2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Stability Criteria for Linear Discrete-Time Systems with Interval-Like Time-Varying Delay

Xiefu Jiang, Qing-Long Han and Xinghuo Yu

Abstract— **This paper is concerned with the stability problem for a class of uncertain linear discrete-time systems with time-varying delay. The delay is of an interval-like type, which means that both lower and upper bounds for the time-varying delay are available. The uncertainty under consideration is norm-bounded uncertainty. Based on Lyapunov-Krasovskii functional approach, delay-dependent stability criteria are obtained using a sum inequality which is first introduced and plays an important role in deriving stability conditions. The criteria are formulated in the form of linear matrix inequalities (LMIs). A numerical example is given to show the effectiveness of the proposed criteria.**

I. INTRODUCTION

During the last two decades the stability problem of linear continuous-time systems with time-delay has received considerable attention. The practical examples of time-delay systems include engineering, communications and biological systems [5]. The existence of delay in a practical system may induce instability, oscillation and poor performance [8]. For recent achievements, see [3] and reference therein. Compared with linear continuous-time systems with time-delay, less attention has been paid to linear *discrete-time* systems with time-delay. The reason is that for linear discrete-time systems with *constant* time-delay, one can transform them into the delay-free systems via state augmentation approach. However, the augmentation approach can not be applied to linear discrete-time systems with *time-varying* delay.

Similar to the case of linear continuous-time systems with time-delay [3], stability criteria for linear discrete-time systems with time-delay can be classified into two types: delay-independent stability criteria [9], [12] and delay-dependent stability criteria [6], [7]. In general, the delay-dependent stability criteria can provide some less conservative results than delay-independent stability criteria. Therefore, in the recent years, the delay-dependent stability problem of linear discrete-time systems with time-delay, especially with time-varying delay, has attracted some researchers' interest.

For linear continuous-time systems, it is well known that there are some systems which are stable with some *nonzero* delay, but are unstable without delay [1], [2]. For such case, if there is a time-varying perturbation on the nonzero delay, it is of great significance to consider the stability of systems with *interval time-varying* delay. The stability of such kinds of systems was investigated in [4] using the Lyapunov-Krasovskii approach.

In this paper, we will consider the stability problem for a class of linear discrete-time delay systems with **interval-like** time-varying delay, the discrete analogues of linear continuous-time systems with *interval time-varying* delay. The linear discrete-time delay systems with **interval-like** time-varying delay appear in the field of networked control systems [10], [11]. Based on Lyapunov-Krasovskii functional approach, delay-dependent stability criteria will be derived by using a sum inequality which will be first estabilished. A numerical example will be given to show the effectiveness of the criteria.

Notation: For symmetric matrices X and Y, the notation $X > Y$ ($X \geq Y$) means that matrix $X - Y$ is positive definite (positive semi-definite). I is an identity matrix of appropriate dimensions. Matrices, if not explicitly stated, are assumed to have compatible dimensions. For any real matrix A, A^T denotes the transpose of matrix A. For any nonsingular matrix A, A^{-1} denotes the inverse of matrix A. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{R}^{m\times n}$ is the set of all $m\times n$ matrices. $W^{\frac{1}{2}}$ denotes the square root of symmetric positive semi-definite matrix $W \geq 0$ ($W^{\frac{1}{2}} = V\Lambda^{\frac{1}{2}}V^T$ with V the eigenvector matrix of W satisfying $VV^T = I$ and Λ the diagonal eigenvalues matrix of W).

II. PROBLEM STATEMENT

Consider the following linear discrete-time system with time-varying delay

$$\begin{cases} x(k+1) = [A + \Delta A(k)]x(k) \\ \qquad\qquad +[A_1 + \Delta A_1(k)]x(k - h(k)), \\ x(k) = \phi(k), \ -h_M \leq k \leq 0, \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state, A and A_1 are known real parameter matrices of appropriate dimensions, $\Delta A(k)$ and $\Delta A_1(k)$ are real-valued unknown matrices representing discrete-time varying parameter uncertainties of (1), and are assumed to be of the form

$$\left[\begin{array}{cc} \Delta A(k) & \Delta A_1(k) \end{array}\right] = DF(k)\left[\begin{array}{cc} E & E_1 \end{array}\right], \tag{2}$$

The research work was partially supported by the Central Queensland University for the 2004 Research Advancement Awards Scheme Project "Analysis and synthesis of networked control systems", the Strategic Research Project "Delay effects: analysis, synthesis and applications" (2003-2006), the Seed Grant Scheme of the Research Development and Incentives Program "Robust stability and stabilization for systems with interval time-varying delay and its application to networked control systems"(RDI(S)05/12), and the Faculty of Informatics and Communication for the Small Grant Project "On stability for systems with interval time-varying delay arising in networked control systems".

