Stability for Some Extremal Properties of the Simplex

Rolf Schneider

Abstract. Some geometric inequalities for convex bodies, where the equality cases characterize simplices, are improved in the form of stability estimates. The inequalities all deal with covering by homothetic copies.

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1. Introduction

Generally speaking, a stability version of a geometric inequality with known equality cases is an explicit estimate, exhibiting how closely the extremal situation is approximated if the equality is only satisfied up to some given error. In the geometry of convex bodies, affine-invariant continuous functionals often attain one of their extreme values at the simplices, and corresponding stability results are a challenge. We know of only a few examples, see [2], [3], [7], [8]. This paper adds some more. As a rule, the first step in proving a stability improvement of an inequality consists in optimizing a proof of the inequality to make the identification of the equality cases as easy as possible.

Let X be a Minkowski space, that is, a real vector space of finite dimension $n \geq 2$ with a norm $\|\cdot\|$. In the following, all metric notions refer to this norm and its unit ball, B. By D, d, R, r we denote, respectively, diameter, minimal width, circumradius, and inradius, as functions on the space $\mathcal{K}$ of convex bodies (compact, convex subsets with interior points) of X. The sharp inequalities

$$\frac{R}{D} \leq \frac{n}{n+1} \tag{1.1}$$

and

$$\frac{d}{r} \leq n+1 \tag{1.2}$$

are Minkowski space analogues of classical Euclidean results, due to Jung and to Steinhagen, respectively. Inequality (1.1) was established by Bohnenblust [1]. Proofs of both inequalities together with a discussion of the equality cases were given by Leichtweiss [9], and independent proofs of the inequalities are due to Eggleston [5]. Equality in (1.1) and (1.2) holds if the body K under consideration is a simplex and the unit ball B is the difference body of this simplex. Conversely, if equality holds in (1.1) or (1.2) for a body K, then K is a simplex, but the possible unit balls B are not unique (up to affine transformations); they have been described by Leichtweiss [9].

The quantities D, d, R, r appearing here can be expressed in terms of a single function of two convex bodies. For $K, L \in \mathcal{K}$ (not necessarily symmetric), let

$$\rho(K,L) := \min\{\lambda > 0 : \exists\, x \in X : K + x \subset \lambda L\}.$$

If $K^* = \frac{1}{2}(K-K)$ denotes the Minkowski symmetral (half the difference body) of K, then

$$D = 2\rho(K^*,B), \qquad d = \frac{2}{\rho(B,K^*)}, \tag{1.3}$$

$$R = \rho(K,B), \qquad r = \frac{1}{\rho(B,K)}. \tag{1.4}$$

The first relation in (1.3) follows from

$$\forall\, x,y \in K : \|x-y\| \leq \lambda \Leftrightarrow \forall\, z \in K-K : \|z\| \leq \lambda \Leftrightarrow 2K^* \subset \lambda B,$$

and the second (using the support function h with respect to some Euclidean structure, or defined on the dual space) from

$$\forall\, u : h(K,u) + h(K,-u) \geq \lambda h(B,u) \Leftrightarrow 2K^* \supset \lambda B.$$

Relations (1.4) are essentially the definitions. Inequalities (1.1) and (1.2) now read as

$$\frac{\rho(K,B)}{\rho(K^*,B)} \leq \frac{2n}{n+1} \tag{1.5}$$

and

$$\frac{\rho(B,K)}{\rho(B,K^*)} \leq \frac{n+1}{2}, \tag{1.6}$$

and thus as inequalities for affine invariant-functionals of a pair of convex bodies.

Both inequalities can neatly be expressed in terms of covering. For sets $K, M \subset X$, we say that K *can be covered by* M (or that M *can cover* K) if $K \subset M + x$ for a suitable vector $x \in X$. Recall that B is centrally symmetric. Inequality (1.5) is equivalent (by homogeneity) to the following assertion.

(A) *If any two-point subset of K can be covered by B (equivalently, $K^* \subset B$), then K can be covered by $\frac{2n}{n+1}B$.*

By a supporting slab of a convex body K we understand the region of X bounded by two parallel supporting hyperplanes of K. Then inequality (1.6) can be expressed as follows.

