STABILITY FOR BORELL-BRASCAMP-LIEB INEQUALITIES

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ABSTRACT. We study stability issues for the so-called Borell-Brascamp-Lieb inequalities, proving that when near equality is realized, the involved functions must be L^1-close to be p-concave and to coincide up to homotheties of their graphs.

1. Introduction

The aim of this paper is to study the stability of the so-called *Borell-Brascamp-Lieb inequality* (BBL inequality below), which we recall hereafter.

Proposition 1.1 (BBL inequality). *Let* $0 < \lambda < 1, -\frac{1}{n} \leqslant p \leqslant +\infty$, $0 \leqslant f, g, h \in L^1(\mathbb{R}^n)$ *and assume the following holds*

$$h((1-\lambda)x + \lambda y) \geqslant \mathcal{M}_p(f(x), g(y); \lambda) \tag{1.1}$$

for every x, $y \in \mathbb{R}^n$. *Then*

$$\int_{\mathbb{R}^n} h\,dx \geqslant \mathcal{M}_{\frac{p}{np+1}}\left(\int_{\mathbb{R}^n} f\,dx, \int_{\mathbb{R}^n} g\,dx\,; \lambda\right). \tag{1.2}$$

Here the number $p/(np+1)$ has to be interpreted in the obvious way in the extremal cases (i.e. it is equal to $-\infty$ when $p = -1/n$ and to $1/n$ when $p = +\infty$) and the quantity $\mathcal{M}_q(a, b; \lambda)$ represents the (λ-weighted) *q-mean* of two nonnegative numbers a and b, that is $\mathcal{M}_q(a, b; \lambda) = 0$ if $ab = 0$ for every $q \in \mathbb{R} \cup \{\pm\infty\}$ and

$$\mathcal{M}_q(a, b; \lambda) = \begin{cases} \max\{a, b\} & q = +\infty\,, \\ [(1-\lambda)a^q + \lambda b^q]^{\frac{1}{q}} & 0 \neq q \in \mathbb{R}\,, \\ a^{1-\lambda} b^{\lambda} & q = 0\,, \\ \min\{a, b\} & q = -\infty\,, \end{cases} \quad \text{if } ab > 0\,. \tag{1.3}$$

The BBL inequality was first proved (in a slightly different form) for $p > 0$ by Henstock and Macbeath (with $n = 1$) in [25] and by Dinghas in [14]. Then it was generalized by Brascamp and Lieb in [6] and by Borell in [4]. The case $p = 0$ is usually known as *Prékopa-Leindler inequality*, as it was previously proved by Prékopa [29] and Leindler [27] (later rediscovered by Brascamp and Lieb in [5]).

In this paper we deal only with the case $p > 0$ and are particularly interested in the equality conditions of BBL, that are discussed in [16] (see Theoreme 12 therein). To avoid triviality, if not otherwise explicitly declared, we will assume throughout the paper that f, $g \in L^1(\mathbb{R}^n)$ are nonnegative compactly supported functions (with supports Supp(f) and Supp(g)) such that

$$F = \int_{\mathbb{R}^n} f\,dx > 0 \quad \text{and} \quad G = \int_{\mathbb{R}^n} g\,dx > 0\,.$$

Let us restate a version of the BBL inequality including its equality condition in the case

$$p = \frac{1}{s} > 0\,,$$

adopting a slightly different notation.

Proposition 1.2. *Let $s > 0$ and f, g be as said above. Let $\lambda \in (0,1)$ and h be a nonnegative function belonging to $L^1(\mathbb{R}^n)$ such that*

$$h((1-\lambda)x + \lambda y) \geqslant \Big((1-\lambda)f(x)^{1/s} + \lambda g(y)^{1/s}\Big)^s \tag{1.4}$$

for every $x \in \operatorname{Supp}(f)$, $y \in \operatorname{Supp}(g)$.

Then

$$\int_{\mathbb{R}^n} h\;dx \geqslant \mathcal{M}_{\frac{1}{n+s}}(F, G; \lambda)\;. \tag{1.5}$$

Moreover equality holds in (1.5) if and only if there exists a nonnegative concave function φ such that

$$\varphi(x)^s = a_1\, f(b_1 x - \bar{x}_1) = a_2\, g(b_2 x - \bar{x}_2) = a_3\, h(b_3 x - \bar{x}_3) \quad a.e.\ x \in \mathbb{R}^n\,, \tag{1.6}$$

for some $\bar{x}_1, \bar{x}_2, \bar{x}_3 \in \mathbb{R}^n$ and suitable $a_i, b_i > 0$ for $i = 1, 2, 3$.

Notice that, given f and g, the smallest function satisfying (1.4) (hence the smallest function to which Proposition 1.2 possibly applies to) is their p-Minkowksi sum (or (p,λ)-supremal convolution), defined as follows (for $p = \frac{1}{s}$)

$$h_{s,\lambda}(z) = \sup\Big\{\Big((1-\lambda)f(x)^{1/s} + \lambda g(y)^{1/s}\Big)^s \;:\; z = (1-\lambda)x + \lambda y\Big\} \tag{1.7}$$

for $z \in (1-\lambda)\operatorname{Supp}(f) + \lambda \operatorname{Supp}(g)$ and $h_{s,\lambda}(z) = 0$ if $z \notin (1-\lambda)\operatorname{Supp}(f) + \lambda \operatorname{Supp}(g)$.

When dealing with a rigid inequality, a natural question arises about the stability of the equality case; here the question at hand is the following: if we are close to equality in (1.5), must the functions f, g and h be close (in some suitable sense) to satisfy (1.6)?

The investigation of stability issues in the case $p = 0$ was started by Ball and Böröczky in [2, 3] and new related results are in [7]. The general case $p > 0$ has been very recently faced in [22]. But the results of [22], as well as the quoted results for $p = 0$, hold only in the restricted class of p-concave functions, hence answering only a half of the question. Let us recall here the definition of p-concave function: a nonnegative function u is *p-concave* for some $p \in \mathbb{R} \cup \{\pm\infty\}$ if

$$u((1-\lambda)x + \lambda y) \geqslant \mathcal{M}_p(u(x), u(y); \lambda) \quad \text{for every } x, y \in \mathbb{R}^n \text{ and every } \lambda \in (0,1)\,.$$

Roughly speaking, u is p-concave if it has convex support Ω and: (i) u^p is concave in Ω for $p > 0$; (ii) $\log u$ is concave in Ω for $p = 0$; (iii) u^p is convex in Ω for $p < 0$; (iv) u is quasi-concave, i.e. all its superlevel sets are convex, for $p = -\infty$; (v) u is a positive constant in Ω, for $p = +\infty$.

Here we want to remove this restriction, proving that near equality in (1.5) is possible if and only if the involved functions are close to coincide up to homotheties of their graphs and they are also nearly p-concave, in a suitable sense. But before stating our main result in detail, we need to introduce some notation: for $s > 0$, we say that two functions $v, \hat{v} : \mathbb{R}^n \to [0, +\infty)$ are *s-equivalent* if there exist $\mu_v > 0$ and $\bar{x} \in \mathbb{R}^n$ such that

$$\hat{v}(x) = \mu_v^s \, v\left(\frac{x - \bar{x}}{\mu_v}\right) \qquad \text{a.e. } x \in \mathbb{R}^n. \tag{1.8}$$

Now we are ready to state our main result, which regards the case $s = 1/p \in \mathbb{N}$. Later (see §4) we will extend the result to the case $0 < s \in \mathbb{Q}$ in Corollary 4.3 and finally (see Corollary 5.1 in §5) we will give a slightly weaker version, valid for every $s > 0$.

Theorem 1.3. *Let f, g, h as in Proposition 1.2 with*

$$0 < s \in \mathbb{N}\,.$$

Assume that

$$\int_{\mathbb{R}^n} h \; dx \leqslant \mathcal{M}_{\frac{1}{n+s}}(F, G\,; \lambda) + \varepsilon \tag{1.9}$$

for some $\varepsilon > 0$ small enough.