Xiefu Jiang and Qing-Long Han are with the Faculty of Informatics and Communication, Central Queensland University, Rockhampton, QLD 4702, Australia. E-mail: q.han@cqu.edu.au

Xinghuo Yu is with the Science, Engineering and Technology Portfolio Office, Royal Melbourne Institute of Technology, Melbourne, Vic 3001, Australia.

where D, E and E_1 are known real constant matrices of appropriate dimensions. $F(k) \in \mathbb{R}^{\alpha\times\beta}$ is a discrete-time varying uncertainty matrix satisfying

$$F^T(k)F(k) \leq I, \tag{3}$$

$\phi(k)$ is the initial condition of the system (1). $h(k)$ is a positive integer function representing the time-varying delay of the system (1) satisfying

$$0 < h_m \leq h(k) \leq h_M, \tag{4}$$

where h_m and h_M are two known positive integers, for this case, the $h(k)$ is called an **interval-like** time-varying delay.

The purpose of this paper is to develop delay-dependent robust stability criteria for the system (1) for any **interval-like** time-varying delay $h(k)$ satisfying (4).

To end this section, we introduce the following lemma that will play an important role in deriving the stability criteria.

Lemma 1: For any constant positive semi-definite symmetric matrix $W \in \mathbb{R}^{n\times n}$, $W^T = W \geq 0$, two positive integers r and r_0 satisfying $r \geq r_0 \geq 1$, the following inequality holds

$$\left(\sum_{i=r_0}^{r} x(i)\right)^T W \left(\sum_{i=r_0}^{r} x(i)\right) \leq \tilde{r}\sum_{i=r_0}^{r} x^T(i)Wx(i).$$

where $\tilde{r} = r - r_0 + 1$.

Proof: It is easy to see that

$$\begin{aligned}
&\left(\sum_{i=r_0}^{r} x(i)\right)^T W \left(\sum_{i=r_0}^{r} x(i)\right)\\
=\ & \frac{1}{2}\sum_{i=r_0}^{r}\sum_{j=r_0}^{r} 2x^T(i)Wx(j)\\
=\ & \frac{1}{2}\sum_{i=r_0}^{r}\sum_{j=r_0}^{r} 2\left(W^{\frac{1}{2}}x(i)\right)^T\left(W^{\frac{1}{2}}x(j)\right)\\
\leq\ & \frac{1}{2}\sum_{i=r_0}^{r}\sum_{j=r_0}^{r}\left(x^T(i)Wx(i) + x^T(j)Wx(j)\right)\\
=\ & (r - r_0 + 1)\sum_{i=r_0}^{r} x^T(i)Wx(i).
\end{aligned}$$

The proof is complete. ■

III. MAIN RESULT

Defining $h_{av} = \begin{cases} \frac{1}{2}(h_M + h_m), \\ \quad \text{if } h_M + h_m \text{ is an even integer} \\ \frac{1}{2}(h_M + h_m - 1), \\ \quad \text{if } h_M + h_m \text{ is an odd integer} \end{cases}$

and $\delta = \max\{h_{av} - h_m, h_M - h_{av}\}$, then $h(k)$ is a discrete-time time-varying sequence satisfying $h_{av} - \delta \leq h(k) \leq h_{av} + \delta$, where δ can be taken as the range of variation of the time delay $h(k)$.

In this section, employing the sum inequality in Lemma 1, a delay-dependent stability criterion in terms of an LMI form is first presented for the following nominal system with interval-like time-varying delay $h(k)$ satisfying (4).

$$\begin{cases} x(k+1) = Ax(k) + A_1x(k - h(k)), \\ x(k) = \phi(k), \ -h_M \leq k \leq 0. \end{cases} \tag{5}$$

Proposition 1: For some given positive integers h_m and h_M, the system (5) is asymptotically stable for any $h(k)$ satisfying (4), if there exist some matrices $P > 0$, $Q > 0$, $R > 0$ and $S > 0$ of appropriate dimensions such that the following LMI holds

$$\Xi \triangleq \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{12}^T & \Xi_{22} \end{bmatrix} < 0, \tag{6}$$

where

$$\begin{aligned}
\Xi_{11} &= \begin{bmatrix} -P + Q - R & R & 0 \\ R & -Q - R & 0 \\ 0 & 0 & -\delta S \end{bmatrix},\\
\Xi_{12} &= \begin{bmatrix} A^TP & h_{av}(A - I)^TR & \tilde{\delta}(A - I)^TS \\ A_1^TP & h_{av}A_1^TR & \tilde{\delta}A_1^TS \\ \delta A_1^TP & \delta h_{av}A_1^TR & \tilde{\delta}\delta A_1^TS \end{bmatrix},\\
\Xi_{22} &= diag\{\ -P \quad -R \quad -\tilde{\delta}S\ \},\\
\tilde{\delta} &= 2\delta + 1.
\end{aligned}$$

