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Research Article

Stability for Caputo Fractional Differential Systems

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We introduce the notion of h-stability for fractional differential systems. Then we investigate the boundedness and h-stability of solutions of Caputo fractional differential systems by using fractional comparison principle and fractional Lyapunov direct method. Furthermore, we give examples to illustrate our results.

1. Introductions and Preliminaries

Lakshmikantham et al. [1–5] investigated the basic theory of initial value problems for fractional differential equations involving Riemann-Liouville differential operators of order $0 < q < 1$. They followed the classical approach of the theory of differential equations of integer order in order to compare and contrast the differences as well as the intricacies that might result in development [6, Vol. I]. Li et al. [7] obtained some results about stability of solutions for fractional-order dynamic systems using fractional Lyapunov direct method and fractional comparison principle. Choi and Koo [8] improved on the monotone property of Lemma 1.7.3 in [5] for the case $g(t,u) = \lambda u$ with a nonnegative real number λ. Choi et al. [9] also investigated Mittag-Leffler stability of solutions of fractional differential equations by using the fractional comparison principle.

In this paper we introduce the notion of h-stability for fractional differential equations. Then, we investigate the boundedness and h-stability of solutions of Caputo fractional differential systems by using fractional comparison principle and fractional Lyapunov direct method. Furthermore, we give some examples to illustrate our results.

For the basic notions and theorems about fractional calculus, we mainly refer to some books [5, 10, 11].

We recall the notions of Mittag-Leffler functions which were originally introduced by Mittag-Leffler in 1903 [12]. Similar to the exponential function frequently used in the solutions of integer-order systems, a function frequently used in the solutions of fractional order systems is the Mittag-Leffler function, defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \tag{1}$$

where $\alpha > 0$ and Γ is the Gamma function [11]. The Mittag-Leffler function with two parameters has the following form:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \tag{2}$$

where $\alpha > 0$ and $\beta > 0$. For $\beta = 1$, we have $E_\alpha(z) = E_{\alpha,1}(z)$. Also, $E_{1,1}(z) = e^z$.

Note that the exponential function e^{at} possesses the semigroup property (i.e., $e^{a(t+s)} = e^{at}e^{as}$ for all $t, s \geq 0$), but the Mittag-Leffler function $E_\alpha(at^\alpha)$ does not satisfy the semigroup property unless $\alpha = 1$ or $a = 0$ [13].

We recall briefly the notions and basic properties about fractional integral operators and fractional derivatives of functions [5, 10]. Let $J = [t_0, \infty) \subset \mathbb{R}^+ = [0, \infty)$.

Definition 1 (see [5]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f \in L_1(J, \mathbb{R})$ is defined as

$$I_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s)\, ds, \tag{3}$$

where $t_0 \in \mathbb{R}$ (provided that the integral exists in the Lebesgue sense).

Definition 2 (see [5]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : J \to \mathbb{R}$ is given by

$$D_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d^n}{dt^n} \int_{t_0}^{t} (t-s)^{n-\alpha-1} f(s)\, ds \right), \quad n-1 < \alpha < n, n \in \mathbb{N}, \tag{4}$$

provided that the right side is pointwise defined on J.

If $0 < \alpha < 1$, then the Riemann-Liouville fractional derivative of order α of a function f reduces to

$$D_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{-\alpha} f(s)\, ds. \tag{5}$$

Note that the Riemann-Liouville fractional derivatives have singularity at t_0 and the fractional equations in the Riemann-Liouville sense require initial conditions at some point different from t_0. To overcome this issue, Caputo [14] defined the fractional derivative in the following way.

Definition 3 (see [10]). Let α be a positive real number such that $n-1 < \alpha \le n$ for $n \in \mathbb{N}$. The Caputo fractional derivative of order α of a function f is defined by

$${}^{C}D_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds, \tag{6}$$

where $f^{(n)}(s) = d^n/ds^n f(s)$.