Then there exist a $\frac{1}{s}$-concave function $u : \mathbb{R}^n \longrightarrow [0, +\infty)$ and two functions $\hat{f}$ and $\hat{g}$, s-equivalent to f and g in the sense of (1.8) *(with suitable μ_f and μ_g given in* (3.15)*) and with support sets Ω_0 and Ω_1 respectively, such that the following hold:*

$$\mathrm{Supp}(u) \supseteq (\Omega_0 \cup \Omega_1)\,, \qquad u \geqslant \hat{f} \text{ in } \Omega_0, \qquad u \geqslant \hat{g} \text{ in } \Omega_1\,, \tag{1.10}$$

$$\int_{\mathbb{R}^n} (u - \hat{f}) \; dx \; + \; \int_{\mathbb{R}^n} (u - \hat{g}) \; dx \; \leqslant \; C_{n+s}\left(\frac{\varepsilon}{\mathcal{M}_{\frac{1}{n+s}}(F, G\,; \lambda)}\right)\,, \tag{1.11}$$

where $C_{n+s}(\eta)$ is an infinitesimal function for $\eta \longrightarrow 0$ (whose explicit expression is given later, see (2.4)*).*

Notice that the function u is bounded, hence as a byproduct of the proof we obtain that the functions f and g have to be bounded as well (see Remark 3.1). The proof of the above theorem is based on a proof of the BBL inequality due to Klartag [26], which directly connects the BBL inequality to the Brunn-Minkowski inequality, and the consequent application of a recent stability result for the Brunn-Minkowski inequality by Figalli and Jerison [18], which does not require any convexity assumption of the involved sets. Indeed [18] is the first paper, at our knowledge, investigating on stability issues for the Brunn-Minkowski inequality outside the realm of convex bodies. Noticeably, Figalli and Jerison ask therein for a functional counterpart of their result, pointing out that *"at the moment some stability estimates are known for the Prékopa-Leindler inequality only in one dimension or for some special class of functions* [2, 3]*, and a general stability result would be an important direction of future investigations."*. Since BBL inequality is the functional counterpart of the Brunn-Minkowksi inequality (for any $p > 0$ as much as for $p = 0$), this paper can be considered a first answer to the question by Figalli and Jerison.

The paper is organized as follows. The Brunn-Minkowski inequality and the stability result of [18] are recalled in §2, where we also discuss the equivalence between the Brunn-Minkowski and the BBL inequality. In §3 we prove Theorem 1.3. Finally §4 contains the already mentioned generalization to the case of rational s, namely Corollary 4.3, while §5 is devoted to Corollary 5.1, where we prove a stability for every $s > 0$ under a suitable normalization for $\int f$ and $\int g$. The paper ends with an Appendix (§6) where we give the proofs of some easy technical lemmas for the reader's convenience.

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2. Preliminaries

2.1. **Notation.** Throughout the paper the symbol $|\cdot|$ is used to denote different things and we hope this is not going to cause confusion. In particular: for a real number a we denote by $|a|$ its absolute value, as usual; for a vector $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ we denote by $|x|$ its euclidean norm, that is $|x| = \sqrt{x_1^2 + \cdots + x_m^2}$; for a set $A \subset \mathbb{R}^m$ we denote by $|A|$ its (m-dimensional) Lebesgue measure or, sometimes, its outer measure if A is not measurable.

The support set of a nonnegative function $f : \mathbb{R}^m \to [0, +\infty)$ is denoted by $\mathrm{Supp}(f)$, that is $\mathrm{Supp}(f) = \overline{\{x \in \mathbb{R}^m : f(x) > 0\}}$.

Let $\lambda \in (0,1)$, the Minkowski convex combination (of coefficient λ) of two nonempty sets $A, B \subseteq \mathbb{R}^n$ is given by

$$(1-\lambda)A + \lambda B = \{(1-\lambda)a + \lambda b : \ a \in A, \ b \in B\}.$$

2.2. **About the Brunn-Minkowski inequality.** The classical form of the Brunn-Minkowski inequality (BM in the following) regards only convex bodies and it is at the core of the related theory (see [32]). Its validity has been extended later to the class of measurable sets and we refer to the beautiful paper by Gardner [21] for a throughout presentation of BM inequality, its history and its intriguing relationships with many other important geometric and analytic inequalities. Let us now recall it (in its general form).

Proposition 2.1 (Brunn-Minkowski inequality)**.** *Given $\lambda \in (0,1)$, let $A, B \subseteq \mathbb{R}^n$ be nonempty measurable sets. Then*

$$|(1-\lambda)A + \lambda B|^{1/n} \geqslant (1-\lambda)\,|A|^{1/n} + \lambda\,|B|^{1/n} \tag{2.1}$$

(where $|\cdot|$ possibly means outer measure if $(1-\lambda)A + \lambda B$ is not measurable).

In addition, if $|A|, |B| > 0$, then equality in (2.1) holds if and only if there exist a convex set $K \subseteq \mathbb{R}^n$, $v_1, v_2 \in \mathbb{R}^n$ and $\lambda_1, \lambda_2 > 0$ such that

$$\lambda_1 A + v_1 \subseteq K, \quad \lambda_2 B + v_2 \subseteq K, \quad |K \setminus (\lambda_1 A + v_1)| = |K \setminus (\lambda_2 B + v_2)| = 0. \tag{2.2}$$

We remark that equality holds in (2.1) if and only if the involved sets are convex (up to a null measure set) and homothetic.

The stability of BM inequality was first investigated only in the class of convex sets, see for instance [15, 23, 19, 20, 31]. Very recently Christ [10, 11] started the investigation without convexity assumptions, and its qualitative results have been made quantitative and sharpened by Figalli and Jerison in [18]; here is their result, for $n \geqslant 2$.

Proposition 2.2. *Let $n \geqslant 2$, and $A, B \subset \mathbb{R}^n$ be measurable sets with $|A| = |B| = 1$. Let $\lambda \in (0,1)$, set $\tau = \min\{\lambda, 1-\lambda\}$ and $S = (1-\lambda)A + \lambda B$. If*

$$|S| \ \leqslant \ 1 + \delta \tag{2.3}$$

for some $\delta \leqslant e^{-M_n(\tau)}$*, then there exists a convex* $K \subset \mathbb{R}^n$ *such that, up to a translation,*

$$A, B \subseteq K \qquad \text{and} \qquad |K \setminus A| + |K \setminus B| \leqslant \tau^{-N_n} \delta^{\sigma_n(\tau)}.$$

The constant N_n *can be explicitly computed and we can take*

$$M_n(\tau) = \frac{2^{3^{n+2}} n^{3^n} |\log \tau|^{3^n}}{\tau^{3^n}}, \qquad \sigma_n(\tau) = \frac{\tau^{3^n}}{2^{3^{n+1}} n^{3^n} |\log \tau|^{3^n}}.$$

Remark 2.3. As already said, the proof of our main result is based on Proposition 2.2 and now we can give the explicit expression of the infinitesimal function C_{n+s} of Theorem 1.3:

$$C_{n+s}(\eta) = \frac{\eta^{\sigma_{n+s}(\tau)}}{\omega_s \, \tau^{N_{n+s}}}, \tag{2.4}$$

where ω_s denotes the measure of the unit ball in $\mathbb{R}^s$.

Next, for further use, we rewrite Proposition 2.2 without the normalization constraint about the measures of the involved sets A and B.

Corollary 2.4. *Let* $n \geqslant 2$ *and* $A, B \subset \mathbb{R}^n$ *be measurable sets with* $|A|, |B| \in (0, +\infty)$*. Let* $\lambda \in (0,1)$*, set* $\tau = \min\{\lambda, 1-\lambda\}$ *and* $S = (1-\lambda)A + \lambda B$*. If*

$$\frac{|S| - \left[(1-\lambda)|A|^{1/n} + \lambda |B|^{1/n}\right]^n}{\left[(1-\lambda)|A|^{1/n} + \lambda |B|^{1/n}\right]^n} \leqslant \delta \tag{2.5}$$

for some $\delta \leqslant e^{-M_n(\tau)}$*, then there exist a convex* $K \subset \mathbb{R}^n$ *and two homothetic copies* $\tilde{A}$ *and* $\tilde{B}$ *of* A *and* B *such that*

$$\tilde{A}, \tilde{B} \subseteq K \qquad \text{and} \qquad \left|K \setminus \tilde{A}\right| + \left|K \setminus \tilde{B}\right| \leqslant \tau^{-N_n} \delta^{\sigma_n(\tau)}.$$

Proof. The proof is standard and we give it just for the sake of completeness. First we set

$$\tilde{A} = \frac{A}{|A|^{1/n}}, \qquad \tilde{B} = \frac{B}{|B|^{1/n}}$$

so that $|\tilde{A}| = |\tilde{B}| = 1$. Then we define

$$\tilde{S} := \mu \tilde{A} + (1-\mu)\tilde{B} \qquad \text{with} \quad \mu = \frac{(1-\lambda)|A|^{1/n}}{(1-\lambda)|A|^{1/n} + \lambda |B|^{1/n}},$$

and observe that $|\tilde{S}| \geqslant 1$ by the Brunn-Minkowski inequality. It is easily seen that

$$\tilde{S} = \frac{S}{(1-\lambda)|A|^{1/n} + \lambda |B|^{1/n}}.$$

Now we see that the hypothesis (2.3) holds for $\tilde{A}, \tilde{B}, \tilde{S}$, indeed

$$\left|\tilde{S}\right| - 1 = \frac{|S| - \left[(1-\lambda)|A|^{1/n} + \lambda |B|^{1/n}\right]^n}{\left[(1-\lambda)|A|^{1/n} + \lambda |B|^{1/n}\right]^n} \leqslant \delta,$$

by (2.5). Finally Proposition 2.2 applied to $\tilde{A}, \tilde{B}$ and $\tilde{S}$ implies the result and this concludes the proof. □

2.3. **The equivalence between BBL and BM inequalities.** The equivalence between the two inequalities is well known and it becomes apparent as soon as one notices that the (p,λ)-supremal convolution defined in (1.7) corresponds to the Minkowski linear combinations of the graphs of f^p and g^p. In particular, for $p = 1$, (1.2) coincides with (2.1) where $A = \{(x,t) \in \mathbb{R}^{n+1} \,:\, 0 \leqslant t \leqslant f(x)\}$ and $B = \{(x,t) \in \mathbb{R}^{n+1} \,:\, 0 \leqslant t \leqslant g(x)\}$.

