PERGAMON

Nonlinear Analysis 43 (2001) 485–507

Nonlinear Analysis

www.elsevier.nl/locate/na

Stability enhancement by boundary control in the Kuramoto–Sivashinsky equation ☆

Wei-Jiu Liu, Miroslav Krstić *

Department of AMES, University of California at San Diego, La Jolla, CA 92093-0411, USA

Received 7 September 1998; accepted 11 March 1999

Keywords: Kuramoto–Sivashinsky equation; Boundary control; Stabilization; Spectral analysis

1. Introduction

In this article, we address the problem of global boundary control of the Kuramoto–Sivashinsky equation (KS-equation)

$$u_t + u_{xxxx} + \lambda u_{xx} + uu_x = 0, \quad 0 < x < 1, \;\; t > 0, \tag{1.1}$$

where we refer to $\lambda > 0$ as the "anti-diffussion" parameter. Note that a more general form $u_t + \lambda_1 u_{xxxx} + \lambda_2 u_{xx} + \lambda_3 uu_x = 0$ can always be reduced to (1.1) by appropriate rescaling of t, x and u. Eq. (1.1) was derived independently by Kuramoto et al. [20–22] as a model for phase turbulence in reaction–diffusion systems and by Sivashinsky [33] as a model for plane flame propagation, describing the combined influence of diffusion and thermal conduction of the gas on stability of a plane flame front. So far, it has been well understood that the KS-equation can also serve as a mathematical model for cellular instabilities in a variety of situations: the flow of thin liquid films on inclined planes [30] (in the limit of large surface tension), dendritic fronts in dilute binary alloys [31], and Alfven drift waves in plasmas [23] (as a nonlinear saturation mechanism of the dissipative trapped ion modes).

☆ This work was supported by grants from Office of Naval Research, Air Force Office of Scientific Research, and National Science Foundation.

* Corresponding author. Tel.: +1-858-822-1374; fax: +1-858-822-3107.

E-mail addresses: weiliu@cherokee.ucsd.edu (W.-J. Liu), krstic@ucsd.edu (M. Krstić).

PII: S0362-546X(99)00215-1

The problem of large-time behavior of this nonlinear fourth-order dissipative equation has been extensively studied. The pioneering work appears to be due to Foias et al. [10] and Nicolaenko et al. [28–30], who described the global attractors and inertial manifords of the KS-equation. Since then, there has been an impressive amount of progress on analysis of the KS-equation [2,3,5–9,11–16,18,19,24,27,32,35–38].

At this stage, control problems for the KS-equation are largely unexplored. He et al. [17] have studied numerical aspects of controllability and optimal control. Christofides [4] has developed linear controllers based on a Galerkin truncation which achieve local stabilization. Both [17,4] employ distributed control and periodic boundary conditions.

In this paper we are concerned with boundary control. We start by showing that, under Dirichlet boundary conditions, the trivial solution $u(x,t)\equiv 0$ is unstable for $\lambda > 4\pi^2$ and asymptotically stable for $\lambda < 4\pi^2$ (for the latter case we derive global exponential decay estimates). Then we move to the problem with Neumann boundary control. The uncontrolled system is not asymptotically stable even for $\lambda < 4\pi^2$. We introduce nonlinear boundary feedback and prove that it guarantees L^2-global exponential stability, H^2-global asymptotic stability, and H^2-semiglobal exponential stability if $\lambda < 4\pi^2$. By constructing a Green function and using the Banach contraction mapping principle, we prove that the closed-loop system has a global unique and infinitely differentiable solution.

We point out that the boundary stabilization problem for $\lambda > 4\pi^2$, i.e., when the uncontrolled system is unstable under both Dirichlet and Neumann boundary conditions, remains open. In our opinion, this problem requires a radically different approach than the one presented in this paper.

The nonlinear boundary conditions that we design as a feedback law are motivated by the above physical problems, especially the problem of boundary stabilization of flame front instabilities. An example of experimental setup is a combustor consisting of two concentric cylinders with a narrow gap filled with combustible gas. In the absence of control, the flame front would develop "wrinkles" governed by Kuramoto–Sivashinsky dynamics. While one could stabilize the flame by actuating the fuel supply all around the base of the combustor (distributed control), the problem that this paper solves with boundary actuation would require fuel modulation only on a small section of the base of the combustor. An alternative actuation would be with a moving flame holder. The sensing of the flame front can be accomplished using various photo-detecting, laser, and video devices. An important property of the control law derived in the paper is that it can be implemented by actuating any two of the four variables u, u_x, u_{xx}, u_{xxx} at the boundary, and sensing the remaining two variables. This property is achieved by selecting the control laws (boundary conditions) as invertible functions. The consequence is that the control laws will be implementable whatever the variables accessible for actuation may be for a physical problem at hand. Apart from the physical motivation for this work, its mathematical motivation should not be overlooked. The KS equation can be regarded as a nonlinear, higher-dimensional extension of the heat equation, in which Neumann control (actuated via heat flux) is a natural choice. Since the uncontrolled KS equation with Neumann boundary conditions is not asymptotically stable, the objective of designing Neumann boundary feedback is justified. We point out that, while, in a second-order problem like heat equation the term "Neumann" would be referring to boundary conditions on u_x, in a fourth-order problem like KS we are referring to u_{xx} and u_{xxx} (the term "Dirichlet" is used in reference to u, u_x).

We present our main results in Section 2. Section 3 is dedicated to the spectral analysis of the linear problem, from which we find the critical value $\lambda^* = 4\pi^2$. In order to prove the main results, we first establish a new differential inequality of Gronwall type in Section 4, and then prove the main results by using Lyapunov techniques for the stability theorems and the Banach contraction mapping principle for the well-posedness theorem.

2. Main results

We first consider the following uncontrolled equation with Dirichlet boundary conditions:

$$\begin{aligned} &u_t + u_{xxxx} + \lambda u_{xx} + u u_x = 0, \quad 0 < x < 1, \ t > 0, \\ &u(0,t) = u(1,t) = 0, \quad t > 0, \\ &u_x(0,t) = u_x(1,t) = 0, \quad t > 0, \\ &u(x,0) = u^0(x), \quad 0 < x < 1. \end{aligned} \tag{2.1}$$

The energy $E(t)$ of solutions of (2.1) is defined by

$$E(t) = \int_0^1 u(x,t)^2 \, \mathrm{d}x \tag{2.2}$$

and its higher-order energy $V(t)$ is defined by

$$V(t) = \int_0^1 u_{xx}(x,t)^2 \, \mathrm{d}x. \tag{2.3}$$

The stability of system (2.1) significantly depends on the anti-diffusion parameter $\lambda > 0$. Roughly speaking, the system is asymptotically stable if λ is small enough and unstable if λ sufficiently large. To locate the boundary value λ^* between stability and instability, we need to analyze the following eigenvalue problem: [1]

$$\begin{aligned} &\varphi_{xxxx} + \lambda \varphi_{xx} = \sigma \varphi, \quad 0 < x < 1, \\ &\varphi(0) = \varphi(1) = \varphi_x(0) = \varphi_x(1) = 0. \end{aligned} \tag{2.4}$$

For a given $\lambda \in \mathbf{R}$, since $\partial^4/\partial x^4 + \lambda \partial^2/\partial x^2$ with the above Dirichlet boundary conditions is a self-adjoint operator, the eigenvalues are real numbers. Moreover, since the inverse of the operator is compact, the eigenvalues are a sequence $\{\sigma_n\}$ such that

[1] Note that σ corresponds to $-\partial/\partial t$ rather than the more common $+\partial/\partial t$. This convention is followed for subsequent notational convenience. One needs to keep this in mind in interpreting stability results: $\operatorname{Re} \sigma > 0$ means stability and $\operatorname{Re} \sigma < 0$ means instability.

$\lim_{n\to\infty} \sigma_n = +\infty$. Let us define

$$\sigma(\lambda) = \min_n \sigma_n(\lambda). \tag{2.5}$$

Then we have

Lemma 2.1. *The function $\sigma(\lambda)$ is strictly decreasing on $\mathbb{R}$ and*

$$\sigma(4\pi^2) = 0. \tag{2.6}$$

Thus the critical value λ^* is equal to $4\pi^2$. If $\lambda < 4\pi^2$, system (2.1) is stable. More precisely, we have the following stability theorem.

Theorem 2.1. *Suppose that $\lambda < 4\pi^2$.*

(i) *If the initial data $u^0(x) \in L^2(0,1)$, then the solution of problem* (2.1) *satisfies the following global-exponential stability estimate*:

$$E(t) \le E(0)\mathrm{e}^{-2\sigma(\lambda)t}, \quad \forall t \ge 0. \tag{2.7}$$

(ii) *If the initial data $u^0(x) \in H_0^2(0,1)$, then the solution of problem* (2.1) *satisfies the following global-asymptotic and semiglobal-exponential stability estimate*:

$$V(t) \le [C(\lambda^2 + \sigma(\lambda))E(0) + V(0)]\exp(CE(0))\mathrm{e}^{-\sigma(\lambda)t}, \quad \forall t \ge 0, \tag{2.8}$$

where C is a positive constant independent of u^0 and λ.

