# STABILITY CRITERIA OF 3D INVISCID SHEARS

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**Abstract.** Recent numerical studies in the area of transition to turbulence discovered that the classical plane Couette flow, plane Poiseuille flow, and pipe Poiseuille flow share some universal 3D steady coherent structure in the form of a "streak-roll-critical layer". As the Reynolds number approaches infinity, the steady coherent structure approaches a 3D limiting shear of the form $(U(y,z), 0, 0)$ in velocity variables. All such 3D shears are steady states of the 3D Euler equations. This raises the importance of investigating the stability of such inviscid 3D shears in contrast to the classical Rayleigh theory of inviscid 2D shears. Several general criteria of stability for such inviscid 3D shears are derived. In the Appendix, an argument is given to show that a 2D limiting shear can only be the classical laminar shear.

**1. Introduction.** There has been a lot of continuing interest in searching for 3D steady solutions (or traveling wave solutions in a different frame) in plane Couette flow, plane Poiseuille flow, and pipe Poiseuille flow ([19], [18], [23], [12], [24], [22], [7], [4], [8]). There seems to be confirmation of their existence in experiments [8]. Recent numerical studies of [24], [23], [22] reveal that the so-called lower branch steady states in the plane Couette flow, plane Poiseuille flow, and pipe Poiseuille flow share some universal steady coherent structure in the form of a "streak-roll-critical layer". As the Reynolds number approaches infinity, the steady coherent structure approaches a 3D limiting shear of the form $(U(y,z), 0, 0)$ in velocity variables. All the 3D shears of this form are steady states of the 3D Euler equations. This raises two important questions: (1). What is the special property of the limiting shear? (2). What is the nature of stability of 3D inviscid shears in contrast to the classical Rayleigh theory of 2D inviscid shears? The first question was addressed in [16]. It turns out that the limiting shear satisfies a necessary condition: $\int \Delta U f(U) dy dz = 0$ for any function $f$. We shall address the second question in this study. We shall use the channel flow (plane Couette flow and plane Poiseuille flow) as the example. Some partial results are obtained here. Both questions, especially the second one, deserve much further studies.

As the Reynolds number decreases from infinity, the limiting 3D shear as a steady state deforms into the lower branch steady state, while the 3D shear itself undergoes a slow drifting toward the classical laminar shear. In fact, all the shears (3D and 2D) form a stable submanifold of the classical laminar shear. These shears can play a fundamental role in the transition to turbulence from the classical laminar shear [15].

**2. Inviscid channel flow.** The inviscid channel flow is governed by the 3D Euler equations

$$\partial_t u_i + u_j u_{i,j} = -p_{,i}, \quad u_{i,i} = 0, \tag{2.1}$$

where $(u_1, u_2, u_3)$ are the three components of the fluid velocity along the $(x, y, z)$ directions, and $p$ is the pressure. The boundary condition is the so-called slip condition

$$u_2(x, a, z) = 0, \quad u_2(x, b, z) = 0, \tag{2.2}$$

where $a < b$, and $u_i$ $(i = 1, 2, 3)$ are periodic in the $x$ and $z$ directions with periods $\ell_1$ and $\ell_3$.

We start with the steady shear solutions of the 3D Euler equations:

$$u_1 = U(y, z), \quad u_2 = 0, \quad u_3 = 0, \quad p = p_0 \text{ (a constant)},$$

where $U(y, z)$ is periodic in $z$ with period $\ell_3$. Linearize the 3D Euler equations with the notation

$$u_1 = U(y, z) + \left[ e^{ik(x-ct)} u(y, z) + \text{c.c.} \right], \quad u_2 = e^{ik(x-ct)} v(y, z) + \text{c.c.},$$

$$u_3 = e^{ik(x-ct)} w(y, z) + \text{c.c.}, \quad p \to p_0 + \left[ e^{ik(x-ct)} p(y, z) + \text{c.c.} \right],$$

where $k$ is a real constant and $c$ is a complex constant, we obtain the linearized 3D Euler equations