(B) *If any supporting slab of K can cover the parallel supporting slab of B (equivalently, $K^* \supset B$), then K can cover $\frac{2}{n+1}B$.*

In either case, if the dilatation factor cannot be improved, then K is a simplex.

As remarked in [4, p. 135], it is easy to deduce from (A) a similar assertion about arbitrary (not necessarily symmetric) convex bodies K, L. We formulate it here as the inequality

$$\frac{\rho(K,L)}{\rho(K^*,L^*)} \leq n, \tag{1.7}$$

or in words as follows.

(C) *If any two-point subset of K can be covered by L (equivalently, if any supporting slab of L can cover the parallel supporting slab of K, or equivalently, $K^* \subset L^*$), then K can be covered by nL.*

(We point out that the extension of (1.1), (1.2) to nonsymmetric bodies in [9] is of a different kind.)

In (C), the dilatation factor cannot be improved if and only if K is a simplex and L is its image under reflection in a point.

Theorems 3.1 and 4.1 of this paper provide stability versions for these extremal properties of the simplex. They are derived from a stability estimate for the Minkowski measure of symmetry; this is Theorem 2.1.

2. The Minkowski Measure of Symmetry

For a convex body $K \in \mathcal{K}$, the *Minkowski measure of symmetry*, denoted by $q(K)$, is defined as the smallest number $\lambda > 0$ such that λK can cover $-K$, equivalently

$$q(K) := \rho(K, -K) = \min\{\lambda > 0 : \exists\, x \in X : K + x \subset -\lambda K\}.$$

Obviously, $q(K) \geq 1$, with equality if and only if K is centrally symmetric. It is known that

$$q(K) \leq n, \tag{2.1}$$

with equality if and only if K is a simplex (references are in [6, Section 6.1]).

This extremal property of the simplex can be improved in the form of a stability version. For this, we need an appropriate notion of distance for affine equivalence classes of convex bodies. The extended Banach–Mazur distance $d_{BM}(K, L)$ of not necessarily symmetric convex bodies $K, L \in \mathcal{K}$ is defined as the smallest number

$\lambda > 0$ such that a suitable affine transform of K can cover L and can be covered by λL, thus

$$d_{BM}(K,L) := \min\{\lambda \geq 1 : \exists\, \alpha \in \mathrm{Aff}(n)\, \exists\, x \in X : L \subset \alpha K \subset \lambda L + x\}.$$

Here $\mathrm{Aff}(n)$ denotes the group of affine transformations of X.

Stability estimates for the inequality (2.1) were obtained by Böröczky [2, 3] and Guo [7]. The strongest assertion is that of Böröczky [2], but his proof seems to be inconclusive at the end. We give, therefore, a detailed different proof (Theorem 2.1). Here Δ denotes an n-dimensional simplex.

Theorem 2.1. *Let $0 \leq \epsilon < \frac{1}{n}$. If the convex body K satifies*

$$q(K) > n - \epsilon,$$

then

$$d_{BM}(K,\Delta) < 1 + \frac{(n+1)\epsilon}{1-n\epsilon}.$$

Proof. If we want to show that some convex body is close to a simplex, we must find such a simplex, and thus its vertices. In the present case, the vertices are found by an application of Helly's theorem (this extends the approach in [4, Theorem 2.7], which goes back to Yaglom and Boltyanskii [10, problem 19]). The connection to covering is made by defining

$$K(x,q) := \frac{q}{q+1}(K-x) + x$$

for $0 \leq q \leq n$ and $x \in K$, and observing that

$$c \in \bigcap_{x \in K} K(x,q) \Leftrightarrow -(K-c) \subset q(K-c). \tag{2.2}$$

The proof follows from

$$\begin{aligned} c \in K(x,q) \quad &\Leftrightarrow \quad \exists\, k \in K : c = \frac{q}{q+1}(k-x) + x \\ &\Leftrightarrow \quad \exists\, k \in K : -(x-c) = q(k-c) \\ &\Leftrightarrow \quad -(x-c) \in q(K-c). \end{aligned}$$