In this theorem and in the sequel, $H^s(0,1)$ denotes the usual Sobolev space (see [1,25]) for any $s \in \mathbf{R}$. For $s \ge 0$, $H_0^s(0,1)$ denotes the completion of $C_0^\infty(0,1)$ in $H^s(0,1)$, where $C_0^\infty(0,1)$ denotes the space of all infinitely differentiable functions on $(0,1)$ with compact support in $(0,1)$.

Remark 1. If $\lambda > 4\pi^2$, the corresponding linear system of (2.1)

$$\begin{aligned} &u_t + u_{xxxx} + \lambda u_{xx} = 0, \quad 0 < x < 1, \ t > 0, \\ &u(0,t) = u(1,t) = 0, \quad t > 0, \\ &u_x(0,t) = u_x(1,t) = 0, \quad t > 0, \\ &u(x,0) = u^0(x), \quad 0 < x < 1 \end{aligned} \tag{2.9}$$

is unstable. In fact, problem (2.9) has solution

$$u = \mathrm{e}^{-\sigma(\lambda)t}\varphi(x) \tag{2.10}$$

for the initial data $u^0(x) = \varphi(x)$, where $\varphi(x)$ is normalized eigenvector of (2.4) corresponding to the eigenvalue $\sigma(\lambda)$. Since $\sigma(\lambda) < 0$, the energy

$$E(t) = \int_0^1 \mathrm{e}^{-2\sigma(\lambda)t}\varphi(x)^2\,\mathrm{d}x = \mathrm{e}^{-2\sigma(\lambda)t} \tag{2.11}$$

tends to $+\infty$ as $t \to +\infty$. Although we are not able to find such an explicit solution for problem (2.1), we conjecture that it also may not be stable.

We then look at the uncontrolled Neumann boundary problem

$$
\begin{aligned}
& u_t + u_{xxxx} + \lambda u_{xx} + u u_x = 0, \quad 0 < x < 1, \; t > 0, \\
& u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad t > 0, \\
& u_{xxx}(0,t) = u_{xxx}(1,t) = 0, \quad t > 0, \\
& u(x,0) = u^0(x), \quad 0 < x < 1.
\end{aligned} \tag{2.12}
$$

Unlike the Dirichlet case, the Neumann problem unfortunately is not *asymptotically* stable even for $\lambda < 4\pi^2$. To see this, let us take the initial data $u^0(x) = 1$. Then the solution is $u(x,t) = 1$. This means that the equilibrium point 0 is not asymptotically stable. In fact, the eigenvalues of linearized problem (2.12) include 0 with an eigenfunction $\varphi_0 = 1 + x$. Hence, a boundary feedback is needed to stabilize the problem. There are various feedbacks we can select. Since our goal is to achieve the *global stabilization*, we introduce the following nonlinear boundary feedback:

$$
\begin{aligned}
& u_{xx}(0) = k u_x(0), \qquad u_{xxx}(0) = -k u(0) - u(0)^3, \\
& u_{xx}(1) = -k u_x(1), \qquad u_{xxx}(1) = k u(1) + u(1)^3,
\end{aligned} \tag{2.13}
$$

where k is a sufficiently large constant (the largeness will be made clear in the proof of Theorem 2.2 below in Section 4). The simpler linear feedback

$$
\begin{aligned}
& u_{xx}(0) = k u_x(0), \qquad u_{xxx}(0) = -k u(0), \\
& u_{xx}(1) = -k u_x(1), \qquad u_{xxx}(1) = k u(1)
\end{aligned} \tag{2.14}
$$

guarantees only local stability. In Section 4 we will see how feedback (2.13) is found. With this feedback, problem (2.12) becomes the following closed-loop problem:

$$
\begin{aligned}
& u_t + u_{xxxx} + \lambda u_{xx} + u u_x = 0, \quad 0 < x < 1, \; t > 0, \\
& u_{xx}(0) = k u_x(0), \qquad u_{xxx}(0) = -k u(0) - u(0)^3, \quad t > 0, \\
& u_{xx}(1) = -k u_x(1), \qquad u_{xxx}(1) = k u(1) + u(1)^3, \quad t > 0, \\
& u(x,0) = u^0(x), \quad 0 < x < 1.
\end{aligned} \tag{2.15}
$$

This closed-loop system is L^2-globally exponentially stable and H^2-globally asymptotically stable. In order to state this result more precisely, we introduce the following higher-order energy function including boundary values:

$$
\begin{aligned}
F(t) = {} & k u_x(0)^2 + k u(0)^2 + \tfrac{1}{2} u(0)^4 + k u_x(1)^2 + k u(1)^2 \\
& + \tfrac{1}{2} u(1)^4 + \int_0^1 u_{xx}^2 \, \mathrm{d}x.
\end{aligned} \tag{2.16}
$$

Theorem 2.2. *If* $\lambda < 4\pi^2$, *then there exists* $k > 0$ *sufficiently large such that the following holds*:

(i) *If the initial data* $u^0(x) \in L^2(0,1)$, *then the solution of problem* (2.15) *satisfies the following global-exponential stability estimate*:

$$
E(t) \le E(0) \mathrm{e}^{-\sigma(\lambda) t}, \quad \forall t \ge 0. \tag{2.17}
$$

(ii) *If the initial data* $u^0(x) \in H^2(0,1)$, *the solution of problem* (2.15) *satisfies the following global-asymptotic and semiglobal-exponential stability estimate*:

$$F(t) \le [CE(0) + F(0)]\exp(CE(0))e^{-\sigma(\lambda)t/2}, \quad \forall t \ge 0, \tag{2.18}$$

$$\|u(t)\|_{H^2}^2 \le C[E(0) + F(0)]\exp(CE(0))e^{-\sigma(\lambda)t/2}, \quad \forall t \ge 0, \tag{2.19}$$

where C *is a positive constant independent of* u^0.

Remark 2. By the embedding theorem (see [1, p. 97]), Theorems 2.1 and 2.2 show that

$$\max_{0\le x\le 1}\{|u(x,t)|, |u_x(x,t)|\} \le \|u(t)\|_{H^2} \le C(u^0, u_x^0, u_{xx}^0)e^{-\delta t}, \quad \forall t \ge 0, \tag{2.20}$$

where δ is a positive constant independent of u^0 and $C(u^0, u_x^0, u_{xx}^0)$ is a positive constant independent of u.

Remark 3. The case of $\lambda \ge 4\pi^2$ remains open. We do not know whether or not we can find a boundary feedback to stablize the KS-equation with Dirichlet or Neumann boundary condition.

Because both the equation and the boundary condition are nonlinear, the well-posedness of problem (2.15) is challenging. We first note that even though problem (2.1) falls in the category of general abstract equations discussed in [34, p. 115] problem (2.15) does not since the boundary condition is nonlinear. Also, the method we used in [26] to deal with the Korteweg–de Vries–Burgers equation cannot be applied since the solutions u of the linear boundary value problem

$$\begin{aligned}
&u_t + u_{xxxx} + \lambda u_{xx} + wu_x = 0, \quad 0 < x < 1, \ t > 0,\\
&u_{xx}(0) = ku_x(0), \qquad u_{xxx}(0) = -ku(0) - u(0)w(0)^2, \quad t > 0,\\
&u_{xx}(1) = -ku_x(1), \qquad u_{xxx}(1) = ku(1) + u(1)w(1)^2, \quad t > 0,\\
&u(x,0) = u^0(x), \quad 0 < x < 1
\end{aligned} \tag{2.21}$$

are less regular than w, where $w = w(x,t)$ is a given function belonging to an appropriate function space. For example, if $w \in C^{0,1}([0,1] \times [0,T])$, we can just have $u \in C([0,1]\times[0,T]) \cap C^1([0,T], L^2(0,1))$, where $C^{0,1}([0,1]\times[0,T])$ denotes the space of all functions $u(x,t)$ continuous with respect to x on $[0,1]$ and continuously differentiable with respect to t on $[0,T]$, and $C^1([0,T], L^2(0,1))$ denotes the space of all continuously differentiable functions $u(t)$ defined on $[0,T]$ with values in $L^2(0,1)$. This may be due to the Neumann-type boundary condition (for the Korteweg–de Vries–Burgers equation discussed in [26], the boundary condition is of mixed Dirichlet–Neumann type). Thus, the mapping defined by

$$Aw = u \tag{2.22}$$

does not map $C^{0,1}([0,1] \times [0,T])$ into itself. Therefore we are forced to find a new approach. For this, we will construct the Green function of the corresponding homogeneous boundary value problem of (2.15) and then transform problem (2.15) into an integral equation so that the Banach contraction mapping principle can be applied. In this way, we will prove

Theorem 2.3. *Let* $u^0(x) \in H^2(0,1)$.

