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STABILITY IN $SO(n+3)/SO(3) \times SO(n)$ BRANCHING

By

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Abstract. The branching rule for $SO(n+3)/SO(3) \times SO(n)$ is discussed. An effective bound for the stability in the branching is given.

1. Introduction

Let G be a compact connected Lie group, K its closed subgroup, and V_K an irreducible K-module. The space of smooth sections $C^\infty(G \times_K V_K)$ of the associated vector bundle $G \times_K V_K$ is a G-module, which has the decomposition into irreducible finite dimensional G-modules; we have a sum of irreducible finite dimensional G-submodules as its dense subspace. Frobenius' reciprocity law shows us that a G-submodule isomorphic to an irreducible G-module V_G appears in the decomposition if and only if there exists a non-vanishing K-homomorphism from V_G to V_K, and the multiplicity of the appearance is equal to the dimension of the space of K-homomorphisms $\mathrm{Hom}_K(V_G, V_K)$. If we assume that all modules are over the complex number field $\mathbf{C}$, by Schur's lemma, the dimension $\dim \mathrm{Hom}_K(V_G, V_K)$ is equal to the multiplicity of K-submodules isomorphic to V_K in the decomposition of V_G as a sum of K-irreducible K-modules. Therefore the decomposition of $C^\infty(G \times_K V_K)$ into G-irreducible G-modules is computed by the knowledge how a G-irreducible G-module V_G decomposes into a sum of K-irreducible K-modules, or, more precisely, by the knowledge which G-irreducible G-module V_G includes a K-submodule isomorphic to V_K in its decomposition into K-irreducible K-submodules and how many times V_G includes V_K. In our setting, the irreducible G-modules and K-modules are determined by their highest weight. When V_G is the irreducible G-module $V_G(\Lambda_G)$ with the highest weight Λ_G and V_K is the irreducible K-module $V_K(\Lambda_K)$ with the highest weight Λ_K, we define the multiplicity $m(\Lambda_G, \Lambda_K)$ by

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$$m(\Lambda_G, \Lambda_K) = \dim \mathrm{Hom}_G(V_G(\Lambda_G), C^\infty(G \times_K V_K(\Lambda_K)))$$
$$= \dim \mathrm{Hom}_K(V_G(\Lambda_G), V_K(\Lambda_K)).$$

The knowledge of $m(\Lambda_G, \Lambda_K)$ for every pair of highest weights Λ_G and Λ_K is called the branching rule for G/K. When we use it for the computation of the decomposition of $C^\infty(G \times_K V_K)$, it is not enough to compute $m(\Lambda_G, \Lambda_K)$ for some randomly taken Λ_G. For a fixed Λ_K, we should determine all the Λ_G for which $m(\Lambda_G, \Lambda_K)$ is positive and the precise value of $m(\Lambda_G, \Lambda_K)$ for them.

In [4], the author gave the branching rule for $SO(n+3)/SO(3) \times SO(n)$ $(n \geq 3)$, but then it was not clear how it can effectively be used for the computation of the decomposition of the space of smooth sections of the associated vector bundle.

In this paper, we shall show the effectiveness by establishing the bound for stability in the branching rule. First, we clarify what is the stability in the branching rule.

Assume that (G, K) is a symmetric pair. We denote by r the rank of (G, K). The G-module decomposition of the space of C^∞-functions $C^\infty(G/K)$ is well-known and clearly described by the theory of spherical functions. There are r fundamental weights $\Lambda_1, \Lambda_2, \ldots, \Lambda_r$, and each G-module whose highest weight is their linear combination over non-negative integers appears in the decomposition just once. Since $C^\infty(G/K)$ is $C^\infty(G \times_K V_K(0))$, the above means that, if we take the highest weight $\Lambda_0 = \sum_{i=1}^r p_i \Lambda_i$ with non-negative integral coefficients p_i $(1 \leq i \leq r)$, we always have

$$m(\Lambda_0, 0) = 1.$$

We fix a non-zero element Φ of $\mathrm{Hom}_G(V_G(\Lambda_0), C^\infty(G/K))$.