To be precise, that Proposition 1.1 implies (2.1) is easily seen by applying (1.2) to the case $f = \chi_A$, $g = \chi_B$, $h = \chi_{(1-\lambda)A+\lambda B}$, $p = +\infty$. The opposite implication can be proved in several ways; hereafter we present a proof due to Klartag [26], which is particularly useful for our goals.

To begin, given two integers $n, s > 0$, let $f : \mathbb{R}^n \longrightarrow [0,+\infty)$ be an integrable function with nonempty support (to avoid the trivial case in which f is identically zero). Following Klartag's notations and ideas [26] (see also [1]), we associate with f the nonempty measurable set

$$K_{f,s} = \left\{(x,y) \in \mathbb{R}^{n+s} = \mathbb{R}^n \times \mathbb{R}^s :\ x \in \mathrm{Supp}(f),\ |y| \leqslant f(x)^{1/s}\right\}, \tag{2.6}$$

where obviously $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^s$. In other words, $K_{f,s}$ is the subset of $\mathbb{R}^{n+s}$ obtained as union of the s-dimensional closed balls of center $(x,0)$ and radius $f(x)^{1/s}$, for x belonging to the support of f, or, if you prefer, the set in $\mathbb{R}^{n+s}$ obtained by rotating with respect to $y = 0$ the $(n+1)$-dimensional set $\{(x,y) \in \mathbb{R}^{n+s} : 0 \leqslant y_1 \leqslant f(x)^{1/s},\ y_2 = \cdots = y_s = 0\}$.
We observe that $K_{f,s}$ is convex if and only if f is $(1/s)$-concave (that is for us a function f having compact convex support such that $f^{1/s}$ is concave on $\mathrm{Supp}\,(f)$). If $\mathrm{Supp}(f)$ is compact, then $K_{f,s}$ is bounded if and only if f is bounded.
Moreover, thanks to Fubini's Theorem, it holds

$$|K_{f,s}| = \int_{\mathrm{Supp}(f)} \omega_s \cdot \left(f(x)^{1/s}\right)^s\, dx = \omega_s \int_{\mathbb{R}^n} f(x)\, dx. \tag{2.7}$$

In this way, the integral of f coincides, up to the constant ω_s, with the volume of $K_{f,s}$. Now we will use this simple identity to prove Proposition 1.2 as a direct application of the BM inequality.

Although of course the set $K_{f,s}$ depends heavily on s, for simplicity from now on we will remove the subindex s and just write K_f for $K_{f,s}$.

Let us start with the simplest case, when $p = 1/s$ with s positive integer.

Proposition 2.5 (BBL, case $1/p = s \in \mathbb{N}$)**.** *Let n, s be positive integers, $\lambda \in (0,1)$ and $f, g, h : \mathbb{R}^n \longrightarrow [0,+\infty)$ be integrable functions, with $\int f > 0$ and $\int g > 0$. Assume that for any $x_0, x_1 \in \mathbb{R}^n$*

$$h\left((1-\lambda)x_0 + \lambda x_1\right) \geqslant \left[(1-\lambda)f(x_0)^{1/s} + \lambda g(x_1)^{1/s}\right]^s. \tag{2.8}$$

Then

$$\left(\int_{\mathbb{R}^n} h\, dx\right)^{\frac{1}{n+s}} \geqslant (1-\lambda)\left(\int_{\mathbb{R}^n} f\, dx\right)^{\frac{1}{n+s}} + \lambda\left(\int_{\mathbb{R}^n} g\, dx\right)^{\frac{1}{n+s}}. \tag{2.9}$$

Proof. Since the integrals of f and g are positive, the sets K_f and K_g have positive measure. Let Ω_λ be the Minkowski convex combination (with coefficient λ) of $\Omega_0 = \mathrm{Supp}(f)$ and $\Omega_1 = \mathrm{Supp}(g)$. Now consider the function $h_{s,\lambda}$ as defined by

(1.7); to simplify the notation, we will denote $h_{s,\lambda}$ by h_λ from now on. First notice that the support of h_λ is Ω_λ. Then it is easily seen that

$$K_{h_\lambda} = (1-\lambda)K_f + \lambda K_g\,. \tag{2.10}$$

Moreover, since $h \geqslant h_\lambda$ by assumption (2.8), we have

$$K_h \supseteq K_{h_\lambda}\,. \tag{2.11}$$

By applying Proposition 2.1 to K_{h_λ}, K_f, K_g we get

$$|K_h|^{\frac{1}{n+s}} \geqslant |K_{h_\lambda}|^{\frac{1}{n+s}} \geqslant (1-\lambda)\,|K_f|^{\frac{1}{n+s}} + \lambda\,|K_g|^{\frac{1}{n+s}}\,, \tag{2.12}$$

where $|K_{h_\lambda}|$ possibly means the outer measure of the set K_{h_λ}.
Finally (2.7) yields

$$|K_h| = \omega_s \int_{\mathbb{R}^n} h\ dx, \qquad |K_f| = \omega_s \int_{\mathbb{R}^n} f\ dx, \qquad |K_g| = \omega_s \int_{\mathbb{R}^n} g\ dx,$$

thus dividing (2.12) by $\omega_s^{\frac{1}{n+s}}$ we get (2.9). □

Next we show how it is possible to generalize Proposition 2.5 to a positive rational index s. The idea is to apply again the Brunn-Minkowski inequality to sets that generalize those of the type (2.6). What follows is a slight variant of the proof of Theorem 2.1 in [26].

The case of a positive rational index s requires the following definition. Given $f : \mathbb{R}^n \longrightarrow [0,+\infty)$ integrable and a positive integer q (it will be the denominator of the rational s) we consider the auxiliary function $\tilde{f} : \mathbb{R}^{nq} \longrightarrow [0,+\infty)$ defined as

$$\tilde{f}(x) = \tilde{f}(x_1, ..., x_q) = \prod_{j=1}^{q} f(x_j), \tag{2.13}$$

where $x = (x_1, ..., x_q) \in (\mathbb{R}^n)^q$. We observe that, by construction,

$$\int_{\mathbb{R}^{nq}} \tilde{f}\ dx = \left(\int_{\mathbb{R}^n} f\ dx\right)^q ; \tag{2.14}$$

moreover $\quad \operatorname{Supp} \tilde{f} = (\operatorname{Supp} f) \times ... \times (\operatorname{Supp} f) = (\operatorname{Supp} f)^q\,.$

As just done, from now on we write A^q to indicate the Cartesian product of q copies of a set A.

Remark 2.6. Let A, B be nonempty sets, $q > 0$ be an integer, μ a real. Clearly

$$(A+B)^q = A^q + B^q, \qquad\qquad (\mu A)^q = \mu A^q.$$

To compare products of real numbers of the type (2.13) the following lemma is useful. It's a consequence of Hölder's inequality (see [24], Theorem 10) for families of real numbers (in our case for two sets of q positive numbers).

Lemma 2.7. *Given an integer $q > 0$, let $\{a_1, ..., a_q\}$, $\{b_1, ..., b_q\}$ be two sets of q real numbers. Then*

$$\left|\prod_{j=1}^{q} a_j\right| + \left|\prod_{j=1}^{q} b_j\right| \leqslant \left[\prod_{j=1}^{q} (|a_j|^q + |b_j|^q)\right]^{1/q}.$$

From this lemma we deduce the following.