(i) *Suppose* λ *and* k *are any constants* (*not required to satisfy the conditions imposed in Theorem* 2.2). *Then there exists a time* $T = T(u^0)$ *such that problem* (2.15) *has a unique and infinitely differentiable solution* u *on* $[0,1] \times (0,T)$ *satisfying*

$$u \in C([0,T), H^2(0,1)). \tag{2.23}$$

(ii) *If* $\lambda < 4\pi^2$, *then there exists* $k > 0$ *sufficiently large such that problem* (2.15) *has a unique and infinitely differentiable solution* u *on* $[0,1] \times (0,\infty)$ *satisfying*

$$u \in C([0,\infty), H^2(0,1)). \tag{2.24}$$

Before we close this section, we point out that feedback (2.13) can also be expressed as

$$\begin{aligned}
u(0) =& \left(-\frac{u_{xxx}(0)}{2} + \sqrt{\left(\frac{k}{3}\right)^3 + \left(\frac{u_{xxx}(0)}{2}\right)^2}\right)^{1/3} \\
&- \left(\frac{u_{xxx}(0)}{2} + \sqrt{\left(\frac{k}{3}\right)^3 + \left(\frac{u_{xxx}(0)}{2}\right)^2}\right)^{1/3}, \\
u(1) =& \left(\frac{u_{xxx}(1)}{2} + \sqrt{\left(\frac{k}{3}\right)^3 + \left(\frac{u_{xxx}(1)}{2}\right)^2}\right)^{1/3} \\
&+ \left(\frac{u_{xxx}(1)}{2} - \sqrt{\left(\frac{k}{3}\right)^3 + \left(\frac{u_{xxx}(1)}{2}\right)^2}\right)^{1/3}, \\
u_x(0) =& \frac{1}{k} u_{xx}(0), \\
u_x(1) =& -\frac{1}{k} u_{xx}(1),
\end{aligned} \tag{2.25}$$

that is, it can be implemented as Dirichlet boundary control.

3. Spectral analysis of the linearized problem

This section is dedicated to the proof of Lemma 2.1 and some other technical lemmas used in the proofs of the main results.

Lemma 3.1. *For every* $\varphi \in H_0^2(0,1)$, *we have*

$$\|\varphi_{xx}\|_{L^2}^2 - \lambda \|\varphi_x\|_{L^2}^2 \geq \sigma(\lambda) \|\varphi\|_{L^2}^2. \tag{3.1}$$

Proof. Let $\{\sigma_n\}$ be the eigenvalues of (2.4) and $\{\varphi_n\}$ the corresponding orthonormal eigenvectors. Then every $\varphi \in H_0^2(0,1)$ can be expanded as

$$\varphi = \sum_{n=1}^{\infty} c_n \varphi_n. \tag{3.2}$$

By calculation, we obtain

$$\|\varphi_{xx}\|_{L^2}^2 - \lambda \|\varphi_x\|_{L^2}^2 = \sum_{n=1}^{\infty} c_n^2 \sigma_n \ge \sigma(\lambda) \sum_{n=1}^{\infty} c_n^2 = \sigma(\lambda)\|\varphi\|_{L^2}^2. \quad \square \tag{3.3}$$

We now prove Lemma 2.1.

Proof of Lemma 2.1. We first prove that $\sigma(\lambda)$ is strictly decreasing. Let λ_1 and λ_0 be such that $-\infty < \lambda_1 < \lambda_0 < \infty$ and φ_* the normalized eigenfunction corresponding to $\sigma(\lambda_1)$. Since $\varphi_* \in H_0^2(0,1)$, by Lemma 3.1, we have

$$\int_0^1 (\varphi_{*xx}^2 - \lambda_0 \varphi_{*x}^2)\,\mathrm{d}x \ge \sigma(\lambda_0) \int_0^1 \varphi_*^2\,\mathrm{d}x. \tag{3.4}$$

It therefore follows that

$$\begin{aligned}
\sigma(\lambda_1) &= \sigma(\lambda_1) \int_0^1 \varphi_*^2\,\mathrm{d}x \\
&= \int_0^1 (\varphi_{*xx}^2 - \lambda_1 \varphi_{*x}^2)\,\mathrm{d}x \\
&= \int_0^1 (\varphi_{*xx}^2 - \lambda_0 \varphi_{*x}^2)\,\mathrm{d}x + \int_0^1 (\lambda_0 - \lambda_1)\varphi_{*x}^2\,\mathrm{d}x \\
&> \sigma(\lambda_0) \int_0^1 \varphi_*^2\,\mathrm{d}x \\
&= \sigma(\lambda_0).
\end{aligned} \tag{3.5}$$

We then locate the λ^* such that

$$\sigma(\lambda^*) = 0. \tag{3.6}$$

Thus, we consider the following eigenvalue problem:

$$\begin{aligned}
&\varphi_{xxxx} + \lambda \varphi_{xx} = 0, \quad 0 < x < 1, \\
&\varphi(0) = \varphi(1) = \varphi_x(0) = \varphi_x(1) = 0.
\end{aligned} \tag{3.7}$$

Obviously, in order that (3.7) has a nonzero solution, λ must be larger than 0. The corresponding characteristic equation of (3.7) is

$$\xi^4 + \lambda \xi^2 = 0 \tag{3.8}$$

and its solutions are given by

$$\xi_1 = 0, \qquad \xi_2 = 0, \qquad \xi_3 = \mathrm{i}\alpha, \qquad \xi_4 = -\mathrm{i}\alpha, \tag{3.9}$$

where

$$\alpha = \sqrt{\lambda}. \tag{3.10}$$

Thus, the general solution of (3.7) is given by

$$\varphi(x) = C_1 + C_2 x + C_3 \cos(\alpha x) + C_4 \sin(\alpha x). \tag{3.11}$$

By the boundary condition of (3.7), we infer that

$$\begin{aligned} &C_1 + C_3 = 0,\\ &C_2 + \alpha C_4 = 0,\\ &C_1 + C_2 + C_3 \cos\alpha + C_4 \sin\alpha = 0,\\ &C_2 - C_3 \alpha \sin\alpha + C_4 \alpha \cos\alpha = 0. \end{aligned} \tag{3.12}$$

It is easy to see that the determinant of the coefficient matrix of the above system is equal to

$$2\alpha - 2\alpha\cos\alpha - \alpha^2 \sin\alpha. \tag{3.13}$$

Thus, in order to ensure that (3.7) has a *nonzero* solution, α must satisfy

$$2 - 2\cos\alpha - \alpha\sin\alpha = 0. \tag{3.14}$$

This equation has infinitely many solutions

$$\alpha_n = 2n\pi, \quad n = 1, 2, \ldots \tag{3.15}$$

and consequently,

$$\lambda_n = 4n^2\pi^2, \quad n = 1, 2, \ldots . \tag{3.16}$$

Since we have proved that $\sigma(\lambda)$ is strictly decreasing, the λ^* we are looking for is

$$\lambda^* = 4\pi^2, \tag{3.17}$$

which gives $\sigma(4\pi^2) = 0$. This completes the proof. □

The following two lemmas will be used in the proofs of Theorems 2.1 and 2.2.

Lemma 3.2. *If $\lambda < 4\pi^2$, then, for every $\varphi \in H_0^2(0,1)$, we have*

$$C_1 \|\varphi_{xx}\|_{L^2}^2 \le \|\varphi_{xx}\|_{L^2}^2 - \lambda \|\varphi_x\|_{L^2}^2 \le C_2 \|\varphi_{xx}\|_{L^2}^2, \tag{3.18}$$

where C_1, C_2 are positive constants independent of φ.

Proof. We define two operators Λ_1 and Λ_2 by

$$\Lambda_1 \varphi = \varphi_{xxxx} + \lambda \varphi_{xx} \quad \text{for any } \varphi \in H^4(0,1) \cap H_0^2(0,1), \tag{3.19}$$

$$\Lambda_2 \varphi = \varphi_{xxxx} \quad \text{for any } \varphi \in H^4(0,1) \cap H_0^2(0,1). \tag{3.20}$$

By Lemma 3.1, both Λ_1 and Λ_2 are positive and self-adjoint operators in $L^2(0,1)$ with domain $H^4(0,1) \cap H_0^2(0,1)$. It therefore follows from Remark 2.3 of Lions and Magenes [25, p. 10] that

$$D(\Lambda_1^{1/2}) = D(\Lambda_2^{1/2}) = H_0^2(0,1) \tag{3.21}$$

with equivalent norms $\|\Lambda_1^{1/2}\varphi\|_{L^2}$ and $\|\Lambda_2^{1/2}\varphi\|_{L^2}$, where $D(\Lambda_1^{1/2})$ and $D(\Lambda_2^{1/2})$ denote the domains of $\Lambda_1^{1/2}$ and $\Lambda_2^{1/2}$, respectively. This implies (3.18). □

Lemma 3.3. *If $\lambda < 4\pi^2$, then, for every $u \in H^2(0,1)$, we have*

$$\frac{\sigma(\lambda)}{2}\|u\|_{L^2}^2 \le \|u_{xx}\|_{L^2}^2 - \lambda\|u_x\|_{L^2}^2 + C_1[u_x(1)^2 + u_x(0)^2 + u(1)^2 + u(0)^2], \tag{3.22}$$

$$C_2\|u_{xx}\|_{L^2}^2 \le \|u_{xx}\|_{L^2}^2 - \lambda\|u_x\|_{L^2}^2 + C_3[u_x(1)^2 + u_x(0)^2 + u(1)^2 + u(0)^2], \tag{3.23}$$

where C_1, C_2, C_3 are positive constants depending on λ, but independent of u.

