, ds_q. \end{aligned}$$

using the induction assumption we obtain the desired inequality for $n = q + 1$:

$$\begin{aligned} V_0(\tau_{k+1}, x(\tau_{k+1})) &\le V_0(\tau_k + 0, x(\tau_k + 0)) + \sum_{p=1}^{q-1} V_p(\tau_k + 0, x(\tau_k + 0)) \frac{(\tau_{k+1} - \tau_k)^p}{p!} \\ &\quad + \int_{\tau_k}^{\tau_{k+1}} \int_{\tau_k}^{s_1} \dots \int_{\tau_k}^{s_{q-2}} V_q(s_{q-1}, x(s_{q-1}))\, ds_0 \dots ds_{q-1} + \sum_{p=1}^{q} \frac{\eta_{p-1}(\|d_1\|_{\mathcal{U}_1})\theta^p}{p!} \\ &\le V_0(\tau_k + 0, x(\tau_k + 0)) + \sum_{p=1}^{q-1} V_p(\tau_k + 0, x(\tau_k + 0)) \frac{(\tau_{k+1} - \tau_k)^p}{p!} \\ &\quad + \int_{\tau_k}^{\tau_{k+1}} \int_{\tau_k}^{s_1} \dots \int_{\tau_k}^{s_{q-2}} V_q(\tau_k + 0, x(\tau_k + 0))\, ds_0 \dots ds_{q-1} \\ &\quad + \int_{\tau_k}^{\tau_{k+1}} \int_{\tau_k}^{s_1} \dots \int_{\tau_k}^{s_{q-1}} V_{q+1}(s_q, x(s_q)) ds_0 \dots ds_q \\ &\quad + \int_{\tau_k}^{\tau_{k+1}} \int_{\tau_k}^{s_1} \dots \int_{\tau_k}^{s_{q-1}} \eta_q(\|d_1(s_q)\|_{U_1}) ds_0 \dots ds_q \\ &\quad + \sum_{p=1}^{q} \frac{\eta_{p-1}(\|d_1\|_{\mathcal{U}_1})\theta^p}{p!} \\ &\le V_0(\tau_k + 0, x(\tau_k + 0)) + \sum_{p=1}^{q} V_p(\tau_k + 0, x(\tau_k + 0)) \frac{(\tau_{k+1} - \tau_k)^p}{p!} \\ &\quad + \int_{\tau_k}^{\tau_{k+1}} \int_{\tau_k}^{s_1} \dots \int_{\tau_k}^{s_{q-1}} V_{q+1}(s_q, x(s_q)) ds_0 \dots ds_q + \sum_{p=1}^{q+1} \frac{\eta_{p-1}(\|d_1\|_{\mathcal{U}_1})\theta^p}{p!}. \end{aligned}$$

The Lemma is proved.

7.3 Proof of Proposition 1

We take $V(u(\cdot,t)) = \int\limits_0^l u^2(z,t)\,dz$ as a Lyapunov function, and calculate its derivative along the flow ϕ_c for $t \in [0,\tau]$:

$$\dot{V}(u(\cdot,t)) = 2\int\limits_0^l (a^2 u_{zz}(z,t) + b(z)u(z,t))u(z,t)\,dz$$
$$= -2\int\limits_0^l a^2 u_z^2(z,t)\,dz + 2\int\limits_0^l b(z)u^2(z,t) \le 2b_{\max} V(u(\cdot,t)),$$

which implies the first of the needed inequalities: $\|u(\cdot,t)\|_{L^2[0,l]} \le e^{\tau b_{\max}} \|u(\cdot,0)\|_{L^2[0,l]}$. The second inequality becomes trivial.

7.4 Proof of Proposition 2

We consider the eigenvalue problem (49) where b depends on ε treated as a small parameter. It is easy to see that $\lambda < \max_{z\in[0,\pi]} b(z) = 1 + \varepsilon$. For $\varepsilon = 0$ we have $\lambda_{\max}(0) = 0$, hence for the largest eigenvalue, we can write $\lambda_{\max}(\varepsilon) = \lambda_1\varepsilon + O(\varepsilon^2)$. The solution of (49) for $\varepsilon > 0$ small enough is given by

$$\psi(z) = \begin{cases} C_l \sin(\omega_l z), & z \in [0, 2\delta), \\ C_r \sin(\omega_r(\pi - z)), & z \in (2\delta, \pi] \end{cases},$$

where C_l and C_r are some real constants, $\omega_l := \sqrt{1 - 2\varepsilon - \lambda}$, $\omega_r := \sqrt{1 + \varepsilon - \lambda}$, and must satisfy the continuity conditions

$$\psi(2\delta - 0) = \psi(2\delta + 0), \quad \psi'(2\delta - 0) = \psi'(2\delta + 0).$$

These conditions are satisfied if $\omega_l \mathrm{ctg}(2\omega_l\delta) = -\omega_r \mathrm{ctg}(\omega_r(\pi - 2\delta))$.