Proof: Choose a Lyapunov-Krasovskii functional candidate as follows

$$V(k) \triangleq V_1(k) + V_2(k) + V_3(k) + V_4(k), \tag{7}$$

where

$$\begin{aligned}
V_1(k) &= x^T(k)Px(k),\\
V_2(k) &= \sum_{i=k-h_{av}}^{k-1} x^T(i)Qx(i),\\
V_3(k) &= h_{av}\sum_{i=1}^{h_{av}}\sum_{j=k-i}^{k-1} e^T(j)Re(j),\\
V_4(k) &= \sum_{i=h_{av}-\delta}^{h_{av}+\delta}\sum_{j=k-i}^{k-1} e^T(j)Se(j),\\
e(j) &= x(j) - x(j+1),
\end{aligned}$$

where $P > 0$, $Q > 0$, $R > 0$ and $S > 0$. It is easy to see that the system (5) can be rewritten as

$$\begin{aligned}
&x(k+1) = Ax(k) + A_1x(k - h_{av})\\
&+A_1[x(k - h(k)) - x(k - h_{av})]\\
&= \begin{cases} Ax(k) + A_1x(k - h_{av}) \\ +A_1\sum\limits_{i=h_{av}+1}^{h(k)} e(k - i), & \text{if } h(k) > h_{av} \\ Ax(k) + A_1x(k - h_{av}), & \text{if } h(k) = h_{av} \\ Ax(k) + A_1x(k - h_{av}) \\ -A_1\sum\limits_{i=h(k)}^{h_{av}-1} e(k - i - 1). & \text{if } h(k) < h_{av} \end{cases}
\end{aligned} \tag{8}$$

Case I: $h(k) > h_{av}$.

Taking the difference of $V_1(k)$, the increment of $V_1(k)$ is

$$\begin{aligned}\Delta V_1(k) &= V_1(k+1)-V_1(k)\\ &= x^T(k)A^TPAx(k)\\ &\quad +2x^T(k)A^TPA_1x(k-h_{av})\\ &\quad +2x^T(k)A^TPA_1\sum_{i=h_{av}+1}^{h(k)}e(k-i)\\ &\quad +x^T(k-h_{av})A_1^TPA_1x(k-h_{av})\\ &\quad +2x^T(k-h_{av})A_1^TPA_1\sum_{i=h_{av}+1}^{h(k)}e(k-i)\\ &\quad +\left(\sum_{i=h_{av}+1}^{h(k)}e(k-i)\right)^T A_1^TPA_1\times\\ &\quad \left(\sum_{i=h_{av}+1}^{h(k)}e(k-i)\right)-x^T(k)Px(k). \end{aligned}\tag{9}$$

The increment of $V_2(k)$ is easily computed as

$$\begin{aligned}&\Delta V_2(k)=V_2(k+1)-V_2(k)\\ &=x^T(k)Qx(k)-x^T(k-h_{av})Qx(k-h_{av}).\end{aligned}\tag{10}$$

The increment of $V_3(k)$ is

$$\begin{aligned}\Delta V_3(k) &= V_3(k+1)-V_3(k)\\ &= h_{av}^2e^T(k)Re(k)\\ &\quad -h_{av}\sum_{i=1}^{h_{av}}e^T(k-i)Re(k-i)\end{aligned}\tag{11}$$

The increment of $V_4(k)$ is

$$\begin{aligned}\Delta V_4(k) &= V_4(k+1)-V_4(k)\\ &= (2\delta+1)e^T(k)Se(k)\\ &\quad -\sum_{i=h_{av}-\delta}^{h_{av}+\delta}e^T(k-i)Se(k-i).\end{aligned}\tag{12}$$

From (9)-(12) we have

$$\begin{aligned}\Delta V(k) &= V(k+1)-V(k)\\ &= x^T(k)A^TPAx(k)\\ &\quad +2x^T(k)A^TPA_1x(k-h_{av})\\ &\quad +2x^T(k)A^TPA_1\sum_{i=h_{av}+1}^{h(k)}e(k-i)\\ &\quad +x^T(k-h_{av})A_1^TPA_1x(k-h_{av})\\ &\quad +2x^T(k-h_{av})A_1^TPA_1\sum_{i=h_{av}+1}^{h(k)}e(k-i)\\ &\quad +\left(\sum_{i=h_{av}+1}^{h(k)}e(k-i)\right)^T A_1^TPA_1\times\\ &\quad \left(\sum_{i=h_{av}+1}^{h(k)}e(k-i)\right)-x^T(k)Px(k)\\ &\quad +x^T(k)Qx(k)-x^T(k-h_{av})Qx(k-h_{av})\\ &\quad +e^T(k)\left(h_{av}^2R+(2\delta+1)S\right)e(k)\\ &\quad -h_{av}\sum_{i=1}^{h_{av}}e^T(k-i)Re(k-i)\\ &\quad -\sum_{i=h_{av}-\delta}^{h_{av}+\delta}e^T(k-i)Se(k-i).\end{aligned}\tag{13}$$