Proof. Let

$$\begin{aligned} w(x) = {} & [u_x(1) + u_x(0) - 2u(1) + 2u(0)]x^3 + [-u_x(1) - 2u_x(0) \\ & +3u(1) - 3u(0)]x^2 + u_x(0)x + u(0). \end{aligned} \tag{3.24}$$

Then we have

$$w(0) = u(0), \qquad w(1) = u(1), \qquad w_x(0) = u_x(0), \qquad w_x(1) = u_x(1) \tag{3.25}$$

and

$$u - w \in H_0^2(0,1). \tag{3.26}$$

It therefore follows from Lemma 3.1 that

$$\sigma(\lambda)\int_0^1 (u-w)^2\,\mathrm{d}x \le \int_0^1 (u-w)_{xx}^2\,\mathrm{d}x - \lambda\int_0^1 (u-w)_x^2\,\mathrm{d}x. \tag{3.27}$$

Moreover, we have (the following C's denoting various positive constants that may vary from line to line)

$$\begin{aligned} \int_0^1 (u-w)_{xx}^2\,\mathrm{d}x &= \int_0^1 u_{xx}^2\,\mathrm{d}x - 2\int_0^1 u_{xx}w_{xx}\,\mathrm{d}x + \int_0^1 w_{xx}^2\,\mathrm{d}x \\ &= \int_0^1 u_{xx}^2\,\mathrm{d}x - 2u_xw_{xx}|_0^1 + 2uw_{xxx}|_0^1 + \int_0^1 w_{xx}^2\,\mathrm{d}x \\ &\le \int_0^1 u_{xx}^2\,\mathrm{d}x + C[u_x(1)^2 + u_x(0)^2 + u(1)^2 + u(0)^2], \end{aligned} \tag{3.28}$$

$$\begin{aligned} -\lambda\int_0^1 (u-w)_x^2\,\mathrm{d}x &= -\lambda\int_0^1 u_x^2\,\mathrm{d}x + 2\lambda\int_0^1 u_xw_x\,\mathrm{d}x - \lambda\int_0^1 w_x^2\,\mathrm{d}x \\ &= -\lambda\int_0^1 u_x^2\,\mathrm{d}x + 2\lambda uw_x|_0^1 - 2\lambda\int_0^1 uw_{xx}\,\mathrm{d}x - \lambda\int_0^1 w_x^2\,\mathrm{d}x \\ &\le -\lambda\int_0^1 u_x^2\,\mathrm{d}x + \frac{\sigma(\lambda)}{4}\int_0^1 u^2\,\mathrm{d}x \\ &\quad +C[u_x(1)^2 + u_x(0)^2 + u(1)^2 + u(0)^2] \end{aligned} \tag{3.29}$$

and

$$\int_0^1 uw\,\mathrm{d}x \le \frac{1}{4}\int_0^1 u^2\,\mathrm{d}x + C[u_x(1)^2 + u_x(0)^2 + u(1)^2 + u(0)^2]. \tag{3.30}$$

It therefore follows from (3.27) that

$$\frac{\sigma(\lambda)}{2}\int_0^1 u^2\,\mathrm{d}x \le \int_0^1 u_{xx}^2\,\mathrm{d}x - \lambda\int_0^1 u_x^2\,\mathrm{d}x + C[u_x(1)^2 + u_x(0)^2 + u(1)^2 + u(0)^2]. \tag{3.31}$$

In a similar way, by using Lemma 3.2, we can prove (3.23). □

In order to use the Banach contraction mapping principle to prove Theorem 2.3, we need to construct the Green function which depends on the eigenfunctions of the following eigenvalue problem:

$$\begin{aligned} &\varphi_{xxxx} = \sigma\varphi, \quad 0 < x < 1, \\ &\varphi_{xx}(0) = \varphi_{xxx}(0) = \varphi_{xx}(1) = \varphi_{xxx}(1) = 0. \end{aligned} \tag{3.32}$$

Thus, we discuss this problem here.

Lemma 3.4. *The eigenvalues* σ_n $(n = 1, 2, \ldots)$ *of* (3.2) *are given by*

$$\sigma_n = \alpha_n^4, \quad n = 0, 1, 2, \ldots, \tag{3.33}$$

where

$$\alpha_0 = 0, \tag{3.34}$$

$$2n\pi - \frac{\pi}{2} < \alpha_{2n-1} < 2n\pi, \quad n = 1, 2, \ldots, \tag{3.35}$$

$$2n\pi < \alpha_{2n} < 2n\pi + \frac{\pi}{2}, \quad n = 1, 2, \ldots, \tag{3.36}$$

$$\lim_{n\to\infty}\left(2n\pi - \frac{\pi}{2} - \alpha_{2n-1}\right) = \lim_{n\to\infty}\left(2n\pi + \frac{\pi}{2} - \alpha_{2n}\right) = 0. \tag{3.37}$$

The corresponding orthonormal eigenvectors are given by

$$\varphi_0 = 1, \tag{3.38}$$

$$\begin{aligned} \varphi_n = \frac{1}{\beta_n}\Bigg[&\frac{\cos\alpha_n - \sin\alpha_n - \mathrm{e}^{-\alpha_n}}{\mathrm{e}^{\alpha_n} - \cos\alpha_n - \sin\alpha_n}\mathrm{e}^{\alpha_n x} + \frac{\mathrm{e}^{\alpha_n} - \mathrm{e}^{-\alpha_n} - 2\sin\alpha_n}{\mathrm{e}^{\alpha_n} - \cos\alpha_n - \sin\alpha_n}\cos(\alpha_n x) \\ &- \frac{\mathrm{e}^{\alpha_n} + \mathrm{e}^{-\alpha_n} - 2\cos\alpha_n}{\mathrm{e}^{\alpha_n} - \cos\alpha_n - \sin\alpha_n}\sin(\alpha_n x) + \mathrm{e}^{-\alpha_n x}\Bigg], \quad n = 1, 2, \ldots, \end{aligned} \tag{3.39}$$

where β_n *are the normalizing constants such that*

$$\int_0^1 \varphi_n^2\,\mathrm{d}x = 1, \quad n = 1, 2, \ldots. \tag{3.40}$$

Proof. The corresponding characteristic equation of (3.32) is

$$\xi^4 = \sigma \tag{3.41}$$

and its solutions are

$$\xi_1 = \alpha, \qquad \xi_2 = \mathrm{i}\alpha, \qquad \xi_3 = -\alpha, \qquad \xi_1 = -\mathrm{i}\alpha, \tag{3.42}$$

where

$$\alpha = \sigma^{1/4}. \tag{3.43}$$

Thus, the general solutions of (3.32) are given by

$$\varphi(x) = C_1 \mathrm{e}^{\alpha x} + C_2 \cos \alpha x + C_3 \sin \alpha x + C_4 \mathrm{e}^{-\alpha x}. \tag{3.44}$$

By the boundary condition of (3.32), it follows that

$$\begin{aligned} &C_1 - C_3 - C_4 = 0,\\ &C_1 - C_2 + C_4 = 0,\\ &C_1 \mathrm{e}^{\alpha} - C_2 \cos\alpha - C_3 \sin\alpha + C_4 \mathrm{e}^{-\alpha} = 0,\\ &C_1 \mathrm{e}^{\alpha} + C_2 \sin\alpha - C_3 \cos\alpha - C_4 \mathrm{e}^{-\alpha} = 0. \end{aligned} \tag{3.45}$$

It is easy to see that the determinant of the coefficient matrix of the above system is equal to $2\cos\alpha(\mathrm{e}^{\alpha} + \mathrm{e}^{-\alpha}) - 4$. Thus, in order to ensure that (3.32) has a nonzero solution, α must satisfy

$$\mathrm{ch}\,\alpha \cos\alpha = 1. \tag{3.46}$$

This equation has infinite solutions α_n $(n = 1, 2, \ldots)$ and their properties listed in the lemma can been clearly seen by plotting the functions $1/\mathrm{ch}\,\alpha$ and $\cos\alpha$ on the same figure and determining the points of intersection. Hence the eigenvalues σ_n $(n=1, 2, \ldots)$ of (3.32) are given by

$$\sigma_n = \alpha_n^4. \tag{3.47}$$

Solving system (3.45), we obtain the corresponding eigenfunctions as given in the lemma. □

4. Proofs of the results

We first establish a differential inequality of Gronwall type, which is frequently used in this section.