Corollary 2.8. *Let* $\lambda \in (0,1),\ s=\frac{p}{q}$ *with integers* $p,q>0$.
Given $f,g:\mathbb{R}^n \longrightarrow [0,+\infty),\ x_1,...,x_q,x'_1,...,x'_q \in \mathbb{R}^n$, *it holds*

$$(1-\lambda)\prod_{j=1}^{q} f(x_j)^{1/p} + \lambda \prod_{j=1}^{q} g(x'_j)^{1/p} \leqslant \prod_{j=1}^{q}\Big[(1-\lambda)f(x_j)^{1/s}+\lambda g(x'_j)^{1/s}\Big]^{1/q}.$$

Proof. Observing that

$$(1-\lambda)\prod_{j=1}^{q} f(x_j)^{1/p} + \lambda \prod_{j=1}^{q} g(x'_j)^{1/p} = \prod_{j=1}^{q}(1-\lambda)^{1/q} f(x_j)^{1/p} + \prod_{j=1}^{q} \lambda^{1/q} g(x'_j)^{1/p},$$

the result follows directly from Lemma 2.7 applied to $\{a_1,...,a_q\}$, $\{b_1,...,b_q\}$ with

$$a_j=(1-\lambda)^{1/q}f(x_j)^{1/p}, \qquad b_j=\lambda^{1/q}g(x'_j)^{1/p}, \qquad j=1,...,q.$$

□

Let

$$s=\frac{p}{q}$$

with integers $p,q>0$ that we can assume are coprime.
Given an integrable function $f:\mathbb{R}^n \longrightarrow [0,+\infty)$ not identically zero, we define the nonempty measurable subset of $\mathbb{R}^{nq+p}$

$$W_{f,s}=K_{\tilde f,p}=\Big\{(x,y)\in(\mathbb{R}^n)^q\times\mathbb{R}^p:\ x\in \mathrm{Supp}(\tilde f),\ |y|\leqslant \tilde f(x)^{1/p}\Big\}= \tag{2.15}$$

$$\left\{(x_1,...,x_q,y)\in(\mathbb{R}^n)^q\times\mathbb{R}^p:\ x_j\in\mathrm{Supp}(f)\ \forall\, j=1,...,q,\quad |y|\leqslant\prod_{j=1}^{q} f(x_j)^{1/p}\right\}.$$

We notice that this definition naturally generalizes (2.6), since in the case of an integer $s>0$ it holds $s=p,\ q=1$, so in this case $\tilde f=f$ and $W_{f,s}=K_f$.

As for $K_{f,s}$, for simplicity we will remove systematically the subindex s and write W_f in place of $W_{f,s}$ if there is no possibility of confusion. Clearly
(2.16)

$$|W_f|=\int_{\mathrm{Supp}(\tilde f)}\omega_p\cdot\Big(\tilde f(x)^{1/p}\Big)^p\ dx=\omega_p\int_{\mathbb{R}^{nq}}\tilde f(x)\ dx=\omega_p\left(\int_{\mathbb{R}^n} f(x)\ dx\right)^q$$

where the last equality is given by (2.14).
Moreover we see that W_f is convex if and only if $\tilde f$ is $\frac{1}{p}$-concave (that is, if and only if f is $\frac{1}{s}$-concave, see Lemma 4.1 later on). Next we set

$$W=(1-\lambda)W_f+\lambda W_g\,. \tag{2.17}$$

Finally, we notice that, by (2.10), we have

$$W=K_{\tilde h_{p,\lambda},p}\,,$$

where $\tilde h_{p,\lambda}$ is the $(1/p,\lambda)$-supremal convolution of $\tilde f$ and $\tilde g$ as defined in (1.7). In other words, W is the set made by the elements $(z,y)\in(\mathbb{R}^n)^q\times\mathbb{R}^p$ such that $z\in(1-\lambda)\,\mathrm{Supp}(\tilde f)+\lambda\,\mathrm{Supp}(\tilde g)$ and

$$|y|\leqslant\ \sup\big\{(1-\lambda)\tilde f(x)^{1/p}+\lambda\tilde g(x')^{1/p}:\ z=(1-\lambda)x+\lambda x',\, x\in\mathrm{Supp}(\tilde f),\, x'\in\mathrm{Supp}(\tilde g)\big\}. \tag{2.18}$$

Lemma 2.9. *With the notations introduced above, it holds*

$$W \subseteq W_{h_\lambda} \subseteq W_h\,,$$

where h_λ *is the* $(1/s,\lambda)$*-supremal convolution of* f,g, *and* h *is as in Proposition 1.2.*

Proof. The second inclusion is obvious, since $h \geqslant h_\lambda$ by assumption (1.4). Regarding the other inclusion, first we notice that (2.15) and Remark 2.6 yield

$$W_{h_\lambda} = \left\{(z,y)\in(\mathbb{R}^n)^q\times\mathbb{R}^p:\ z\in \operatorname{Supp}(\tilde{h_\lambda}),\ |y|\leqslant \tilde{h_\lambda}(z)^{1/p}\right\} =$$

$$= \left\{(z,y)\in(\mathbb{R}^n)^q\times\mathbb{R}^p:\ z\in ((1-\lambda)\operatorname{Supp}(f)+\lambda\operatorname{Supp}(g))^q\,,\ |y|\leqslant \tilde{h_\lambda}(z)^{1/p}\right\} =$$

$$= \left\{(z,y)\in(\mathbb{R}^n)^q\times\mathbb{R}^p:\ z\in (1-\lambda)\operatorname{Supp}(\tilde{f})+\lambda\operatorname{Supp}(\tilde{g}),\ |y|\leqslant \tilde{h_\lambda}(z)^{1/p}\right\}\,,$$

where $\tilde{h_\lambda}$ is the function associated to h_λ by (2.13). To conclude it is sufficient to compare this with the condition given by (2.18).
For every $z\in(1-\lambda)\operatorname{Supp}(\tilde{f})+\lambda\operatorname{Supp}(\tilde{g})$ consider

$$\sup\left\{(1-\lambda)\tilde{f}(x)^{1/p}+\lambda\tilde{g}(x')^{1/p}\right\} = \sup\left\{(1-\lambda)\prod_{j=1}^q f(x_j)^{1/p}+\lambda\prod_{j=1}^q g(x'_j)^{1/p}\right\}\,,$$

where the supremum is made with respect to $x\in\operatorname{Supp}(\tilde{f}),\ x'\in\operatorname{Supp}(\tilde{g})$ such that $z=(1-\lambda)x+\lambda x'$. Corollary 2.8 then implies

$$\sup\left\{(1-\lambda)\tilde{f}(x)^{1/p}+\lambda\tilde{g}(x')^{1/p}\right\} \leqslant \sup\left\{\prod_{j=1}^q\left[(1-\lambda)f(x_j)^{1/s}+\lambda g(x'_j)^{1/s}\right]^{1/q}\right\}\leqslant$$

$$\leqslant\prod_{j=1}^q\left\{\sup\left[(1-\lambda)f(x_j)^{1/s}+\lambda g(x'_j)^{1/s}\right]^{1/q}\right\} = \prod_{j=1}^q\left\{h_\lambda\left((1-\lambda)x_j+\lambda x'_j\right)^{1/qs}\right\} =$$

$$= \tilde{h_\lambda}\left((1-\lambda)x+\lambda x'\right)^{1/p} = \tilde{h_\lambda}(z)^{1/p},$$

having used the definition (2.13) in the penultimate equality. Therefore if

$$|y|\leqslant\sup\left\{(1-\lambda)\tilde{f}(x)^{1/p}+\lambda\tilde{g}(x')^{1/p}\right\}\,,$$

that is if $(z,y)\in W$ by (2.18), then

$$|y|\leqslant \tilde{h_\lambda}(z)^{1/p}\,,$$

i.e. $(z,y)\in W_{h_\lambda}$. This concludes the proof. □

We are ready to prove the following version of the Borell-Brascamp-Lieb inequality, which holds for any positive real index s (and in fact also for $s=0$).

Proposition 2.10 (BBL for $p>0$)**.** *Let* $s>0$, $\lambda\in(0,1)$, *let* $n>0$ *be integer. Given* $f,g,h:\mathbb{R}^n\longrightarrow[0,+\infty)$ *integrable such that* $\int f>0$ *and* $\int g>0$, *assume that for any* $x_0,x_1\in\mathbb{R}^n$

$$h\left((1-\lambda)x_0+\lambda x_1\right)\geqslant\left[(1-\lambda)f(x_0)^{1/s}+\lambda g(x_1)^{1/s}\right]^s. \tag{2.19}$$

Then

$$\left(\int_{\mathbb{R}^n}h\ dx\right)^{\frac{1}{n+s}}\geqslant(1-\lambda)\left(\int_{\mathbb{R}^n}f\ dx\right)^{\frac{1}{n+s}}+\lambda\left(\int_{\mathbb{R}^n}g\ dx\right)^{\frac{1}{n+s}}. \tag{2.20}$$

Proof. Assume first that $s > 0$ is rational and let $s = \frac{p}{q}$ with p, q coprime positive integers. Thanks to (2.17) we can apply Proposition 2.1 to W_f, W_g (that are nonempty measurable subsets of $\mathbb{R}^{nq+p}$), so

$$|W|^{\frac{1}{nq+p}} \geqslant (1-\lambda)\,|W_f|^{\frac{1}{nq+p}} + \lambda\,|W_g|^{\frac{1}{nq+p}}\,,$$

where $|W|$ possibly means the outer measure of the set W. On the other hand Lemma 2.9 implies $|W_h| \geqslant |W|$, thus

$$|W_h|^{\frac{1}{nq+p}} \geqslant (1-\lambda)\,|W_f|^{\frac{1}{nq+p}} + \lambda\,|W_g|^{\frac{1}{nq+p}}\,.$$