Use Lemma 1 to obtain

$$\begin{aligned}&\left(\textstyle\sum_{i=h_{av}+1}^{h(k)}e(k-i)\right)^T A_1^TPA_1\textstyle\sum_{i=h_{av}+1}^{h(k)}e(k-i)\\ &\le (h(k)-h_{av})\textstyle\sum_{i=h_{av}+1}^{h(k)}e^T(k-i)A_1^TPA_1e(k-i),\end{aligned}\tag{14}$$

and

$$\begin{aligned}&-h_{av}\textstyle\sum_{i=1}^{h_{av}}e^T(k-i)Re(k-i)\\ &\le -\left(\textstyle\sum_{i=1}^{h_{av}}e(k-i)\right)^T R\left(\textstyle\sum_{i=1}^{h_{av}}e(k-i)\right)\\ &= -[x(k-h_{av})-x(k)]^TR[x(k-h_{av})-x(k)].\end{aligned}\tag{15}$$

Noting that (8) we have

$$\begin{aligned}&e^T(k)\Upsilon e(k)\\ =\ &x^T(k)(A-I)^T\Upsilon(A-I)x(k)\\ &+2x^T(k)(A-I)^T\Upsilon A_1x(k-h_{av})\\ &+2x^T(k)(A-I)^T\Upsilon A_1\sum_{i=h_{av}+1}^{h(k)}e(k-i)\\ &+x^T(k-h_{av})A_1^T\Upsilon A_1x(k-h_{av})\\ &+2x^T(k-h_{av})A_1^T\Upsilon A_1\sum_{i=h_{av}+1}^{h(k)}e(k-i)\\ &+\left(\sum_{i=h_{av}+1}^{h(k)}e(k-i)\right)^T A_1^T\Upsilon A_1\sum_{i=h_{av}+1}^{h(k)}e(k-i),\end{aligned}$$

where $\Upsilon = h_{av}^2R+(2\delta+1)S$. Using Lemma 1 again in the last term of the right hand of the above equality yields

$$\begin{aligned}&e^T(k)\Upsilon e(k)\\ &\le x^T(k)(A-I)^T\Upsilon(A-I)x(k)\\ &+2x^T(k)(A-I)^T\Upsilon A_1x(k-h_{av})\\ &+2x^T(k)(A-I)^T\Upsilon A_1\textstyle\sum_{i=h_{av}+1}^{h(k)}e(k-i)\\ &+x^T(k-h_{av})A_1^T\Upsilon A_1x(k-h_{av})\\ &+2x^T(k-h_{av})A_1^T\Upsilon A_1\textstyle\sum_{i=h_{av}+1}^{h(k)}e(k-i)\\ &+(h(k)-h_{av})\textstyle\sum_{i=h_{av}+1}^{h(k)}e^T(k-i)A_1^T\Upsilon A_1e(k-i).\end{aligned}\tag{16}$$

In addition, it is easy to see that

$$
\begin{aligned}
& -\sum_{i=h_{av}-\delta}^{h_{av}+\delta} e^T(k-i)Se(k-i) \\
\leq\ & -\sum_{i=h_{av}+1}^{h(k)} e^T(k-i)Se(k-i). \qquad (17)
\end{aligned}
$$

