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STABILITY IN DISTRIBUTION FOR A CLASS OF SINGULAR DIFFUSIONS

BY GOPAL K. BASAK AND RABI N. BHATTACHARYA[1]

University of California, Berkeley and Indiana University

A verifiable criterion is derived for the stability in distribution of singular diffusions, that is, for the weak convergence of the transition probability $p(t; x, dy)$, as $t \to \infty$, to a unique invariant probability. For this we establish the following: (i) tightness of $\{p(t; x, dy)$: $t \geq 0\}$; and (ii) asymptotic flatness of the stochastic flow. When specialized to highly nonradial nonsingular diffusions the results here are often applicable where Has'minskii's well-known criterion fails. When applied to traps, a sufficient condition for stochastic stability of nonlinear diffusions is derived which supplements Has'minskii's result for linear diffusions. We also answer a question raised by L. Stettner (originally posed to him by H. J. Kushner): Is the diffusion stable in distribution if the drift is Bx where B is a stable matrix, and $\sigma(\cdot)$ is Lipschitzian, $\sigma(\underline{0}) \neq 0$? If not, what additional conditions must be imposed?

1. Introduction. Consider a diffusion $\{X^x(t)$: $t \geq 0\}$ on $\Re^k$ satisfying Itô's equation

$$X^x(t) = x + \int_0^t BX^x(s)\,ds + \int_0^t \sigma(X^x(s))\,dW(s), \tag{1.1}$$

where B is a $k \times k$ matrix, $\sigma(\cdot)$ is a Lipschitzian $(k \times l)$-matrix-valued function on $\Re^k$ and $\{W(t)$: $t \geq 0\}$ is a standard l-dimensional Brownian motion. Let $p(t; x, dy)$ denote the transition probability of the diffusion. The following definitions apply to general diffusions, and not only to those of the form (1.1) with linear drifts.

DEFINITION 1.1. A diffusion is *stable in distribution* if its transition probability $p(t; x, dy)$ converges weakly to some probability measure $\pi(dy)$, as $t \to \infty$, for every x.

It is clear that stability in distribution implies the existence of a unique invariant probability. It is simple to check that stability in distribution follows from the following: (i) tightness of $\{p(t; x, dy)$: $0 \leq t < \infty\}$; and (ii) the following notion of asymptotic flatness.

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DEFINITION 1.2. The stochastic flow $\{X^x(t)\colon t \geq 0,\ x \in \Re^k\}$ is *asymptotically flat* (in probability) *uniformly on compacts* if

$$\sup_{x,y\in K} P(|X^x(t) - X^y(t)| > \varepsilon) \to 0 \quad \text{as } t \to \infty, \tag{1.2}$$

for every $\varepsilon > 0$ and every compact set K.

We will actually derive a stronger property than (1.2) called *asymptotic flatness of the stochastic flow in the δth mean* ($\delta > 0$), which means that for every compact K,

$$\lim_{t\to\infty} \sup_{x,y\in K} E|X^x(t) - X^y(t)|^\delta = 0. \tag{1.3}$$

In the special case when a *trap* x^* exists, that is, $X^{x^*}(t) = x^*$ for all $t \geq 0$, (1.2) with $y = x^*$ implies stochastic stability defined as follows.

DEFINITION 1.3. Let x^* be a trap. Then $\{X^x(t)\colon t \geq 0,\ x \in \Re^k\}$ is *stochastically stable* (in probability) if for all $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ there exists $\delta > 0$ such that

$$\sup_{0\leq t<\infty} \sup_{\{x\colon |x-x^*|\leq\delta\}} P(|X^x(t) - x^*| > \varepsilon_1) < \varepsilon_2. \tag{1.4}$$

Consider the question: Under what conditions on B and $\sigma(\cdot)$ is the diffusion stable in distribution? If $\sigma(\cdot)\sigma(\cdot)'$ is nonsingular, then the existence of an invariant probability is equivalent to stability in distribution, and Has'minskii's well-known criteria apply (Has'minskii [4]; Bhattacharya [2]). Our main interest lies in the *singular* case: $\sigma(x)\sigma(x)'$ is of rank less than k for some x. In this case the existence of a unique invariant probability does not necessarily imply stability in distribution, as may be shown by examples [e.g., $k = 1$, $B = 1$, $\sigma(x) = x$]. If $\sigma(\cdot) \equiv \sigma$ is a constant matrix, then a well-known necessary and sufficient condition for stability is that all eigenvalues of B have negative real parts (see, e.g., Arnold [1], pages 178–187). If $\sigma(\cdot)$ is linear, that is, every element of $\sigma(\cdot)$ is a linear function, then $x = \underline{0}$ is a *trap* and stability in distribution is equivalent to stochastic stability (in probability), which has been extensively studied by Has'minskii [5], Chapter 6. We are primarily interested in the case $\sigma(\underline{0}) \neq 0$, that is, $\underline{0}$ is not a trap. In this case if the diffusion is stable in distribution, the invariant probability has no discrete component. However, one may also derive criteria for stochastic stability by the method used in this article (see Remark 2.4).