Lemma 4.1. *Let g, h, y be three positive and integrable functions on $(t_0, +\infty)$ such that y' is locally integrable on $(t_0, +\infty)$. Assume that*

$$\frac{\mathrm{d}y}{\mathrm{d}t} \le gy + h \quad \textit{for } t \ge t_0, \tag{4.1}$$

$$\int_{t_0}^{\infty} g(s)\,\mathrm{d}s \le C_1, \tag{4.2}$$

$$\int_{t_0}^{\infty} e^{\delta s} h(s)\,ds \le C_2, \tag{4.3}$$

$$\int_{t_0}^{\infty} e^{\delta s} y(s)\,ds \le C_3, \tag{4.4}$$

where δ, C_1, C_2, C_3 are positive constants. Then

$$y(t) \le [C_2 + \delta C_3 + y(t_0)]e^{C_1} e^{-\delta(t-t_0)} \quad \textit{for } t \ge t_0. \tag{4.5}$$

Proof. Multiplying (4.1) by $e^{\delta t}$, we obtain

$$\frac{d}{dt}(e^{\delta t} y) \le e^{\delta t} g y + e^{\delta t} h + \delta e^{\delta t} y \quad \text{for } t \ge t_0. \tag{4.6}$$

By Gronwall's inequality (see, e.g., [34, p. 90]), we deduce

$$\begin{aligned} e^{\delta t} y(t) &\le e^{\delta t_0} y(t_0) \exp\left(\int_{t_0}^{t} g(s)\,ds\right) \\ &\quad + \int_{t_0}^{t} (e^{\delta s} h(s) + \delta e^{\delta s} y(s)) \exp\left(-\int_{t}^{s} g(\tau)\,d\tau\right)\,ds \\ &\le (C_2 + \delta C_3)e^{C_1} + e^{\delta t_0 + C_1} y(t_0) \end{aligned} \tag{4.7}$$

which implies (4.5). □

In what follows, the C's denote generic positive constants that may vary from line to line.

Proof of Theorem 2.1. (i) By a straightforward calculation, we have

$$\dot{E}(t) = -2\int_0^1 u_{xx}^2\,dx + 2\lambda \int_0^1 u_x^2\,dx. \tag{4.8}$$

It therefore follows from Lemma 3.1 that

$$\dot{E}(t) \le -2\sigma(\lambda)E(t) \tag{4.9}$$

which implies (2.7).

(ii) By (4.8) and Lemma 3.2, we deduce that there exists a constant $C > 0$ such that

$$\dot{E}(t) + CV(t) \le 0. \tag{4.10}$$

Multiplying (4.10) by $e^{\sigma(\lambda)t}$, we obtain

$$\frac{d}{dt}(e^{\sigma(\lambda)t}E(t)) + Ce^{\sigma(\lambda)t}V(t) \le \sigma(\lambda)e^{\sigma(\lambda)t}E(t) \le \sigma(\lambda)E(0)e^{-\sigma(\lambda)t}. \tag{4.11}$$

Integrating from 0 to t gives

$$e^{\sigma(\lambda)t}E(t) + C\int_0^t e^{\sigma(\lambda)s}V(s)\,ds \le E(0)(2 - e^{-\sigma(\lambda)t}) \tag{4.12}$$

which implies

$$C\int_0^t e^{\sigma(\lambda)s}V(s)\,ds \le 2E(0), \quad \forall t \ge 0. \tag{4.13}$$

Multiplying the first equation of (2.1) by u_{xxxx}, integrating from 0 to 1 by parts and noting that

$$2\int_0^1 u_x^2\,dx \le \int_0^1 u_{xx}^2\,dx, \tag{4.14}$$

$$u(x)^2 \le \int_0^1 u_x^2\,dx, \tag{4.15}$$

we obtain with Young's inequality

$$\begin{aligned}\dot V(t) &= -2\int_0^1 u_{xxxx}^2\,dx - 2\lambda\int_0^1 u_{xxxx}u_{xx}\,dx - 2\int_0^1 uu_xu_{xxxx}\,dx\\ &\le \lambda^2\int_0^1 u_{xx}^2\,dx + \int_0^1 u^2u_x^2\,dx\\ &\le \lambda^2V(t) + \tfrac14 V(t)^2.\end{aligned} \tag{4.16}$$

Using (4.13) and applying Lemma 4.1 with

$$g = \tfrac14 V, \qquad h = \lambda^2 V, \qquad y = V, \qquad \delta = \sigma(\lambda), \tag{4.17}$$

we obtain

$$V(t) \le [C(\lambda^2 + \sigma(\lambda))E(0) + V(0)]\exp(CE(0))e^{-\sigma(\lambda)t}. \tag{4.18}$$

Now let us explain how feedback (2.13) is found. First, we obtain

$$\begin{aligned}\dot E(t) &= 2\int_0^1 uu_t\,dx\\ &= 2\int_0^1 u[-u_{xxxx} - \lambda u_{xx} - uu_x]\,dx\\ &= -2\int_0^1 u_{xx}^2\,dx + 2\lambda\int_0^1 u_x^2\,dx\\ &\quad - 2u(1)u_{xxx}(1) + 2u_x(1)u_{xx}(1) - 2\lambda u(1)u_x(1) - \tfrac23 u(1)^3\\ &\quad + 2u(0)u_{xxx}(0) - 2u_x(0)u_{xx}(0) + 2\lambda u(0)u_x(0) + \tfrac23 u(0)^3.\end{aligned} \tag{4.19}$$

It therefore follows from (3.22) and (4.19) that

$$\begin{aligned}\dot{E}(t) \le & - \sigma(\lambda)E(t) + C[u_x(1)^2 + u_x(0)^2 + u(1)^2 + u(0)^2] \\ & - 2u(1)u_{xxx}(1) + 2u_x(1)u_{xx}(1) + \lambda[u(1)^2 + u_x(1)^2] + \tfrac{1}{3}[u(1)^2 + u(1)^4] \\ & + 2u(0)u_{xxx}(0) - 2u_x(0)u_{xx}(0) + \lambda[u(0)^2 + u_x(0)^2] + \tfrac{1}{3}[u(0)^2 + u(0)^4] \\ = & - \sigma(\lambda)E(t) + u_x(0)[Cu_x(0) - 2u_{xx}(0)] + u(0)[2u_{xxx}(0) + Cu(0) + \tfrac{1}{3}u(0)^3] \\ & + u_x(1)[Cu_x(1) + 2u_{xx}(1)] + u(1)[- 2u_{xxx}(1) + Cu(1) + \tfrac{1}{3}u(1)^3]. \qquad (4.20)\end{aligned}$$

This leads us to take feedback (2.13) because, with this feedback, we have

$$\begin{aligned}\dot{E}(t) \le & -\sigma(\lambda)E(t) + (C - 2k)u_x(0)^2 + (-2k + C)u(0)^2 \\ & -\tfrac{5}{3}u(0)^4 + (C - 2k)u_x(1)^2 + (-2k + C)u(1)^2 - \tfrac{5}{3}u(1)^4 \\ \le & -\sigma(\lambda)E(t) \qquad (4.21)\end{aligned}$$

if k is large enough. This implies (2.17).

Proof of Theorem 2.2. (i) (2.17) has been proved above.

(ii) By (2.13), (3.23) and (4.19), we have

$$\begin{aligned}\dot{E}(t) \le & -C_2 \int_0^1 u_{xx}^2 \,\mathrm{d}x + (C - 2k)u_x(0)^2 + (-2k + C)u(0)^2 - \tfrac{5}{3}u(0)^4 \\ & +(C - 2k)u_x(1)^2 + (-2k + C)u(1)^2 - \tfrac{5}{3}u(1)^4. \qquad (4.22)\end{aligned}$$

Set

$$m = \min\{C_2, 2k - C, 5/3\} > 0, \qquad (4.23)$$

$$G(t) = \int_0^1 u_{xx}^2 \,\mathrm{d}x + u(0)^2 + u(0)^4 + u_x(0)^2 + u(1)^2 + u(1)^4 + u_x(1)^2. \qquad (4.24)$$

Then we have

$$\dot{E}(t) + mG(t) \le 0. \qquad (4.25)$$

As in (4.12), we obtain

$$\exp(\sigma(\lambda)t/2)E(t) + m \int_0^t \exp(\sigma(\lambda)s/2)G(s)\,\mathrm{d}s \le 2E(0). \qquad (4.26)$$

On the other hand, we have

$$\begin{aligned} & 2\int_0^1 u_t u_{xxxx} \,\mathrm{d}x \\ & \quad = 2u_t u_{xxx}\big|_0^1 - 2u_{xt}u_{xx}\big|_0^1 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 u_{xx}^2 \,\mathrm{d}x\right)\end{aligned}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(ku_x(0)^2 + ku(0)^2 + \tfrac{1}{2}u(0)^4 + ku_x(1)^2 + ku(1)^2 + \tfrac{1}{2}u(1)^4 + \int_0^1 u_{xx}^2\,\mathrm{d}x\right)$$

$$= \dot{F}(t) \quad \text{(note the definition (2.16) of } F\text{)}. \tag{4.27}$$