$$ik(U - c)u + vU_y + wU_z = -ikp, \tag{2.3}$$

$$ik(U - c)v = -p_y, \tag{2.4}$$

$$ik(U - c)w = -p_z, \tag{2.5}$$

$$iku + v_y + w_z = 0. \tag{2.6}$$

Two forms of simplified systems can be derived:

$$k^2(U - c)v = \partial_y \left[ (U - c)(v_y + w_z) - (U_y v + U_z w) \right], \tag{2.7}$$

$$k^2(U - c)w = \partial_z \left[ (U - c)(v_y + w_z) - (U_y v + U_z w) \right], \tag{2.8}$$

with boundary condition $v(a, z) = v(b, z) = 0$ and $v, w$ are periodic in $z$; also,

$$(U - c)^2 \nabla \cdot \left[ (U - c)^{-2} \nabla p \right] = k^2 p, \tag{2.9}$$

with boundary condition $\partial_y p(a, z) = \partial_y p(b, z) = 0$ and $p$ is periodic in $z$. We are not successful in utilizing the system (2.7)-(2.8). System (2.9) turns out to be fruitful. The first result that can be derived from system (2.9) is the Howard semicircle theorem.

THEOREM 2.1 ([9], [3]). The unstable eigenvalues (if they exist) lie inside the semicircle in the complex plane:

$$\left(c_r - \frac{M+m}{2}\right)^2 + c_i^2 \leq \left(\frac{M-m}{2}\right)^2, \ c = c_r + ic_i, \ c_i > 0,$$

where $M = \max_{y,z} U$ and $m = \min_{y,z} U$.

*Proof.* Multiplying (2.9) with $\bar{p}$, integrating by parts, and splitting into real and imaginary parts, we obtain that

$$\int_0^{\ell_3} \int_a^b UG dy dz = c_r \int_0^{\ell_3} \int_a^b G dy dz, \tag{2.10}$$

$$\int_0^{\ell_3} \int_a^b U^2 G dy dz = (c_r^2 + c_i^2) \int_0^{\ell_3} \int_a^b G dy dz, \tag{2.11}$$

where

$$G = |U - c|^{-4} \left[|\nabla p|^2 + k^2 |p|^2\right].$$

Let

$$M = \max_{y,z} U, \quad m = \min_{y,z} U.$$

Then

$$\int_0^{\ell_3} \int_a^b (U - m)(M - U) G dy dz \geq 0.$$

Expanding this inequality and utilizing (2.10)-(2.11), we arrive at the semicircle inequality in the theorem. $\square$

Our next goal is to find a counterpart of the Rayleigh criterion [2]. For that goal, we need to introduce the transform

$$\tilde{p} = (U - c)^{-1} p. \tag{2.12}$$

Then $\tilde{p}$ satisfies

$$\Delta \tilde{p} + \left[\frac{\Delta U}{U - c} - \frac{2\nabla U \cdot \nabla U}{(U - c)^2}\right] \tilde{p} = k^2 \tilde{p}, \tag{2.13}$$

with the boundary condition

$$\partial_y \tilde{p} + \frac{U_y}{U - c} \tilde{p} = 0, \ \text{at } y = a, b. \tag{2.14}$$

It turns out that we can only derive results when $U$ satisfies the constraint

$$U_y = 0, \ \text{at } y = a, b; \tag{2.15}$$

in this case, $\tilde{p}$ satisfies the simplified boundary condition

$$\partial_y \tilde{p} = 0, \ \text{at } y = a, b. \tag{2.16}$$

THEOREM 2.2. For $U(y, z)$ satisfying the constraint $U_y = 0$, at $y = a, b$, if $U(y, z)$ has an (inviscid) unstable eigenvalue, then

(1)
$$\nabla \cdot \left( \frac{1}{|U - c|^4} \nabla U \right) = 0,$$

at some point $(y_*, z_*)$ in the interior of the domain, for some $c$ ($c_i > 0$) in the semi-circle $\left( c_r - \frac{M+m}{2} \right)^2 + c_i^2 \leq \left( \frac{M-m}{2} \right)^2$, where $M = \max_{y,z} U$ and $m = \min_{y,z} U$;

(2)
$$2 \frac{|c|^2 - U^2}{|U - c|^2} |\nabla U|^2 + U \Delta U > 0,$$

at some point $(y_*, z_*)$ in the interior of the domain, for some $c$ ($c_i > 0$) in the semi-circle $\left( c_r - \frac{M+m}{2} \right)^2 + c_i^2 \leq \left( \frac{M-m}{2} \right)^2$, where $M = \max_{y,z} U$ and $m = \min_{y,z} U$.