The main distinction between nonsingular diffusions and singular ones in the present context is that for nonsingular diffusions *tightness* of $\{p(t; x, dy)\colon t \geq 0\}$ for some x is equivalent to stability in distribution, while this is far from being true in the singular case. Here is a simple but interesting example.

EXAMPLE 1.1. Let $k = 2$, $B = \text{diag}(-1, -1)$ and

$$\sigma(x) = c\begin{pmatrix} x_2 & 0 \\ -x_1 & 0 \end{pmatrix}.$$

Then $R^2(t) := X_1^2(t) + X_2^2(t)$ satisfies $dR^2(t) = (c^2 - 2)R^2(t)\,dt$, so that $R^2(t) = R^2(0) \times \exp\{(c^2 - 2)t\}$. Consider the case $|c| = \sqrt{2}$. Then $R^2(t) = R^2(0)$ for all t, which in particular implies tightness of $\{p(t; x, dy)\colon t \geq 0\}$ for every x. On the other hand, there is an invariant probability on every circle and the angular motion on the circle is a periodic diffusion. If $|c| \neq \sqrt{2}$, the only invariant probability is the point mass at the origin. If $|c| > \sqrt{2}$ and $X(0) \neq \underline{0}$, then $R^2(t) \to \infty$ a.s. as $t \to \infty$. If $|c| < \sqrt{2}$, then the diffusion is stochastically stable a.s. and, therefore, stable in distribution. One may modify this example by taking $\sigma(\cdot)$ so that $\sigma(\cdot)\sigma(\cdot)'$ is nonsingular on $\{|x| < r\}$ for some $r > 0$, and letting $\sigma(\cdot)$ be as above with $c = \pm\sqrt{2}$ on $\{|x| \geq r\}$. In this case the diffusion starting in $\{|x| \leq r\}$ has the limit cycle property, converging in distribution to the invariant probability on $\{|x| = r\}$, but still has infinitely many invariant probabilities—one on each circle $\{|x| = r'\}$, $r' \geq r$.

The following simple example shows that (1.3) alone is not enough and that tightness is needed along with (1.2) [or (1.3)] to establish stability in distribution.

EXAMPLE 1.2. Let $k = 1$, $b(x) = e^{-x}$, $\sigma(x) \equiv 0$. Then $X^x(t) = \ln(t + e^x) \to \infty$ as $t \to \infty$, but $X^x(t) - X^y(t) = \ln((t + e^x)/(t + e^y)) \to 0$ as $t \to \infty$, uniformly for x, y in a compact set K.

Finally, stable singular diffusions are not in general Harris recurrent, nor strongly mixing. To derive central limit theorems and laws of the iterated logarithm for processes such as $\int_0^t f(X(s))\,ds$, a convenient method in this case is to show that f belongs to the range of the infinitesimal generator on $L^2(\mathfrak{R}^k, \pi)$ (Bhattacharya [3]). Estimates of asymptotic flatness such as (2.17) enable one to identify a broad subset of the range.

Some qualitative aspects of asymptotics of singular diffusions have been studied by Kliemann [7].

2. The main result. Assume that, for some $\lambda_0 \geq 0$,

$$\|\sigma(x) - \sigma(y)\| \leq \lambda_0|x - y|, \quad \text{for all } x, y. \tag{2.1}$$

Throughout $\cdot$ (dot) and $|\ |$ denote euclidean inner product and norm, while $\|\ \|$ denotes matrix norm with respect to $|\ |$. Write

$$\begin{aligned} a(x) &= \sigma(x)\sigma(x)', \\ a(x,y) &= (\sigma(x) - \sigma(y))(\sigma(x) - \sigma(y))'. \end{aligned} \tag{2.2}$$

Write tr(A) for the *trace* of the matrix A. Our main result is:

THEOREM 2.1. *Suppose $\sigma(\cdot)$ is Lipschitzian.*

(a) *If there exist a symmetric positive definite matrix C and a positive constant γ such that*

$$\begin{aligned}(2.3)\qquad & 2C(x-y)\cdot B(x-y) - \frac{2C(x-y)\cdot a(x,y)C(x-y)}{(x-y)\cdot C(x-y)} \\ & \quad + \operatorname{tr}(a(x,y)C) \le -\gamma|x-y|^2, \qquad x \ne y,\end{aligned}$$

then the diffusion (1.1) *is stable in distribution.*

(b) *If there exist a symmetric positive definite matrix C and a constant $\beta > 0$ such that*

$$(2.4)\qquad 2Cx\cdot Bx - \frac{2Cx\cdot a(x)Cx}{x\cdot Cx} + \operatorname{tr}(a(x)C) \le -\beta|x|^2 \quad \text{for all sufficiently large } |x|,$$

then there exists an invariant probability.

The inequalities (2.3) and (2.4) arise from the use of the Liapounov function $v(x) := (x\cdot Cx)^{1-\varepsilon}$ [for a suitable $\varepsilon \in [0,1)$] applied, respectively, to the stochastic processes $Z^{x,y}(t) := X^x(t) - X^y(t)$ and $X^x(t)$.

As a corollary we have

COROLLARY 2.2. *Assume $\sigma(\cdot)$ is Lipschitzian and all eigenvalues of B have negative real parts. Assume in addition that*

$$(2.5)\qquad (k-1)\lambda_0^2 < \frac{1}{\Lambda_P},$$

where λ_0 is as in (2.1) *and Λ_P is the largest eigenvalue of*

$$(2.6)\qquad P := \int_0^\infty \exp\{sB'\}\exp\{sB\}\,ds.$$

Then the diffusion (1.1) *is stable in distribution.*

PROOF. In order to deduce Corollary 2.2 from Theorem 2.1, let $C = P$. It is not difficult to check that

$$(2.7)\qquad B'P + PB = -I,$$

where I is the $k\times k$ identity matrix. Using this, we get $2Px\cdot Bx = x\cdot(PB + B'P)x = -|x|^2$. Also,

$$\operatorname{tr}(a(x,y)P) = \operatorname{tr}\big(\sqrt{P}\,a(x,y)\sqrt{P}\big)$$

and

$$\frac{2P(x-y)\cdot a(x,y)P(x-y)}{(x-y)\cdot P(x-y)} = \frac{2\sqrt{P}(x-y)\cdot\sqrt{P}a(x,y)\sqrt{P}\sqrt{P}(x-y)}{\sqrt{P}(x-y)\cdot\sqrt{P}(x-y)}$$
$$\geq 2\big(\text{smallest eigenvalue of } \sqrt{P}a(x,y)\sqrt{P}\big).$$

Therefore,

$$(2.8)\qquad \begin{aligned} &-\frac{2P(x-y)\cdot a(x,y)P(x-y)}{(x-y)\cdot P(x-y)} + \operatorname{tr}(a(x,y)P) \\ &\leq \sum_{i=2}^{k}\lambda_i(x,y) - \lambda_1(x,y), \end{aligned}$$

where $\lambda_1(x,y) \leq \lambda_2(x,y) \leq \cdots \leq \lambda_k(x,y)$ are the eigenvalues of $\sqrt{P}a(x,y)\sqrt{P}$. The right side of (2.8) is clearly no larger than

$$(k-1)\|\sqrt{P}a(x,y)\sqrt{P}\| \leq (k-1)(\|P\|)(\|a(x,y)\|) \leq (k-1)\Lambda_P\lambda_0^2|x-y|^2.$$

Now let $\gamma = (1-(k-1)\Lambda_P\lambda_0^2)$ to obtain (2.3). □

REMARK 2.1. Before proceeding with the proof of Theorem 2.1, let us note that if $\sigma(\cdot)$ is Lipschitzian, then (2.3) implies (2.4) for every $\beta \in (0,\lambda)$. To see this simply take $y = 0$ in (2.3) and use the estimate

$$(2.9)\qquad \begin{aligned} -\gamma|x|^2 &\geq 2Cx\cdot Bx - \frac{2Cx\cdot a(x,0)Cx}{x\cdot Cx} + \operatorname{tr}(a(x,0)C) \\ &= 2Cx\cdot Bx - \frac{2Cx\cdot a(x)Cx}{x\cdot Cx} + \operatorname{tr}(a(x)C) + O(|x|) \quad \text{as } |x|\to\infty. \end{aligned}$$

As we shall see in the course of the proof of Theorem 2.1, (2.4) implies the *existence* of an invariant probability, but not *uniqueness*. The stronger condition (2.3) also implies the asymptotic flatness (1.3). The existence of an invariant probability and asymptotic flatness together immediately yield uniqueness and stability.