Now let $K \subset X$ be a convex body with $q(K) > n - \epsilon$, where $0 \leq \epsilon < 1/n$, and put $q := n - \epsilon$. Since $q < q(K)$, no point $c \in K$ satisfies the right-hand side of (2.2). By Helly's theorem, there must exist $n+1$ points $v_0, v_1, \dots, v_n \in K$ such that

$$\bigcap_{i=0}^{n} K(v_i, q) = \emptyset. \tag{2.3}$$

Since the set of all $(n+1)$-tuples $(v_0, \dots, v_n)$ satisfying (2.3) is open in K^{n+1}, we can assume that $v_0, \dots, v_n$ are affinely independent. Then $\Delta := \mathrm{conv}\,\{v_0, \dots, v_n\}$ is an n-simplex contained in K. We put

$$\lambda := 1 + \frac{(n+1)\epsilon}{1-n\epsilon}$$

and assert that
$$K \subset \operatorname{int} \lambda\Delta. \tag{2.4}$$
Suppose that (2.4) were false. Then some facet of $\lambda\Delta$, say the one opposite to λv_0, contains a point $p \in K$. There is a unique representation
$$p = \sum_{i=1}^{n} \gamma_i \lambda v_i \quad \text{with} \quad \gamma_i \geq 0, \quad \sum_{i=1}^{n} \gamma_i = 1.$$

Now we put
$$r := \frac{q}{q+1} = \frac{n-\epsilon}{n+1-\epsilon}$$
and
$$z := [1 - n(1-r)]v_0 + (1-r)\sum_{i=1}^{n} v_i.$$
With
$$\alpha_0 := \frac{1-n(1-r)}{r} = \frac{1-\epsilon}{n-\epsilon}, \quad \alpha_1 := 0, \quad \alpha_j := \frac{1-r}{r} = \frac{1}{n-\epsilon}$$
for $j = 2, \ldots, n$ we have $\alpha_i \geq 0$ and $\sum_{i=0}^{n} \alpha_i = 1$, hence
$$\begin{aligned} z &= \sum_{i=0}^{n} \alpha_i [r v_i + (1-r) v_1] \in \operatorname{conv}\{r(v_i - v_1) + v_1 : i = 0, \ldots, n\} \\ &= r(\Delta - v_1) + v_1 = \Delta(v_1, q). \end{aligned}$$
Similarly, $z \in \Delta(v_i, q)$ for $i = 1, \ldots, n$. Since $\Delta(v_i, q) \subset K(v_i, q)$, it follows from (2.2) that
$$z \notin K(v_0, q).$$

Without loss of generality (namely, after applying a translation), we can assume that
$$\sum_{i=0}^{n} v_i = o, \tag{2.5}$$
then $z = [1 - (n+1)(1-r)]v_0$. From $z \notin r(K - v_0) + v_0$ we see that the point
$$z_0 := -\tau v_0 \quad \text{with} \quad \tau := \frac{1+\epsilon}{n-\epsilon}$$
satisfies $z_0 \notin K$. By (2.5),
$$z_0 = \tau(v_1 + \cdots + v_n). \tag{2.6}$$

Using the point p introduced above, there is a unique affine representation
$$z_0 = \sum_{i=1}^{n} \beta_i v_i + \beta_{n+1} p \quad \text{with} \quad \sum_{i=1}^{n+1} \beta_i = 1,$$

thus

$$z_0 = \sum_{i=1}^{n} [\beta_i + \beta_{n+1}\gamma_i\lambda] v_i.$$

Comparing this with (2.6), we get

$$\beta_i + \beta_{n+1}\gamma_i\lambda = \tau \quad \text{for } i = 1, \dots, n,$$

hence

$$\sum_{i=1}^{n} \beta_i + \lambda\beta_{n+1} = n\tau$$

and thus

$$\beta_{n+1} = \frac{n\tau - 1}{\lambda - 1} = \frac{1 - n\epsilon}{n - \epsilon} \geq 0.$$

For $i = 1, \dots, n$, we obtain

$$\beta_i = \frac{1+\epsilon}{n-\epsilon}(1 - \gamma_i) \geq 0.$$

This yields

$$z_0 \in \operatorname{conv}\{v_1, \dots, v_n, p\} \subset K,$$

a contradiction. This shows that (2.4) holds, which implies $d_{BM}(K, \Delta) < \lambda$. $\square$