*Proof.* Multiplying (2.13) with $\bar{\tilde{p}}$, integrating by parts, and splitting into real and imaginary parts, we obtain that

$$\int_0^{\ell_3} \int_a^b \left[ \frac{U - c_r}{|U - c|^2} \Delta U - 2 \frac{(U - c_r)^2 - c_i^2}{|U - c|^4} |\nabla U|^2 \right] |\tilde{p}|^2 dy dz > 0, \qquad (2.17)$$

$$\int_0^{\ell_3} \int_a^b \left[ \frac{1}{|U - c|^2} \Delta U - 4 \frac{U - c_r}{|U - c|^4} |\nabla U|^2 \right] |\tilde{p}|^2 dy dz = 0. \qquad (2.18)$$

Equation (2.18) directly implies the first claim in the theorem. The second claim is along the spirit of the Fjortoft theorem [2]. Multiplying (2.18) by $c_r$ and adding (2.17), we obtain the second claim. $\qquad \square$

Next we will derive a relation between $c_i$ and $k$.

THEOREM 2.3. *For $U(y, z)$ satisfying the constraint $U_y = 0$ at $y = a, b$, let $A = \max_{y,z} |\nabla U|$, $B = \max_{y,z} |\Delta U|$. Then the unstable eigenvalue (if it exists) and the wave number $k$ satisfy the condition*

$$(kc_i)^2 \leq 2A^2 + Bc_i \quad (c_i > 0),$$

$$\left[ \text{equivalently,} \ \left( kc_i - \frac{B}{2k} \right)^2 \leq 2A^2 + \left( \frac{B}{2k} \right)^2 \right].$$

*Proof.* Multiplying (2.13) with $\bar{\tilde{p}}$ and integrating by parts, we obtain that

$$\int_0^{\ell_3} \int_a^b \left[ \frac{\Delta U}{U - c} - 2 \frac{|\nabla U|^2}{(U - c)^2} \right] |\tilde{p}|^2 dy dz = \int_0^{\ell_3} \int_a^b \left[ |\nabla \tilde{p}|^2 + k^2 |\tilde{p}|^2 \right] dy dz. \qquad (2.19)$$

The left-hand side of (2.19) is less than or equal to

$$\left( \frac{B}{|c_i|} + 2 \frac{A^2}{|c_i|^2} \right) \int_0^{\ell_3} \int_a^b |\tilde{p}|^2 dy dz.$$

The right-hand side of (2.19) is greater than or equal to

$$k^2 \int_0^{\ell_3} \int_a^b |\tilde{p}|^2 dy dz.$$

Thus

$$\frac{B}{c_i} + 2 \frac{A^2}{c_i^2} \geq k^2 \quad (\text{when } c_i > 0),$$

which leads to the claim of the theorem.                                     □

It is obvious that Theorems 2.2 and 2.3 apply to the 2D shears $U(y)$ too. Theorem 2.2 is not the exact 3D counterpart of the 2D Rayleigh criterion. The exact counterpart seems elusive. Next we will derive a variation formula for the unstable eigenvalue. This type of formula was initially derived by Tollmien [21], [17] for 2D shears. They are useful in deriving unstable eigenvalues near neutral eigenvalues. For 3D shears in atmosphere problems [1], specific approximations can make the stability problem very similar to the 2D Rayleigh problem. In such a case, a similar variation formula can also be derived to predict unstable eigenvalues near neutral eigenvalues [1]. In our current case, no approximation can be made, and we have a much harder problem. We have to work with the pressure variable of which the singularity nature is not clear even for 2D shears. We can derive a formula near an unstable eigenvalue, but its limit to a neutral eigenvalue is elusive and finding a neutral eigenvalue here is more challenging than finding an unstable eigenvalue in contrast to the 2D shear problem. Let $(p, c, k)$ and $(p_1, c_1, k_1)$ be two unstable eigenfunctions (if they exist) to (2.9). Then

$$\nabla \cdot \left[ (U - c)^{-2} \nabla p \right] = k^2 (U - c)^{-2} p, \tag{2.20}$$

$$\nabla \cdot \left[ (U - c_1)^{-2} \nabla p_1 \right] = k_1^2 (U - c_1)^{-2} p_1. \tag{2.21}$$