When $0 < \alpha < 1$, then the Caputo fractional derivative of order α of f reduces to

$${}^{C}D_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{f'(s)}{(t-s)^{\alpha}} ds. \tag{7}$$

When $0 < \alpha < 1$, we have

$$\begin{aligned} {}^{C}D_{t_0}^{\alpha} f(t) &= D_{t_0}^{\alpha} \left[f(t) - f(t_0) \right] \\ &= D_{t_0}^{\alpha} f(t) - \frac{f(t_0)}{\Gamma(1-\alpha)} (t-t_0)^{-\alpha}. \end{aligned} \tag{8}$$

In particular, if $f(t_0) = 0$, then we have

$${}^{C}D_{t_0}^{\alpha} f(t) = D_{t_0}^{\alpha} f(t). \tag{9}$$

Hence, we can see that the Caputo derivative is defined for functions for which the Riemann-Liouville derivative exists. Also, we note that the Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\alpha}(z)$ satisfy the more general differential relations

$$\begin{aligned} {}^{C}D_{t_0}^{\alpha} E_{\alpha} \left(\lambda (t-t_0)^{\alpha} \right) &= \lambda E_{\alpha} \left(\lambda (t-t_0)^{\alpha} \right), \\ D_{t_0}^{\alpha} E_{\alpha,\alpha} \left(\lambda (t-t_0)^{\alpha} \right) &= \lambda E_{\alpha,\alpha} \left(\lambda (t-t_0)^{\alpha} \right), \end{aligned} \tag{10}$$

respectively, for $\lambda \in \mathbb{R}$.

We can obtain the following asymptotic property for $E_\alpha(\lambda t^\alpha)$ and $E_{\alpha,\alpha+1}(\lambda t^\alpha)$ from the result [10, page 51].

Lemma 4 (see [10]). *When $\alpha > 0$, then $E_{\alpha,\beta}(z)$ has different asymptotic behavior at infinity for $0 < \alpha < 2$ and $\alpha \ge 2$.*

(1) *If $0 < \alpha < 2$ and μ is a real number such that*

$$\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}, \tag{11}$$

then, for $p \in \mathbb{N} \setminus \{1\}$, the following asymptotic expansions are valid:

$$\begin{aligned} E_{\alpha,\beta}(z) = {} & \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp\left(z^{1/\alpha}\right) \\ & - \sum_{k=1}^{p} \frac{1}{\Gamma(\beta - \alpha k)} \frac{1}{z^k} + O\left(\frac{1}{z^{p+1}}\right), \end{aligned} \tag{12}$$

with $|z| \to \infty, |\arg(z)| \le \mu$; and

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{p} \frac{1}{\Gamma(\beta - \alpha k)} \frac{1}{z^k} + O\left(\frac{1}{z^{p+1}}\right), \tag{13}$$

with $|z| \to \infty, \mu \le |\arg(z)| \le \pi$.

(2) *When $\alpha \ge 2$, then, for $p \in \mathbb{N} \setminus \{1\}$, the following asymptotic estimate holds:*

$$\begin{aligned} & E_{\alpha,\beta}(z) \\ & = \frac{1}{\alpha} \sum_{n} \left(z^{1/\alpha} \exp\left[\frac{2n\pi i}{\alpha}\right] \right)^{1-\beta} \exp\left[\exp\left(\frac{2n\pi i}{\alpha}\right) z^{1/\alpha} \right] \\ & \quad - \sum_{k=1}^{p} \frac{1}{\Gamma(\beta - \alpha k)} \frac{1}{z^k} + O\left(\frac{1}{z^{p+1}}\right), \end{aligned} \tag{14}$$

with $|z| \to \infty, |\arg(z)| \le \alpha\pi/2$, and where the first sum is taken over all integer n such that

$$\left| \arg(z) + 2\pi n \right| \le \frac{\alpha\pi}{2}. \tag{15}$$

Lemma 5. *Let $0 < \alpha \le 1$ and $\lambda < 0$. Then, $E_{\alpha,\alpha}(\lambda t^\alpha)$ and $E_{\alpha,\alpha+1}(\lambda t^\alpha)$ tend monotonically to zero as $t \to \infty$.*