Then from (13)—(17) we have

$$
\begin{aligned}
&\Delta V(k) = V(k+1) - V(k) \\
&\leq x^T(k)A^TPAx(k) + 2x^T(k)A^TPA_1x(k-h_{av}) \\
&+2x^T(k)A^TPA_1 \sum_{i=h_{av}+1}^{h(k)} e(k-i) \\
&+x^T(k-h_{av})A_1^TPA_1x(k-h_{av}) \\
&+2x^T(k-h_{av})A_1^TPA_1 \sum_{i=h_{av}+1}^{h(k)} e(k-i) \\
&+(h(k)-h_{av}) \sum_{i=h_{av}+1}^{h(k)} e^T(k-i)A_1^TPA_1e(k-i) \\
&-x^T(k)Px(k) + x^T(k)Qx(k) \\
&-x^T(k-h_{av})Qx(k-h_{av}) \\
&-[x(k-h_{av})-x(k)]^TR[x(k-h_{av})-x(k)] \\
&- \sum_{i=h_{av}+1}^{h(k)} e^T(k-i)Se(k-i) \\
&+x^T(k)(A-I)^T\Upsilon(A-I)x(k) \\
&+2x^T(k)(A-I)^T\Upsilon A_1x(k-h_{av}) \\
&+2x^T(k)(A-I)^T\Upsilon A_1 \sum_{i=h_{av}+1}^{h(k)} e(k-i) \\
&+x^T(k-h_{av})A_1^T\Upsilon A_1x(k-h_{av}) \\
&+2x^T(k-h_{av})A_1^T\Upsilon A_1 \sum_{i=h_{av}+1}^{h(k)} e(k-i) \\
&+(h(k)-h_{av}) \sum_{i=h_{av}+1}^{h(k)} e^T(k-i)A_1^T\Upsilon A_1e(k-i) \\
&= \frac{1}{h(k)-h_{av}} \sum_{i=h_{av}+1}^{h(k)} \xi^T(k,i)\Phi\xi(k,i), \qquad (18)
\end{aligned}
$$

where

$$
\begin{aligned}
\xi^T(k,i) &= \begin{bmatrix} x^T(k) & x^T(k-h_{av}) & e^T(k-i) \end{bmatrix}, \\
\Phi &= \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{12}^T & \Phi_{22} & \Phi_{23} \\ \Phi_{13}^T & \Phi_{23}^T & \Phi_{33} \end{bmatrix}, \\
\Phi_{11} &= A^TPA - P + Q - R \\
&\quad +(A-I)^T(h_{av}^2R + (2\delta+1)S)(A-I), \\
\Phi_{12} &= A^TPA_1 + R \\
&\quad +(A-I)^T(h_{av}^2R + (2\delta+1)S)A_1, \\
\Phi_{13} &= (h(k)-h_{av})[A^TPA_1 \\
&\quad +(A-I)^T(h_{av}^2R + (2\delta+1)S)A_1], \\
\Phi_{22} &= A_1^TPA_1 - Q - R \\
&\quad +A_1^T(h_{av}^2R + (2\delta+1)S)A_1, \\
\Phi_{23} &= (h(k)-h_{av})[A_1^TPA_1 \\
&\quad +A_1^T(h_{av}^2R + (2\delta+1)S)A_1], \\
\Phi_{33} &= (h(k)-h_{av})^2[A_1^TPA_1 + A_1^T(h_{av}^2R \\
&\quad +(2\delta+1)S)A_1] - (h(k)-h_{av})S.
\end{aligned}
$$

Case II: $h(k) = h_{av}$.

For this case it is easy to get

$$
\begin{aligned}
\Delta V(k) &= V(k+1) - V(k) \\
&\leq x^T(k)A^TPAx(k) \\
&\quad +2x^T(k)A^TPA_1x(k-h_{av}) \\
&\quad +x^T(k-h_{av})A_1^TPA_1x(k-h_{av}) \\
&\quad -x^T(k)Px(k) + x^T(k)Qx(k) \\
&\quad -x^T(k-h_{av})Qx(k-h_{av}) \\
&\quad -[x(k-h_{av})-x(k)]^TR[x(k-h_{av}) \\
&\quad -x(k)] + x^T(k)(A-I)^T\Upsilon(A-I)x(k) \\
&\quad +2x^T(k)(A-I)^T\Upsilon A_1x(k-h_{av}) \\
&\quad +x^T(k-h_{av})A_1^T\Upsilon A_1x(k-h_{av}) \\
&= \eta^T(k)\hat{\Phi}\eta(k), \qquad (19)
\end{aligned}
$$

where

$$
\begin{aligned}
\eta^T(k) &= \begin{bmatrix} x^T(k) & x^T(k-h_{av}) \end{bmatrix}, \\
\hat{\Phi} &= \begin{bmatrix} \hat{\Phi}_{11} & \hat{\Phi}_{12} \\ \hat{\Phi}_{12}^T & \hat{\Phi}_{22} \end{bmatrix}, \\
\hat{\Phi}_{11} &= A^TPA - P + Q - R \\
&\quad +(A-I)^T(h_{av}^2R + (2\delta+1)S)(A-I), \\
\hat{\Phi}_{12} &= A^TPA_1 + R \\
&\quad +(A-I)^T(h_{av}^2R + (2\delta+1)S)A_1, \\
\hat{\Phi}_{22} &= A_1^TPA_1 - Q - R \\
&\quad +A_1^T(h_{av}^2R + (2\delta+1)S)A_1.
\end{aligned}
$$

Case III: $h(k) < h_{av}$.