Multiplying (2.20) by $p_1$ and (2.21) by $p$, integrating and subtracting, we get

$$(c_1 - c) \int_0^{\ell_3} \int_a^b \frac{c_1 + c - 2U}{(U - c_1)^2 (U - c)^2} \left[ \nabla p \cdot \nabla p_1 + k^2 p p_1 \right] dy dz$$

$$= (k_1 - k)(k_1 + k) \int_0^{\ell_3} \int_a^b \frac{p p_1}{(U - c_1)^2} dy dz,$$

from which we obtain the variational formula

$$\frac{dc}{dk} = -k \int_0^{\ell_3} \int_a^b \frac{p^2}{(U - c)^2} dy dz$$

$$\times \left[ \int_0^{\ell_3} \int_a^b \frac{1}{(U - c)^3} \left[ \nabla p \cdot \nabla p + k^2 p^2 \right] dy dz \right]^{-1}. \tag{2.22}$$

The merit of this formula is that it does not involve $dp$. This formula is valid at an unstable eigenvalue ($c = c_r + i c_i$, $c_i > 0$). If the unstable eigenvalue lies on a curve $c = c(k)$ that leads to a neutral eigenvalue $c^0 = c(k^0)$, $c_i^0 = 0$ as in the case of a 2D shear, then by the semicircle theorem 2.1, $c^0 = U(y_0, z_0)$ for some $(y_0, z_0)$. In such a case, the limit $k \to k^0$ of (2.22) is still very attractive even though finding the neutral eigenvalue $c^0$ here is more challenging than finding an unstable eigenvalue in contrast to the case of 2D shears. On the other hand, the limit seems very singular (even for 2D shears). First of all, in the limit $k \to k^0$, equation (2.9) is singular, so the limiting eigenfunction $p$ will be singular too. The following simple equation

$$t^2 q'' + \alpha t q' + \beta q = 0 \quad (\alpha, \beta \text{ constants})$$

already shows a variety of singular solutions near $t = 0$. Moreover, the singularities generated by $(U - c^0)$ in the integrals in (2.22) add to the challenge.

Similarly, starting from (2.13), (2.15) and (2.16), we can derive the following:

$$(c_1 - c) \int_0^{\ell_3} \int_a^b \left[ \frac{\Delta U}{(U - c_1)(U - c)} + \frac{2\nabla U \cdot \nabla U(c_1 + c - 2U)}{(U - c_1)^2(U - c)^2} \right] \tilde{p}\tilde{p}_1 dydz$$

$$= (k_1 - k)(k_1 + k) \int_0^{\ell_3} \int_a^b \tilde{p}\tilde{p}_1 dydz,$$

from which we obtain the variational formula

$$\frac{dc}{dk} = 2k \int_0^{\ell_3} \int_a^b \tilde{p}^2 dydz$$

$$\times \left[ \int_0^{\ell_3} \int_a^b \left[ \frac{\Delta U}{(U - c)^2} - \frac{4\nabla U \cdot \nabla U}{(U - c)^3} \right] \tilde{p}^2 dydz \right]^{-1}. \tag{2.23}$$

