# Stability Conditions for Multiclass Fluid Queueing Networks

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*Abstract*—**We introduce a new method to investigate stability of work-conserving policies in multiclass queueing networks. The method decomposes feasible trajectories and uses linear programming to test stability. We show that this linear program is a necessary and sufficient condition for the stability of all work-conserving policies for multiclass fluid queueing networks with two stations. Furthermore, we find new sufficient conditions for the stability of multiclass queueing networks involving any number of stations and conjecture that these conditions are also necessary. Previous research had identified sufficient conditions through the use of a particular class of (piecewise linear convex) Lyapunov functions. Using linear programming duality, we show that for two-station systems the Lyapunov function approach is equivalent to ours and therefore characterizes stability exactly.**

## I. Introduction

The problem of establishing conditions under which a multiclass queueing network is stable under a particular policy has attracted a great deal of attention in recent years. It is known that for single class [2], [16], [19] and multiclass acyclic queueing networks [11], a necessary and sufficient condition for stability of all work-conserving policies is that the traffic intensity at each station of the network is less than one. For multiclass networks with feedback, [13], [14], and [17] have identified particular priority policies that lead to instability even if the traffic intensity at each station of the network is less than one. More surprisingly, [3] and [18] have shown that these instability phenomena are present even for the standard first-in/first-out (FIFO) policy. It is, therefore, a rather interesting problem to identify the right set of necessary and sufficient conditions for stability of multiclass queueing networks under work-conserving policies.

In recent years, researchers have identified progressively sharper sufficient conditions for stability of all work-conserving policies through the use of Lyapunov functions. Kumar and Meyn [12] used quadratic Lyapunov functions, while Botvich and Zamyatin [4], Dai and Weiss [8], and Down and Meyn [9] used piecewise linear convex Lyapunov functions. Chen and Zhang [6] have found some sufficient (but not necessary) conditions for the stability of multiclass queueing networks under FIFO. In all cases, it was established that a multiclass network is stable if certain linear programming problems are feasible. To the best of our knowledge, the sharpest such conditions are those of [8] and [9] obtained through the use of piecewise linear convex Lyapunov functions. For some specific examples (for example in [4]), the conditions obtained are indeed sharp. In general, however, the problem of establishing the exact stability region, i.e., sharp necessary and sufficient conditions for stability, is open. Furthermore, it is not known whether the Lyapunov function method with piecewise linear convex functions (or with any convex function) has the power of establishing the exact stability region.

Dai [7] has shown that a stochastic multiclass network is stable if the associated fluid limit (a deterministic network) is stable. Meyn [15] has proven a partial converse result. For this reason, the exact stability conditions obtained in this paper for the fluid model are suspected to hold for stochastic queueing networks as well.

The contributions as well as the structure of this paper are as follows.

1) We introduce, in Section III, a new method to investigate the stability of work-conserving policies in multiclass fluid networks. The method looks at the detailed structure of possible trajectories. We find the exact stability region for two-station multiclass networks. The stability condition is expressed in terms of a linear program.
2) We demonstrate, in Section IV, a duality relationship between our linear program from Section III and the linear program proposed in [9] using Lyapunov function methods. We, therefore, establish that piecewise linear, convex Lyapunov functions have the power of checking stability exactly for networks with two stations.
3) We find, in Section V, new sufficient conditions for multiclass networks with more than two stations that we believe are necessary, although we were unable to establish necessity. The conditions are again expressed in terms of a linear program with a small number of variables and constraints.

## II. Notation

We introduce a fluid model $(\alpha, \mu, P, C)$ consisting of $n$ classes $C_1, \cdots, C_n$ and $J$ service stations $1, \cdots, J$, as follows. Each class is served at a particular station. Let

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$\sigma_j$ be the set of classes served in station $j$. The external arrival rate for class $i$ is $\alpha_i$, and the service rate is $\mu_i$. Let $\alpha = (\alpha_1, \cdots, \alpha_n)'$ and $\mu = (\mu_1, \cdots, \mu_n)'$. After a service completion, a fraction $p_{ij}$ of class $i$ customers becomes class $j$ and a fraction $1 - \sum_j p_{ij}$ exits the system. Let $P$ be the substochastic matrix $P = (P_{ij})_{1\le i, j\le n}$. Finally, we define the $J \times n$ matrix $C$ as follows: $c_{jk} = 1$ if class $k$ is served at station $j$ and $c_{jk} = 0$ otherwise. We let $M = \text{diag}\{\mu_1, \cdots, \mu_n\}$ and assume that the matrix $P$ has spectral radius less than one.

Any scheduling policy can be described in terms of the variables $T_k(t)$ defined as the amount of time class $k$ is being served in the interval $[0, t]$ and $Q_k(t)$ defined as the queue length for class $k$ at time $t$. We let $T(t) = [T_1(t), \cdots, T_n(t)]'$ and $Q(t) = [Q_1(t), \cdots, Q_n(t)]'$.

Throughout the paper we call $Q(t)$ the trajectory of the fluid process under the allocation process $T(t)$. Given the initial condition $Q(0)$, the dynamics of the queue length process are as follows:

$$\begin{aligned} Q_k(t) &= Q_k(0) + \alpha_k t + \sum_{i=1}^{n} \mu_i T_i(t) p_{ik} - \mu_k T_k(t) \\ &\ge 0, \qquad k = 1, \cdots, n \end{aligned}$$

or in matrix form

$$\begin{aligned} Q(t) &= Q(0) + \alpha t + [P' - I] M T(t) \\ &\ge 0. \end{aligned}$$

We assume that the allocation process satisfies the following conditions.

1) $T(0) = 0$.
2) (Feasibility) For any $t_2 > t_1 \ge 0$ and any station $i$
$$\sum_{k\in\sigma_i} [T_k(t_2) - T_k(t_1)] \le t_2 - t_1 \tag{1}$$
and $T_k(t)$ is nondecreasing.
3) (Work-conservation) If for all $t \in [t_1, t_2]$ we have $\sum_{k\in\sigma_i} Q_k(t) > 0$ for some station $i$, then
$$\sum_{k\in\sigma_i} [T_k(t_2) - T_k(t_1)] = t_2 - t_1. \tag{2}$$
Any scheduling policy satisfying all the above properties is called a (feasible) work-conserving policy.

An alternative characterization of the above requirements is to introduce for any station $i$, the cumulative idling process

$$U_i(t) = t - \sum_{k\in\sigma_i} T_k(t).$$

Feasibility condition (1) then requires that $U_i(t)$ be nonnegative and nondecreasing, while the work-conservation condition is rewritten as follows: if for all $t \in [t_1, t_2]$ we have $\sum_{k\in\sigma_i} Q_k(t) > 0$, then

$$U_i(t_1) = U_i(t_2). \tag{3}$$

Following Chen [5], a fluid network $(\alpha, \mu, P, C)$ is said to be (globally) stable for all work-conserving policies if for every work-conserving allocation process $T(t)$ and every initial condition $Q(0)$, there exists a finite time $t_0$ such that $Q(t) = 0$ for all $t \ge t_0$. Rybko and Stolyar [17] show that this is equivalent to the weaker condition: for every work-conserving allocation process $T(t)$ and every initial condition $Q(0)$, there exists a finite time $t_0$ such that $Q(t_0) = 0$. We will use this as our working definition of stability.

A necessary condition for stability (see Chen [5]) is that the traffic intensity vector $\rho$ defined by $\rho = CM^{-1}[I - P']^{-1}\alpha$ satisfies

$$\rho < e \tag{4}$$

where $e = (1, \cdots, 1)'$. As mentioned in the introduction, for general multiclass networks with feedback, this condition is not sufficient. Our goal in the next section is to establish necessary and sufficient conditions for the stability of a multiclass fluid network with two stations, given that $\rho < e$. In preparation for this analysis, we introduce some further notation.

We refer to $Q(t) \in R_+^n$ as the state of the system at time $t \ge 0$. We partition the set $R_+^n - \{0\}$ of nonzero states into the following finite family of subspaces. For any nonempty set of service stations $S \subset \{1, 2, \cdots, J\}$, we let

$$R_S = \left\{ x \in R_+^n : \forall i \in S, \sum_{k\in\sigma_i} x_k > 0, \text{ and } \forall i \notin S, \sum_{k\in\sigma_i} x_k = 0 \right\}$$

i.e., $R_S$ corresponds to states for which all stations in $S$ are busy, while all other stations have empty buffers.

## III. Stability Conditions for Multiclass Two-Station Fluid Networks

In this section, we establish necessary and sufficient conditions for stability for the case where $J = 2$, i.e., for multiclass networks with two stations. Throughout this section, we assume that $\rho < e$, since otherwise the system is unstable.

We denote by $R_1$, $R_2$, and $R_{12}$ the subspaces corresponding to $S = \{1\}, \{2\}, \{1, 2\}$, respectively, as defined at the end of Section II. In particular, for $Q \in R_1$ station 2 has no customers, for $Q \in R_2$ station 1 has no customers, while for $Q \in R_{12}$ both stations have customers in queue. The proposition that follows states that a trajectory can be broken down into subtrajectories of four different types.