The space $C^\infty(G \times_K V_K)$ is a $C^\infty(G/K)$-module, and the module structure is compatible with the G-module structure. For $\Psi \in \mathrm{Hom}_G(V_G(\Lambda_G), C^\infty(G \times_K V_K))$, we have $\Psi \otimes \Phi \in \mathrm{Hom}_G(V_G(\Lambda_G) \otimes V_G(\Lambda_0), C^\infty(G \times_K V_K))$, where we used $V_K \otimes V_K(0) = V_K$. Since $V_G(\Lambda_G + \Lambda_0)$ is the G-submodule of $V_G(\Lambda_G) \otimes V_G(\Lambda_0)$ containing the tensor product of the highest weight vectors of $V_G(\Lambda_G)$ and $V_G(\Lambda_0)$, we have the restriction of $\Psi \otimes \Phi$ to $\mathrm{Hom}_G(V_G(\Lambda_G + \Lambda_0), C^\infty(G \times_K V_K))$, and we denote it also by $\Psi \otimes \Phi$. Let $v_G(\Lambda_G)$ be the vector corresponding to the highest weight in $V_G(\Lambda_G)$ and $v_G(\Lambda_0)$ that in $V_G(\Lambda_0)$. If the image $\Psi(v_G(\Lambda_G))$ is a non-zero section in $C^\infty(G \times_K V_K)$, the image $(\Psi \otimes \Phi)(v_G(\Lambda_G) \otimes v_G(\Lambda_0))$ is also a non-zero section, since it is a multiplication of $\Psi(v_G(\Lambda_G))$ by a non-zero function $\Phi(v_G(\Lambda_0))$ that does not vanish on an open dense set of G/K.

Therefore, if $\mathrm{Hom}_G(V_G(\Lambda_G), C^\infty(G \times_K V_K))$ has the dimension q, we can make q independent realizations of $V_G(\Lambda_G + \Lambda_0)$ in $C^\infty(G \times_K V_K)$. We conclude the following proposition.

Proposition 1. *Let Λ_K be the highest weight of an irreducible K-module, and Λ_G the highest weight of an irreducible G-module. For any highest weight $\Lambda_0 = \sum_{i=1}^r p_i \Lambda_i$ with non-negative integers p_i $(1 \le i \le r)$, we have*

$$m(\Lambda_G + \Lambda_0, \Lambda_K) \ge m(\Lambda_G, \Lambda_K).$$

Therefore, the value $m(\Lambda_G, \Lambda_K)$ is non-decreasing with respect to the addition of the highest weight Λ_0.

It is generally known to stabilize for the large Λ_G; the value $m(\Lambda_G, \Lambda_K)$ stops increasing. See, for example, Sato [3]. But, for our application, the effective bound for Λ_G so that the equality $m(\Lambda_G + \Lambda_0, \Lambda_K) = m(\Lambda_G, \Lambda_K)$ should hold is needed. With such bound, we can select Λ_G for which we should compute $m(\Lambda_G, \Lambda_K)$, and can finally compute the decomposition of the space $C^\infty(G \times_K V_K(\Lambda_K))$. Thus, the effectiveness of a branching rule is evaluated by how it gives the stability bound. We shall show the effecitve bound is obtained from the branching rule in [4].

The author is led to consider the stability bound for the branching rule of $SO(n+3)/SO(3) \times SO(n)$ after Professor Mashimo's works [1], [2] and thanks him for the valuable discussions on this theme.

2. The Case n is Even

We first recall the branching rule given in [4] for $G = SO(2m+3)$ and $K = SO(3) \times SO(2m)$ with the integer $m \ge 2$. We use the same notation for weights given there.

The hightest weight Λ_G of an irreducible G-module is of the form $\Lambda_G = h_0\lambda_0 + h_1\lambda_1 + \cdots + h_m\lambda_m$, where $h_0, h_1, \dots, h_m$ are integers satisfying $h_0 \ge h_1 \ge \cdots \ge h_m \ge 0$. The highest weight Λ_K of an irreducible K-module is of the form $\Lambda_K = p_0\lambda_0 + p_1\lambda_1 + \cdots + p_{m-1}\lambda_{m-1} + \varepsilon p_m \lambda_m$, where $p_0, p_1, \dots, p_{m-1}, p_m$ are integers satisfying $p_0 \ge 0$ and $p_1 \ge \cdots \ge p_{m-1} \ge p_m \ge 0$, and ε is $+1$ or -1.

An irreducible G-module $V_G(\Lambda_G)$ is always the complexification of a real vector space with G-action. On the other hand, an irreducible K-module $V_K(\Lambda_K)$ is the complexification of a real vector space with K-action, when $p_m = 0$. (In this case, ε is irrelevant.) The complexification of a real vector space with irreducible

K-action is $V_K(\Lambda_K)$ with $p_m = 0$ or a direct sum $V_K(\Lambda_K) + V_K(\overline{\Lambda_K})$ with $p_m > 0$, where $\overline{\Lambda_K}$ is the Λ_K the sign ε of which is reversed. In the decomposition of $V_G(\Lambda_G)$, there appear only $V_K(\Lambda_K)$'s with $p_m = 0$ or direct sums $V_K(\Lambda_K) + V_K(\overline{\Lambda_K})$ with $p_m > 0$, since they must be the complexifications of real vector spaces with irreducible K-action. In this respect, we may restrict our attention to Λ_K with $\varepsilon = 1$ (or $p_m = 0$), for, if $V_K(\Lambda_K)$ appears in the decomposition of $V_G(\Lambda_G)$, $V_K(\overline{\Lambda_K})$ also appears with the same multiplicity.