Multiplying the first equation of (2.15) by u_{xxxx}, integrating from 0 to 1 by parts and noting that

$$u(x)^2 \le 2u(0)^2 + 2\int_0^1 u_x^2\,\mathrm{d}x, \tag{4.28}$$

we obtain

$$\begin{aligned}
\dot{F}(t) &= -2\int_0^1 u_{xxxx}^2\,\mathrm{d}x - 2\lambda\int_0^1 u_{xx}u_{xxxx}\,\mathrm{d}x - 2\int_0^1 uu_xu_{xxxx}\,\mathrm{d}x \\
&\le \lambda^2\int_0^1 u_{xx}^2\,\mathrm{d}x + \int_0^1 u^2u_x^2\,\mathrm{d}x \\
&\le \lambda^2\int_0^1 u_{xx}^2\,\mathrm{d}x + 2u(0)^2\int_0^1 u_x^2\,\mathrm{d}x + 2\left(\int_0^1 u_x^2\,\mathrm{d}x\right)^2 \\
&\le \lambda^2\int_0^1 u_{xx}^2\,\mathrm{d}x + u(0)^4 + 3\left(\int_0^1 u_x^2\,\mathrm{d}x\right)^2 \\
&= \lambda^2\int_0^1 u_{xx}^2\,\mathrm{d}x + u(0)^4 + 3\left(2u_x(0)^2 + \int_0^1 u_{xx}^2\,\mathrm{d}x\right)^2 \\
&\le CF(t) + CF(t)^2.
\end{aligned} \tag{4.29}$$

Using (4.26) and applying Lemma 4.1 with

$$g = CF, \qquad h = CF, \qquad y = F, \qquad \delta = \sigma(\lambda)/2, \tag{4.30}$$

we deduce (2.18).

Estimate (2.19) is a direct consequence of (2.17) and (2.18) and the equivalence of the norms $\|u\|_{H^2}$ and $(\|u\|_{L^2} + \|u_{xx}\|_{L^2})^{1/2}$ (see, e.g., the interpolation theorem, [1, p. 79, Corollary 4.16]). □

In order to use the Banach contraction mapping principle to prove Theorem 2.3, we construct the Green function of the following problem:

$$\begin{aligned}
&u_t + u_{xxxx} = 0, \quad 0 < x < 1,\ t > 0, \\
&u_{xx}(0,t) = u_{xxx}(0,t) = u_{xx}(1,t) = u_{xxx}(1,t) = 0, \quad t > 0.
\end{aligned} \tag{4.31}$$

It seems that such a Green function is not known in the existing literature.

Setting

$$u = \varphi(x)\psi(t) \tag{4.32}$$

then we have

$$-\frac{\psi_t(t)}{\psi(t)}=\frac{\varphi_{xxxx}(x)}{\varphi(x)}\equiv\sigma \quad \text{(a constant)}. \tag{4.33}$$

Accordingly, the Green function of (4.31) is given by

$$G(x,y,t,\tau)=1+\sum_{n=1}^{\infty}\varphi_n(x)\varphi_n(y)\mathrm{e}^{-\alpha_n^4(t-\tau)}, \quad 0\le x,\ y\le 1,\ t>\tau, \tag{4.34}$$

where $\sigma_n=\alpha_n^4$ are the eigenvalues of (3.32) and φ_n are the corresponding orthonormal eigenfunctions which are given in Lemma 3.4.

Lemma 4.2. *There exists a constant $C>0$ such that*

$$\int_0^t\int_0^1 |G(x,y,t,\tau)|\,\mathrm{d}y\,\mathrm{d}\tau\le t+C\sum_{n=1}^{\infty}\frac{1-\mathrm{e}^{-\alpha_n^4 t}}{\alpha_n^4}, \tag{4.35}$$

$$\int_0^t |G(x,y,t,\tau)|\,\mathrm{d}\tau\le t+C\sum_{n=1}^{\infty}\frac{1-\mathrm{e}^{-\alpha_n^4 t}}{\alpha_n^4}, \tag{4.36}$$

$$\int_0^t\int_0^1 \left|\frac{\partial^{n_1+n_2}}{\partial x^{n_1}\partial y^{n_2}}G(x,y,t,\tau)\right|\,\mathrm{d}y\,\mathrm{d}\tau\le C\sum_{n=1}^{\infty}\frac{1-\mathrm{e}^{-\alpha_n^4 t}}{\alpha_n^{4-n_1-n_2}}, \quad 1\le n_1+n_2\le 2, \tag{4.37}$$

$$\int_0^t \left|\frac{\partial^{n_1+n_2}}{\partial x^{n_1}\partial y^{n_2}}G(x,y,t,\tau)\right|\,\mathrm{d}\tau\le C\sum_{n=1}^{\infty}\frac{1-\mathrm{e}^{-\alpha_n^4 t}}{\alpha_n^{4-n_1-n_2}}, \quad 1\le n_1+n_2\le 2. \tag{4.38}$$

Proof. Since there exists a constant $C>0$ such that

$$\left|\frac{\cos\alpha_n-\sin\alpha_n-\mathrm{e}^{-\alpha_n}}{\mathrm{e}^{\alpha_n}-\cos\alpha_n-\sin\alpha_n}\mathrm{e}^{\alpha_n x}\right|\le C, \tag{4.39}$$

$$\left|\frac{\mathrm{e}^{\alpha_n}-\mathrm{e}^{-\alpha_n}-2\sin\alpha_n}{\mathrm{e}^{\alpha_n}-\cos\alpha_n-\sin\alpha_n}\cos(\alpha_n x)\right|\le C, \tag{4.40}$$

$$\left|\frac{\mathrm{e}^{\alpha_n}+\mathrm{e}^{-\alpha_n}-2\cos\alpha_n}{\mathrm{e}^{\alpha_n}-\cos\alpha_n-\sin\alpha_n}\sin(\alpha_n y)\right|\le C, \tag{4.41}$$

we have

$$|\varphi_n(x)|\le C, \quad 0\le x\le 1 \tag{4.42}$$

from which (4.35)–(4.38) follow. $\square$

Using the Green function, we can transform the following non-homogeneous initial boundary value problem:

$$\begin{aligned}
&u_t+u_{xxxx}=f(x,t,u,u_x,u_{xx},u_{xxx}), \quad 0<x<1,\ t>0,\\
&u_{xx}(0,t)=b_1(t), \qquad u_{xxx}(0,t)=b_2(t), \quad t>0,\\
&u_{xx}(1,t)=b_3(t), \qquad u_{xxx}(1,t)=b_4(t), \quad t>0,\\
&u(x,0)=u^0(x), \quad 0<x<1
\end{aligned} \tag{4.43}$$

into an integral equation, where f, $b_1(t)$, $b_2(t)$, $b_3(t)$, $b_4(t)$ are given functions.

Lemma 4.3. *Problem* (4.43) *is equivalent to the following integral equation*:

$$\begin{aligned} u = &\int_0^1 G(x,y,t,0)u^0(y)\,\mathrm{d}y \\ &+ \int_0^t \int_0^1 G(x,y,t,\tau) f(y,\tau,u(y,\tau),u_y(y,\tau),u_{yy}(y,\tau),u_{yyy}(y,\tau))\,\mathrm{d}y\,\mathrm{d}\tau \\ &+ \int_0^t [b_3(\tau)G_y(x,1,t,\tau) - b_1(\tau)G_y(x,0,t,\tau)]\tau \\ &- \int_0^t [b_4(\tau)G(x,1,t,\tau) - b_2(\tau)G(x,0,t,\tau)]\,\mathrm{d}\tau. \end{aligned} \tag{4.44}$$

Proof. Multiplying the first equation of (4.43) by G and integrating over $(0,1)\times(0,t)$, we obtain

$$\begin{aligned} &\int_0^t \int_0^1 G(x,y,t,\tau)u_\tau(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau + \int_0^t \int_0^1 G(x,y,t,\tau)u_{yyyy}(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \\ &\quad = \int_0^t \int_0^1 G(x,y,t,\tau) f(y,\tau,u(y,\tau),u_y(y,\tau),u_{yy}(y,\tau),u_{yyy}(y,\tau))\,\mathrm{d}y\,\mathrm{d}\tau. \end{aligned} \tag{4.45}$$

Integrating by parts, we have

$$\begin{aligned} &\int_0^t \int_0^1 G(x,y,t,\tau)u_{yyyy}(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \\ &\quad = \int_0^t \int_0^1 G_{yyyy}(x,y,t,\tau)u(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \quad (\text{note that } G_{yy} = G_{yyy} = 0 \text{ at } y = 0,1) \\ &\qquad + \int_0^t [b_1(\tau)G_y(x,0,t,\tau) - b_3(\tau)G_y(x,1,t,\tau)]\,\mathrm{d}\tau \\ &\qquad + \int_0^t [b_4(\tau)G(x,1,t,\tau) - b_2(\tau)G(x,0,t,\tau)]\,\mathrm{d}\tau, \end{aligned} \tag{4.46}$$

$$\begin{aligned} \int_0^t \int_0^1 G(x,y,t,\tau)u_\tau(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau = &\int_0^1 G(x,y,t,t)u(y,t)\,\mathrm{d}y \\ &- \int_0^1 G(x,y,t,0)u^0(y)\,\mathrm{d}y \\ &- \int_0^t \int_0^1 G_\tau(x,y,t,\tau)u(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \end{aligned}$$

$$= u(x,t) - \int_0^1 G(x,y,t,0)u^0(y)\,\mathrm{d}y - \int_0^t \int_0^1 G_\tau(x,y,t,\tau)u(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau. \tag{4.47}$$