Similar to Case I, we have

$$
\begin{aligned}
&\Delta V(k) = V(k+1) - V(k) \\
&\leq x^T(k)A^TPAx(k) + 2x^T(k)A^TPA_1x(k-h_{av}) \\
&-2x^T(k)A^TPA_1 \sum_{i=h(k)}^{h_{av}-1} e(k-i-1) \\
&+x^T(k-h_{av})A_1^TPA_1x(k-h_{av}) \\
&-2x^T(k-h_{av})A_1^TPA_1 \sum_{i=h(k)}^{h_{av}-1} e(k-i-1) \\
&+(h_{av}-h(k)) \sum_{i=h(k)}^{h_{av}-1} e^T(k-i-1)]A_1^TPA_1e(k-i-1) \\
&-x^T(k)Px(k) + x^T(k)Qx(k) \\
&-x^T(k-h_{av})Qx(k-h_{av}) \\
&-[x(k-h_{av})-x(k)]^TR[x(k-h_{av})-x(k)] \\
&- \sum_{i=h(k)}^{h_{av}-1} e^T(k-i-1)S(k-i-1) \\
&+x^T(k)(A-I)^T\Upsilon(A-I)x(k) \\
&+2x^T(k)(A-I)^T\Upsilon A_1x(k-h_{av}) \\
&-2x^T(k)(A-I)^T\Upsilon A_1 \sum_{i=h(k)}^{h_{av}-1} e(k-i-1) \\
&+x^T(k-h_{av})A_1^T\Upsilon A_1x(k-h_{av})
\end{aligned}
$$

$$-2x^T(k-h_{av})A_1^T\Upsilon A_1\sum_{i=h(k)}^{h_{av}-1}e(k-i-1)$$
$$+(h_{av}-h(k))\sum_{i=h(k)}^{h_{av}-1}e^T(k-i-1)A_1^T\Upsilon A_1 e(k-i-1)$$
$$=\frac{1}{h_{av}-h(k)}\sum_{i=h(k)}^{h_{av}-1}\zeta^T(k,i)\tilde{\Phi}\zeta(k,i), \quad (20)$$

where

$$\begin{aligned}
\zeta^T(k,i) &= \begin{bmatrix} x(k) \\ x(k-h_{av}) \\ [x(k-i)-x(k-i-1)] \end{bmatrix}, \\
\tilde{\Phi} &= \begin{bmatrix} \tilde{\Phi}_{11} & \tilde{\Phi}_{12} & \tilde{\Phi}_{13} \\ \tilde{\Phi}_{12}^T & \tilde{\Phi}_{22} & \tilde{\Phi}_{23} \\ \tilde{\Phi}_{13}^T & \tilde{\Phi}_{23}^T & \tilde{\Phi}_{33} \end{bmatrix}, \\
\tilde{\Phi}_{11} &= A^TPA-P+Q-R \\
&\quad +(A-I)^T(h_{av}^2R+(2\delta+1)S)(A-I), \\
\tilde{\Phi}_{12} &= A^TPA_1+R \\
&\quad +(A-I)^T(h_{av}^2R+(2\delta+1)S)A_1, \\
\tilde{\Phi}_{13} &= (h_{av}-h(k))[A^TPA_1 \\
&\quad +(A-I)^T(h_{av}^2R+(2\delta+1)S)A_1], \\
\tilde{\Phi}_{22} &= A_1^TPA_1-Q-R \\
&\quad +A_1^T(h_{av}^2R+(2\delta+1)S)A_1, \\
\tilde{\Phi}_{23} &= (h_{av}-h(k))[A_1^TPA_1 \\
&\quad +A_1^T(h_{av}^2R+(2\delta+1)S)A_1], \\
\tilde{\Phi}_{33} &= (h_{av}-h(k))^2[A_1^TPA_1+A_1^T(h_{av}^2R \\
&\quad +(2\delta+1)S)A_1]-(h_{av}-h(k))S.
\end{aligned}$$

Summarizing the above discussions, from (18), (19) and (20) we obtain

$$\Delta V(k)=\begin{cases} \frac{1}{h(k)-h_{av}}\sum_{i=h_{av}+1}^{h(k)}\xi^T(k,i)\Phi\xi(k,i), & \text{if } h(k)>h_{av} \\ \eta^T(k)\hat{\Phi}\eta(k), & \text{if } h(k)=h_{av} \\ \frac{1}{h_{av}-h(k)}\sum_{i=h(k)}^{h_{av}-1}\zeta^T(k,i)\tilde{\Phi}\zeta(k,i). & \text{if } h(k)<h_{av} \end{cases} \quad (21)$$