Proof. If we set $\beta = \alpha$ and $z = \lambda t^\alpha$ in Lemma 4, then it follows from Lemma 4 that for $p = 2$ we have

$$\begin{aligned} E_{\alpha,\alpha}(\lambda t^\alpha) &= -\sum_{k=1}^{2} \frac{1}{\Gamma(\alpha - \alpha k)} \frac{1}{\lambda^k t^{k\alpha}} + O\left(\frac{1}{\lambda^3 t^{3\alpha}}\right) \\ &= -\frac{1}{\Gamma(-\alpha)} \frac{1}{\lambda^2 t^{2\alpha}} \\ & \quad + O\left(\frac{1}{\lambda^3 t^{3\alpha}}\right) \longrightarrow 0, \quad \text{as } t \longrightarrow \infty, \\ & \quad \mu \le \left|\arg(\lambda)\right| \le \pi. \end{aligned} \tag{16}$$

Thus, we have

$$E_{\alpha,\alpha}(\lambda t^\alpha) \longrightarrow 0, \quad \text{as } t \longrightarrow \infty, \qquad \left|\arg(\lambda)\right| > \frac{\pi\alpha}{2}. \tag{17}$$

For $\beta = \alpha + 1$, we also have

$$E_{\alpha,\alpha+1}(\lambda t^{\alpha}) \longrightarrow 0, \quad \text{as } t \longrightarrow \infty, \qquad |\arg(\lambda)| > \frac{\pi\alpha}{2}, \tag{18}$$

by the above similar argument. This completes the proof. □

Corollary 6. *Let $0 < \alpha < 1$ and $|\arg(\lambda)| > \pi\alpha/2$. Then, one has*

$$t^{\alpha}E_{\alpha,\alpha+1}(\lambda t^{\alpha}) = -\frac{1}{\lambda} - \frac{1}{\Gamma(1-\alpha)\lambda^2 t^{\alpha}} + O\left(\frac{1}{\lambda^3 t^{2\alpha}}\right) \quad \text{as } t \longrightarrow \infty. \tag{19}$$

2. Main Results

Let $0 < q < 1$ and $p = 1 - q$. Denote by $C_p([t_0, T], \mathbb{R}^n)$ the function space

$$C_p([t_0,T],\mathbb{R}^n) = \left\{x \in C((t_0,T],\mathbb{R}^n) \mid x(t)(t-t_0)^p \in C([t_0,T],\mathbb{R}^n)\right\}. \tag{20}$$

Let $\Omega \subset \mathbb{R}^n$ be a domain and $f \in C([t_0, t_0 + a] \times \Omega, \mathbb{R}^n)$. We consider the Caputo fractional differential system with the initial value

$${}^{C}D^{q}_{t_0}x = f(t,x), \qquad x(t_0) = x_0, \tag{21}$$

where $f(t, 0) = 0$. If $x \in C_p([t_0, t_0 + a], \mathbb{R}^n)$ satisfies (21), it also satisfies the Volterra fractional integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}f(s,x(s))\,ds, \quad t_0 \le t \le t_0 + a, \tag{22}$$

and vice versa.

In the sequential we assume that the solution $x(t)$ of (21) exists globally on $J = [t_0, \infty)$. See [5, Theorem 2.10.1] for the existence and uniqueness result.

Next, we consider the nonhomogeneous linear fractional differential equation with Caputo fractional derivative

$${}^{C}D^{q}_{t_0}x = \lambda x + h(t), \qquad x(t_0) = x_0, \tag{23}$$

where $h \in C_p(J, \mathbb{R})$ is Hölder continuous with exponent q. Then, we get the unique solution of (23) as

$$x(t) = x_0E_q\left(\lambda(t-t_0)^q\right) + \int_{t_0}^{t}(t-s)^{q-1}E_{q,q}\left(\lambda(t-s)^q\right)h(s)\,ds \tag{24}$$

for each $t \in J$.