*Proposition 1:* Consider a stable work-conserving trajectory $Q(t)$ and let $\tau$ be the smallest time such that $Q(\tau) = 0$. There exists a (finite or infinite) nondecreasing sequence $t_i$ such that $\sup_i t_i = \tau$ and such that for all times less than $\tau$ the following hold:

$$Q(t_{4m+1}) \in R_1 \quad \text{and for} \quad t \in [t_{4m+1}, t_{4m+2}], \quad Q(t) \in R_1 \cup R_{12}$$
$$Q(t_{4m+2}) \in R_1 \quad \text{and for} \quad t \in (t_{4m+2}, t_{4m+3}), \quad Q(t) \in R_{12}$$

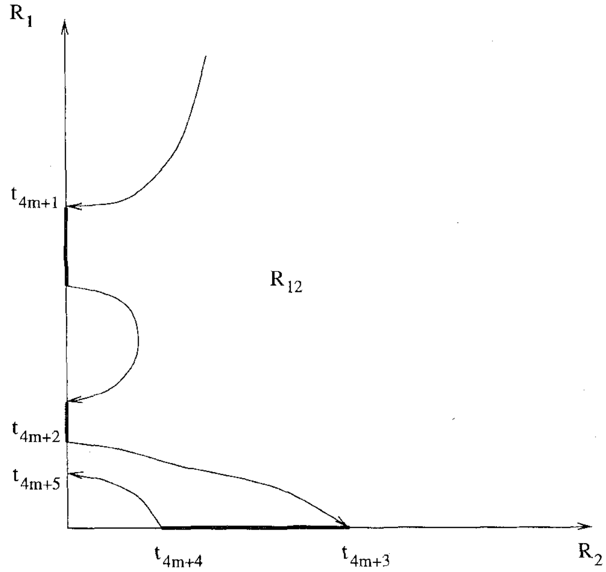


Fig. 1. The times $t_i$ for a typical trajectory.

$$Q(t_{4m+3}) \in R_2 \quad \text{and for} \quad t \in [t_{4m+3}, t_{4m+4}], \quad Q(t) \in R_2 \cup R_{12}$$
$$Q(t_{4m+4}) \in R_2 \quad \text{and for} \quad t \in (t_{4m+4}, t_{4m+5}), \quad Q(t) \in R_{12}.$$

*Proof:* This is a simple consequence of the fact that starting in $R_1$, the system can get to $R_2$ only by first going through $R_{12}$, and vice versa; see Fig. 1. In particular, once $t_{4m+1}$ has been defined, we may let $t_{4m+3} = \min\{t > t_{4m+1} | Q(t) \in R_2\}$ and $t_{4m+2} = \max\{t < t_{4m+3} | Q(t) \in R_1\}$. [In case $Q(t)$ never enters $R_2$ after time $t_{4m+1}$, then the preceding definition of $t_{4m+3}$ is inapplicable; however, in this case, the system gets to $Q(\tau) = 0$ without ever leaving $R_1 \cup R_{12}$. Thus, $[t_{4m+1}, \tau)$ can be taken as the last interval.] Having thus defined $t_{4m+3}$, the times $t_{4m+4}$ and $t_{4m+5}$ are defined similarly. □

### A. Bounds for the Strong Busy Period of Stable Work-Conserving Policies

In this subsection, we find an upper bound on the time that stable work-conserving policies take to empty the fluid network starting with an initial condition $Q(0)$. This time is usually called the strong busy period. This result is of independent interest as it contributes to our understanding of the performance of the network; it is also the key to our stability analysis in the next subsection.

*Proposition 2:* Consider a stable work-conserving policy $T(t)$ starting with initial condition $Q(0) \neq 0$. Let $\tau$ be the smallest time such that $Q(\tau) = 0$. Then, $\tau$ is bounded above by the optimal value of the following linear program to be called $LP[Q(0)]$:

maximize

$$\sum_{j=1}^{4} \tau_j$$

subject to

$$\tau_1 = \sum_{k\in\sigma_1} \tau_k^1, \quad \tau_1 \geq \sum_{k\in\sigma_2} \tau_k^1$$
$$\tau_2 = \sum_{k\in\sigma_1} \tau_k^2, \quad \tau_2 = \sum_{k\in\sigma_2} \tau_k^2$$
$$\tau_3 \geq \sum_{k\in\sigma_1} \tau_k^3, \quad \tau_3 = \sum_{k\in\sigma_3} \tau_k^3$$
$$\tau_4 = \sum_{k\in\sigma_1} \tau_k^4, \quad \tau_4 = \sum_{k\in\sigma_2} \tau_k^4$$

$\forall\, k \in \sigma_2$:

$$\alpha_k \tau_1 + \sum_{i=1}^{n} \mu_i p_{ik} \tau_i^1 - \mu_k \tau_k^1 = 0$$
$$\alpha_k \tau_2 + \sum_{i=1}^{n} \mu_i p_{ik} \tau_i^2 - \mu_k \tau_k^2 \geq 0$$
$$\alpha_k \tau_4 + \sum_{i=1}^{n} \mu_i p_{ik} \tau_i^4 - \mu_k \tau_k^4 \leq 0$$

$\forall\, k \in \sigma_1$:

$$\alpha_k \tau_3 + \sum_{i=1}^{n} \mu_i p_{ik} \tau_i^3 - \mu_k \tau_k^3 = 0$$
$$\alpha_k \tau_4 + \sum_{i=1}^{n} \mu_i p_{ik} \tau_i^4 - \mu_k \tau_k^4 \geq 0$$
$$\alpha_k \tau_2 + \sum_{i=1}^{n} \mu_i p_{ik} \tau_i^2 - \mu_k \tau_k^2 \leq 0$$

$\forall\, k \in \{1, \cdots, n\}$:

$$\alpha_k \sum_{j=1}^{4} \tau_j + \sum_{i=1}^{n} \mu_i p_{ik} \sum_{j=1}^{4} \tau_i^j - \mu_k \sum_{j=1}^{4} \tau_k^j = -Q_k(0)$$
$$\tau_j \geq 0, \; \tau_k^j \geq 0. \tag{5}$$

*Proof:* Consider a stable work conserving policy with initial condition $Q(0) \neq 0$. Without loss of generality, we only provide the proof for the case $Q(0) \in R_1$; the proof for the other cases is essentially identical. Let $t_1 = 0$ and let the times $t_j$ be as in the statement of Proposition 1. For $j = 1, \cdots, 4$ we introduce the following variables:

$$\tau_j = \sum_{m=0}^{\infty} (t_{4m+j+1} - t_{4m+j}) \tag{6}$$

and

$$\tau_k^j = \sum_{m=0}^{\infty} [T_k(t_{4m+j+1}) - T_k(t_{4m+j})]. \tag{7}$$

Intuitively, $\tau_1$ is the total amount of time the trajectory spends in $R_1$ as well as in excursions from $R_1$ into $R_{12}$ and back into $R_1$; $\tau_2$ is the total amount of time the trajectory spends in $R_{12}$ coming from $R_1$ and going to $R_2$; $\tau_3$ is the total amount of time the trajectory spends in $R_2$ as well as in excursions from $R_2$ into $R_{12}$ and back into $R_2$; finally, $\tau_4$ is the total amount of time the trajectory spends in $R_{12}$, coming from $R_2$ and going to $R_1$. Clearly $\tau_j \geq 0$ and the first time that $Q(t)$ becomes zero is given by $\tau = \tau_1 + \tau_2 + \tau_3 + \tau_4$. Note that for every class $k$, $\tau_k^1$, $\tau_k^2$, $\tau_k^2$, and $\tau_k^4$ is the total work allocated to class $k$ during the time intervals that enter in the definitions of $\tau_1$, $\tau_2$, $\tau_3$, $\tau_4$, respectively.

For all $t \in [t_{4m+1}, t_{4m+2}]$, we have $Q(t) \in R_1 \cup R_{12}$, and therefore $\sum_{k\in\sigma_1} Q_k(t) > 0$. Because the policy is work-conserving

$$t_{4m+2} - t_{4m+1} = \sum_{k\in\sigma_1} [T_k(t_{4m+2}) - T_k(t_{4m+1})]. \quad (8)$$

By summing over $m \geq 0$ we obtain that

$$\tau_1 = \sum_{k\in\sigma_1} \tau_k^1$$

which simply expresses the work conservation in station 1, while the trajectory is in $R_1 \cup R_{12}$ (station 1 busy). Similarly, work conservation for station 2, while the trajectory is in $R_2 \cup R_{12}$ (station 2 busy) leads to

$$\tau_3 = \sum_{k\in\sigma_2} \tau_k^2.$$

Moreover, for $t \in (t_{4m+2}, t_{4m+3}) \cup (t_{4m+4}, t_{4m+5})$, we have $Q(t) \in R_{12}$, and work conservation for both stations leads to

$$\begin{aligned}\tau_2 &= \sum_{k\in\sigma_1} \tau_k^2 \\ &= \sum_{k\in\sigma_2} \tau_k^2 \\ \tau_4 &= \sum_{k\in\sigma_1} \tau_k^4 \\ &= \sum_{k\in\sigma_2} \tau_k^4.\end{aligned}$$

For every station $j$, we have

$$\sum_{k\in\sigma_j} [T_k(t_{i+1}) - T_k(t_i)] \leq t_{i+1} - t_i$$

leading to

$$\begin{aligned}\tau_1 &\geq \sum_{k\in\sigma_2} \tau_k^1 \\ \tau_3 &\geq \sum_{k\in\sigma_1} \tau_k^2.\end{aligned}$$

By definition of the times $t_i$, we have $Q(t_{4m+1}) \in R_1$ and $Q(t_{4m+2}) \in R_1$. Thus, for all $k \in \sigma_2$ we have

$$\begin{aligned}Q_k(t_{4m+1}) &= Q_k(t_{4m+2}) \\ &= 0\end{aligned}$$

which leads to

$$\alpha_k(t_{4m+2} - t_{4m+1}) + \sum_{i=1}^n \mu_i p_{ik}[T_i(t_{4m+2}) - T_i(t_{4m+1})] - \mu_k[T_k(t_{4m+2}) - T_k(t_{4m+1})] = 0, \qquad k \in \sigma_2.$$

Summing over all $m \geq 0$, we obtain

$$\alpha_k \tau_1 + \sum_{i=1}^n \mu_i p_{ik} \tau_i^1 - \mu_k \tau_k^1 = 0, \quad k \in \sigma_2.$$

Similarly, for $k \in \sigma_1$, we have $Q_k(t_{4m+3}) = Q_k(t_{4m+4}) = 0$ which yields

$$\alpha_k(t_{4m+4} - t_{4m+3}) + \sum_{i=1}^n \mu_i p_{ik}[T_i(t_{4m+4}) - T_i(t_{4m+3})] - \mu_k[T_k(t_{4m+4}) - T_k(t_{4m+3})] = 0, \qquad k \in \sigma_1$$

and leads to

$$\alpha_k \tau_3 + \sum_{i=1}^n \mu_i p_{ik} \tau_i^3 - \mu_k \tau_k^3 = 0, \qquad k \in \sigma_1.$$