Combining the above equalities, it follows that problem (4.43) is equivalent to the integral Eq. (4.44). □

Proof of Theorem 2.3. By Lemma 4.3, problem (2.15) is equivalent to

$$\begin{aligned} u = {} & \int_0^1 G(x,y,t,0)u^0(y)\,\mathrm{d}y \\ & - \int_0^t \int_0^1 G(x,y,t,\tau)(\lambda u_{yy}(y,\tau) + u(y,\tau)u_y(y,\tau))\,\mathrm{d}y\,\mathrm{d}\tau \\ & - \int_0^t [ku_y(1,\tau)G_y(x,1,t,\tau) + ku_y(0,\tau)G_y(x,0,t,\tau)]\,\mathrm{d}\tau \\ & - \int_0^t [ku(1,\tau) + u(1,\tau)^3]G(x,1,t,\tau)\,\mathrm{d}\tau \\ & - \int_0^t [ku(0,\tau) + u(0,\tau)^3]G(x,0,t,\tau)\,\mathrm{d}\tau \\ = {} & \int_0^1 G(x,y,t,0)u^0(y)\,\mathrm{d}y \\ & + \lambda \int_0^t \int_0^1 G_y(x,y,t,\tau)u_y(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau - \int_0^t \int_0^1 G(x,y,t,\tau)u(y,\tau)u_y(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \\ & - \int_0^t [ku_y(1,\tau)G_y(x,1,t,\tau) + ku_y(0,\tau)G_y(x,0,t,\tau)]\,\mathrm{d}\tau \\ & - \int_0^t [\lambda u_y(1,\tau) + ku(1,\tau) + u(1,\tau)^3]G(x,1,t,\tau)\,\mathrm{d}\tau \\ & - \int_0^t [ku(0,\tau) + u(0,\tau)^3 - \lambda u_y(0,\tau)]G(x,0,t,\tau)\,\mathrm{d}\tau. \end{aligned} \tag{4.48}$$

We define a nonlinear operator A as

$$\begin{aligned} Au(x,t) = {} & \int_0^1 G(x,y,t,0)u^0(y)\,\mathrm{d}y \\ & + \lambda \int_0^t \int_0^1 G_y(x,y,t,\tau)u_y(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \\ & - \int_0^t \int_0^1 G(x,y,t,\tau)u(y,\tau)u_y(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \\ & - \int_0^t [ku_y(1,\tau)G_y(x,1,t,\tau) + ku_y(0,\tau)G_y(x,0,t,\tau)]\,\mathrm{d}\tau \end{aligned}$$

$$- \int_0^t [\lambda u_y(1,\tau) + ku(1,\tau) + u(1,\tau)^3] G(x,1,t,\tau)\,\mathrm{d}\tau$$

$$- \int_0^t [ku(0,\tau) + u(0,\tau)^3 - \lambda u_y(0,\tau)] G(x,0,t,\tau)\,\mathrm{d}\tau \tag{4.49}$$

and note that

$$\frac{\partial}{\partial x}(Au(x,t)) = \int_0^1 G_x(x,y,t,0)u^0(y)\,\mathrm{d}y$$

$$+ \lambda \int_0^t \int_0^1 G_{xy}(x,y,t,\tau)u_y(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau$$

$$- \int_0^t \int_0^1 G_x(x,y,t,\tau)u(y,\tau)u_y(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau$$

$$- \int_0^t [ku_y(1,\tau)G_{xy}(x,1,t,\tau) + ku_y(0,\tau)G_{xy}(x,0,t,\tau)]\,\mathrm{d}\tau$$

$$- \int_0^t [\lambda u_y(1,\tau) + ku(1,\tau) + u(1,\tau)^3] G_x(x,1,t,\tau)\,\mathrm{d}\tau$$

$$- \int_0^t [ku(0,\tau) + u(0,\tau)^3 - \lambda u_y(0,\tau)] G_x(x,0,t,\tau)\,\mathrm{d}\tau. \tag{4.50}$$

Let $C^{1,0}([0,1]\times[0,T])$ denote the set of all functions $u(x,t)$ continuously differentiable with respect to x on $[0,1]$ and continuous with respect to t on $[0,T]$. $C^{1,0}([0,1]\times[0,T])$ is a Banach space with the following norm:

$$\|u\|_\infty = \max_{\substack{0\le x\le 1\\ 0\le t\le T}} \{|u(x,t)|, |u_x(x,t)|\}. \tag{4.51}$$

Set

$$M_0 = \max_{\substack{0\le x\le 1\\ 0\le t\le T}} \left\{ \left| \int_0^1 G(x,y,t,0)u^0(y)\,\mathrm{d}y \right|, \left| \int_0^1 G_x(x,y,t,0)u^0(y)\,\mathrm{d}y \right| \right\} \tag{4.52}$$

$$B(0,2M_0) = \{u \in C^{1,0}([0,1]\times[0,T]) \colon \|u\|_\infty \le 2M_0\}. \tag{4.53}$$

Since $u^0 \in H^2(0,1)$, we have

$$\|u^0_{xx}\|^2_{L^2} = \sum_{n=1}^{\infty} c_n^2 \sigma_n = \sum_{n=1}^{\infty} c_n^2 \alpha_n^4 < +\infty, \tag{4.54}$$

where

$$c_n = \int_0^1 \varphi_n u^0\,\mathrm{d}x, \quad n = 0,1,2,\ldots \tag{4.55}$$

and $\sigma_n = \alpha_n^4$ are the eigenvalues of (3.32) and φ_n are the corresponding orthonormal eigenfunctions which are given in Lemma 3.4. Consequently, we obtain

$$\begin{aligned}\left|\int_0^1 G(x,y,t,0)u^0(y)\,\mathrm{d}y\right| &= \left|\int_0^1 \left[1+\sum_{n=1}^{\infty}\varphi_n(x)\varphi_n(y)\mathrm{e}^{-\alpha_n^4 t}\right]u^0(y)\,\mathrm{d}y\right| \\ &\le |c_0| + C\sum_{n=1}^{\infty}|c_n| \\ &\le |c_0| + C\left(\sum_{n=1}^{\infty}c_n^2\alpha_n^4\right)^{1/2}\left(\sum_{n=1}^{\infty}\alpha_n^{-4}\right)^{1/2} \\ &< +\infty \end{aligned} \tag{4.56}$$

and

$$\begin{aligned}\left|\int_0^1 G_x(x,y,t,0)u^0(y)\,\mathrm{d}y\right| &= \left|\int_0^1 \left[\sum_{n=1}^{\infty}\varphi_{nx}(x)\varphi_n(y)\mathrm{e}^{-\alpha_n^4 t}\right]u^0(y)\,\mathrm{d}y\right| \\ &\le C\sum_{n=1}^{\infty}|c_n\alpha_n| \\ &\le |c_0| + C\left(\sum_{n=1}^{\infty}c_n^2\alpha_n^4\right)^{1/2}\left(\sum_{n=1}^{\infty}\alpha_n^{-2}\right)^{1/2} \\ &< +\infty. \end{aligned} \tag{4.57}$$

Hence, we have $M_0 < \infty$. We are going to prove that A maps $B(0, 2M_0)$ into itself and is a contractive operator if T is small enough. For this, we set

$$\begin{aligned}\delta(T) = \max_{0\le n_1+n_2\le 2}\ \max_{\substack{0\le x,y\le 1\\ 0\le t\le T}} \Bigg\{ &\int_0^t\int_0^1 \left|\frac{\partial^{n_1+n_2}}{\partial x^{n_1}\partial y^{n_2}}G(x,y,t,\tau)\right|\mathrm{d}y\,\mathrm{d}\tau, \\ &\int_0^t \left|\frac{\partial^{n_1+n_2}}{\partial x^{n_1}\partial y^{n_2}}G(x,y,t,\tau)\right|\mathrm{d}\tau\Bigg\}. \end{aligned} \tag{4.58}$$