3. Stability for Assertions (A) and (B)

For $K \in \mathcal{K}$, we define

$$p(K) := \rho(K, K^*) = \min\{\lambda > 0 : \exists x \in X : K + x \subset \lambda K^*\}, \tag{3.1}$$

$$s(K) := \rho(K^*, K) = \min\{\lambda > 0 : \exists x \in X : K^* + x \subset \lambda K\}. \tag{3.2}$$

In the following, we write $q(K) = q$, $p(K) = p$, $s(K) = s$. For $\lambda > 0$ we have

$$K + x \subset -\lambda K \Leftrightarrow K + \lambda K + x \subset 2\lambda K^* \Leftrightarrow K + \frac{1}{\lambda+1}x \subset \frac{2\lambda}{\lambda+1}K^*.$$

In particular, with suitable x, y,

$$K + x \subset -qK \Rightarrow K + \frac{1}{q+1}x \subset \frac{2q}{q+1}K^* \Rightarrow p \leq \frac{2q}{q+1},$$

$$K + y \subset pK^* \Rightarrow K + \frac{2}{2-p}y \subset -\frac{p}{2-p}K \Rightarrow q \leq \frac{p}{2-p},$$

thus

$$p = \frac{2q}{q+1}. \tag{3.3}$$

Similarly,

$$K + x \subset -\lambda K \Leftrightarrow K - K + x \subset -(\lambda+1)K \Leftrightarrow 2K^* + x \subset -(\lambda+1)K,$$

which yields

$$s = \frac{q+1}{2}. \tag{3.4}$$

As in the introduction, we assume that X is equipped with a norm, and its unit ball is denoted by B.

Let K be a convex body of diameter $D > 0$ and circumradius R, and let $q = q(K)$, $p = p(K)$, $s = s(K)$. The preceding identities together with (2.1) allow us to give a very short version of Eggleston's [5] proofs of (1.1) and (1.2).

Suppose that $\lambda \geq p$, hence $K + x \subset \lambda K^*$ for some $x \in X$. After a translation, we may assume that $K \subset \lambda K^*$. We cannot have $\lambda K^* \subset \operatorname{int} RB$, since then $K \subset \operatorname{int} RB$, and R would not be the circumradius of K. Hence, λK^* has a point x with $\|x\| \geq R$. Therefore, λK contains two points at distance at least $2R$, which means that $\lambda D \geq 2R$. Since this holds for all $\lambda \geq p$, we get

$$\frac{\rho(K,B)}{2\rho(K^*,B)} = \frac{R}{D} \leq \frac{p}{2} = \frac{q}{q+1} \leq \frac{n}{n+1} \tag{3.5}$$

and thus (1.5).

Suppose that $\lambda \geq s$, hence $K^* + x \subset \lambda K$ for some $x \in X$. Applying a homothety, we may assume that $\frac{1}{\lambda}K^* \subset K$. The inclusion $rB \subset \operatorname{int}\frac{1}{\lambda}K^*$ would imply that r is not the inradius of K, hence is impossible. Therefore, rB has some supporting halfspace that contains $\frac{1}{\lambda}K^*$. This means that r is not less than $\frac{1}{2\lambda}$ times the width of K in the corresponding direction, hence $r \geq d/2\lambda$. Since this holds for all $\lambda \geq s$, we get

$$\frac{2\rho(B,K)}{\rho(B,K^*)} = \frac{d}{r} \leq 2s = q + 1 \leq n + 1 \tag{3.6}$$

and thus (1.6).

Turning to stability estimates, we assume that, for some convex body K and some unit ball B, inequality (1.1) holds with approximate equality, namely

$$\frac{R}{D} > \frac{n-\epsilon}{n-\epsilon+1}$$

with some ϵ satisfying $0 < \epsilon < 1/n$. Since the function $q \mapsto q/(q+1)$ is strictly increasing, (3.5) gives $q > n - \epsilon$.