Since $Q(t_{4m+2}) \in R_1$ and $Q(t_{4m+3}) \in R_2$, we obtain

$$0 = Q_k(t_{4m+2}), \qquad k \in \sigma_2$$

and

$$0 \leq Q_k(t_{4m+3}), \qquad k \in \sigma_2$$

which implies that for all $k \in \sigma_2$, $Q_k(t_{4m+3}) - Q_k(t_{4m+2}) \geq 0$, leading to

$$\alpha_k(t_{4m+3} - t_{4m+2}) + \sum_{i=1}^n \mu_i p_{ik}[T_i(t_{4m+3}) - T_i(t_{4m+2})] - \mu_k[T_k(t_{4m+3}) - T_k(t_{4m+2})] \geq 0, \qquad k \in \sigma_2.$$

Summing over all $m \geq 0$, we obtain

$$\alpha_k \tau_2 + \sum_{i=1}^n \mu_i p_{ik} \tau_i^2 - \mu_k \tau_k^2 \geq 0, \qquad k \in \sigma_2.$$

Similarly, for all $k \in \sigma_1$, $Q_k(t_{4m+3}) - Q_k(t_{4m+2}) \leq 0$, leading to

$$\alpha_k(t_{4m+3} - t_{4m+2}) + \sum_{i=1}^n \mu_i p_{ik}[T_i(t_{4m+3}) - T_i(t_{4m+2})] - \mu_k[T_k(t_{4m+3}) - T_k(t_{4m+2})] \leq 0, \qquad k \in \sigma_1$$

and therefore

$$\alpha_k \tau_2 + \sum_{i=1}^n \mu_i p_{ik} \tau_i^2 - \mu_k \tau_k^2 \leq 0, \qquad k \in \sigma_1.$$

Finally, since $Q(t_{4m+4}) \in R_2$ and $Q(t_{4m+5}) \in R_1$, we obtain

$$\alpha_k(t_{4m+5} - t_{4m+4}) + \sum_{i=1}^n \mu_i p_{ik}[T_i(t_{4m+5}) - T_i(t_{4m+4})] - \mu_k[T_k(t_{4m+5}) - T_k(t_{4m+4})] \geq 0, \qquad k \in \sigma_1$$

$$\alpha_k(t_{4m+5} - t_{4m+4}) + \sum_{i=1}^n \mu_i p_{ik}[T_i(t_{4m+5}) - T_i(t_{4m+4})] - \mu_k[T_k(t_{4m+5}) - T_k(t_{4m+4})] \leq 0, \qquad k \in \sigma_2$$

leading, respectively, to

$$\alpha_k \tau_4 + \sum_{i=1}^n \mu_i p_{ik} \tau_i^4 - \mu_k \tau_k^4 \geq 0, \qquad k \in \sigma_1$$

$$\alpha_k \tau_4 + \sum_{i=1}^n \mu_i p_{ik} \tau_i^4 - \mu_k \tau_k^4 \leq 0, \qquad k \in \sigma_2.$$

Recall that $\tau = \sum_{j=1}^{4} \tau_j$. Then, from the dynamics of the network

$$Q_k(\tau) = Q_k(0) + \alpha_k \tau + \sum_{i=1}^{n} \mu_i p_{ik} \sum_{j=1}^{4} \tau_i^j - \mu_k \sum_{j=1}^{4} \tau_k^j.$$

Since $Q(\tau) = 0$, we obtain

$$\alpha_k \tau + \sum_{i=1}^{n} \mu_i p_{ik} \sum_{j=1}^{4} \tau_i^j - \mu_k \sum_{j=1}^{4} \tau_k^j = -Q_k(0), \quad k = 1, \cdots, n.$$

We have shown that all of the constraints of the linear program $LP[Q(0)]$ must be satisfied, and therefore $\tau$ must be bounded above by the value of this linear program. □

The linear program $LP[Q(0)]$ gives an upper bound on the strong busy period of all stable work-conserving policies. Similarly, if we minimize $\sum_{i=1}^{4} \tau_i$ we find a lower bound on the time it takes for the network to empty using a work-conserving policy starting from an initial condition $Q(0)$. The lower bound is particularly interesting as it gives information on the least possible emptying time.

### B. Sufficient Conditions for Stability

In this subsection, we derive sufficient conditions for stability of the fluid network. The sufficient conditions involve the linear program $LP[0]$ which is defined exactly as the linear program $LP[Q(0)]$ of the preceding subsection, except that the right-hand side variables $Q_k(0)$ in (5) are set to zero.

*Theorem 1—Sufficient Conditions for Stability:* Consider the following set of linear inequalities in $4(n+1)$ variables:

$$\tau_1 = \sum_{k \in \sigma_1} \tau_k^1, \; \tau_1 \geq \sum_{k \in \sigma_2} \tau_k^1 \tag{9}$$

$$\tau_2 = \sum_{k \in \sigma_1} \tau_k^2, \; \tau_2 = \sum_{k \in \sigma_2} \tau_k^2 \tag{10}$$

$$\tau_3 \geq \sum_{k \in \sigma_1} \tau_k^2, \; \tau_3 = \sum_{k \in \sigma_2} \tau_k^3 \tag{11}$$

$$\tau_4 = \sum_{k \in \sigma_1} \tau_k^4, \; \tau_4 = \sum_{k \in \sigma_2} \tau_k^4 \tag{12}$$

$\forall k \in \sigma_2$:

$$\alpha_k \tau_1 + \sum_{i=1}^{n} \mu_i p_{ik} \tau_i^1 - \mu_k \tau_k^1 = 0 \tag{13}$$

$$\alpha_k \tau_2 + \sum_{i=1}^{n} \mu_i p_{ik} \tau_i^2 - \mu_k \tau_k^2 \geq 0 \tag{14}$$

$$\alpha_k \tau_4 + \sum_{i=1}^{n} \mu_i p_{ik} \tau_i^4 - \mu_k \tau_k^4 \leq 0 \tag{15}$$

$\forall k \in \sigma_1$:

$$\alpha_k \tau_3 + \sum_{i=1}^{n} \mu_i p_{ik} \tau_i^3 - \mu_k \tau_k^3 = 0 \tag{16}$$

$$\alpha_k \tau_4 + \sum_{i=1}^{n} \mu_i p_{ik} \tau_i^4 - \mu_k \tau_k^4 \geq 0 \tag{17}$$

$$\alpha_k \tau_2 + \sum_{i=1}^{n} \mu_i p_{ik} \tau_i^2 - \mu_k \tau_k^2 \leq 0 \tag{18}$$

$\forall k \in \{1, \cdots, n\}$:

$$\alpha_k \sum_{j=1}^{4} \tau_j + \sum_{i=1}^{n} \mu_i p_{ik} \sum_{j=1}^{4} \tau_i^j - \mu_k \sum_{j=1}^{4} \tau_k^j = 0, \quad \tau_j \geq 0, \; \tau_k^j \geq 0 \tag{19}$$

to be referred to as $LP[0]$. If $LP[0]$ has zero as the only feasible solution, then the multiclass fluid network $(\alpha, \mu, P, C)$ is stable for all work-conserving policies.

*Proof:* Let us assume that zero is the only feasible solution of $LP[0]$. Let us also assume that there exists an initial condition $Q(0) \neq 0$ and a work-conserving policy such that $Q(t)$ never becomes zero. We will derive a contradiction.

Recall that the constraints in $LP[0]$ and in $LP[Q(0)]$ are the same except that the right-hand side in (5) is changed from $-Q_k(0)$ to zero. Using linear programming theory ([1]) and since zero is the only feasible solution of $LP[0]$, it follows that the feasible set of $LP[Q(0)]$ is bounded. Let $Z$ be the optimal value of the objective function in $LP[Q(0)]$ which is finite.

Let us now consider the unstable policy starting from $Q(0)$. Let us follow this policy up to time $Z$; from then on, let us switch to some stable work-conserving policy (under our standing assumption that $\rho < e$, it is known that such a policy exists). We then obtain a work-conserving policy that, starting from $Q(0)$, eventually leads the state to zero, say at some time $\tau$. By construction $\tau > Z$. On the other hand, Proposition 2 asserts that $\tau \leq Z$. This is a contradiction and the proof is complete. □

### C. Necessary Conditions for Stability

In this section, we show that the conditions of Theorem 1 are also necessary. In particular, we show that if the linear program $LP[0]$ has a nonzero solution $(\tau_j, \tau_k^j)$, $j = 1, \cdots, 4$, $k = 1, \cdots, n$, then there exists a work-conserving policy and an initial condition $Q(0) \neq 0$ such that for some time $\tau > 0$, $Q(\tau) = Q(0)$. By repeating the same policy each time that the state $Q(0)$ is revisited, the system never empties and therefore the fluid network is unstable. In preparation of the instability theorem we prove the following proposition.

*Proposition 3:* If $(\tau_j, \tau_k^j)$, $j = 1, \cdots, 4$, $k = 1, \cdots, n$ is a nonzero solution of $LP[0]$, then $\tau_j > 0$ for all $j = 1, \cdots, 4$.