Lemma 7 (see [9, Lemma 3.2]). *If one sets $h(t) \equiv d$ in (23) with a constant d, then the solution of (24) reduces to*

$$x(t) = x_0E_q\left(\lambda(t-t_0)^q\right) + d(t-t_0)^q \times E_{q,q+1}\left(\lambda(t-t_0)^q\right), \quad t \in J. \tag{25}$$

Remark 8. If $h(t) \equiv 0$, then it follows from Lemma 7 that

$$x(t) = x_0E_q\left(\lambda(t-t_0)^q\right), \quad t \in J. \tag{26}$$

We can obtain the following Caputo fractional differential inequality of Gronwall type by Lemma 7.

Lemma 9. *Suppose that $m \in C_p(\mathbb{R}^+, \mathbb{R})$ satisfies*

$${}^{C}D^{q}_{t_0}m(t) \le \lambda m(t) + d, \quad m(t_0) = m_0, \quad t \ge t_0 \ge 0, \tag{27}$$

where $\lambda, d \in \mathbb{R}$. Then one has

$$m(t) \le m(t_0)E_q\left(\lambda(t-t_0)^q\right) + d(t-t_0)^qE_{q,q+1}\left(\lambda(t-t_0)^q\right), \quad t \ge t_0 \ge 0. \tag{28}$$

Proof. There exists a nonnegative function $n(t)$ satisfying

$${}^{C}D^{q}_{t_0}m(t) - \lambda m(t) - d + n(t) = 0, \quad t \ge t_0 \ge 0. \tag{29}$$

It follows from Lemma 7 that

$$m(t) = m(t_0)E_q\left(\lambda(t-t_0)^q\right) + d(t-t_0)^qE_{q,q+1}\left(\lambda(t-t_0)^q\right) - n(t) * t^{q-1}E_{q,q}\left(\lambda(t-t_0)^q\right), \quad t \ge t_0, \tag{30}$$

where $*$ denotes the convolution operator of nonnegative functions $n(t)$ and $t^{q-1}E_{q,q}(\lambda(t-t_0)^q)$. Since $n(t) * t^{q-1}E_{q,q}(\lambda(t-t_0)^q)$ is nonnegative for each $t \ge t_0$, then we have

$$m(t) \le m(t_0)E_q\left(\lambda(t-t_0)^q\right) + d(t-t_0)^qE_{q,q+1}\left(\lambda(t-t_0)^q\right), \quad t \ge t_0 \ge 0. \tag{31}$$

This completes the proof. □

Remark 10. If we set $q = 1$ and $d = 0$ in Lemma 9, then we have

$$m(t) \le m(t_0)E_{1,1}(\lambda(t-t_0)) = m(t_0)e^{\lambda(t-t_0)}, \quad t \ge t_0 \ge 0. \tag{32}$$

We can obtain the following result about fractional integral inequality. It is adapted from the comparison principle regarding nonstrict inequalities in [2, 5].

Lemma 11 (see [8, Lemma 2.11]). *Let $0 < q < 1$ and $g \in C(J \times \mathbb{R}, \mathbb{R}^+)$. Suppose that $w, v \in C(J, \mathbb{R}^+)$ satisfy the fractional integral inequality:*

$$v(t) - I^{q}_{t_0}g(t,v(t)) < w(t) - I^{q}_{t_0}g(t,w(t)), \tag{33}$$

where $I^{q}_{t_0}g(t, v(t)) = 1/(\Gamma(q))\int_{t_0}^{t}(t-s)^{q-1}g(s, v(s))ds$ and $g(t, u)$ is monotonic nondecreasing in u for each $t \ge t_0$. If $v(t_0) < w(t_0)$, then one has $v(t) < w(t)$ on J.

Pinto [15] introduced h-stability which is an important extension of the notions of exponential stability and uniform Lipschitz stability for differential equations.

We will give the notion of h-stability for Caputo fractional differential systems.