Similarly, if (1.2) holds with approximate equality,

$$\frac{d}{r} > n - \epsilon + 1,$$

then (3.6) gives $q > n - \epsilon$. Theorem 2.1 thus yields the following result.

Theorem 3.1. *Let $0 < \epsilon < \frac{1}{n}$. If the diameter D and the circumradius R of a convex body K with respect to some Minkowski metric on X satisfy*

$$\frac{R}{D} > \frac{n-\epsilon}{n-\epsilon+1},$$

or if the minimal width d and the inradius r of K satisfy

$$\frac{d}{r} > n - \epsilon + 1,$$

then

$$d_{BM}(K,\Delta) < 1 + \frac{(n+1)\epsilon}{1-n\epsilon}$$

where Δ is an n-dimensional simplex.

4. Stability for Assertion (C)

For a proof of (1.7), we can assume that $\rho(K^*,L^*) = 1$, hence $K^* \subset L^*$. Using this together with (3.1), (3.2), we get, with suitable x, y,

$$K + x \subset p(K)K^* \subset p(K)L^* \subset p(K)s(L)L + y,$$

hence with (3.3) and (3.4) we obtain

$$\rho(K,L) \le p(K)s(L) = \frac{2q(K)}{q(K)+1}\frac{q(L)+1}{2} \le \frac{n}{n+1}(n+1) = n \tag{4.1}$$

and thus (1.7).

Although the equality case is covered by the stability estimate below, we prove it separately, in the hope that it makes the subsequent stability proof more perspicuous. Suppose that equality holds in (1.7). Then $q(K) = n$ and $q(L) = n$, hence K and L are simplices. After a dilatation of L we can assume that

$$\rho(K^*,L^*) = 1, \tag{4.2}$$

then

$$\rho(K,L) = n. \tag{4.3}$$

We use an auxiliary Euclidean structure with scalar product $\langle\cdot,\cdot\rangle$ and the corresponding notation. For example, $H^-(K,u)$ and $H(K,u)$ are the supporting halfspace and the supporting hyperplane with outer unit normal vector u, respectively, of a convex body K, and $h(K,\cdot)$ denotes the support function of K.

Let $u_0, u_1, \dots, u_n$ be the unit normal vectors of the facets of the simplex L. We may assume that the origin o is the centre of gravity of L, then

$$h(L,-u_i) = nh(L,u_i) \qquad \text{for } i = 0,\dots,n. \tag{4.4}$$

There is a unique translate of K, and w.l.o.g. we may assume that K itself is this translate, such that

$$K \subset \bigcap_{i=1}^{n} H^-(nL,u_i), \qquad K \cap H(nL,u_i) \neq \emptyset \quad \text{for } i = 1,\dots,n. \tag{4.5}$$

From (4.3) it follows that then also

$$K \subset H^-(nL,u_0), \qquad K \cap H(nL,u_0) \neq \emptyset. \tag{4.6}$$

Relation (4.2) implies $K^* \subset L^*$, hence

$$h(K,u_i) + h(K,-u_i) \le h(L,u_i) + h(L,-u_i) \quad \text{for } i = 0,\dots,n.$$

Since $h(K,u_i) = h(nL,u_i)$ by (4.5) and (4.6), together with (4.4) this gives $h(K,-u_i) \le h(L,u_i)$, thus $h(-K,u_i) \le h(L,u_i)$ for $i = 0,\dots,n$ and hence $-K \subset L$. From $h(K,u_i) = h(nL,u_i) = h(L,-u_i)$ we get $h(-K,-u_i) = h(L,-u_i)$. Therefore, $-K$ contains the vertices of L, which yields that $-K = L$. Hence, equality holds in (1.7) if and only if K is a simplex and L is homothetic to $-K$.

Our corresponding stability result reads as follows.

Theorem 4.1. *If K, L are convex bodies satisfying*

$$\frac{\rho(K,L)}{\rho(K^*,L^*)} \ge n - \frac{n}{n+1}\epsilon, \tag{4.7}$$

where $0 \le \epsilon < 1/(5n^3+n)$, then there is a simplex Δ with centroid at the origin such that suitable homothets K', L' of K and L satisfy

$$(1-a_n\epsilon)\Delta \subset -K' \subset \Delta$$

and

$$(1-b_n\epsilon)\Delta \subset L' \subset \Delta$$

with $a_n = 5n^3+n$ and $b_n = 2n$.