*Proof:* Suppose $\tau_1 = 0$. Then from (9) $\tau_k^1 = 0$ for all $k = 1, \cdots, n$, and therefore from (19) we obtain for all $k = 1, \cdots, n$

$$\alpha_k(\tau_2 + \tau_3 + \tau_4) + \sum_{i=1}^{n} \mu_i p_{ik} (\tau_i^2 + \tau_i^3 + \tau_i^4) - \mu_k (\tau_k^2 + \tau_k^2 + \tau_k^4) = 0$$

or in matrix form, with $\tau^j = (\tau_1^j, \cdots, \tau_n^j)'$

$$\alpha(\tau_2 + \tau_3 + \tau_4) + [P' - I] M [\tau^2 + \tau^3 + \tau^4] = 0.$$

Multiplying both sides from the left by $CM^{-1}[I-P']^{-1}$ we obtain

$$\begin{pmatrix}\rho_1-1\\ \rho_2-1\end{pmatrix}(\tau_2+\tau_3+\tau_4) + \begin{bmatrix}\tau_2+\tau_3+\tau_4-\sum_{k\in\sigma_1}(\tau_k^2+\tau_k^3+\tau_k^4)\\ \tau_2+\tau_3+\tau_4-\sum_{k\in\sigma_2}(\tau_k^2+\tau_k^2+\tau_k^4)\end{bmatrix}=0.$$

But from (10)–(12) we obtain

$$\tau_2+\tau_3+\tau_4=\sum_{k\in\sigma_2}(\tau_k^2+\tau_k^3+\tau_k^4).$$

Since $\tau_2+\tau_3+\tau_4>0$, we obtain that $\rho_2=1$, a contradiction. A similar argument shows that $\tau_3>0$.

Suppose now that $\tau_2=0$. From (10), $\tau^2=(\tau_1^2,\cdots,\tau_n^2)=0$, while from (13), (15), and (19), we obtain that

$$\alpha_k\tau_3+\sum_{i=1}^n \mu_i p_{ik}\tau_i^3-\mu_k\tau_k^3\geq 0, \qquad k\in\sigma_2.$$

From (16) we obtain

$$\alpha_k\tau_3+\sum_{i=1}^n \mu_i p_{ik}\tau_i^3-\mu_k\tau_k^3=0, \qquad k\in\sigma_1.$$

Combining these two equations in matrix form, we obtain

$$\alpha\tau_3+[P'-I]M\tau^3\geq 0.$$

Multiplying both sides of the inequality by $CM^{-1}[I-P']^{-1}$, we obtain

$$\begin{pmatrix}\rho_1-1\\ \rho_2-1\end{pmatrix}\tau_3+\begin{pmatrix}\tau_3-\sum_{k\in\sigma_1}\tau_k^3\\ \tau_3-\sum_{k\in\sigma_2}\tau_k^3\end{pmatrix}\geq 0.$$

Since from (11), $\tau_3=\sum_{k\in\sigma_2}\tau_k^3$ and $\tau_3>0$, we obtain that $\rho_2=1$, a contradiction. By a similar argument $\tau_4>0$. □

We next prove that the condition of Theorem 1 is also necessary.

*Theorem 2—Necessary Conditions for Stability:* If the linear program $LP[0]$ has a nonzero solution, then there exists a work-conserving policy under which the multiclass fluid network $(\alpha,\mu,P,C)$ is unstable.

*Proof:* Let $(\tau_j,\tau_k^j)$ be a nonzero solution of the linear program $LP[0]$. We will construct an initial condition $Q(0)\in R_1$ and a work-conserving policy such that for some time $\tau>0$, $Q(\tau)=Q(0)$. It will follow that there exists a work-conserving policy under which the system never empties and therefore the fluid network is unstable.

Let

$$Q_k(0)=-\left(\alpha_k\tau_2+\sum_{i=1}^n \mu_i p_{ik}\tau_i^2-\mu_k\tau_k^2\right), \quad k\in\sigma_1 \tag{20}$$

and

$$Q_k(0)=0, \qquad k\in\sigma_2.$$

Constraint (18) guarantees that $Q(0)\geq 0$. We next show that $\sum_{k\in\sigma_1}Q_k(0)>0$, i.e., $Q(0)\in R_1$. If $Q(0)=0$, then, for all $k\in\sigma_1$

$$\alpha_k\tau_2+\sum_{i=1}^n \mu_i p_{ik}\tau_i^2-\mu_k\tau_k^2=0.$$

Moreover, from (14), for all $k\in\sigma_2$

$$\alpha_k\tau_2+\sum_{i=1}^n \mu_i p_{ik}\tau_i^2-\mu_k\tau_k^2\geq 0.$$

In matrix form, with $\tau^i=(\tau_1^i,\cdots,\tau_n^i)'$, the previous equations become

$$\alpha\tau_2+[P'-I]M\tau^2\geq 0.$$

Multiplying by $CM^{-1}[I-P']^{-1}$, we obtain

$$\begin{pmatrix}\rho_1-1\\ \rho_2-1\end{pmatrix}\tau_2+\begin{pmatrix}\tau_2-\sum_{k\in\sigma_1}\tau_k^2\\ \tau_2-\sum_{k\in\sigma_2}\tau_k^2\end{pmatrix}\geq 0.$$

From (10), we have $\tau_2=\sum_{k\in\sigma_1}\tau_k^2=\sum_{k\in\sigma_2}\tau_k^2$. From Proposition 3, $\tau_2>0$, so $\rho_1,\rho_2\geq 1$, a contradiction and therefore, $Q(0)\neq 0$.

We construct the following allocation process for $k=1,\cdots,n$ as shown in (20a) at the bottom of the page. We show that the above allocation process is both feasible and work-conserving.

We first consider the first interval $[0,\tau_2]$. By the dynamics of the fluid network for this allocation process and starting

$$T_k(t)=\begin{cases}\dfrac{t}{\tau_2}\tau_k^2 & t\in[0,\tau_2];\\[2mm] \tau_k^2+\dfrac{t-\tau_2}{\tau_3}\tau_k^2 & t\in(\tau_2,\tau_2+\tau_3];\\[2mm] \tau_k^2+\tau_k^2+\dfrac{t-\tau_2-\tau_3}{\tau_4}\tau_k^4 & t\in(\tau_2+\tau_3,\tau_2+\tau_3+\tau_4];\\[2mm] \tau_k^2+\tau_k^2+\tau_k^4+\dfrac{t-\tau_2-\tau_3-\tau_4}{\tau_1}\tau_k^1 & t\in(\tau_2+\tau_3+\tau_4,\tau_2+\tau_3+\tau_4+\tau_1]\end{cases} \tag{20a}$$

from the initial condition given above, we obtain from (14) and (20)

$$Q_k(\tau_2) = 0, \qquad k \in \sigma_1$$
$$Q_k(\tau_2) = \alpha_k \tau_2 + \sum_{i=1}^{n} \mu_i p_{ik} \tau_i^2 - \mu_k \tau_k^2 \geq 0, \qquad k \in \sigma_2.$$

We next show that

$$\sum_{k \in \sigma_2} Q_k(\tau_2) > 0$$

so $Q(\tau_2) \in R_2$. If not, then

$$Q_k(\tau_2) = 0, \qquad k \in \sigma_2$$

or

$$\alpha_k \tau_2 + \sum_{i=1}^{n} \mu_i p_{ik} \tau_i^2 - \mu_k \tau_k^2 = 0, \qquad k \in \sigma_2.$$

Then from (13) and (19), we obtain that

$$\alpha_k(\tau_3 + \tau_4) + \sum_{i=1}^{n} \mu_i p_{ik} (\tau_i^3 + \tau_i^4) - \mu_k(\tau_k^3 + \tau_k^4) = 0, \qquad k \in \sigma_2.$$

Also from (16) and (17), we obtain that

$$\alpha_k(\tau_3 + \tau_4) + \sum_{i=1}^{n} \mu_i p_{ik} (\tau_i^3 + \tau_i^4) - \mu_k(\tau_k^3 + \tau_k^4) \geq 0, \qquad k \in \sigma_1.$$

Written in matrix from, the two previous relations become

$$\alpha(\tau_3 + \tau_4) + [P' - I]M(\tau^3 + \tau^4) \geq 0.$$

Multiplying by $CM^{-1}[I - P']^{-1}$, we obtain

$$\begin{pmatrix} \rho_1 - 1 \\ \rho_2 - 1 \end{pmatrix} (\tau_3 + \tau_4) + \begin{bmatrix} \tau_3 + \tau_4 - \sum_{k \in \sigma_1} (\tau_k^2 + \tau_k^4) \\ \tau_3 + \tau_4 - \sum_{k \in \sigma_2} (\tau_k^2 + \tau_k^4) \end{bmatrix} \geq 0.$$

Since $\tau_3 + \tau_4 = \sum_{k \in \sigma_2} (\tau_k^2 + \tau_k^4)$ and $\tau_3 + \tau_4 > 0$, we obtain $\rho_2 \geq 1$, a contradiction, and therefore $\sum_{k \in \sigma_2} Q_k(\tau_2) > 0$.

Since the allocation process is linear, we obtain

$$\forall\, t \in [0, \tau_2], \qquad Q(t) \geq 0$$

and

$$\forall\, t \in (0, \tau_2), \qquad Q(t) \in R_{12}$$

i.e., the allocation process is feasible. We next show that it is also work-conserving. From (10)

$$t = \sum_{k \in \sigma_1} \frac{t}{\tau_2} \tau_k^2 = \sum_{k \in \sigma_2} \frac{t}{\tau_2} \tau_k^2$$

or equivalently

$$\forall t \in [0, \tau_2] : U_1(t) = U_2(t) = U_1(0) = U_2(0) = 0$$

and the process is indeed work-conserving.

In the interval $(\tau_2, \tau_2 + \tau_3]$, we prove similarly that for $k \in \sigma_2$ we have $Q_k(\tau_2 + \tau_3) \geq 0$ and $\sum_{k \in \sigma_2} Q_k(\tau_2 + \tau_3) > 0$. Therefore, $Q(\tau_2 + \tau_3) \in R_2$, and since $Q(\tau_2) \in R_2$, we obtain by linearity that

$$\forall\, t \in [\tau_2, \tau_2 + \tau_3], \qquad Q(t) \in R_2.$$

Work-conservation is shown similarly.

Additionally, we show that in the interval $t \in (\tau_2 + \tau_3, \tau_2 + \tau_3 + \tau_4]$, $Q(t) \in R_{12}$ and in the interval $t \in [\tau_2 + \tau_3 + \tau_4, \tau_2 + \tau_3 + \tau_4 + \tau_1]$, $Q(t) \in R_1$, while the process is work-conserving.