We use the following two basic facts from analysis

$$(1 + x)^{1/2} = 1 + 1/2x + O(x^2), \quad x \to 0,$$
$$\mathrm{ctg}(x) = \mathrm{ctg}(x_0) - (x - x_0)/\sin^2(x_0) + O((x - x_0)^2), \quad x \to x_0,$$

that enable us to write for $\varepsilon \to 0+$ the next equalities.

$$\omega_r \mathrm{ctg}(\omega_r(\pi - 2\delta)) = \big(1 + \frac{1}{2}(1-\lambda_1)\varepsilon\big)\big(\mathrm{ctg}(\pi - 2\delta) - \frac{(\pi - 2\delta)(1-\lambda_1)\varepsilon}{2\sin^2(2\delta)}\big) + O(\varepsilon^2)$$
$$= -\mathrm{ctg}(2\delta) + \frac{1}{2}(1-\lambda_1)\Big(- \mathrm{ctg}(2\delta) - \frac{\pi - 2\delta}{\sin^2(2\delta)}\Big)\varepsilon + O(\varepsilon^2).$$
$$\omega_l \mathrm{ctg}(2\omega_l\delta) = \big(1 - \frac{1}{2}(2+\lambda_1)\varepsilon\big)\big(\mathrm{ctg}(2\delta) + \frac{2\delta(2+\lambda_1)}{2\sin^2(2\delta)}\varepsilon\big) + O(\varepsilon^2)$$
$$= \mathrm{ctg}(2\delta) + \frac{1}{2}(2+\lambda_1)\Big(- \mathrm{ctg}(2\delta) + \frac{2\delta}{\sin^2(2\delta)}\Big)\varepsilon + O(\varepsilon^2).$$

From which we finally conclude that

$$(2+\lambda_1)\Big(- \mathrm{ctg}(2\delta) + \frac{2\delta}{\sin^2(2\delta)}\Big) = (1-\lambda_1)\Big(\mathrm{ctg}(2\delta) + \frac{\pi - 2\delta}{\sin^2(2\delta)}\Big)$$
$$\frac{\pi}{\sin^2(2\delta)}\lambda_1 = \mathrm{ctg}(2\delta) + \frac{\pi - 2\delta}{\sin^2(2\delta)} + 2\mathrm{ctg}(2\delta) - \frac{4\delta}{\sin^2(2\delta)}$$

From which we calculate $\lambda_1 = \frac{2}{\pi}\Big(\frac{3}{4}\sin(4\delta) + \frac{\pi}{2} - 3\delta\Big)$. This proves the statement.

7.5 Proof of Proposition 3

From the condition of Proposition 1, the following estimate follows:

$$\frac{d}{dt}(x^{\mathrm{T}}(t)x(t)) = \frac{2x^{\mathrm{T}} f(x) + 2x^{\mathrm{T}} d_1(t)}{\sqrt{1 + \|x(t)\|^{2m}}} \le \frac{2a_0\|x\|^{m+1} + 2\|x\|\|d_1\|}{\sqrt{1 + \|x(t)\|^{2m}}}.$$

By the Young's inequality (2)

$$\|x\|\|d_1\| \le \frac{\|x\|^{m+1}}{m+1} + \frac{m}{m+1}\|d_1(t)\|^{(m+1)/m},$$

we further obtain

$$\frac{d}{dt}(x^{\mathrm{T}}(t)x(t)) \le 2\left(a_0 + \frac{1}{m+1}\right)\frac{\|x(t)\|^{m-1}}{\sqrt{1 + \|x(t)\|^{2m}}}\|x(t)\|^2 + \frac{2m}{m+1}\|d_1(t)\|^{(m+1)/m}.$$

Since $\sup_{s\in\mathbb{R}_+} \frac{s^{m-1}}{\sqrt{1+s^{2m}}} = \frac{(m-1)^{(m-1)/2m}}{\sqrt{m}} < \infty$, denoting $\mu_0 = \big(a_0 + \frac{1}{m+1}\big)\frac{(m-1)^{(m-1)/2m}}{\sqrt{m}}$, we arrive to the following differential inequality

$$\frac{d}{dt}\|x(t)\|^2 \le 2\mu_0\|x(t)\|^2 + \frac{2m}{m+1}\|d_1(t)\|^{(m+1)/m}.$$

Hence by the comparison principle, we obtain

$$\|x(t)\|^2 \le e^{2\mu_0\tau}\|x_0\|^2 + \frac{m}{\mu_0(m+1)}(e^{2\mu_0\tau} - 1)\|d_1\|_{\mathcal{U}_1}^{(m+1)/m}$$

for $t \in [0, \tau]$. The inequality (5) follows then immediately, if we take $\omega_1(\tau) = e^{\mu_0\tau}$, $\omega_2(\tau) = \sqrt{\frac{m}{\mu_0(m+1)}(e^{2\mu_0\tau} - 1)}$. The estimate (6) for $t = k\theta$ can be obtained easier:

$$\begin{aligned}\|x(t^+)\| &\le \big(1 + \frac{b_0\|x\|^m}{\sqrt{1 + \|x(t)\|^{2m}}}\big)\|x(t)\| + \frac{\|d_2(k)\|}{\sqrt{1 + \|x(t)\|^{2m}}} \\ &\le (1 + b_0)\|x(t)\| + \|d_2\|_{\mathcal{U}_2}.\end{aligned}$$

This finishes the proof.

7.6 Proof of Proposition 4

From the condition of the proposition, we have that $\sigma_4(\tau)\frac{(m-1)^{(m-1)/2m}}{\sqrt{m}} < \sigma_3(\tau)$, is satisfied for $\tau = 0$. By the continuity, the inequality is also true for some small enough $\tau > 0$. A simple calculation shows that $\max_{s\in\mathbb{R}_+} \frac{s^{m-1}}{\sqrt{1+s^{2m}}} = \frac{(m-1)^{(m-1)/2m}}{\sqrt{m}}$, so that from the last inequality, we conclude

$$\sigma_3(\tau) - \sigma_4(\tau)\frac{s^{m-1}}{\sqrt{1 + s^{2m}}} \ge \sigma_3(\tau) - \sigma_4(\tau)\frac{(m-1)^{(m-1)/2m}}{\sqrt{m}} := \varepsilon_1 > 0,$$

which proves the Proposition 4.

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