PROOF OF THEOREM 2.1. Consider the (Liapounov) function

$$(2.10)\qquad v(x) = (x\cdot Cx)^{1-\varepsilon}$$

for some $\varepsilon \in [0,1)$ to be chosen later. Define, for a given pair (x,y) with $x \neq y$,

$$(2.11)\qquad \begin{aligned} Z^{x,y}(t) &:= X^x(t) - X^y(t) \\ &= x - y + \int_0^t BZ^{x,y}(s)\,ds + \int_0^t\big(\sigma(X^x(s)) - \sigma(X^y(s))\big)\,dW(s), \\ \tau_0 &:= \inf\{t \geq 0: Z^{x,y}(t) = 0\}. \end{aligned}$$

By Itô's lemma (see Ikeda and Watanabe [6], pages 66–67)

$$\begin{aligned} v(Z^{x,y}(t)) - v(x-y) &= \int_0^t \tilde{L}(v)(X^x(s), X^y(s))\,ds \\ &\quad + \int_0^t (\operatorname{grad} v)(Z^{x,y}(s)) \\ &\quad \cdot (\sigma(X^x(s)) - \sigma(X^y(s)))\,dW(s), \qquad t < \tau_0, \end{aligned} \tag{2.12}$$

where writing ∂_i for differentiation with respect to the ith coordinate and using (2.3),

$$\begin{aligned} \tilde{L}(v)(x,y) &:= B(x-y)\cdot(\operatorname{grad} v)(x-y) + \frac{1}{2}\sum_{i,j=1}^{k} a_{ij}(x,y)(\partial_i\partial_j v)(x-y) \\ &= (1-\varepsilon)((x-y)\cdot C(x-y))^{-\varepsilon}\Bigg[2B(x-y)\cdot C(x-y) \\ &\qquad -2\varepsilon\frac{(x-y)\cdot Ca(x,y)C(x-y)}{(x-y)\cdot C(x-y)} + \operatorname{tr}(a(x,y)C)\Bigg] \\ &\le (1-\varepsilon)((x-y)\cdot C(x-y))^{-\varepsilon}\Bigg[-\gamma|x-y|^2 \\ &\qquad +2(1-\varepsilon)\frac{(x-y)\cdot Ca(x,y)C(x-y)}{(x-y)\cdot C(x-y)}\Bigg] \\ &\le (1-\varepsilon)((x-y)\cdot C(x-y))^{-\varepsilon} \\ &\qquad \times\left[-\gamma|x-y|^2 + 2(1-\varepsilon)\lambda_0^2\Lambda_C|x-y|^2\right]. \end{aligned} \tag{2.13}$$

Here Λ_C is the largest eigenvalue of C. Now choose $\varepsilon \in [0,1)$ such that

$$-\gamma_1 := -\gamma + 2(1-\varepsilon)\lambda_0^2\Lambda_C < 0. \tag{2.14}$$

Then we have

$$\tilde{L}(v)(x,y) \le -\alpha v(x-y), \tag{2.15}$$

with $\alpha := (\gamma_1(1-\varepsilon))/\Lambda_C$. Consider the process $Y(t) := \exp\{\alpha t\}v(Z^{x,y}(t))$. It follows from (2.12) and (2.15) that $\{Y(t\wedge\tau_0): t \ge 0\}$ is a positive supermartingale. In particular,

$$EY(t\wedge\tau_0) \le EY(0) = v(x-y). \tag{2.16}$$

Since $Z^{x,y}(t) = 0$ a.s. for all $t \ge \tau_0$, so that $Y(t) = 0$ for all $t \ge \tau_0$, (2.16) implies $EY(t) \le v(x-y)$. That is,

$$\begin{aligned} &E(Z^{x,y}(t)\cdot CZ^{x,y}(t))^{1-\varepsilon} \\ &\qquad \le \exp\{-\alpha t\}((x-y)\cdot C(x-y))^{1-\varepsilon} \qquad t \ge 0. \end{aligned} \tag{2.17}$$

This establishes the asymptotic flatness of the stochastic flow [in the $2(1-\varepsilon)$th mean].