(The factor $\frac{n}{n+1}$ in (4.7) is chosen for convenience.) We prepare the proof by a lemma.

Lemma 4.2. *Let $T \subset X$ be a simplex with vertices $v_0,\dots,v_n$ and centroid o. For $0 < r < \frac{1}{n+1}$ define*

$$T_i(r) := r(T - v_i) + v_i, \qquad i = 0,\dots,n.$$

Let $s := 1-(n+1)r$. If $x_i \in T_i(r)$ for $i = 0,\dots,n$, then

$$sT \subset \operatorname{conv}\{x_0,\dots,x_n\}.$$

Proof. Since the assertion is invariant under linear transformations, we can introduce a scalar product $\langle\cdot,\cdot\rangle$ so that the simplex T becomes regular, then $\langle v_i, v_j\rangle = -1/n$ for $i \ne j$. Let

$$H^-(u,t) := \{x \in X : \langle x,u\rangle \le t\}$$

be a supporting halfspace of $\operatorname{conv}\{x_0,\dots,x_n\}$. The vector u is in the positive hull of some n vectors of $v_0,\dots,v_n$, say

$$u = \sum_{k=1}^{n} \alpha_k v_k \qquad \text{with } \alpha_k \ge 0.$$

For $j = 1,\dots,n$,

$$\langle v_j,u\rangle - \langle v_0,u\rangle = \alpha_j\left(\langle v_j,v_j\rangle + \frac{1}{n}\right) \ge 0.$$

Let $i \in \{0,\dots,n\}$. For $j \in \{0,\dots,n\}$, let v_{ij} be the vertex of $T_i(r)$ corresponding to v_j under the homothety that maps T to $T_i(r)$; then also

$$\langle v_{ij},u\rangle \ge \langle v_{i0},u\rangle, \qquad j = 1,\dots,n.$$

Since $x_i \in T_i(r)$ and $x_i \in H^-(u,t)$, we get

$$t \geq \langle x_i, u\rangle \geq \min\{\langle v_{ij}, u\rangle : j = 0,\dots,n\} = \langle v_{i0}, u\rangle, \qquad i = 0,\dots,n.$$

Choose i with

$$\langle v_i, u\rangle \geq \langle v_k, u\rangle, \qquad k = 1,\dots,n,$$

then

$$n\langle v_i, u\rangle \geq \sum_{k=1}^{n} \langle v_k, u\rangle = -\langle v_0, u\rangle.$$

Since $s = 1 - r - nr$, we obtain

$$\begin{aligned}\langle sv_i, u\rangle - \langle v_{i0}, u\rangle &= (1-r)\langle v_i, u\rangle - nr\langle v_i, u\rangle - \langle rv_0 + (1-r)v_i, u\rangle \\ &= -nr\langle v_i, u\rangle - r\langle v_0, u\rangle \leq 0.\end{aligned}$$

Thus, for $k = 1,\dots,n$,

$$\langle sv_0, u\rangle \leq \langle sv_k, u\rangle \leq \langle sv_i, u\rangle \leq \langle v_{i0}, u\rangle \leq t.$$

Hence, all vertices of the simplex sT are contained in $H^-(u,t)$. Since the latter was an arbitrary supporting halfspace of $\mathrm{conv}\{x_0,\dots,x_n\}$, this convex hull contains sT. $\square$

Proof of Theorem 3. Let $0 \leq \epsilon < 1/(5n^3 + n)$, and assume that

$$\frac{\rho(K,L)}{\rho(K^*,L^*)} \geq n - \frac{n}{n+1}\epsilon =: \mu.$$

First we apply a dilatation to L so that $\rho(K^*,L^*) = 1$. Then, by (4.1) and (2.1),

$$n - \frac{n}{n+1}\epsilon \leq \rho(K,L) \leq \frac{q(K)}{q(K)+1}(q(L)+1) \leq \frac{n}{n+1}(q(L)+1),$$

hence

$$q(L) \geq n - \epsilon.$$

Since also $q(\mu L) \geq n - \epsilon$, inspection of the proof of Theorem 2.1 shows that there exists a simplex Δ with centroid at the origin o such that, after replacing L by a suitable translate, we have

$$\Delta \subset \mu L \subset \lambda\Delta \quad \text{with} \quad \lambda = 1 + 2n\epsilon.$$