In addition, because of (19), $Q(\tau_1 + \tau_2 + \tau_3 + \tau_4) = Q(0)$. It follows that the fluid network never empties for this work-conserving feasible policy and is unstable. □

The necessity proof has identified a particular way that an unstable work-conserving trajectory materializes, leading to some insight as to how instability may be reached. In particular, we have shown that if there exists an unstable trajectory, then there exists a periodic trajectory with a particular structure.

Combining Theorems 1 and 2, we obtain the main theorem of this section.

*Theorem 3:* A two-station multiclass fluid network $(\alpha, \mu, P, C)$ is stable for all work conserving policies if and only if the load condition $\rho < e$ holds and the linear program $LP[0]$ has zero as the only feasible solution.

### D. A Special Case

To illustrate the use (as well as the power) of Theorem 3, we prove that a two-station fluid network, in which one of the two stations has only one class, is stable provided that the load condition (4) is satisfied. This generalizes previous results obtained by Kumar [10], Down, and Meyn [9] for a three-class, two-station network.

*Theorem 4:* A fluid network satisfying the load condition $\rho < e$ with two stations and such that only one class is served by station 2 ($|\sigma_2| = 1$) is stable.

*Proof:* We show that the corresponding linear program $LP[0]$ cannot have a nonzero solution. For the purposes of contradiction suppose that $(\tau_j, \tau_k^j)$ is a nonzero solution to $LP[0]$. Let $\sigma_2 = \{l\}$. We distinguish between two cases.

*Case 1:*

$$\alpha_l \tau_3 + \sum_{i=1}^{n} \mu_i p_{il} \tau_i^3 - \mu_l \tau_l^3 \geq 0.$$

From (16)

$$\alpha_k \tau_3 + \sum_{i=1} \mu_i p_{ik} \tau_i^3 - \mu_k \tau_k^2 = 0, \qquad \forall\, k \in \sigma_1.$$

We combine the previous relations in matrix form as follows:

$$\alpha \tau_3 + [P' - I]M\tau^3 \geq 0.$$

We multiply both sides by $CM^{-1}[I - P']^{-1}$ to obtain

$$\begin{pmatrix}\rho_1 - 1\\ \rho_2 - 1\end{pmatrix}\tau_3 + \begin{pmatrix}\tau_3 - \sum_{k\in\sigma_1}\tau_k^2\\ \tau_3 - \tau_l^3\end{pmatrix} \geq 0.$$

But from (11), we obtain $\tau_3 = \tau_l^3$ and from Proposition 3, we obtain $\tau_3 > 0$, leading to $\rho_2 = 1$, a contradiction.

*Case 2:*

$$\alpha_l\tau_3 + \sum_{i=1}^{n} \mu_i p_{il}\tau_i^3 - \mu_l\tau_l^3 \leq 0.$$

From (19), we obtain

$$\alpha_l(\tau_4 + \tau_1 + \tau_2) + \sum_{i=1}^{n} \mu_i p_{il}(\tau_i^4 + \tau_i^1 + \tau_i^2) - \mu_l(\tau_l^4 + \tau_l^1 + \tau_l^2) \geq 0.$$

Moreover, from (16) and (19) we obtain

$$\alpha_k(\tau_4 + \tau_1 + \tau_2) + \sum_{i=1}^{n} \mu_i p_{ik}(\tau_i^4 + \tau_i^1 + \tau_i^2) - \mu_k(\tau_k^4 + \tau_k^1 + \tau_k^2) = 0, \qquad k \in \sigma_1$$

which, in matrix form, becomes

$$\alpha(\tau_4 + \tau_1 + \tau_2) + [P' - I]M(\tau^4 + \tau^1 + \tau^2) \geq 0.$$

Multiplying both sides by $CM^{-1}[I - P']^{-1}$ we obtain

$$\begin{pmatrix}\rho_1 - 1\\ \rho_2 - 1\end{pmatrix}(\tau_4 + \tau_1 + \tau_2) + \begin{bmatrix}\tau_4 + \tau_1 + \tau_2 - \sum_{k\in\sigma_1}(\tau_k^4 + \tau_k^1 + \tau_k^2)\\ \tau_4 + \tau_1 + \tau_2 - (\tau_l^4 + \tau_l^1 + \tau_l^2)\end{bmatrix} \geq 0.$$

From (9), (10), and (12), we obtain

$$\tau_4 + \tau_1 + \tau_2 = \sum_{k\in\sigma_1}(\tau_k^4 + \tau_k^1 + \tau_k^2)$$

and since $\tau_4 + \tau_1 + \tau_2 > 0$, then $\rho_1 = 1$, a contradiction. □

## IV. On the Power of Piecewise Linear Lyapunov Functions

It is well known (see, for example, [9]) that a multiclass fluid network is stable under all work conserving policies if and only if there exists a Lyapunov function which decreases along all possible trajectories. An example of such a function is the maximum (over all work conserving policies) of the time it takes for the system to empty. However, to prove that a system is stable, one needs to explicitly construct such a Lyapunov function, and this can be quite difficult. One possibility that has been investigated recently is to restrict to a class of convex Lyapunov functions (quadratic or piecewise linear) and to use mathematical programming techniques to identify a suitable Lyapunov function within such a class; see Kumar and Meyn [12], Botvich and Zamyatin [4], Dai and Weiss [8], Down and Meyn [9].

These papers, however, leave open the question of whether convex Lyapunov functions have the power to establish (sharp) necessary and sufficient conditions for stability. In other words, is it true that a system is stable under all work conserving policies if and only if there exists a convex Lyapunov function that testifies to this?

In this section we give a positive answer to this question for the case of a piecewise linear, convex Lyapunov function and a two-station multiclass fluid network. Concretely, we will show that a two-station network is stable if and only if the linear program constructed by Down and Meyn in [9] has a feasible solution. This solution (as discussed in [9]), if it exists, provides a certain piecewise linear Lyapunov function which guarantees stability. In particular, we will demonstrate that the dual of this linear program is a relaxation of the linear program $LP[0]$ constructed in the previous section. Finally, we will simplify $LP[0]$ and construct a linear program with only $2n$ variables that exactly characterizes stability.

### *A. Piecewise Linear Lyapunov Functions and Duality*

Consider a multiclass fluid network $(\alpha, \mu, P, C)$, with two stations, which is a reentrant line. Namely, there is only a single arrival stream of customers, i.e., $\alpha_1 = \lambda$, $\alpha_2 = \cdots = \alpha_n = 0$. These customers are processed deterministically from class $k$ to class $k+1$ ($p_{k,k+1} = 1$ for $k = 1, 2, \cdots, n-1$, $p_{ij} = 0$ otherwise). Down and Meyn [9] proved that if the following linear program:

$$\begin{aligned}
\lambda L_1 + \mu_i(L_{i+1} - L_i) &\leq -1 \quad i \in \sigma_1\\
\lambda Q_1 + \mu_j(Q_{j+1} - Q_j) &\leq -1 \quad j \in \sigma_2\\
\lambda L_1 + \mu_i(L_{i+1} - L_i) &\\
+\mu_j(L_{j+1} - L_j) &\leq -1 \quad i \in \sigma_1,\, j \in \sigma_2\\
\lambda Q_1 + \mu_i(Q_{i+1} - Q_i) &\\
+\mu_j(Q_{j+1} - Q_j) &\leq -1 \quad i \in \sigma_1,\, j \in \sigma_2\\
L_i &\geq Q_i \quad i \in \sigma_1\\
L_j &\leq Q_j \quad j \in \sigma_2\\
L \geq 0,\, Q &\geq 0
\end{aligned}$$

is feasible, then the piecewise linear function $\Phi(x) = \max(L'x, Q'x)$, for $x \geq 0$, is a Lyapunov function and therefore the network is stable for all work-conserving policies.

We can easily extend this linear program to a general multiclass two-station fluid network $(\alpha, \mu, P, C)$, i.e., not necessarily a reentrant line. If the following linear program (we call it $LP[\text{dm}]$):

$$(X_i) \quad \sum_{k=1}^{n} L_k\alpha_k + \sum_{k=1}^{n} L_k p_{ik}\,\mu_i - L_i\,\mu_i + V \leq -1 \quad i \in \sigma_1$$

$$(X_j) \quad \sum_{k=1}^{n} L_k p_{jk}\,\mu_j - L_j\,\mu_j \leq V \quad j \in \sigma_2$$

$$(Y_j) \quad \sum_{k=1}^{n} Q_k\alpha_k + \sum_{k=1}^{n} Q_k p_{jk}\,\mu_j - Q_j\,\mu_j + W \leq -1 \quad j \in \sigma_2$$

$$\begin{aligned}
(Y_i) \quad & \sum_{k=1}^{n} Q_k p_{ik}\, \mu_i - Q_i\, \mu_i \\
& \quad \leq W \quad i \in \sigma_1 \\
(m_i) \quad & L_i \geq Q_i \quad i \in \sigma_1 \\
(n_j) \quad & L_j \leq Q_j \quad j \in \sigma_2 \\
& L, Q, V, W \geq 0
\end{aligned}$$

is feasible, then a piecewise linear function $\Phi(x) = \max(L'x, Q'x)$ is a Lyapunov function, and therefore the network is stable for all work-conserving policies (the associated dual variables are indicated in parenthesis).