Finally the latter inequality with the identity (2.16) is equivalent to

$$\omega_p^{\frac{1}{nq+p}}\left(\int_{\mathbb{R}^n} h\;dx\right)^{\frac{q}{nq+p}} \geqslant \omega_p^{\frac{1}{nq+p}}\left[(1-\lambda)\left(\int_{\mathbb{R}^n} f\;dx\right)^{\frac{q}{nq+p}} + \lambda\left(\int_{\mathbb{R}^n} g\;dx\right)^{\frac{q}{nq+p}}\right].$$

Dividing by $\omega_p^{\frac{1}{nq+p}}$ we get (2.20), since

$$\frac{q}{nq+p} = \frac{q}{q(n+s)} = \frac{1}{n+s}$$

is exactly the required index. The case of a real $s > 0$ (and also $s = 0$) follows by a standard approximation argument. □

3. The proof of Theorem 1.3

The idea is to apply the result of Figalli-Jerison, more precisely Corollary 2.4, to the sets K_{h_λ}, K_f, K_g, and then translate the result in terms of the involved functions. We remember that with h_λ we denote the function $h_{s,\lambda}$ given by (1.7).

We also recall that we set $F = \int f$ and $G = \int g$.

Thanks to (2.7), assumption (1.9) is equivalent to

$$\omega_s^{-1}\,|K_h| \leqslant \omega_s^{-1}\left[(1-\lambda)\,|K_f|^{\frac{1}{n+s}} + \lambda\,|K_g|^{\frac{1}{n+s}}\right]^{n+s} + \varepsilon\,,$$

which, by (2.11), implies

$$|K_{h_\lambda}| \leqslant \left[(1-\lambda)\,|K_f|^{\frac{1}{n+s}} + \lambda\,|K_g|^{\frac{1}{n+s}}\right]^{n+s} + \varepsilon\omega_s. \tag{3.1}$$

If ε is small enough, by virtue of (2.10) we can apply Corollary 2.4 to the sets K_{h_λ}, K_f, K_g and from (3.1) we obtain that they satisfy assumption (2.5) with

$$\delta = \frac{\varepsilon\omega_s}{\mathcal{M}_{\frac{1}{n+s}}(|K_f|\,,|K_g|\,;\lambda)} = \frac{\varepsilon}{\mathcal{M}_{\frac{1}{n+s}}(F,G;\lambda)}. \tag{3.2}$$

Then, if $\delta \leqslant e^{-M_{n+s}(\tau)}$, there exist a convex $K \subset \mathbb{R}^{n+s}$ and two homothetic copies $\hat{K}_f$ and $\hat{K}_g$ of K_f and K_g such that $|\hat{K}_f| = |\hat{K}_g| = 1$ and

$$\left(\hat{K}_f \cup \hat{K}_g\right) \subseteq K \tag{3.3}$$

and

$$\left|K \setminus \hat{K}_f\right| + \left|K \setminus \hat{K}_g\right| \leqslant \tau^{-N_{n+s}}\left(\frac{\varepsilon}{\mathcal{M}_{\frac{1}{n+s}}(F,G;\lambda)}\right)^{\sigma_{n+s}(\tau)}. \tag{3.4}$$

Remark 3.1. Since $|\hat{K}_f| = |\hat{K}_g| = 1$, (3.4) implies that the convex set K has finite positive measure. Then it is bounded (since convex), whence (3.3) yields the boundedness of K_f and K_g which in turn implies the boundedness of the functions f and g. For simplicity, we can assume the convex K is compact (possibly substituting it with its closure).

In what follows, we indicate with $(x, y) \in \mathbb{R}^n \times \mathbb{R}^s$ an element of $\mathbb{R}^{n+s}$. When we say (see just before (3.3)) that $\hat{K}_f$ and $\hat{K}_g$ are homothetic copies of K_f and K_g, we mean that there exist $z_0 = (x_0, y_0) \in \mathbb{R}^{n+s}$ and $z_1 = (x_1, y_1) \in \mathbb{R}^{n+s}$ such that

$$\hat{K}_f = |K_f|^{-\frac{1}{n+s}} (K_f - z_0) \quad \text{and} \quad \hat{K}_g = |K_g|^{-\frac{1}{n+s}} (K_g - z_1) \,. \tag{3.5}$$

Clearly, without loss of generality we can take $z_0 = 0$.

To conclude the proof, we want now to show that, up to a suitable symmetrization, we can take $y_1 = 0$ (i.e. the translation of the homothetic copy $\hat{K}_g$ of K_g is horizontal) and that the convex set K given by Figalli and Jerison can be taken of the type K_u for some $\frac{1}{s}$-concave function u.

For this, let us introduce the following Steiner type symmetrization in $\mathbb{R}^{n+s}$ with respect to the n-dimensional hyperspace $y = 0$ (see for instance [9]). Let C be a bounded measurable set in $\mathbb{R}^{n+s}$, for every $\bar{x} \in \mathbb{R}^n$ we set

$$C(\bar{x}) = \{y \in \mathbb{R}^s \,:\, (\bar{x}, y) \in C\}$$

and

$$r_C(\bar{x}) = \left(\omega_s^{-1} |C(\bar{x})|\right)^{1/s} \,. \tag{3.6}$$

Then we define the S-symmetrand of C as follows

$$S(C) = \left\{(\bar{x}, y) \in \mathbb{R}^{n+s} :\; C \cap \{x = \bar{x}\} \neq \emptyset,\; |y| \leqslant r_C(\bar{x})\right\} \,, \tag{3.7}$$

i.e. $S(C)$ is obtained as union of the s-dimensional closed balls of center $(\bar{x}, 0)$ and radius $r_C(\bar{x})$, for $\bar{x} \in \mathbb{R}^n$ such that $C \cap \{x = \bar{x}\}$ is nonempty. Thus, fixed $\bar{x}$, the (s-dimensional) measure of the corresponding section of $S(C)$ is

$$\mathcal{H}^s(S(C) \cap \{x = \bar{x}\}) = \omega_s r_C(\bar{x})^s = |C(\bar{x})| \,. \tag{3.8}$$

We describe the main properties of S-symmetrization, for bounded measurable susbsets of $\mathbb{R}^{n+s}$:
(i) if $C_1 \subseteq C_2$ then $S(C_1) \subseteq S(C_2)$ (obvious by definition);
(ii) $|C| = |S(C)|$ (consequence of (3.8) and Fubini's Theorem) so the S-symmetrization is measure preserving;
(iii) if C is convex then $S(C)$ is convex (the proof is based on the BM inequality in $\mathbb{R}^s$ and, for the sake of completeness, is given in the Appendix).

Now we symmetrize $K, \hat{K}_f, \hat{K}_g$ (and then replace them with $S(K), S(\hat{K}_f), S(\hat{K}_g)$). Clearly

$$S(\hat{K}_f) = \hat{K}_f, \tag{3.9}$$

$$S(\hat{K}_g) = S\left(|K_g|^{-\frac{1}{n+s}} (K_g - (x_1, y_1))\right) = |K_g|^{-\frac{1}{n+s}} (K_g - (x_1, 0)) \,. \tag{3.10}$$

Moreover, (iii) implies that $S(K)$ is convex and by (i) and (3.3) we have

$$(S(\hat{K}_f) \cup S(\hat{K}_g)) \subseteq S(K) \,. \tag{3.11}$$

The latter, (3.4) and Fubini's theorem imply

$$\left|S(K)\setminus S(\hat K_f)\right| + \left|S(K)\setminus S(\hat K_g)\right| \leqslant \tau^{-N_{n+s}}\left(\frac{\varepsilon}{\mathcal{M}_{\frac{1}{n+s}}(F,G;\lambda)}\right)^{\sigma_{n+s}(\tau)}\,. \tag{3.12}$$

Finally we notice that $S(K)$ is a compact convex set of the desired form.