We use a scalar product $\langle\cdot,\cdot\rangle$ for which Δ is regular. Let $u_0, u_1, \dots, u_n$ be the outer unit normal vectors of the facets of Δ. There is translate of K, w.l.o.g. K itself, such that

$$K \subset \bigcap_{i=1}^{n} H^-(\lambda\Delta, u_i), \qquad K \cap H(\lambda\Delta, u_i) \neq \emptyset \quad \text{for } i = 1,\dots,n.$$

For $i = 1,\dots,n$ we then have

$$h(K, u_i) = h(\lambda\Delta, u_i) \geq h(\mu L, u_i) = \mu h(L, u_i). \tag{4.8}$$

Let v_0 be the vertex of $\lambda\Delta$ opposite to the facet with normal vector u_0. Shrinking $\lambda\Delta$ from v_0 by the factor $1/\lambda$, we obtain the simplex $\Delta' = \Delta + (1-\lambda^{-1})v_0$. If $h(K,u_0) < h(\Delta',u_0)$, then a smaller homothet of Δ' contains K, which implies that a smaller homothet of μL can cover K, in contradiction to $\rho(K,L) = \mu$. Hence,

$$\begin{aligned} h(K,u_0) &\geq h(\Delta',u_0) = h(\Delta,u_0) + (1-\lambda^{-1})\langle v_0,u_0\rangle \\ &= h(\Delta,u_0) - (1-\lambda^{-1})nh(\lambda\Delta,u_0) = \beta h(\lambda\Delta,u_0) \qquad (4.9) \\ &\geq \beta h(\mu L,u_0) = \beta\mu h(L,u_0) \qquad (4.10) \end{aligned}$$

with

$$\beta := \frac{n+1}{\lambda} - n \geq 1 - 3n^2\epsilon.$$

The relation $\rho(K^*,L^*) = 1$ implies $K^* \subset L^*$, hence

$$h(K,u_i) + h(K,-u_i) \leq h(L,u_i) + h(L,-u_i) \quad \text{for } i = 0,\dots,n. \qquad (4.11)$$

For $i = 0,\dots,n$ we have

$$h(\mu L,-u_i) \leq h(\lambda\Delta,-u_i) = nh(\lambda\Delta,u_i) = n\lambda h(\Delta,u_i) \leq n\lambda h(\mu L,u_i),$$

hence

$$h(L,-u_i) \leq n\lambda h(L,u_i). \qquad (4.12)$$

For $i = 1,\dots,n$ we obtain from (4.8), (4.11), (4.12) that

$$h(-K,u_i) = h(K,-u_i) \leq (1+n\lambda-\mu)h(L,u_i). \qquad (4.13)$$

Similarly, from (4.10), (4.11), (4.12) we get

$$h(-K,u_0) = h(K,-u_0) \leq (1+n\lambda-\beta\mu)h(L,u_0). \qquad (4.14)$$

Since $h(\mu L,u_i) \leq h(\lambda\Delta,u_i)$ for $i = 0,\dots,n$, we see from (4.13) and (4.14) that

$$\begin{aligned} h(-K,u_i) &\leq (1+n\lambda-\mu)\frac{\lambda}{\mu}h(\Delta,u_i), \qquad i = 1,\dots,n, \\ h(-K,u_0) &\leq (1+n\lambda-\beta\mu)\frac{\lambda}{\mu}h(\Delta,u_i). \end{aligned}$$

These inequalities show that

$$-K \subset \Delta_1,$$

where Δ_1 is the simplex with

$$\begin{aligned} h(\Delta_1,u_i) &= (1+n\lambda-\mu)\frac{\lambda}{\mu}h(\Delta,u_i), \qquad i = 1,\dots,n, \qquad (4.15) \\ h(\Delta_1,u_0) &= (1+n\lambda-\beta\mu)\frac{\lambda}{\mu}h(\Delta,u_i). \qquad (4.16) \end{aligned}$$