Definition 12. The zero solution $x = 0$ of (21) is said to be

(1) an *h-system* if there exist a constant $c \geq 1$ and a positive continuous function $h : J \to \mathbb{R}$ such that

$$|x(t)| \leq c\,|x(a)|\,h(t)\,h(a)^{-1}, \quad t \geq a \geq t_0, \tag{34}$$

for $|x(a)| \leq \delta$. Here $h(a)^{-1} = 1/h(a)$.

(2) h-stable if h is bounded.

We recall the stability in the sense of Mittag-Leffler [8, 16].

Definition 13. The zero solution $x = 0$ of (21) is said to be a Mittag-Leffler system if

$$|x(t)| \leq \left\{m(x(t_0))\,E_q\left(\lambda(t - t_0)^q\right)\right\}^b, \quad t \geq t_0, \tag{35}$$

where $\lambda \in \mathbb{R}, b > 0, m(0) = 0, m(x) \geq 0$, and $m(x)$ are locally Lipschitz on $x \in B \subseteq \mathbb{R}^n$ with Lipschitz constant m_0.

The zero solution $x = 0$ of (21) is called *Mittag-Leffler stable* if the constant λ in (35) is nonpositive.

Note that the Mittag-Leffler stability implies h-stability, but the converse does not hold in general. See Remark 19 for the example.

We can obtain the following result adapted from Theorem 3.4 in [8].

Theorem 14. *Suppose that the function f of* (21) *satisfies*

$$|f(t,x)| \leq g(t,|x|), \tag{36}$$

where $g \in C(J \times \mathbb{R}, \mathbb{R}^+)$ is monotonic increasing in u for each $t \in J$ with $g(t,0) = 0$. One considers the Caputo fractional differential equation

$${}^CD^q_{t_0}u(t) = g(t,u), \quad u(t_0) = u_0, \quad t \geq t_0. \tag{37}$$

If the zero solution $u = 0$ of (37) *is an h-system, then the zero solution $x = 0$ of* (21) *is also an h-system whenever $u_0 > |x(t_0)|$.*

Proof. The equation (21) is equivalent to the following Volterra fractional integral equation:

$$x(t) = x(t_0) + I^q_{t_0}f(t,x(t)), \quad t \geq t_0. \tag{38}$$

Then, we obtain

$$\begin{aligned} |x(t)| &= |x(t_0)| + \left|I^q_{t_0}f(t,x(t))\right| \\ &\leq |x(t_0)| + I^q_{t_0}|f(t,x(t))| \\ &\leq |x(t_0)| + I^q_{t_0}g(t,|x(t)|), \quad t \geq t_0. \end{aligned} \tag{39}$$

Thus we have

$$\begin{aligned} &|x(t)| - I^q_{t_0}g(t,|x(t)|) \\ &\quad\leq |x(t_0)| < u_0 = u(t) \\ &\quad\quad - I^q_{t_0}g(t,u(t)), \quad t \geq t_0, \end{aligned} \tag{40}$$

where $u_0 = u(t_0)$. By Lemma 11, we have $x(t) < u(t)$ for all $t \geq t_0$. Since $u = 0$ of (37) is an h-system, there exist a constant $c_1 \geq 1$ and a positive continuous function $h : J \to \mathbb{R}$ such that

$$|u(t)| \leq c_1\,|u(a)|\,h(t)\,h(t_0)^{-1}, \quad t \geq a \geq t_0, \tag{41}$$

for $|u(a)| \leq \delta$. Thus, we see that

$$\begin{aligned} |x(t)| &< u(t) \\ &\leq c_1\,|u(a)|\,h(t)\,h(a)^{-1} \\ &= c\,|x(a)|\,h(t)\,h(a)^{-1}, \quad t \geq a \geq t_0, \end{aligned} \tag{42}$$

where $u(a) = |x(a)|d$ with $d > 1$ and $c = c_1 d$. This completes the proof. □

Corollary 15. *Suppose that all conditions of Theorem 14 hold. The asymptotic stability of* (37) *implies the corresponding asymptotic stability of* (21).