In view of Remark 2.1, to complete the proof of Theorem 2.1 we need to show that (2.4) implies the existence of an invariant probability. But (2.4) implies $Lv(x) \to -\infty$ as $|x| \to \infty$, and the existence of an invariant probability follows from Has'minskii [5], Theorem 5.1, page 90. Note that v may be modified near the origin to make it twice continuously differentiable on all of $\mathfrak{R}^k$. □

REMARK 2.2 (Almost sure asymptotic flatness). The proof of Theorem 2.1 may be slightly modified to show that if (2.3) holds, then the stochastic flow is asymptotically flat *almost surely*, that is, there exists a finite random variable $V^{x,y}$ such that

$$v(Z^{x,y}(t)) \le V^{x,y}\exp\{-\alpha t\} \quad \text{a.s., } t \ge 0. \tag{2.18}$$

REMARK 2.3 (Nonlinear drift). If instead of a linear drift one has an arbitrary Lipschitzian drift $b(\cdot)$, then Theorem 2.1 holds with $B(x-y)$ [in (2.3)] and Bx [in (2.4)] replaced by $b(x) - b(y)$ and $b(x)$, respectively. There is no essential change in the proof.

REMARK 2.4 (Stochastic stability). If the origin $\underline{0}$ is a *trap*, that is, $\sigma(\underline{0}) = 0$, then $\{p(t; 0, dy): t \ge 0\}$ is trivially tight. In this case the proof of Theorem 2.1 shows that it is enough to check (2.3) with $y = 0$ (for all x). In view of (2.18), the diffusion is then stochastically stable a.s. and in the δth mean, for some $\delta > 0$. More generally, if the drift is $b(\cdot)$ [$b(\cdot)$ and $\sigma(\cdot)$ are assumed to be Lipschitzian] and if x^* is a *trap*, that is, $b(x^*) = 0$, $\sigma(x^*) = 0$, then a sufficient condition that $X_t^x \to x^*$ a.s. and in the δth mean exponentially fast for every x as $t \to \infty$, is

$$\begin{aligned} &2C(x-x^*)\cdot b(x) - \frac{2C(x-x^*)\cdot a(x)C(x-x^*)}{(x-x^*)\cdot C(x-x^*)} + \operatorname{tr}(a(x,x^*)C) \\ &\qquad \le -\gamma|x-x^*|^2 \quad \text{for all } x \ne x^*, \end{aligned} \tag{2.19}$$

for some positive definite matrix C and some $\gamma > 0$. In the case $b(\cdot)$ and $\sigma(\cdot)$ are both linear, this result may also be derived by the method of Has'minskii [5]. Note that $a(x, x^*) = a(x)$ if $\sigma(x^*) = 0$.

REMARK 2.5. Suppose the left side in (2.3) is *greater than or equal* to $\gamma|x-y|^2$ for some $\gamma > 0$ and all $x \ne y$. Then one may show (by the method of proof for asymptotic flatness) that $|X^x(t) - X^y(t)| \to \infty$ a.s. (and in the δth mean) exponentially fast as $t \to \infty$. This is true for the general nonlinear case, if $B(x-y)$ is replaced by $b(x) - b(y)$. Similarly, if the left side of (2.4) is greater than or equal to $\beta|x|^2$ for all x, then $|X^y(t)| \to \infty$ a.s. as $t \to \infty$ for every y that is not a trap.

REMARK 2.6 (Criterion for stability in distribution for nonsingular diffusions). Since (2.4) ensures tightness, a nonsingular diffusion with drift $b(\cdot)$ is stable in distribution if (2.4) holds [with Bx replaced by $b(x)$]. Although for nonsingular diffusions Has'minskii's useful criterion of positive recurrence is available, it is not very suitable if the infinitesimal generator is far from being radial. We give a simple example where Has'minskii's criterion is not satisfied, but (2.4) holds.

EXAMPLE 2.1. Let $k = 2$, $B = \operatorname{diag}(-1, -1)$, $a(x) = (\delta_1 + \delta_2(x_2)^2)I$, where $\delta_1 > 0$, $\delta_2 \geq 0$ are constants. To apply Has'minskii's criterion we compute (see Bhattacharya [2])

$$\underline{\alpha}(r) := \inf_{|x|=r} \frac{x \cdot a(x)x}{|x|^2} = \delta_1,$$

$$\bar{\beta}(r) := \sup_{|x|=r} \frac{2x \cdot Bx + \operatorname{tr}(a(x))}{(x \cdot a(x)x)/|x|^2} - 1 = \left(1 - \frac{2}{\delta_2}\right) + \frac{2\delta_1/\delta_2}{\delta_1 + \delta_2 r^2}, \tag{2.20}$$

$$\bar{I}(r) := \int_1^r \frac{\bar{\beta}(u)}{u}\,du = \left(1 - \frac{2}{\delta_2}\right)\ln r + O(1).$$