Noting that $|h(k)-h_{av}|\le\delta$ for $k=1,2,3,\cdots$, $\Phi<0$, $\hat{\Phi}<0$ and $\tilde{\Phi}<0$ are implied by

$$\Theta=\begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} \\ \Theta_{12}^T & \Theta_{22} & \Theta_{23} \\ \Theta_{13}^T & \Theta_{23}^T & \Theta_{33} \end{bmatrix}<0, \quad (22)$$

where

$$\begin{aligned}
\Theta_{11} &= A^TPA-P+Q-R \\
&\quad +(A-I)^T(h_{av}^2R+(2\delta+1)S)(A-I), \\
\Theta_{12} &= A^TPA_1+R \\
&\quad +(A-I)^T(h_{av}^2R+(2\delta+1)S)A_1, \\
\Theta_{13} &= \delta[A^TPA_1 \\
&\quad +(A-I)^T(h_{av}^2R+(2\delta+1)S)A_1], \\
\Theta_{22} &= A_1^TPA_1-Q-R \\
&\quad +A_1^T(h_{av}^2R+(2\delta+1)S)A_1, \\
\Theta_{23} &= \delta[A_1^TPA_1+A_1^T(h_{av}^2R+(2\delta+1)S)A_1], \\
\Theta_{33} &= \delta^2[A_1^TPA_1+A_1^T(h_{av}^2R \\
&\quad +(2\delta+1)S)A_1]-\delta S.
\end{aligned}$$

So, if (22) holds, then $\Delta V(k)\le-\lambda x^T(k)x(k)$ for some scalar $\lambda>0$. Therefore, the system (5) is asymptotically stable. By Schur complement, (22) is equivalent to (6). The proof is complete. ■

Concerning the norm-bounded uncertainty, by Proposition 1, the following corollary is easily obtained for system (1).

Corollary 1: For some given positive integers h_m and h_M, the system (1) is robustly stable for any time-delay $h(k)$ satisfying (4) and all admissible parameter uncertainties satisfying (2) and (3), if there exist a scalar $\varepsilon>0$, some matrices $P>0$, $Q>0$, $R>0$ and $S>0$ of appropriate dimensions such that the following LMI holds

$$\begin{bmatrix} \alpha_1 & \alpha_2^T \\ \alpha_2 & \alpha_3 \end{bmatrix}<0. \quad (23)$$

where

$$\begin{aligned}
\alpha_1 &= \begin{bmatrix} -P+Q-R & R & 0 \\ R & -Q-R & 0 \\ 0 & 0 & -\delta S \end{bmatrix}, \\
\alpha_2 &= \begin{bmatrix} PA & PA_1 & \delta PA_1 \\ h_{av}R(A-I) & h_{av}RA_1 & \delta h_{av}RA_1 \\ \tilde{\delta}S(A-I) & \tilde{\delta}SA_1 & \tilde{\delta}\delta SA_1 \\ 0 & 0 & 0 \\ \varepsilon E & \varepsilon E_1 & \varepsilon\delta E_1 \end{bmatrix}, \\
\alpha_3 &= \begin{bmatrix} -P & 0 & 0 & PD & 0 \\ 0 & -R & 0 & h_{av}RD & 0 \\ 0 & 0 & -\tilde{\delta}S & \tilde{\delta}SD & 0 \\ D^TP & h_{av}D^TR & \tilde{\delta}D^TS & -\varepsilon I & 0 \\ 0 & 0 & 0 & 0 & -\varepsilon I \end{bmatrix} \\
\tilde{\delta} &= 2\delta+1.
\end{aligned}$$

Proof: Replace A and A_1 with $A+DF(k)E$ and $A_1+DF(k)E_1$in (6), respectively, to obtain

$$\Xi+\vartheta_1F(k)\vartheta_2^T+\vartheta_2F^T(k)\vartheta_1^T<0, \quad (24)$$

where Ξ is defined in Proposition 1 and

$$\vartheta_1^T=\begin{bmatrix} 0 & 0 & 0 & D^TP & h_{av}D^TR^T & (2\delta+1)D^TS^T \end{bmatrix}$$
$$\vartheta_2=\begin{bmatrix} E & E_1 & \delta E_1 & 0 & 0 & 0 \end{bmatrix}$$

It is clear to see that (24) is equivalent to

$$\Xi+\varepsilon^{-1}\vartheta_1\vartheta_1^T+\varepsilon\vartheta_2\vartheta_2^T<0 \quad (25)$$

for a scalar $\varepsilon>0$. By Schur complement and (6), (25) is equivalent to (23). This completes the proof. ■

Remark 1: The scalar $\varepsilon>0$ in (23) can be absorbed by other variables by introducing $\tilde{P}=\varepsilon^{-1}P$, $\tilde{Q}=\varepsilon^{-1}Q$, $\tilde{R}=\varepsilon^{-1}R$ and $\tilde{S}=\varepsilon^{-1}S$.