**3. Viscous channel flow.** The viscous channel flow is governed by the Navier-Stokes equations

$$\partial_t u_i + u_j u_{i,j} = -p_{,i} + \epsilon u_{i,jj}, \quad u_{i,i} = 0, \tag{3.1}$$

where $(u_1, u_2, u_3)$ are the three components of the fluid velocity along the $(x, y, z)$ directions, $p$ is the pressure, and $\epsilon = 1/R$ is the inverse of the Reynolds number $R$. The boundary condition is

$$u_1(x, a, z) = \alpha, \quad u_1(x, b, z) = \beta, \quad u_j(x, a, z) = u_j(x, b, z) = 0 \ (j = 2, 3), \tag{3.2}$$

where $a < b$, $\alpha < \beta$, and $u_i$ $(i = 1, 2, 3)$ are periodic in the $x$ and $z$ directions with periods $\ell_1$ and $\ell_3$. For the viscous channel flow, the 3D shears mentioned above are no longer fixed points; instead they drift slowly in time (sometimes called quasi-steady solutions):

$$\left( e^{\epsilon t \Delta} U(y, z), 0, 0 \right).$$

By ignoring the slow drift and pretending they are still fixed points (or by using artificial body forces to stop the drifting), their unstable eigenvalues will lead to transient nonlinear growths as shown numerically [13]. A better explanation here is to use the theory of geometric singular perturbation ([6], [14]). The slowly drifting 3D shears altogether form a locally invariant slow (center) manifold. The normal direction growth rate (or decay rate) of this slow manifold has a persistence property (i.e. robust). Thus the growth rate can be estimated by ignoring the slow drift. The geometric singular perturbation theory implies the transient nonlinear growth induced by the linear growth rate.

The corresponding linear Navier-Stokes operator at $(U(y, z), 0, 0)$ is given by the following counterpart of (2.3)-(2.6):

$$ik(U - c)u + vU_y + wU_z = -ikp + \epsilon[\Delta - k^2]u, \tag{3.3}$$

$$ik(U - c)v = -p_y + \epsilon[\Delta - k^2]v, \tag{3.4}$$

$$ik(U - c)w = -p_z + \epsilon[\Delta - k^2]w, \tag{3.5}$$

$$iku + v_y + w_z = 0. \tag{3.6}$$

Again, two forms of simplified systems can be derived:

$$\partial_y \left\{ \left[ \Delta - k^2 - ikR(U - c) \right] (v_y + w_z) + ikR(vU_y + wU_z) \right\}$$
$$-k^2(\Delta - k^2)v + ik^3R(U - c)v = 0, \tag{3.7}$$
$$\partial_z \left\{ \left[ \Delta - k^2 - ikR(U - c) \right] (v_y + w_z) + ikR(vU_y + wU_z) \right\}$$
$$-k^2(\Delta - k^2)w + ik^3R(U - c)w = 0, \tag{3.8}$$

with the boundary conditions

$$v(a, z) = v(b, z) = v_y(a, z) = v_y(b, z) = w(a, z) = w(b, z) = 0,$$

where $v, w$ are periodic in $z$; and the other form

$$(U - c)^2 \nabla \cdot \left\{ (U - c)^{-1} [\epsilon(\Delta - k^2) - ik(U - c)]^{-1} \nabla p \right\} = ikp$$
$$+ \frac{\epsilon}{ik}(\Delta - k^2) \nabla \cdot \left\{ [\epsilon(\Delta - k^2) - ik(U - c)]^{-1} \nabla p \right\}, \tag{3.9}$$

the boundary condition of which is complicated.

The system (3.7)-(3.8) looks quite convenient for numerical simulations. Multiplying (3.7) by $\bar{v}$ and (3.8) by $\bar{w}$, integrating and adding the two equations, we obtain the following expression for the eigenvalue in terms of the eigenfunction:

$$c_i = \frac{-1}{kRD}[A + E + H + kR \, \text{Re}(G)], \quad c_r = \frac{1}{D}[B + \, \text{Im}(G)], \tag{3.10}$$

where

$$A = \int \left[ |\nabla v_y|^2 + |\nabla w_z|^2 + k^2|v_y|^2 + k^2|w_z|^2 \right] \geq 0;$$

$$B = \int U \left( |v_y|^2 + |w_z|^2 + \bar{v}_y w_z + v_y \bar{w}_z \right.$$
$$\left. + k^2|v|^2 + k^2|w|^2 \right), \, \text{real};$$