Let the objective function in $LP$[dm] be to maximize $0L + 0Q + 0V + 0W$ and consider the dual $LP$. It is a homogeneous $LP$ in the variables $X_k, Y_k, k = 1, 2, \cdots, n, m_k, k \in \sigma_1, n_k, k \in \sigma_2$ which has the following form:

maximize

$$-\sum_{i\in\sigma_1} X_i - \sum_{j\in\sigma_2} Y_j$$

subject to

$$\begin{aligned}
&\alpha_k \sum_{i\in\sigma_1} X_i + \sum_{i=1}^{n} \mu_i p_{ik} X_i - \mu_k X_k - m_k \leq 0 \quad k \in \sigma_1 \\
&\alpha_k \sum_{j\in\sigma_2} Y_j + \sum_{i=1}^{n} \mu_i p_{ik} Y_i - \mu_k Y_k + m_k \leq 0 \quad k \in \sigma_1 \\
&\alpha_k \sum_{i\in\sigma_1} X_i + \sum_{i=1}^{n} \mu_i p_{ik} X_i - \mu_k X_k + n_k \leq 0 \quad k \in \sigma_2 \\
&\alpha_k \sum_{j\in\sigma_2} Y_j + \sum_{i=1}^{n} \mu_i p_{ik} Y_i - \mu_k Y_k - n_k \leq 0 \quad k \in \sigma_2 \\
&\sum_{i\in\sigma_1} X_i \leq \sum_{j\in\sigma_2} X_j \\
&\sum_{j\in\sigma_2} Y_j \leq \sum_{i\in\sigma_1} Y_i \\
&X, Y, m, n \leq 0.
\end{aligned}$$

The above linear program is equivalent to

maximize

$$\sum_{i\in\sigma_1} X_i + \sum_{j\in\sigma_2} Y_j$$

subject to

$$\begin{aligned}
&\alpha_k \sum_{i\in\sigma_1} X_i + \sum_{i=1}^{n} \mu_i p_{ik} X_i - \mu_k X_k - m_k \geq 0 \quad k \in \sigma_1 \\
&\alpha_k \sum_{j\in\sigma_2} Y_j + \sum_{i=1}^{n} \mu_i p_{ik} Y_i - \mu_k Y_k + m_k \geq 0 \quad k \in \sigma_1 \\
&\alpha_k \sum_{i\in\sigma_1} X_i + \sum_{i=1}^{n} \mu_i p_{ik} X_i - \mu_k X_k + n_k \geq 0 \quad k \in \sigma_2 \\
&\alpha_k \sum_{j\in\sigma_2} Y_j + \sum_{i=1}^{n} \mu_i p_{ik} Y_i - \mu_k Y_k - n_k \geq 0 \quad k \in \sigma_2 \\
&\sum_{i\in\sigma_1} X_i \geq \sum_{j\in\sigma_2} X_j \\
&\sum_{j\in\sigma_2} Y_j \geq \sum_{i\in\sigma_1} Y_i \\
&X, Y, m, n \geq 0
\end{aligned}$$

which we call $DLP$[dm].

*Lemma 5:* $LP$[dm] is feasible if and only if $DLP$[dm] has zero as the only feasible solution.

*Proof:* The proof follows immediately from strong duality of linear programming (see [1]). □

We will gradually simplify $DLP$[dm]. We start with the following lemma.

*Lemma 6:* $DLP$[dm] has a nonzero feasible solution if and only if the following linear program, called $DLP$[1], has a nonzero feasible solution:

maximize

$$\sum_{i\in\sigma_1} X_i + \sum_{j\in\sigma_2} Y_j$$

subject to

$$\alpha_k \sum_{i\in\sigma_1} X_i + \sum_{i=1}^{n} \mu_i p_{ik} X_i - \mu_k X_k \geq 0 \qquad k \in \sigma_1 \quad (21)$$

$$\alpha_k \sum_{j\in\sigma_2} Y_j + \sum_{i=1}^{n} \mu_i p_{ik} Y_i - \mu_k Y_k \geq 0 \qquad k \in \sigma_2 \quad (22)$$

$$\alpha_k \left( \sum_{i\in\sigma_1} X_i + \sum_{j\in\sigma_2} Y_j \right) + \sum_{i=1}^{n} \mu_i p_{ik}(X_i + Y_i) - \mu_k (X_k + Y_k) \geq 0 \qquad \forall k \quad (23)$$

$$\sum_{k\in\sigma_1} X_k \geq \sum_{k\in\sigma_2} X_k \quad (24)$$

$$\sum_{k\in\sigma_2} Y_k \geq \sum_{k\in\sigma_1} Y_k \quad (25)$$

$$X, Y \geq 0.$$

*Proof:* Let $X_k, Y_k, m_k, n_k$ be a feasible nonzero solution to $DLP$[dm]. Since

$$\alpha_k \sum_{i\in\sigma_1} X_i + \sum_{i=1}^{n} \mu_i p_{ik} X_i - \mu_k X_k - m_k \geq 0, \qquad m_k \geq 0$$

(21) follows. Similarly, (22) follows. By adding inequalities in $DLP$[dm] corresponding to stations $\sigma_1$ and $\sigma_2$ separately, we obtain that $X_k, Y_k$ is a feasible nonzero solution to $DLP$[1].

Conversely, if $X_k, Y_k$ is a nonzero solution to $DLP$[1], then by setting

$$\forall k \in \sigma_1 : \alpha_k \sum_{i\in\sigma_1} X_i + \sum_{i=1}^{n} \mu_i p_{ik} X_i - \mu_k X_k = m_k$$

and

$$\forall k \in \sigma_2 : \alpha_k \sum_{j\in\sigma_2} Y_j + \sum_{i=1}^{n} \mu_i p_{ik} Y_i - \mu_k Y_k = n_k$$

we obtain that $X_k, Y_k, m_k, n_k$ is a nonzero solution to $DLP$[dm]. □

The next lemma shows that we can change (23) to an equality.

*Lemma 7:* Let $DLP[2]$ be a linear program obtained from $DLP[1]$ by replacing (23) with equality. Then, if the condition $\rho < e$ holds, $DLP[2]$ has a nonzero feasible solution if and only if $DLP[1]$ has a nonzero feasible solution.

*Proof:* Trivially, if $X, Y$ is a nonzero solution to $DLP[2]$, then it is also a nonzero solution to $DLP[1]$. For the converse part, let $X, Y$ be a nonzero solution to $DLP[1]$. We will construct a nonzero solution to $DLP[2]$.

Let us rewrite (23) in matrix form as follows:

$$\alpha(x+y) + [P' - I]M(X+Y) \geq 0 \tag{26}$$

where we define

$$\begin{aligned} x &= \sum_{i \in \sigma_1} X_i \\ y &= \sum_{j \in \sigma_2} Y_j \\ X &= (X_1, \cdots, X_n) \\ Y &= (Y_1, \cdots, Y_n). \end{aligned} \tag{27}$$

Since $[I - P']^{-1}$ and $M^{-1}$ exist and are nonnegative, (26) is equivalent to

$$M^{-1}[I - P']^{-1}\alpha(x+y) - (X+Y) \geq 0$$

or simply

$$\rho(x+y) - (X+Y) \geq 0.$$

We will increase $X_k$ to $\hat{X}_k$ for all $k \in \sigma_2$ so that for all $k \in \sigma_2$

$$\rho_k(x+y) - (\hat{X}_k + Y_k) = 0.$$

This is possible to do because $x$ is not affected by $X_k$ for $k \in \sigma_2$. Notice also that this change can only increase the left-hand side of (21).

Similarly, we construct $\hat{Y}_k$ for all $k \in \sigma_1$ such that for all $k \in \sigma_1$

$$\rho_k(x+y) - (X_k + \hat{Y}_k) = 0$$

and (22) is still satisfied. Finally, we show that (24) and (25) are still satisfied. We have, by construction

$$\begin{aligned} \sum_{k \in \sigma_2} \hat{X}_k + \sum_{k \in \sigma_2} Y_k &= \sum_{k \in \sigma_2} \rho_k(x+y) \\ &= \rho_{\sigma_2}(x+y) \\ &\leq x + y. \end{aligned}$$

Since by definition, $y = \sum_{k \in \sigma_2} Y_k$, we obtain that

$$\sum_{k \in \sigma_2} \hat{X}_k \leq x$$

i.e., (24) holds. By a similar reason (25) holds, i.e.,

$$\sum_{k \in \sigma_1} \hat{Y}_k \leq y.$$

The new solution $\hat{X}, \hat{Y}$ satisfies $\hat{X} \geq X, \hat{Y} \geq Y$ and, therefore, it is nonzero. By construction, it is a feasible solution to $DLP[2]$. □

In the remaining part of this section we will show that $DLP[2]$ has a nonzero solution if and only if $LP[0]$ (from Section III) has a nonzero solution. We show first that $DLP[2]$ is a relaxation of $LP[0]$.

*Lemma 8:* Let $(\tau_1, \tau_2, \tau_3, \tau_4, \tau_k^1, \tau_k^2, \tau_k^3, \tau_k^4)$, $k = 1, 2, \cdots, n$ be a nonzero feasible solution to $LP[0]$. Let $X_k = \tau_k^4 + \tau_k^1$, $Y_k = \tau_k^2 + \tau_k^3$, $k = 1, 2, \cdots, n$. Then $(X_k, Y_k)$ is a nonzero feasible solution to $DLP[2]$.

*Proof:* Combining (9) with (12), we obtain (24). Combining (10) with (11), we obtain (25). Equation (19) shows that (23) (with equality) holds. Combining (16) with (18), we obtain that

$$\forall k \in \sigma_1 : \alpha_k \sum_{j \in \sigma_2} Y_j + \sum_{i=1}^{n} \mu_i p_{ik} Y_i - \mu_k Y_k \leq 0.$$

By subtracting this from (23) (with equality) we obtain (21). Equation (22) is obtained similarly. By construction, if

$$(\tau_1, \tau_2, \tau_3, \tau_4, \tau_k^1, \tau_k^2, \tau_k^3, \tau_k^4)$$

is nonzero, then the solution $(X_k, Y_k)$ is nonzero as well. □

We next prove the converse part.

*Lemma 9:* If there exists a nonzero solution to $DLP[2]$, then there exists a nonzero solution to $LP[0]$.

*Proof:* Let $(X_k, Y_k, k = 1, 2, \cdots, n)$ be a nonzero solution to $DLP[2]$. Let $x = \sum_{i \in \sigma_1} X_i$ and $y = \sum_{j \in \sigma_2} Y_j$.