From Lemma 4.2 we get that

$$\lim_{T\to 0}\delta(T) = 0. \tag{4.59}$$

For any $u \in B(0, 2M_0)$, we easily deduce that

$$\begin{aligned}\max_{\substack{0\le x\le 1\\ 0\le t\le T}}\{|Au(x,t)|\} &\le M_0 + C\delta(T)(\|u\|_\infty + \|u\|_\infty^2 + \|u\|_\infty^3) \\ &\le M_0 + C\delta(T)(2M_0 + (2M_0)^2 + (2M_0)^3) \end{aligned} \tag{4.60}$$

and

$$\max_{\substack{0\le x\le 1\\ 0\le t\le T}} \{|(Au(x,t))_x|\} \le M_0 + C\delta(T)(\|u\|_\infty + \|u\|_\infty^2 + \|u\|_\infty^3)$$
$$\le M_0 + C\delta(T)(2M_0 + (2M_0)^2 + (2M_0)^3), \tag{4.61}$$

where C is a positive constant independent of T and u. Therefore, if T is small enough, we obtain $Au \in B(0, 2M_0)$. On the other hand, for any $u_1, u_2 \in B(0, 2M_0)$, in a similar way, we deduce that

$$\|Au_1 - Au_2\|_\infty \le C(u)\delta(T)\|u_1 - u_2\|_\infty. \tag{4.62}$$

Thus, if T is small enough, A is a contraction. By the Banach contraction mapping principle, A has a unique fixed point, and then problem (2.15) has a unique solution on $(0,T)$ which is infinitely differentiable on $[0,1]\times(0,T)$ due to the smoothness of the Green function G. Moreover, if $\lambda < 4\pi^2$, $u^0 \in H^2(0,1)$ and k is sufficiently large, then by Theorem 2.2, the solution actually exists on $[0,\infty)$ and

$$u \in C([0,\infty), H^2(0,1)). \quad \square \tag{4.63}$$

References

[1] R. Adams, Sobolev Spaces, Academic Press, New York, 1975.

[2] D. Armbruster, J. Guckenheimer, P. Holmes, Kuramoto–Sivashinsky dynamics on the center-unstable manifold, SIAM J. Appl. Math. 49 (1989) 676–691.

[3] H.S. Brown, I.G. Kevrekidis, A. Oron, P. Rosenau, Bifurcations and pattern formation in the "regularized" Kuramoto–Sivashinsky equation, Phys. Lett. A 163 (1992) 299–308.

[4] P.D. Christofides, Feedback control of the Kuramoto–Sivashinsky equation, Proceedings of the 1998 IEEE Conference on Decision and Control.

[5] P. Collet, J.-P. Eckmann, H. Epstein, J. Stubbe, A global attracting set for the Kuramoto–Sivashinsky equation, Comm. Math. Phys. 152 (1993) 203–214.

[6] A.P.V. Coward, D.T. Papageorgiou, Y.S. Smyrlis, Nonlinear stability of oscillatory core-annular flow: a generalized Kuramoto–Sivashinsky equation with time periodic coefficients, Z. Angew. Math. Phys. 46 (1995) 1–39.

[7] J. Duan, V.J. Ervin, Dynamics of a nonlocal Kuramoto–Sivashinsky equation, J. Differential Equations 143 (1998) 243–266.

[8] J. Elezgaray, G. Berkooz, H. Dankowicz, P. Holmes, M. Myers, Local models and large scale statistics of the Kuramoto–Sivashinsky equation, in: W. Dahmen, A. Kurdia, P. Oswald (Eds.), Multiscale Wavelet Methods for Partial Differential Equations, Wavelet Anal. Appl., vol. 6, Academic Press, San Diego, CA, 1997, pp. 441–471.

[9] J.N. Elgin, X. Wu, Stability of cellular states of the Kuramoto–Sivashinsky equation, SIAM J. Appl. Math. 56 (1996) 1621–1638.

[10] C. Foias, B. Nicolaenko, G.R. Sell, R. Temam, Inertial manifolds for the Kuramoto–Sivashinsky equation and an estimate of their lowest dimension, J. Math. Pures Appl. (9) 67 (1988) 197–226.

[11] J. Goodman, Stability of the Kuramoto–Sivashinsky and related systems, Comm. Pure Appl. Math. 47 (1994) 293–306.

[12] B.L. Guo, The global attractors for the periodic initial value problem of generalized Kuramoto–Sivashinsky type equations, Progr. Natur. Sci. (English Ed.) 3 (1993) 327–340.

[13] B.L. Guo, Inertial manifolds for equations of generalized Kuramoto–Sivashinsky type (Chinese), J. Math. Study 28 (1995) 50–62.

[14] B.L. Guo, H.J. Gao, Global attractor for axially symmetric Kuramoto–Sivashinsky equation in annular domains, J. Math. Study 29(2) (1996) 1–4.

[15] B.L. Guo, F.Q. Su, The global attractors for the periodic initial value problem of generalized Kuramoto–Sivashinsky type equations in multi-dimensions, J. Partial Differential Equations 6 (1993) 217–236.

[16] B. Guo, X.M. Xiang, The large time convergence of spectral method for generalized Kuramoto–Sivashinsky equations, J. Comput. Math. 15(3) (1997) 1–1.

[17] J.W. He, R. Glowinski, M. Gorman, J. Periaux, Some results on the controllability and the stabilization of Kuramoto–Sivashinsky equation, Preprint.

[18] M.S. Jolly, I.G. Kevrekidis, E.S. Titi, Approximate inertial manifolds for the Kuramoto–Sivashinsky equation: analysis and computations, Phys. D 44 (1990) 38–60.

[19] I. Kukavica, On the behavior of the solutions of the Kuramoto–Sivashinsky equation for negative time, J. Math. Anal. Appl. 166 (1992) 601–606.

[20] Y. Kuramoto, Diffusion-induced chaos in reaction systems, Suppl. Prog. Theor. Phys. 64 (1978) 346–367.

[21] Y. Kuramoto, T. Tsuzuki, On the formation of dissipative structures in reaction–diffusion systems, Prog. Theor. Phys. 54 (1975) 687–699.

[22] Y. Kuramoto, T. Tsuzuki, Persistent propagation of concentration waves in dissipative media far from thermal equilibrium, Prog. Theor. Phys. 55 (1976) 356–369.

[23] R.E. Laquey, S.M. Mahajan, P.H. Rutherford, W.M. Tang, Nonlinear saturation of the trapped-ion mode, Phys. Rev. Lett. 34 (1975) 391–394.

[24] I. Lc, Z. Yang, Y. Wu, Bifurcation and stability of nontrivial solution to Kuramoto–Sivashinsky equation, J. Shanghai Univ. 1 (1997) 95–97.

[25] J.L Lions, E. Magenes, Non-homogeneous Boundary value Problems and Applications, vol. I, Springer, Berlin, 1972.

[26] W.J. Liu, M. Krstić, Global boundary stabilization of the Korteweg–de Vries–Burgers equation, SIAM J. Control Optim., submitted for publication.

[27] D. Michelson, Stability of the Bunsen flame profiles in the Kuramoto–Sivashinsky equation, SIAM J. Math. Anal. 27 (1996) 765–781.

[28] B. Nicolaenko, B. Scheurer, R. Temam, Attractors for the Kuramoto–Sivashinsky equations, in Nonlinear systems of partial differential equations in applied mathematics, Part 2 (Santa Fe, N.M., 1984), Lectures in Appl. Math., vol. 23, Amer. Math. Soc., Providence, R.I., 1986, pp. 149–170.

[29] B. Nicolaenko, B. Scheurer, R. Temam, Quelques propriétés des attracteurs pour l'équation de Kuramoto–Sivashinsky, C. R. Acad. Sci. Paris Sér. I Math. 298 (1984) 23–25.

[30] B. Nicolaenko, B. Scheurer, R. Temam, Some global dynamical properties of the Kuramoto–Sivashinsky equations: nonlinear stability and attractors, Phys. D 16 (1985) 155–183.

[31] A. Novick-Cohen, G.I. Sivashinsky, Instability in solidification fronts of dilute binary alloys: the Kuramoto–Sivashinsky equation, Phys. D. 20 (1986) 237–258.

[32] G.R. Sell, M. Taboada, Local dissipativity and attractors for the Kuramoto–Sivashinsky equation in thin 2D domains, Nonlinear Anal. 18 (1992) 671–687.

[33] G.I. Sivashinsky, Nonlinear analysis of hydrodynamic instability in laminar flames — I: Derivation of basic equations, Acta Astronaut. 4 (1977) 1177–1206.

[34] R. Temam, Infinite-dimensional Dynamical Systems in Mechanics and Physics, 2nd Edition, Springer, New York, 1997.

[35] R. Temam, X.M. Wang, Estimates on the lowest dimension of inertial manifolds for the Kuramoto–Sivashinsky equation in the general case, Differential Integral Equations 7 (1994) 1095–1108.

[36] T.-S. Yang, On traveling-wave solutions of the Kuramoto–Sivashinsky equation, Phys. D 110 (1997) 25–42.

[37] G.S. Yang, Z.D. Dai, Global attractors of a class of degenerate Kuramoto–Sivashinsky equations (Chinese), J. Math. Study 29 (1996) 52–62.

[38] L.H. Zhang, Decay of solutions of the multidimensional generalized Kuramoto–Sivashinsky system, IMA J. Appl. Math. 50 (1993) 29–42.