$$D = \int \left( |v_y|^2 + |w_z|^2 + \bar{v}_y w_z + v_y \bar{w}_z \right.$$
$$\left. + k^2|v|^2 + k^2|w|^2 \right) \geq 0;$$

$$E = \int \left[ \bar{v}_{yy} w_{zy} + \bar{v}_{yz} w_{zz} + k^2 \bar{v}_y w_z + \text{c.c.} \right], \, \text{real};$$

$$G = -i \int (\bar{v}_y + \bar{w}_z)(vU_y + wU_z), \, \text{complex};$$

$$H = k^2 \int \left[ |\nabla v|^2 + |\nabla w|^2 \right] + k^4 \int \left[ |v|^2 + |w|^2 \right] \geq 0.$$

Notice that the expression of $c_r$ has no explicit dependence upon the Reynolds number $R$, but it does depend on $R$ implicitly via the eigenfunction. An unstable eigenvalue corresponds to $kc_i > 0$. Without loss of generality, we assume $k > 0$.

THEOREM 3.1. Let $g = \max\{\|U_y\|_{L^\infty}, \|U_z\|_{L^\infty}\}$, $(k, c)$ be an eigenmode; if $Rg < \min(k, k^3)$ (where $k > 0$), then $c_i < 0$, i.e. a stable eigenvalue.

*Proof.* In the expression of $c_i$ (3.10), notice that $A + E \geq 0$; if $H + kR \operatorname{Re}(G) > 0$, then $c_i < 0$. Notice also that

$$|G| \leq g \int [|v_y|^2 + |w_z|^2 + |v|^2 + |w|^2],$$

where

$$g = \max\{\|U_y\|_{L^\infty}, \|U_z\|_{L^\infty}\}.$$

Thus

$$H + kR \operatorname{Re}(G) \geq (k^2 - kRg) \int \left[ |\nabla v|^2 + |\nabla w|^2 \right] + (k^4 - kRg) \int \left[ |v|^2 + |w|^2 \right].$$

Therefore if $k^2 - kRg > 0$ and $k^4 - kRg > 0$, then $H + kR \operatorname{Re}(G) > 0$. The two conditions here are equivalent to the one in the theorem, and the theorem is proved.   $\square$

The type of claims in Theorem 3.1 and their improvements have been investigated intensively for 2D shears ([20], [10], [11]). On the other hand, as mentioned at the beginning, we are more interested in unstable eigenvalues and for a large Reynolds number $R$ as in [15].

## 4. Appendix: 2D limiting shear can only be the classical laminar shear.

We take the plane Couette flow as the example; for plane/pipe Poiseuille flow, the argument is the same. Assume that as the Reynolds number $R \to +\infty$, a 2D steady state of the plane Couette flow approaches the limiting shear $(U(y), 0)$. Denote by

$$(U(y) + u(x, y), v(x, y))$$

the steady state which is periodic in $x$. Assume that $(u, v)$ and their spatial derivatives are of order $o(R^{-1/2})$ as $R \to +\infty$. Then to the leading order $O(R^{-1})$,

$$U\partial_x u + v\partial_y U = -\partial_x p + \frac{1}{R}\partial_y^2 U,$$

$$U\partial_x v = -\partial_y p,$$

$$\partial_x u + \partial_y v = 0.$$

Taking an average in the $x$-direction (over the period), we get

$$\overline{v}\partial_y U = \frac{1}{R}\partial_y^2 U,$$

$$0 = -\partial_y \overline{p},$$

$$\partial_y \overline{v} = 0.$$

By the boundary condition of $v$ in the $y$-direction, $\overline{v} = 0$. Thus

$$\partial_y^2 U = 0, \quad \text{i.e. } U = c_1 + c_2 y.$$

That is, $U$ has to be the laminar linear shear.

As shown in [16], the corresponding 3D limiting shear $U(y, z)$ does not have to be the linear shear, but rather satisfies a constraint

$$\int \Delta U f(U) \, dydz = 0$$

for any $f$. In fact, the lower branch 3D limiting shear is far away from the linear shear ([24], [23], [22]). Does this hint that the lower branch steady state does not exist in 2D? Numerical simulations could not find any 2D steady state other than the linear shear [5].

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