We will construct a nonzero solution to $LP[0]$.

We select a number $\gamma \in [0, 1]$; we specify how $\gamma$ is selected later. Combining (22) and (23) (with equality), we obtain

$$\alpha_k x + \sum_{i=1}^{n} \mu_i p_{ik} X_i - \mu_k X_k \leq 0, \qquad k \in \sigma_2.$$

Then

$$\alpha_k \gamma x + \sum_{i=1}^{n} \mu_i p_{ik} \gamma X_i - \mu_k \gamma X_k \leq 0, \qquad k \in \sigma_2. \tag{28}$$

Let us rewrite this as follows:

$$\begin{aligned} \alpha_k \gamma x &+ \sum_{i \in \sigma_1} \mu_i p_{ik} \gamma X_i \\ &+ \sum_{j \in \sigma_2} \mu_i p_{ik} \gamma X_i - \mu_k \gamma X_k \leq 0, \qquad k \in \sigma_2. \end{aligned} \tag{29}$$

We introduce the following notation. For any vector $W \in R_+^n$ let $W_{\sigma_1}$ and $W_{\sigma_2}$ be the portion of the vector $W$ corresponding to the indexes in $\sigma_1$ and $\sigma_2$, respectively. We partition the matrix $P$ as follows:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}.$$

The matrices $P_{12}$ and $P_{11}$ are portions of the matrix $P$ corresponding to flows of classes from station 1 to station 2 and from station 1 to itself. Similarly, the matrices $P_{12}$ and $P_{22}$ are the portions of the matrix $P$ corresponding to flows going from station 1 to station 2 and from station 2 to itself.

We rewrite (29) in matrix form

$$\alpha_{\sigma_2} \gamma x + P_{12} M_{\sigma_1} \gamma X_{\sigma_1} + [P_{22} - I_{\sigma_2}] M_{\sigma_2} \gamma X_{\sigma_2} \leq 0. \tag{30}$$

The matrix $P_{22}$ is nonnegative and has spectral radius less than one. Therefore the matrix $[I_{\sigma_2} - P_{22}]^{-1}$ exists and is nonnegative. We rewrite (30) as follows:

$$M_{\sigma_2}^{-1}[I_{\sigma_2} - P_{22}]^{-1}\alpha_{\sigma_2}\gamma x + M_{\sigma_2}^{-1}[I_{\sigma_2} - P_{22}]^{-1}P_{12}M_{\sigma_1}\gamma X_{\sigma_1} - \gamma X_{\sigma_2} \leq 0. \quad (31)$$

We next introduce $|\sigma_2|$-dimensional vectors $\tau_{\sigma_2}^1$, $Z_{\sigma_2}$

$$\begin{aligned}\tau_{\sigma_2}^1 &= M_{\sigma_2}^{-1}[I_{\sigma_2} - P_{22}]^{-1}\alpha_{\sigma_2}\gamma x \\ &\quad + M_{\sigma_2}^{-1}[I_{\sigma_2} - P_{22}]^{-1}P_{12}M_{\sigma_1}\gamma X_{\sigma_1} \\ &= \gamma Z_{\sigma_2} \\ &\geq 0.\end{aligned} \quad (32)$$

From (31) it follows that

$$\begin{aligned}\tau_{\sigma_2}^1 &= \gamma Z_{\sigma_2} \\ &\leq \gamma X_{\sigma_2}.\end{aligned} \quad (33)$$

Having defined the variables $\tau_k^1$ for $k \in \sigma_2$, we let $\tau_k^1 = \gamma X_k$, for $k \in \sigma_1$. Let $\tau_1 = \gamma x$. From (32), (13) follows.

From (24), we obtain

$$\begin{aligned}\gamma x &= \sum_{k\in\sigma_1} \gamma X_k \\ &\geq \sum_{k\in\sigma_2} \gamma X_k.\end{aligned}$$

Then from (33), it follows that (9) is satisfied.

We next let $\tau_k^4 = X_k - \tau_k^1 = (1-\gamma)X_k$ for $k \in \sigma_1$, $\tau_k^4 = X_k - \tau_k^1$ for $k \in \sigma_2$ and $\tau_4 = (1-\gamma)x$. It follows from (33) that $\tau_k^4$ are nonnegative for $k \in \sigma_2$ and, therefore, all the new variables $\tau_k^4$ are nonnegative. Since $x = \sum_{i\in\sigma_1} X_i$, it follows that the first part of (12) is satisfied.

We next show that we can select $\gamma \in [0, 1]$ so that the second part of (12), i.e.,

$$\sum_{k\in\sigma_1} \tau_k^4 = \sum_{k\in\sigma_2} \tau_k^4 \quad (34)$$

is satisfied as well. Recall that $\tau_k^1 = \gamma X_k$, $k \in \sigma_1$, $\tau_k^1 = \gamma Z_k$, $k \in \sigma_2$ [from (32)], $\tau_k^4 = X_k - \tau_k^1 = (1-\gamma)X_k$, $k \in \sigma_1$, $\tau_k^4 = X_k - \tau_k^1$, $k \in \sigma_2$. Then

$$\sum_{k\in\sigma_1} \tau_k^4 = (1-\gamma)\sum_{k\in\sigma_1} X_k$$

and

$$\sum_{k\in\sigma_2} \tau_k^4 = \sum_{k\in\sigma_2}(X_k - \tau_k^1).$$

From (33) $Z_k \leq X_k$, $k \in \sigma_2$ and from (24)

$$\sum_{k\in\sigma_1} X_k \geq \sum_{k\in\sigma_2} X_k.$$

Therefore

$$\begin{aligned}\sum_{k\in\sigma_2} Z_k &\leq \sum_{k\in\sigma_2} X_k \\ &\leq \sum_{k\in\sigma_1} X_k.\end{aligned}$$

In case the first sum is strictly less than the third sum, we take $\gamma$ to be

$$\gamma = \frac{\sum_{k\in\sigma_1} X_k - \sum_{k\in\sigma_2} X_k}{\sum_{k\in\sigma_1} X_k - \sum_{k\in\sigma_2} Z_k}.$$

This guarantees

$$(1-\gamma)\sum_{k\in\sigma_1} X_k = \sum_{k\in\sigma_2}(X_k - \gamma Z_k)$$

or

$$\sum_{k\in\sigma_1} \tau_k^4 = \sum_{k\in\sigma_2} \tau_k^4.$$

From the inequalities above, this value of $\gamma$ satisfies $\gamma \in [0, 1]$. If, on the other hand, all sums are equal, then we take $\gamma$ to be any number in $[0, 1]$ and (34) is still satisfied.

Therefore, we have satisfied (9), (12), and (13). We next prove that (15) and (17) are satisfied as well.

Subtracting (22) from (23) (with equality), we obtain

$$\forall\, k \in \sigma_2 : \alpha_k x + \sum_{i=1}^{n} \mu_i p_{ik} X_i - \mu_k X_k \leq 0$$

which in terms of the variables $\tau_1^1, \cdots, \tau_n^1, \tau_1^4, \cdots, \tau_n^4$ reduces to

$$\forall\, k \in \sigma_2 : \alpha_k(\tau_4 + \tau_1) + \sum_{i=1}^{n} \mu_i p_{ik}(\tau_i^4 + \tau_i^1) - \mu_k(\tau_k^4 + \tau_k^1) \leq 0.$$

This combined with (13) proves (15). Also from (21)

$$\forall\, k \in \sigma_1 : \alpha_k(1-\gamma)x + \sum_{i=1}^{n} \mu_i p_{ik}(1-\gamma)X_i - \mu_k(1-\gamma)X_k \geq 0.$$

From (33) we obtain for $k \in \sigma_2$

$$\begin{aligned}\tau_k^4 &= X_k - \tau_k^1 \\ &\geq X_k - \gamma X_k \\ &= (1-\gamma)X_k.\end{aligned}$$

Therefore

$$\begin{aligned}\forall\, k \in \sigma_1 : \alpha_k(1-\gamma)x &+ \sum_{i\in\sigma_1} \mu_i p_{ik}(1-\gamma)X_i \\ &+ \sum_{i\in\sigma_2} \mu_i p_{ik}\tau_i^4 - \mu_k(1-\gamma)X_k \geq \alpha_k(1-\gamma)x \\ &+ \sum_{i=1}^{n} \mu_i p_{ik}(1-\gamma)X_i - \mu_k(1-\gamma)X_k \geq 0\end{aligned}$$

or, equivalently

$$\forall\, k \in \sigma_1 :\ \alpha_k \tau^4 + \sum_{i=1}^{n} \mu_i p_{ik}\tau_i^4 - \mu_k \tau_k^4 \geq 0$$

which is (17).

We have constructed $\tau_1, \tau_4, \tau_k^1, \tau_k^4, k = 1, 2, \cdots, n$ which satisfy (9), (12), (13), (15), and (17). The construction of $\tau_2, \tau_3, \tau_k^2, \tau_k^3, k = 1, 2, \cdots, n$ is symmetric. Finally, (19) is a simple implication of (23) (with equality). If the initial solution $(x, y, X_k, Y_k)$ is nonzero, then the solution $(\tau_1, \tau_2, \tau_3, \tau_4, \tau_k^1, \tau_k^2, \tau_k^3, \tau_k^4)$, $k = 1, \cdots, n$ is also nonzero. This concludes the proof of the lemma. □

We now summarize the results obtained in this and the previous section.

*Corollary 1:* A multiclass fluid network $(\alpha, \mu, P, C)$ with two stations is stable for all work-conserving policies if and only if one of the following equivalent conditions hold.

1) Linear program $LP$[dm] constructed in [9] is feasible.
2) Linear program $DLP$[2] constructed in this section has zero as the only feasible solution.
3) Linear program $LP$[0] constructed in the previous section has zero as the only feasible solution.

From the above three equivalent tests for stability, $DLP$[2] is the most economical. Unlike $LP$[dm], it can be interpreted physically, with variables corresponding to times arising from a decomposition of trajectories. On the other hand, it has half as many variables compared to $LP$[0].