Remark 3.2. Consider the set K_u associated to a function $u:\mathbb{R}^n\to[0,+\infty)$ by (2.6) and let $\bar x\in\mathbb{R}^n$, $\bar z=(\bar x,0)\in\mathbb{R}^{n+s}$, $\mu>0$ and

$$H=\mu\,(K_u-\bar z)\,.$$

Then

$$H=K_v$$

(the set associated to v by (2.6)) where

$$v(x)=\mu^s u\left(\frac{x-\bar x}{\mu}\right)\,. \tag{3.13}$$

From the previous remarks, we see that the sets $S(\hat K_f)$ and $S(\hat K_g)$ are in fact associated via (2.6) to two functions $\hat f$ and $\hat g$, such that

$$S(\hat K_f)=K_{\hat f}\,,\quad S(\hat K_g)=K_{\hat g}\,, \tag{3.14}$$

and $\hat f$ and $\hat g$ are s-equivalent to f and g respectively, in the sense of (1.8) with

$$\mu_f=(\omega_s F)^{\frac{-1}{n+s}}\,,\qquad \mu_g=(\omega_s G)^{\frac{-1}{n+s}}\,. \tag{3.15}$$

We notice that the support sets Ω_0 and Ω_1 of $\hat f$ and $\hat g$ are given by

$$\Omega_0=\{x\in\mathbb{R}^n\,:(x,0)\in S(\hat K_f)\}\,,\qquad \Omega_1=\{x\in\mathbb{R}^n\,:\,(x,0)\in S(\hat K_g)\}$$

and that they are in fact homothetic copies of the support sets of the original functions f and g.

Now we want to find a $\frac{1}{s}$-concave function u such that $S(K)$ is associated to u via (2.6). We define $u:\mathbb{R}^n\longrightarrow[0,+\infty)$ as follows

$$u(x)=\begin{cases} r_K(x)^s & \text{if }\ (x,0)\in S(K),\\ 0 & \text{otherwise}\,,\end{cases}$$

and prove that

$$K_u=S(K)\,. \tag{3.16}$$

First notice that

$$\mathrm{Supp}(u)=\{x\in\mathbb{R}^n:\ (x,0)\in S(K)\}\,. \tag{3.17}$$

Indeed we have $\{z\in\mathbb{R}^n:\ u(z)>0\}\subseteq\{x\in\mathbb{R}^n:\ (x,0)\in S(K)\}$, whence $\mathrm{Supp}(u)=\overline{\{z\in\mathbb{R}^n:\ u(z)>0\}}\subseteq\{x\in\mathbb{R}^n:\ (x,0)\in S(K)\}$, since the latter is closed. Vice versa let x such that $(x,0)\in S(K)$. If $r_K(x)>0$ (see (3.6)) then $x\in\mathrm{Supp}(u)$ obviously. Otherwise suppose $r_K(x)=0$, then, by the convexity of $S(K)$ and the fact that $S(K)$ is not contained in $\{y=0\}$, evidently

$$[(U\setminus\{x\})\cap\{z\in\mathbb{R}^n:\ r_K(z)>0\}]\neq\emptyset$$

for every neighborhood U of x, i.e. $x\in\mathrm{Supp}(u)$.

By the definition of u and (2.6), using (3.17), we get

$$K_u = \Big\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^s : \ x \in \mathrm{Supp}(u), \ |y| \leqslant u(x)^{1/s}\Big\} =$$
$$= \Big\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^s : \ (x,0) \in S(K), \ |y| \leqslant u(x)^{1/s}\Big\} =$$
$$= \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^s : \ (x,0) \in S(K), \ |y| \leqslant r_K(x)\} = S(K)\,.$$

Therefore we have shown (3.16) and from the convexity of K follows that u is a $\frac{1}{s}$-concave function. Being $K_u \supseteq \Big(K_{\hat f} \cup K_{\hat g}\Big)$, clearly

$$\mathrm{Supp}(u) \supseteq (\Omega_0 \cup \Omega_1)\,, \qquad u \geqslant \hat f \text{ in } \Omega_0, \qquad u \geqslant \hat g \text{ in } \Omega_1\,.$$

The final estimate can be deduced from (3.12). Indeed, thanks to (2.7), we get

$$\Big|K_u \setminus K_{\hat f}\Big| = |K_u| - \Big|K_{\hat f}\Big| = \omega_s \int_{\mathbb{R}^n} (u - \hat f)\ dx,$$

and the same equality holds for $|K_u \setminus K_{\hat g}|$. So (3.12) becomes

$$\int_{\mathbb{R}^n} (u - \hat f)\ dx \ + \ \int_{\mathbb{R}^n} (u - \hat g)\ dx \ \leqslant \omega_s^{-1} \tau^{-N_{n+s}} \left(\frac{\varepsilon}{\mathcal{M}_{\frac{1}{n+s}}(F,G;\lambda)} \right)^{\sigma_{n+s}(\tau)},$$

that is the desired result.

4. A generalization to the case s positive rational

We explain how Theorem 1.3 can be generalized to a positive rational index s. Given $f : \mathbb{R}^n \longrightarrow [0,+\infty)$ and an integer $q > 0$, we consider the auxiliary function $\tilde f : \mathbb{R}^{nq} \longrightarrow [0,+\infty)$ given by (2.13), i.e.

$$\tilde f(x) = \tilde f(x_1, ..., x_q) = \prod_{j=1}^{q} f(x_j),$$

with $x = (x_1, ..., x_q) \in (\mathbb{R}^n)^q$. Clearly f is bounded if and only if $\tilde f$ is bounded. We study further properties of functions of type (2.13).

Lemma 4.1. *Given an integer $q > 0$, and a real $t > 0$ let $\tilde u : \mathbb{R}^{nq} \longrightarrow [0,+\infty)$ be a function of the type* (2.13)*. Then $\tilde u$ is t-concave if and only if the function $u : \mathbb{R}^n \longrightarrow [0,+\infty)$ is (qt)-concave.*

Proof. Suppose first that $\tilde u^t$ is concave. Fixed $\lambda \in (0,1), \ x, x' \in \mathbb{R}^n$, we consider the element of $\mathbb{R}^{nq}$ which has all the q components identical to $(1-\lambda)x + \lambda x'$. From hypothesis it holds

$$\tilde u^t\left((1-\lambda)x + \lambda x', ..., (1-\lambda)x + \lambda x'\right) \geqslant (1-\lambda)\tilde u^t\left(x, ..., x\right) + \lambda \tilde u^t\left(x', ..., x'\right),$$

i.e. (thanks to (2.13))

$$u^{qt}\left((1-\lambda)x + \lambda x'\right) \geqslant (1-\lambda) u^{qt}(x) + \lambda u^{qt}(x').$$

Thus u^{qt} is concave.

Vice versa assume that u^{qt} is concave, and fix $\lambda \in (0,1), \ x = (x_1, ..., x_q), \ x' = \left(x'_1, ..., x'_q\right) \in (\mathbb{R}^n)^q$. We have

$$\tilde u^t\left((1-\lambda)x + \lambda x'\right) = \prod_{j=1}^{q} u^t\left((1-\lambda)x_j + \lambda x'_j\right) = \prod_{j=1}^{q} \left[u^{qt}\left((1-\lambda)x_j + \lambda x'_j\right)\right]^{1/q} \geqslant$$

$$\geqslant \prod_{j=1}^{q} \left[(1-\lambda)u^{qt}(x_j) + \lambda u^{qt}(x'_j)\right]^{1/q} \geqslant \prod_{j=1}^{q}(1-\lambda)^{1/q}u^t(x_j) + \prod_{j=1}^{q} \lambda^{1/q}u^t(x'_j) =$$

$$= (1-\lambda)\prod_{j=1}^{q} u^t(x_j) + \lambda \prod_{j=1}^{q} u^t(x'_j) = (1-\lambda)\tilde{u}^t(x) + \lambda\tilde{u}^t(x'),$$

where the first inequality holds by concavity of u^{qt}, while in the second one we have used Lemma 2.7 with $a_j = (1-\lambda)^{1/q}u^t(x_j),\ b_j = \lambda^{1/q}u^t(x'_j)$. Hence u^t is concave. □

Lemma 4.2. *Let $q > 0$ integer and $u \geqslant f \geqslant 0$ in $\mathbb{R}^n$. Then*

$$\tilde{u} - \tilde{f} \ \geqslant\ \widetilde{u-f}.$$

Proof. The proof is by induction on the integer $q \geqslant 1$. The case $q = 1$ is trivial, because in such case $\tilde{u} = u,\ \tilde{f} = f,\ \widetilde{u-f} = u - f$. For the inductive step assume that the result is true until the index q, and denote with $\tilde{\tilde{u}}, \tilde{\tilde{f}}, \widetilde{\widetilde{u-f}}$ the respective functions of index $q+1$. By the definition (2.13)

$$\left(\tilde{\tilde{u}} - \tilde{\tilde{f}}\right)(x_1,...,x_{q+1}) = \tilde{u}(x_1,...,x_q)u(x_{q+1}) - \tilde{f}(x_1,...,x_q)f(x_{q+1}),$$

$$\widetilde{\widetilde{u-f}}(x_1,...,x_{q+1}) = \widetilde{u-f}(x_1,...,x_q)\cdot(u-f)(x_{q+1}).$$