For $i = 1,\dots,n$ we have

$$h(-K,-u_i) = h(K,u_i) = h(\lambda\Delta,u_i) = \frac{\lambda}{n}h(\Delta,-u_i)$$

and for $i = 0$ similarly, by (4.9),

$$h(-K, -u_0) = h(K, u_0) \geq \beta h(\lambda\Delta, u_0) = \frac{\beta\lambda}{n} h(\Delta, -u_0).$$

Writing

$$H^+(M, u) := \{x \in X : \langle x, u \rangle \geq h(M, u)\}$$

for a convex body M, we see from the preceding that $-K$ contains points in each of the sets

$$\begin{aligned} \Delta_i' &:= \Delta_1 \cap H^+\left(\frac{\lambda}{n}\Delta, -u_i\right), \qquad i = 1, \ldots, n, \\ \Delta_0' &:= \Delta_1 \cap H^+\left(\frac{\beta\lambda}{n}\Delta, -u_0\right), \end{aligned}$$

The simplex Δ_1 is homothetic to Δ, thus $\Delta_1 = s\Delta + t$ with a homothety factor s and a vector t. For a regular simplex T with unit normal vectors $u_0, \ldots, u_n$ (hence $\sum_{i=0}^n u_i = o$), the quantity $\sum_{i=0}^n h(T, u_i)$ is invariant under translations. Using this fact, one finds from (4.15) and (4.16) that the homothety factor is given by

$$s = (1 + n\lambda - \mu)\frac{\lambda}{\mu} + \frac{(1-\beta)\lambda}{n+1}.$$

Each simplex Δ_i', $i = 0, \ldots, n$, is homothetic to Δ and thus to Δ_1, say $\Delta_i' = r_i\Delta_1 + t_i$. To determine the homothety factor r_i, we write $h(\Delta, u_i) =: a$ and obtain from

$$h(\Delta_1, u_i) + h(\Delta_1, -u_i) = s(n+1)a$$

and (4.15), (4.16) that

$$h(\Delta_1, -u_i) = a\frac{n\lambda}{\mu}(1 + n\lambda - \mu) + a(1-\beta)\lambda$$

for $i = 1, \ldots, n$ and

$$h(\Delta_1, -u_0) = a\frac{n\lambda}{\mu}(1 + n\lambda - \mu).$$

From this, we get

$$\begin{aligned} r_i &= \frac{h(\Delta_1, -u_i) - h(\frac{\lambda}{n}\Delta, -u_i)}{(n+1)as} \\ &= \frac{n(1 + n\lambda - \mu) - \beta\mu}{(n+1)(1 + n\lambda - \mu) + (1-\beta)\mu} \end{aligned} \tag{4.17}$$

for $i = 1, \ldots, n$ and

$$r_0 = \frac{n(1 + n\lambda - \mu) - \mu}{(n+1)(1 + n\lambda - \mu) + (1-\beta)\mu}.$$

In particular, $r_i \geq r_0$ for $i = 1, \ldots, n$. Therefore, we can replace Δ_0' by a simplex with homothety factor r_1 instead of r_0 and then apply Lemma 4.2. From this we conclude that $-K$ contains the simplex Δ_2 that is obtained from Δ_1 by shrinking it from its centroid by the factor $s = 1 - (n+1)r_1$.

Altogether we have found that suitable homothetic copies K' of K and L' of L satisfy

$$\frac{1}{\lambda}\Delta \subset L' \subset \Delta, \qquad s\Delta \subset -K' \subset \Delta.$$

Here $1/\lambda \geq 1 - 2n\epsilon$, and for the estimation of s we use (4.17), where we estimate the denominator from below by $n+1$ and the numerator from above by using $\beta \geq 1 - 3n^2\epsilon$. This yields the assertion. $\square$

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Rolf Schneider
Mathematisches Institut
Albert-Ludwigs-Universität
Eckerstr. 1
D-79104 Freiburg i. Br.
Germany

e-mail: `rolf.schneider@math.uni-freiburg.de`