According to Has'minskii's criterion (see [2] and [4]) a sufficient condition for stability in distribution is

$$\int_1^\infty \exp\{-\bar{I}(r)\}\,dr = \infty, \qquad \int_1^\infty \frac{\exp\{\bar{I}(r)\}}{\underline{\alpha}(r)}\,dr < \infty. \tag{2.21}$$

In the present example,

$$\int_1^\infty \exp\{-\bar{I}(r)\}\,dr = \infty \quad \text{for all } \delta_2 \geq 0, \qquad \int_1^\infty \frac{\exp\{\bar{I}(r)\}}{\underline{\alpha}(r)}\,dr < \infty \quad \text{iff } \delta_2 < 1. \tag{2.22}$$

Thus, according to Has'minskii's test, the diffusion is stable in distribution if $\delta_2 \in [0, 1)$. On the other hand, taking $C = I$, the left side of (2.4) is $-2|x|^2$. Thus the criterion (2.4) is satisfied and the diffusion is stable in distribution no matter what the value of the nonnegative constant δ_2 is.

REMARK 2.7. A specific question raised by L. Stettner to one of us during a visit to the IMA in 1986 at the University of Minnesota was: Is the diffusion (1.1) stable in distribution if $\sigma(\cdot)$ is Lipschitzian, $\sigma(\underline{0}) \neq 0$, and all eigenvalues of B have negative real parts? It is obvious from (2.5) that the answer is yes for $k = 1$. A counterexample is contained in Example 1.1 for the case $k = 2$ [with $\sigma(\cdot)$ modified near the origin], which can be extended to $k > 2$. The two examples below show that even when restricted to nonsingular diffusions, the answer is "yes" for $k = 1$ and "no" for $k > 1$.

EXAMPLE 2.2 ($k > 2$). Let $B = -\delta I$ ($\delta > 0$), $a(x) = dr^2 I$ ($d > 0$) for $r \equiv |x| \geq 1$; then $a(\cdot)$ is nonsingular and Lipschitzian on $\mathfrak{R}^k$. In this case Has'minskii's criterion (2.21) is *necessary* as well as sufficient. But $\bar{I}(r) = (k - 1 - (2\delta/d))\ln r$ ($r \geq 1$). If $\delta/d < (k-2)/2$, then the first integral in (2.21) converges, implying that the diffusion is *transient*. If $\delta/d = (k-2)/2$, then the first integral in (2.21) diverges, as does the second integral, and the diffusion is *null recurrent*. If $\delta/d > (k-2)/2$, then (2.21) holds so that the diffusion is *positive recurrent* and, therefore, stable in distribution.

EXAMPLE 2.3 ($k = 2$). Let $B = -\delta I$,

$$a(x) = \begin{pmatrix} \lambda_1 x_2^2 + \varepsilon & -\lambda_1 x_1 x_2 \\ -\lambda_1 x_1 x_2 & \lambda_1 x_1^2 + \varepsilon \end{pmatrix},$$

where δ, λ_1 and ε are positive constants. Note that the positive definite square root of $a(x)$ is Lipschitzian in this case. Then

$$\begin{aligned} \bar{\beta}(r) = \underline{\beta}(r) &:= \inf_{|x|=r} \frac{(\lambda_1 - 2\delta)|x|^2 + \varepsilon}{\varepsilon} = 1 + \frac{\lambda_1 - 2\delta}{\varepsilon} r^2, \\ \bar{I}(r) = \underline{I}(r) &:= \int_1^r \frac{\underline{\beta}(u)}{u}\, du. \end{aligned} \tag{2.23}$$

Has'minskii's criterion for recurrence (or transience) is necessary as well as sufficient here. If $\lambda_1 = 2\delta$, then $\underline{\beta}(r) = 1$ and both the integrals in (2.21) diverge, which implies the diffusion is null recurrent. If $\lambda_1 > 2\delta$, then $\int_1^\infty \exp\{-\underline{I}(r)\}\, dr < \infty$, so that the diffusion is transient and no invariant probability exists.

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DEPARTMENT OF STATISTICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720

DEPARTMENT OF MATHEMATICS
INDIANA UNIVERSITY
BLOOMINGTON, INDIANA 47405