These two equalities imply

$$\left(\tilde{\tilde{u}} - \tilde{\tilde{f}}\right)(x_1,...,x_{q+1}) =$$

$$= \widetilde{\widetilde{u-f}}(x_1,...,x_{q+1}) - \widetilde{u-f}(x_1,...,x_q)\cdot[u(x_{q+1}) - f(x_{q+1})] +$$
$$+\tilde{u}(x_1,...,x_q)u(x_{q+1}) - \tilde{f}(x_1,...,x_q)f(x_{q+1}) \geqslant$$
$$\geqslant \widetilde{\widetilde{u-f}}(x_1,...,x_{q+1}) - \left(\tilde{u} - \tilde{f}\right)(x_1,...,x_q)\left[u(x_{q+1}) - f(x_{q+1})\right] +$$
$$+\tilde{u}(x_1,...,x_q)u(x_{q+1}) - \tilde{f}(x_1,...,x_q)f(x_{q+1}) =$$
$$= \widetilde{\widetilde{u-f}}(x_1,...,x_{q+1}) + f(x_{q+1})\left[\tilde{u}(x_1,...,x_q) - \tilde{f}(x_1,...,x_q)\right] +$$
$$+\tilde{f}(x_1,...,x_q)\left[u(x_{q+1}) - f(x_{q+1})\right] \geqslant$$
$$\geqslant \widetilde{\widetilde{u-f}}(x_1,...,x_{q+1}),$$

having used the inductive hypothesis and the assumption $u \geqslant f \geqslant 0$. □

Corollary 4.3. *Given an integer $n > 0$, $\lambda \in (0,1)$, $s = \frac{p}{q}$ with p, q positive integers, let $f,\ g \in L^1(\mathbb{R}^n)$ be nonnegative compactly supported functions such that*

$$F = \int_{\mathbb{R}^n} f\ dx > 0 \quad \text{and} \quad G = \int_{\mathbb{R}^n} g\ dx > 0.$$

Let $h : \mathbb{R}^n \longrightarrow [0,+\infty)$ satisfy (2.8) and suppose there exists $\varepsilon > 0$ small enough such that

$$\left(\int_{\mathbb{R}^n} h\ dx\right)^q \leqslant \left[\mathcal{M}_{\frac{1}{n+s}}(F,G\,;\lambda)\right]^q + \varepsilon. \tag{4.1}$$

Then there exist a $\frac{1}{p}$-concave function $u' : \mathbb{R}^{nq} \longrightarrow [0, +\infty)$ and two functions $\hat{f},\ \hat{g} : \mathbb{R}^{nq} \longrightarrow [0, +\infty)$, p-equivalent to $\tilde{f}$ and $\tilde{g}$ (given by (2.13)) in the sense of (1.8) with

$$\mu_{\tilde{f}} = \omega_p^{\frac{-1}{nq+p}} F^{\frac{-1}{n+s}}, \qquad \mu_{\tilde{g}} = \omega_p^{\frac{-1}{nq+p}} G^{\frac{-1}{n+s}},$$

and with support sets Ω_0 and Ω_1 respectively, such that the following hold:

$$\mathrm{Supp}(u') \supseteq (\Omega_0 \cup \Omega_1)\,, \qquad u' \geqslant \hat{f} \ \text{in } \Omega_0, \qquad u' \geqslant \hat{g} \ \text{in } \Omega_1\,,$$

and

$$\int_{\mathbb{R}^{nq}} (u' - \hat{f})dx + \int_{\mathbb{R}^{nq}} (u' - \hat{g})\,dx \ \leqslant\ C_{nq+p}\left(\frac{\varepsilon}{\mathcal{M}_{\frac{1}{nq+p}}(F^q, G^q\,;\lambda)}\right). \tag{4.2}$$

Proof. We can assume $h = h_\lambda$. Since f and g are nonnegative compactly supported functions belonging to $L^1(\mathbb{R}^n)$, thus by (2.13) $\tilde{f}, \tilde{g}$ are nonnegative compactly supported functions belonging to $L^1(\mathbb{R}^{nq})$. The assumption (4.1) is equivalent, considering the corresponding functions $\tilde{f}, \tilde{g}, \tilde{h} : \mathbb{R}^{nq} \longrightarrow [0, +\infty)$ and using (2.14), to

$$\int_{\mathbb{R}^{nq}} \tilde{h}\ dx \leqslant \left[(1-\lambda)\left(\int_{\mathbb{R}^{nq}} \tilde{f}\ dx\right)^{\frac{1}{nq+qs}} + \lambda\left(\int_{\mathbb{R}^{nq}} \tilde{g}\ dx\right)^{\frac{1}{nq+qs}}\right]^{nq+qs} + \varepsilon$$

$$\text{i.e.} \qquad \int_{\mathbb{R}^{nq}} \tilde{h}\ dx \leqslant \mathcal{M}_{\frac{1}{nq+p}}(F^q, G^q\,;\lambda) + \varepsilon. \tag{4.3}$$

We notice that the index $qs = p$ is integer, while nq is exactly the dimension of the space in which $\tilde{f}, \tilde{g}, \tilde{h}$ are defined. To apply Theorem 1.3, we have to verify that $\tilde{f}, \tilde{g}, \tilde{h}$ satisfy the corresponding inequality (2.8) of index qs. Given $x_1, ..., x_q,\ x'_1, ..., x'_q \in \mathbb{R}^n$, let $x = (x_1, ..., x_q),\ x' = (x'_1, ..., x'_q) \in (\mathbb{R}^n)^q$. By hypothesis, we know that f, g, h satisfy (2.8), in particular for every $j = 1, ..., q$

$$h\left((1-\lambda)x_j + \lambda x'_j\right) \geqslant \left[(1-\lambda)f(x_j)^{1/s} + \lambda g(x'_j)^{1/s}\right]^s.$$

This implies

$$\prod_{j=1}^{q} h\left((1-\lambda)x_j + \lambda x'_j\right) \geqslant \left[\prod_{j=1}^{q}\left[(1-\lambda)f(x_j)^{1/s} + \lambda g(x'_j)^{1/s}\right]\right]^s \geqslant$$

$$\geqslant \left[(1-\lambda)\left(\prod_{j=1}^{q} f(x_j)\right)^{1/qs} + \lambda\left(\prod_{j=1}^{q} g(x'_j)\right)^{1/qs}\right]^{qs}, \tag{4.4}$$

where the last inequality is due to Corollary 2.8. By definition of (2.13), (4.4) means

$$\tilde{h}\left((1-\lambda)x + \lambda x'\right) \geqslant \left[(1-\lambda)\tilde{f}(x)^{1/qs} + \lambda\tilde{g}(x')^{1/qs}\right]^{qs},$$

i.e. the functions $\tilde{f}, \tilde{g}, \tilde{h} : \mathbb{R}^{nq} \longrightarrow [0, +\infty)$ satisfy the hypothesis (2.8) with the required index qs. Therefore we can apply Theorem 1.3 and conclude that there exist a $\frac{1}{p}$-concave function $u' : \mathbb{R}^{nq} \longrightarrow [0, +\infty)$ and two functions $\hat{f}, \hat{g}$, p-equivalent to $\tilde{f}$ and $\tilde{g}$, with the required properties. The estimate (1.11), applied to (4.3), implies

$$\int_{\mathbb{R}^{nq}} (u' - \hat{f})\, dx \;+\; \int_{\mathbb{R}^{nq}} (u' - \hat{g})\; dx \;\leqslant\; C_{nq+p}\left(\frac{\varepsilon}{\mathcal{M}_{\frac{1}{nq+p}}(F^q, G^q\,; \lambda)}\right).$$

□

Remark 4.4. Assume $F = G$ and, for simplicity, suppose that $\hat{f} = \tilde{f},\ \hat{g} = \tilde{g}$ in Corollary 4.3 (as it is true up to a p-equivalence). Moreover assume that the $\frac{1}{p}$-concave function $u' : \mathbb{R}^{nq} \longrightarrow [0, +\infty)$, given by Corollary 4.3, is of the type (2.13), i.e. $u' = \tilde{u}$ where $u : \mathbb{R}^n \longrightarrow [0, +\infty)$ has to be $\frac{1}{s}$-concave by Lemma 4.1. In this case Corollary 4.3 assumes a simpler statement, which naturally extends the result of Theorem 1.3. Indeed (4.2), thanks to Lemma 4.2, becomes

$$\int_{\mathbb{R}^{nq}} \widetilde{u - f}\; dx \;+\; \int_{\mathbb{R}^{nq}} \widetilde{u - g}\; dx \;\leqslant\; C_{nq+p}\left(\frac{\varepsilon}{\mathcal{M}_{\frac{1}{nq+p}}(F^q, G^q\,; \lambda)}\right), \qquad \text{i.e.}$$

$$\left[\int_{\mathbb{R}^n} (u - f)\; dx\right]^q \;+\; \left[\int_{\mathbb{R}^n} (u - g)\; dx\right]^q \;\leqslant\; C_{nq+p}\left(\frac{\varepsilon}{\mathcal{M}_{\frac{1}{nq+p}}(F^q, G^q\,; \lambda)}\right). \tag{4.5}$$

Unfortunately the function u' constructed in Theorem 1.3 is not necessarily of the desired form, that is in general we can not find a function $u : \mathbb{R}^n \longrightarrow [0, +\infty)$ such that $u' = \tilde{u}$ (a counterexample can be explicitly given). Then our proof can not be easily extended to the general case $s \in \mathbb{Q}$ to get (4.5).