We can obtain an upper bound of solutions for Caputo fractional differential equations via fractional Gronwall's inequality. The following result is adapted from Theorem 5.1 in [7] and Theorem 3.15 in [9].

Lemma 16. *Suppose that $D \subset \mathbb{R}^n$ is a domain containing the origin and $q \in (0,1)$. Let $V : \mathbb{R}^+ \times D \to \mathbb{R}$ be a continuously differentiable function and locally Lipschitz with respect to x satisfying*

$$\alpha_1|x|^a \leq V(t,x) \leq \alpha_2|x|^{ab}, \tag{43}$$

$${}^CD^q_{t_0}V(t,x) \leq -\alpha_3|x|^{ab} + L, t \geq t_0, \tag{44}$$

where $L \in \mathbb{R}$ and $\alpha_1, \alpha_2, \alpha_3, a, b$ are positive constants. Then one has

$$\begin{aligned} |x(t)| \leq \Bigg\{&\frac{\alpha_2}{\alpha_1}|x_0|^{ab}E_q\left(-\frac{\alpha_3}{\alpha_2}(t - t_0)^q\right) \\ &+ \frac{L}{\alpha_1}(t - t_0)^q E_{q,q+1}\left(-\frac{\alpha_3}{\alpha_2}(t - t_0)^q\right)\Bigg\}^{1/a}, \\ &t \geq t_0, \end{aligned} \tag{45}$$

where $x(t) = x(t, t_0, x_0)$ is any solution of (21).

Proof. It follows from (43) and (44) that

$$\begin{aligned} {}^CD^q_{t_0}V(t,x) &\leq -\alpha_3|x|^{ab} + L \\ &\leq -\frac{\alpha_3}{\alpha_2}V(t,x) + L, \quad t \geq t_0. \end{aligned} \tag{46}$$

It follows from Lemma 9 that

$$V(t,x(t)) \le V(t_0,x(t_0)) E_q\left(-\frac{\alpha_3}{\alpha_2}(t-t_0)^q\right) + L(t-t_0)^q E_{q,q+1}\left(-\frac{\alpha_3}{\alpha_2}(t-t_0)^q\right), \quad t \ge t_0. \tag{47}$$

Substituting (47) into (43) yields

$$|x(t)| \le \left\{\frac{\alpha_2}{\alpha_1}|x_0|^{ab} E_q\left(-\frac{\alpha_3}{\alpha_2}(t-t_0)^q\right) + \frac{L}{\alpha_1}(t-t_0)^q E_{q,q+1}\left(-\frac{\alpha_3}{\alpha_2}(t-t_0)^q\right)\right\}^{1/a}, \quad t \ge t_0. \tag{48}$$

This complete the proof. □

We can obtain the boundedness of solutions for Caputo fractional differential equations via the fractional Lyapunov direct method.

Theorem 17. *Under the same assumptions of Lemma 16, all solutions of* (21) *are eventually bounded on J.*

Proof. Let $x(t) = x(t,t_0,x_0)$ be any solution of (21). Then it follows from Lemma 16 that

$$|x(t)| \le \left\{\frac{\alpha_2}{\alpha_1}|x_0|^{ab} E_q\left(-\frac{\alpha_3}{\alpha_2}(t-t_0)^q\right) + \frac{L}{\alpha_1}(t-t_0)^q E_{q,q+1}\left(-\frac{\alpha_3}{\alpha_2}(t-t_0)^q\right)\right\}^{1/a}, \tag{49}$$

for each $t \ge t_0$. In view of Lemma 5 and Corollary 6, we note that $E_q(-(\alpha_3/\alpha_2)(t-t_0)^q)$ tends monotonically zero as $t \to \infty$ and $(t-t_0)^q E_{q,q+1}(-(\alpha_3/\alpha_2)(t-t_0)^q)$ is eventually bounded on $[t_0,\infty)$. Hence, there exist a positive constant $M_0 = M(|x_0|)$ and $t_1 > t_0$ such that

$$|x(t)| \le M_0, \quad t \ge t_1. \tag{50}$$

This completes the proof. □

We can obtain the following result [7, Theorem 5.1] about Mittag-Leffler stability of (21) as a corollary of Lemma 16.