Remark 2: Proposition 1 and Corollary 1 provide the delay-dependent stability conditions which are formulated in an LMI form. Hence, it is easy to compute the maximum bound of the allowable length δ of the interval-like of time-varying delay for given h_{av} or the maximum bound of h_{av} for given δ using efficient convex optimization algorithms.

Remark 3: Based on the obtained stability criteria, one can easily handle the synthesis problem for uncertain linear discrete-time systems with interval-like time-varying delay.

IV. A NUMERICAL EXAMPLE

To show the effectiveness of the proposed delay-dependent stability criteria, consider the system described by (1), (2) and (3) with

$$\begin{aligned} A &= \begin{bmatrix} 0.8 & 0 \\ 0 & 0.91 \end{bmatrix}, A_1 = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}, \\ E &= \begin{bmatrix} 0.02 & 0 \\ 0 & 0.01 \end{bmatrix}, E_1 = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, \\ D &= \alpha I, \alpha \geq 0. \end{aligned}$$

Case I $h_m = h_M = h(k) = h$ is a constant.

For $\alpha = 0$, it is pointed out in [7] that any delay-independent stability criteria fail to verify the stability of the system. Using Proposition 1, the maximum allowed bound h is obtained as $h_m = h_M = 42$ which is less conservative than the one in [7].

For $\alpha > 0$, we consider the effect of the uncertainty bound α on the maximum allowed bound h for robust stability. Numerical results are listed in Table I using Proposition 2. It is clear to see that as α increases, h decreases.

Table. Bound h calculated for different α

α	0	0.5	1	1.5	2	2.5	3
h	42	35	29	25	21	18	16

Case II $h(k)$ is a time-varying delay.

For $\alpha = 0$, by Proposition 1 the considered system is asymptotically stable for $h(k)$ satisfying $7 \leq h(k) \leq 13$, while for $\alpha = 1$, applying Proposition 2 one can guarantee that the system is robustly stable for $h(k)$ satisfying $8 \leq h(k) \leq 12$.

V. CONCLUSION

This paper has proposed stability criteria for a class of uncertain linear discrete-time systems with interval-like time-varying delay. A sum inequality has been established and employed to derive the criteria which are dependent on the lower and upper bounds of the time-varying delay. A numerical example has demonstrated the effectiveness of the criteria.

REFERENCES

[1] K. Gu, Discretization schemes for Lyapunov-Krasovskii functions in time-delay systems, *Kybernetica*, vol. 37, 2001, pp. 479-504.

[2] K. Gu, Q.-L. Han, A.C.J. Luo, and S.-I. Niculescu, Discretized Lyapunov functional for systems with distributed delay and piecewise constant coefficient, *International Journal of Control*, vol. 74, 2001, pp. 734-744.

[3] K. Gu, L. Kharitonov, and J. Chen, *Stability of time-delay systems*, Boston: Birkhauser, 2003.

[4] Q.-L. Han and K. Gu, Stability of linear systems with time-varying delay: A generalized discretized Lyapunov functional approach, *Asian Journal of Control*, vol. 3, 2001, pp. 170-180.

[5] J. K. Hale and S. M. Verduyn Lunel, Introduction to Functional Differential Equations, New York: Springer-Verlag, 1993.

[6] B. Lee and J.G. Lee, Delay-dependent stability criteria for discrete-time delay systems, *in Proceedings of the American Control Conference*, San Diego, California, USA, June 1999.

[7] Y.S. Lee and W.H. Kwon, Delay-dependent robust stabilization of uncertain discrete-time state-delayed systems,*in Preprints of the 15th IFAC World Congress*, Barcelona, Spain, July 2002.

[8] M. Malek-Zavarei, and M. Jamshidi, Time Delay Systems: Analysis, Optimization and Applications, *Amsterdam: North-Holland*, 1987.

[9] T. Mori, N. Fukuma, and M. Kuwakara, Delay-independent stability criteria for discrete-time systems,*IEEE Transactions on Automatic Control*, vol. 27, 1982, pp. 964-966.

[10] M.Y. Chow and Y. Tipsuwan, Network-based control systems: A tutorial, *in Proceedings of IECON'01: The 27th Annual Conference of the IEEE Industrial Electronics Society*, 2001, pp. 1593-1602.

[11] D. Yue, Q.-L. Han, and C. Peng, State feedback controller design of networked control systems, *IEEE Transactions on Circuits and Systems-II: Express Briefs*, vol. 51, 2004, pp. 640-644.

[12] J.W. Wu and K.-S. Hong, Delay-independent exponential stability criteria for time-varying discrete delay systems, *IEEE Transactions on Automatic Control*, vol. 39, 1994, pp. 811-814.