5. A stability for $s > 0$

To complete the paper, we give a (weaker) version of our main stability result Theorem 1.3 which works for an arbitrary real index $s > 0$. For this, let us denote by $[s]$ the integer part of s, i.e. the largest integer not greater than s. Obviously $[s] + 1 > s \geqslant [s]$, whereby (by the monotonicity of p-means with respect to p, i.e. $\mathcal{M}_p(a, b; \lambda) \leqslant \mathcal{M}_q(a, b; \lambda)$ if $p \leqslant q$) for every $a, b \geqslant 0,\ \lambda \in (0, 1)$

$$\left[(1-\lambda)a^{\frac{1}{s}} + \lambda b^{\frac{1}{s}}\right]^s \geqslant \left[(1-\lambda)a^{\frac{1}{[s]+1}} + \lambda b^{\frac{1}{[s]+1}}\right]^{[s]+1}, \tag{5.1}$$

$$\left[(1-\lambda)a^{\frac{1}{n+s}} + \lambda b^{\frac{1}{n+s}}\right]^{n+s} \geqslant \left[(1-\lambda)a^{\frac{1}{n+[s]+1}} + \lambda b^{\frac{1}{n+[s]+1}}\right]^{n+[s]+1}. \tag{5.2}$$

We arrive to the following corollary for every index $s > 0$.

Corollary 5.1. *Given $s > 0,\ \lambda \in (0, 1)$, let $f, g : \mathbb{R}^n \longrightarrow [0, +\infty)$ be integrable functions such that*

$$\int_{\mathbb{R}^n} f\; dx = \int_{\mathbb{R}^n} g\; dx = 1\,. \tag{5.3}$$

Assume $h : \mathbb{R}^n \longrightarrow [0, +\infty)$ satisfies (2.19) *and there exists $\varepsilon > 0$ small enough such that*

$$\int_{\mathbb{R}^n} h\; dx \leqslant 1 + \varepsilon. \tag{5.4}$$

Then there exist a $\frac{1}{[s]+1}$*-concave function* $u:\mathbb{R}^n \longrightarrow [0,+\infty)$ *and two functions* $\hat{f}$ *and* $\hat{g}$*,* $([s]+1)$*-equivalent to* f *and* g *in the sense of* (3.13) *(with* $\mu_f = \mu_g = \left(\omega_{[s]+1}\right)^{\frac{-1}{n+[s]+1}}$*) and with compact supports* Ω_0 *and* Ω_1 *respectively, such that*

$$\mathrm{Supp}(u) \supseteq (\Omega_0 \cup \Omega_1), \qquad u \geqslant \hat{f} \text{ in } \Omega_0, \qquad u \geqslant \hat{g} \text{ in } \Omega_1,$$

and

$$\int_{\mathbb{R}^n} (u-\hat{f})\ dx \ + \ \int_{\mathbb{R}^n} (u-\hat{g})\ dx \ \leqslant \ C_{n+[s]+1}(\varepsilon).$$

Proof. We notice that the assumption (2.19) (i.e. the hypothesis of BBL of index $\frac{1}{s}$), through (5.1), implies that for every $x_0, x_1 \in \mathbb{R}^n$

$$h\left((1-\lambda)x_0+\lambda x_1\right) \geqslant \left[(1-\lambda)f(x_0)^{\frac{1}{[s]+1}} + \lambda g(x_1)^{\frac{1}{[s]+1}}\right]^{[s]+1},$$

i.e. the corresponding hypothesis of BBL for the index $\frac{1}{[s]+1}$. Therefore, thanks to the assumptions (5.3) and (5.4), it holds $\int h \leqslant 1+\varepsilon = \mathcal{M}_{\frac{1}{n+[s]+1}}(\int f, \int g; \lambda) + \varepsilon$, so we can apply directly Theorem 1.3 using the integer $[s]+1$ as index. This concludes the proof. □

Remark 5.2. If we don't use the normalization (5.3) and want to write a result for generic unrelated $F = \int f$ and $G = \int g$, we can notice that assumption (5.4) should be replaced by

$$\int_{\mathbb{R}^n} h\ dx \leqslant \mathcal{M}_{\frac{1}{n+[s]+1}}(F,G;\lambda) + \varepsilon\,.$$

On the other hand, thanks to assumption (2.19), we can apply Proposition 2.10 and obtain

$$\int_{\mathbb{R}^n} h\ dx \geqslant \mathcal{M}_{\frac{1}{n+s}}(F,G;\lambda).$$

Then we would have

$$\mathcal{M}_{\frac{1}{n+s}}(F,G;\lambda) \leqslant \mathcal{M}_{\frac{1}{n+[s]+1}}(F,G;\lambda) + \varepsilon\,.$$

The latter inequality is possible only if F and G are close to each others, thanks to the stability of the monotonicity property of p-means, which states

$$\mathcal{M}_{\frac{1}{n+[s]+1}}(F,G;\lambda) \leqslant \mathcal{M}_{\frac{1}{n+s}}(F,G;\lambda),$$

with equality if and only if $F = G$. In this sense the normalization (5.3) cannot be completely avoided and the result obtained in Corollary 5.1 is weaker than what desired. Indeed notice in particular that it does not coincide with Theorem 1.3 even in the case when s is integer, since $[s]+1 > s$ in that case as well.

6. Appendix

Here we show that the S-symmetrization, introduced in Remark 3.1, preserves the convexity of the involved set (that is the property (iii) therein).
We use the notations of Remark 3.1, in particular we refer to (3.6) and (3.7), and remember that C is a bounded measurable set in $\mathbb{R}^{n+s}$. We need the following preliminary result, based on the Brunn-Minkowski inequality in $\mathbb{R}^s$.

Lemma 6.1. *If C is a bounded convex set in $\mathbb{R}^{n+s}$, then for every $t \in (0,1)$ and every $x_0, x_1 \in \mathbb{R}^n$ such that $C(x_0)$ and $C(x_1)$ are nonempty, it holds*

$$(1-t)r_C(x_0) + tr_C(x_1) \leqslant r_C((1-t)x_0 + tx_1). \tag{6.1}$$

Proof. By defintion of (3.6)

$$r_C(x_0) = {\omega_s}^{-1/s}|C(x_0)|^{1/s}, \quad r_C(x_1) = {\omega_s}^{-1/s}|C(x_1)|^{1/s},$$

thus

$$(1-t)r_C(x_0) + tr_C(x_1) = \omega_s^{-1/s}\left[(1-t)|C(x_0)|^{1/s} + t|C(x_1)|^{1/s}\right]. \tag{6.2}$$

Since C is convex, we notice that $C(x_0), C(x_1)$ are (nonempty) convex sets in $\mathbb{R}^s$ such that

$$(1-t)C(x_0) + tC(x_1) \subseteq C((1-t)x_0 + tx_1). \tag{6.3}$$

Applying BM inequality (i.e. Proposition 2.1) to the sets $C(x_0), C(x_1) \subset \mathbb{R}^s$, (6.2) implies

$$(1-t)r_C(x_0) + tr_C(x_1) \leqslant \omega_s^{-1/s} \left|(1-t)C(x_0) + tC(x_1)\right|^{1/s} \leqslant$$
$$\leqslant \omega_s^{-1/s} \left|C((1-t)x_0 + tx_1)\right|^{1/s} = r_C((1-t)x_0 + tx_1),$$

where in the last inequality we use (6.3). □

Proposition 6.2. *If C is convex then $S(C)$ is convex.*

Proof. Let $t \in (0,1)$, and let $P = (x_0, y_0), Q = (x_1, y_1)$ be two distinct points belonging to $S(C)$, i.e. $C(x_0), C(x_1)$ are nonempty sets and

$$|y_0| \leqslant r_C(x_0), \qquad |y_1| \leqslant r_C(x_1). \tag{6.4}$$

We prove that

$$(1-t)P + tQ = ((1-t)x_0 + tx_1, (1-t)y_0 + ty_1) \in S(C).$$

By assumptions and (6.3) the set $C((1-t)x_0 + tx_1)$ is nonempty. Furthermore by the triangle inequality, (6.4) and Lemma 6.1 we obtain

$$|(1-t)y_0 + ty_1| \leqslant (1-t)\,|y_0| + t\,|y_1| \leqslant (1-t)r_C(x_0) + tr_C(x_1) \leqslant r_C((1-t)x_0 + tx_1).$$

Then $(1-t)P + tQ \in S(C)$, i.e. $S(C)$ is convex. □

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