## V. Sufficient Stability Conditions for a General Multiclass Fluid Network

In this section, we derive new sufficient conditions for stability of a general multiclass fluid network involving an arbitrary number $J$ of stations. We follow the notation of Section II. We consider an arbitrary stable trajectory with $\tau$ being the emptying time.

A time $\hat{t} \le \tau$ will be called an "emptying time for station $\sigma$" if

$$\sum_{k\in\sigma} Q_k(\hat{t}) = 0$$

and there exists an $\epsilon > 0$ such that for all $t \in (\hat{t} - \epsilon, \hat{t})$

$$\sum_{k\in\sigma} Q_k(t) > 0$$

namely, $\hat{t}$ is exactly the time at which station $\sigma$ becomes empty. The set of all "emptying times" $\Lambda$ is clearly a countable set. Let $\Lambda = \{t_1, t_2, \cdots, t_m, \cdots\}$. For any $t, t' \in \Lambda$, we will say that an interval $(t, t')$ is of type $\sigma_r$, $r = 1, 2, \cdots, J$ or a $\sigma_r$-interval if $t'$ is an "emptying time" of station $\sigma_r$ [and no other "emptying times" are located strictly within the interval $(t, t')$]. Consider the example of Fig. 2. In this example, there are three stations and we denote by $t_{l_1}, t_{l_2} \cdots, t_{l_6}$ the first six emptying times. The reason we use a double subscript is that it is possible for the emptying times of two stations to alternate countably many times followed by another countable alternation of the emptying times of two other stations. This situation cannot arise with two stations. It also does not arise when the number of emptying times is finite. So, we can take $t_{l_i} = t_i$ in the example. Here, $t_{l_3}, t_{l_5}$ are the times that station 1 becomes empty, times $t_{l_1}, t_{l_6}$ are the times that station 2 becomes empty, and times $t_{l_2}, t_{l_4}$ are the times that station 3 becomes empty. If there is a time $t_{l_i}$ that two stations become empty at the same time, we assign time $t_{l_i}$ arbitrarily to one of these stations. Notice that by definition, $Q_k(t_{l_i}) = 0$ for all $k \in \sigma_r$ if $(t_{l_{i-1}}, t_{l_i})$ is an interval of type $\sigma_r$.

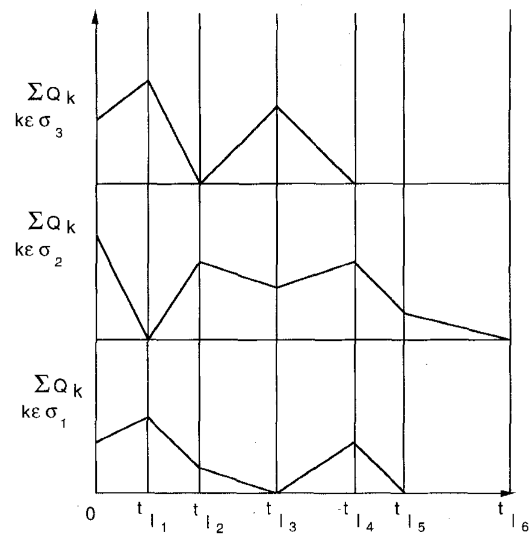


Fig. 2. The emptying times $t_{l_i}$ for a typical trajectory.

By writing the dynamics of the system during a $\sigma_r$ interval $(t_{l_{i-1}}, t_{l_i}]$, we obtain for $k \in \sigma_r$

$$Q_k(t_{l_i}) - Q_k(t_{l_{i-1}}) = \alpha_k(t_{l_i} - t_{l_{i-1}}) + \sum_{j=1}^{n} \mu_j p_{jk}[T_j(t_{l_i}) - T_j(t_{l_{i-1}})] - \mu_k[T_k(t_{l_i}) - T_k(t_{l_{i-1}})].$$

Since $Q_k(t_{l_i}) = 0$ and $Q_k(t_{l_{i-1}}) \ge 0$, we obtain that

$$\alpha_k(t_{l_i} - t_{l_{i-1}}) + \sum_{j=1}^{n} \mu_j p_{jk}[T_j(t_{l_i}) - T_j(t_{l_{i-1}})] - \mu_k[T_k(t_{l_i}) - T_k(t_{l_{i-1}})] \le 0.$$

Summing over all $\sigma_r$ intervals and introducing the new variables

$$\tau_r = \sum_{(t_{l_{i-1}}, t_{l_i}] \text{ is a } \sigma_r\text{-interval}} (t_{l_i} - t_{l_{i-1}}), \quad r = 1, \cdots, J,$$

$$\tau_{jr} = \sum_{(t_{l_{i-1}}, t_{l_i}] \text{ is a } \sigma_r\text{-interval}} [T_j(t_{l_i}) - T_j(t_{l_{i-1}})], \quad j = 1, \cdots n, \quad r = 1, \cdots, J$$

we obtain

$$\alpha_k \tau_r + \sum_{j=1}^{n} \mu_j p_{jk} \tau_{jr} - \mu_k \tau_{kr} \le 0, \qquad \forall\, k \in \sigma_r.$$

Since by definition, during a $\sigma_r$-interval, station $\sigma_r$ is busy, we obtain from work-conservation that

$$\sum_{k\in\sigma_r} [T_k(t_{l_i}) - T_k(t_{l_{i-1}})] = t_{l_i} - t_{l_{i-1}}.$$

Summing over all $\sigma_r$ intervals we obtain that

$$\sum_{k\in\sigma_r} \tau_{kr} = \tau_r, \qquad r = 1, \cdots, n.$$

Since the trajectory is feasible

$$\sum_{k\in\sigma_j} [T_k(t_{l_i}) - T_k(t_{l_{i-1}})] \le t_{l_i} - t_{l_{i-1}}.$$

Summing over all $\sigma_r$ intervals we obtain that

$$\sum_{k\in\sigma_j} \tau_{kr} \le \tau_r, \qquad j \ne r.$$

Finally, since we consider a stable trajectory, all the stations become empty for $T = \max\, t_{l_i} = t_L$. Writing the dynamics of the trajectory we obtain that for all $k = 1, \cdots, n$

$$\begin{aligned} Q_k(t_L) - Q_k(0) = {} & \alpha_k \sum_{i=1}^{L} (t_{l_i} - t_{l_{i-1}}) \\ & + \sum_{i=1}^{L}\sum_{j=1}^{n} \mu_j p_{jk} [T_j(t_{l_i}) - T_j(t_{l_{i-1}})] \\ & - \mu_k \sum_{i=1}^{L} [T_k(t_{l_i}) - T_k(t_{l_{i-1}})]. \end{aligned}$$

Using $Q_k(t_L) = 0$ and decomposing the sums $\sum_{i=1}^{L}$ over $\sigma_r$ intervals we obtain

$$\begin{aligned} & \alpha_k \sum_{r=1}^{J} \tau_r + \sum_{r=1}^{J}\sum_{j=1}^{n} \mu_j p_{jk} \tau_{jr} \\ & \quad - \mu_k \sum_{r=1}^{J} \tau_{kr} = -Q_k(0), \qquad k = 1, \cdots, n. \end{aligned}$$

Using as variables the quantities $\tau_r$ and $\tau_{jr}$ and arguing exactly as in Proposition 2, we obtain the following upper bound on the duration of the strong busy period.

*Proposition 4:* Consider a stable work-conserving policy starting with initial condition $Q(0) \ne 0$. Let $\tau$ be the smallest time such that $Q(\tau) = 0$. Then, $\tau$ is bounded above by the optimal value of the following linear program to be called $G[Q(0)]$:

maximize

$$\sum_{r=1}^{J} \tau_r$$

subject to

$$\alpha_k \tau_r + \sum_{j=1}^{n} \mu_j p_{jk} \tau_{jr} - \mu_k \tau_{kr} \le 0, \qquad \forall\, k \in \sigma_r,$$

$$r = 1, \cdots, J \quad (35)$$

$$\sum_{k\in\sigma_r} \tau_{kr} = \tau_r, \qquad r = 1, \cdots, J \quad (36)$$

$$\sum_{k\in\sigma_j} \tau_{kr} \le \tau_r, \qquad j \ne r \quad (37)$$

$$\begin{aligned} & \alpha_k \sum_{r=1}^{J} \tau_r + \sum_{r=1}^{J}\sum_{j=1}^{n} \mu_j p_{jk} \tau_{jr} \\ & \quad - \mu_k \sum_{r=1}^{J} \tau_{kr} = -\,Q_k(0), \\ & \qquad k = 1, \cdots, n \\ & \qquad \tau_r,\, \tau_{jr} \ge 0. \end{aligned} \quad (38)$$

We conclude this section by stating the sufficient conditions for stability.

*Theorem 10—Sufficient Conditions for Stability:* Suppose that the load condition $\rho < e$ holds. Consider the linear program $G[0]$ obtained by setting $Q(0) = 0$ in $G[Q(0)]$. If $G[0]$ has zero as the only feasible solution, then the multiclass network $(\alpha, \mu, P, C)$ is stable for all work-conserving policies.

*Proof:* The argument is identical with the proof of Theorem 1. □

Since the variables $\tau_r$ can be eliminated using (36), the proposed test for stability involves only $nJ$ variables and $2n + J(J-1)$ constraints, which is efficiently solvable. The linear program $G[0]$ is the direct generalization of the linear program $DLP[2]$ in Lemma for two stations, where we have subtracted (35) from (38).

## VI. Conclusions

For two-station multiclass fluid networks we have established necessary and sufficient conditions for stability of all work-conserving policies. We have also proved that piecewise linear Lyapunov functions establish stability sharply.

For networks with more than two stations, we have established sufficient conditions for stability and we conjecture that they are also necessary. Given that in terms of stability the equivalence of fluid and stochastic networks is not fully proven (although highly suspected), our results do not yet imply necessary and sufficient conditions for stochastic networks as well.

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