Corollary 18. *If one sets $L = 0$ in the assumption of Lemma 16, then the zero solution $x = 0$ of* (21) *is Mittag-Leffler stable.*

3. Examples

In this section we give tow examples which illustrate some results in the previous section.

Example 1 (see [8]). To illustrate Theorem 14, we consider the Caputo fractional differential equation

$$^{C}D_{t_0}^{1/2}x(t) = \frac{x}{t(1+x^2)}, \quad t \ge t_0 > 0, \tag{51}$$

where $f(t,x) = x/t1 + x^2$. Then the zero solution $x = 0$ of (51) is h-stable.

Proof. The function f satisfies

$$|f(t,x)| \le g(t,|x|) = \frac{|x|}{t}, \quad t > 0, \tag{52}$$

and the solution of the Caputo fractional differential equation

$$^{C}D_0^{1/2}u(t) = \frac{1}{t}u, \qquad u(t_0) = u_0 \tag{53}$$

is given by $u(t) = ct^{-1/2}e^{-1/t}$, $t > 0$. We have

$$u(t) = u_0\sqrt{t_0}e^{1/t_0}\frac{1}{\sqrt{t}e^{1/t}} = u_0 h(t) h(t_0)^{-1}, \quad t \ge t_0 > 0, \tag{54}$$

where $h(t) = t^{-1/2}e^{-1/t}$. Thus, the zero solution $u = 0$ of (53) is h-stable. Hence, the zero solution $x = 0$ of (51) is h-stable by Theorem 14. □

Remark 19. We note that the fractional differential equation (53) given in the proof of Example 1 is h-stable but not Mittag-Leffler stable.

Proof. Let $u(t)$ be any solution of (53). Then, it follows from [8, Example 2.2] that $u(t)$ is neither monotonic nondecreasing in t nor monotonic nonincreasing in t. Furthermore, we easily see that

$$\lim_{t\to 0^+} u(t) = 0 = \lim_{t\to\infty} u(t). \tag{55}$$

Suppose that (53) is Mittag-Leffler stable; that is, there exist positive constants λ and b satisfying

$$|u(t)| \le \left\{m(u(t_0)) E_{1/2}\left(-\lambda(t-t_0)^{1/2}\right)\right\}^b, \quad t \ge t_0 > 0, \tag{56}$$

where $m(0) = 0, m(x) \ge 0$, and $m(x)$ is locally Lipschitz in x. Since $E_{1/2}(-\lambda(t-t_0)^{1/2})$ is monotonic nonincreasing in t [17], we see that the right-hand function of (56) also is monotonic decreasing in t. This contradicts the fact that $u(t)$ has neither monotonic nondecreasing property nor monotonic nonincreasing property. □

Next, we will give an example to illustrate Theorem 17.

Example 2. Let $0 < q < 1$. We consider the Caputo fractional differential equation

$$^{C}D_0^q|x(t)| = -|x(t)| + f(t), \quad t \ge 0, \tag{57}$$

where $f \in C(\mathbb{R}^+,\mathbb{R})$ is bounded by a constant d. Let $V(t,x) = |x|$. Then it follows that

$$\begin{aligned} ^{C}D_0^qV(t,x) = {}^{C}D_0^q|x| &= -|x| + f(t) \\ &\le -V(t,x) + d, \quad t \ge 0. \end{aligned} \tag{58}$$

Let $\alpha_1 = \alpha_2 = \alpha_3 = a = b = 1$. Applying them in Lemma 16 gives

$$|x(t)| \le |x(0)| E_q(-t^q) + dt^q E_{q,q+1}(-t^q), \quad t \ge 0. \tag{59}$$

Hence, all solutions of (57) are eventually bounded by Theorem 17.

Conflict of Interests

The authors declare no conflict of interests.

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