In the following, we set $s(\lambda) = \exp(\lambda) - \exp(-\lambda)$ and $c(\lambda) = \exp(\lambda) + \exp(-\lambda)$.

THEOREM 2. *The irreducible K-module* $V_K(\Lambda_K)$ *with the highest weight* $\Lambda_K = p_0\lambda_0 + p_1\lambda_1 + \cdots + p_m\lambda_m$ *appears in the decomposition of the irreducible G-module* $V_G(\Lambda_G)$ *with the highest weight* $\Lambda_G = h_0\lambda_0 + h_1\lambda_1 + \cdots + h_m\lambda_m$ *if and only if the following conditions are satisfied.*

1. $p_m \le h_{m-1}$, $p_{m-1} \le h_{m-2}$, $h_{i+2} \le p_i \le h_{i-1}$ $(1 \le i \le m-2)$.
2. *In the following expression, in which we calculate the left hand side and arrange them as in the right hand side,* m_{p_0} *does not vanish*:

$$\sum_{(k_1,\dots,k_m)} \frac{\prod_{i=0}^{m} s(l_i\lambda_0)}{(s(\lambda_0))^m} = \sum_{p\ge 0} m_p s\left(\left(p+\frac{1}{2}\right)\lambda_0\right),$$

where the sum in the left hand side is taken over all the sequences of integers $k_1,\dots,k_m$ *satisfying*

$$k_1 \ge \cdots \ge k_m \ge 0,$$

$$p_m \le k_m \le \min\{p_{m-1}, h_{m-1}\},$$

$$\max\{p_i, h_{i+1}\} \le k_i \le \min\{p_{i-1}, h_{i-1}\} \quad (2 \le i \le m-1),$$

$$\max\{p_1, h_2\} \le k_1 \le h_0,$$

and $l_0, l_1, \dots, l_m$ *are given by*

$$l_0 = h_0 - \max\{h_1, k_1\} + 1,$$

$$l_i = \min\{h_i, k_i\} - \max\{h_{i+1}, k_{i+1}\} + 1 \quad (1 \le i \le m-1),$$

$$l_m = \min\{h_m, k_m\} + \frac{1}{2}.$$

We have $m(\Lambda_G, \Lambda_K) = m_{p_0}$.

The fundamental weights for the pair (G, K) are given by

$$\Lambda_1 = 2\lambda_0,$$

$$\Lambda_2 = 2\lambda_0 + 2\lambda_1,$$

$$\Lambda_3 = \lambda_0 + \lambda_1 + \lambda_2.$$

Let Λ_0 be any linear combination of Λ_1, Λ_2, and Λ_3 with non-negative integral coefficients. Our main theorem on the stability bound is given as follows:

Theorem 3. *Assume that* $h_0 - h_1 \geq p_0 + p_1$, $h_1 - h_2 \geq p_0 + p_1$, *and* $h_2 \geq p_1$. *Then we have* $m(\Lambda_G + \Lambda_0, \Lambda_K) = m(\Lambda_G, \Lambda_K)$.

For the proof, we first assume $h_2 \geq p_1$. Then the summation over $(k_1, \ldots, k_m)$ in Theorem 2 splits into the product of two parts.

$$\sum_{(k_1,\ldots,k_m)} \frac{\prod_{i=0}^{m} s(l_i\lambda_0)}{(s(\lambda_0))^m} = \sum_{h_2 \leq k_1 \leq h_0} \frac{s(l_0\lambda_0)s(l_1\lambda_0)}{(s(\lambda_0))^2} s\left(\frac{1}{2}\lambda_0\right)$$

$$\times \sum_{\substack{\max\{p_2, h_3\} \leq k_2 \leq p_1 \\ \max\{p_i, h_{i+1}\} \leq k_i \leq \max\{p_{i-1}, h_{i-1}\} \ (3 \leq i \leq m-1) \\ p_m \leq k_m \leq \max\{p_{m-1}, h_{m-1}\}}} \prod_{i=2}^{m-1} \frac{s(l_i\lambda_0)}{s(\lambda_0)} \cdot \frac{s(l_m\lambda_0)}{s(\frac{1}{2}\lambda_0)},$$

where we have

$$l_0 = h_0 - \max\{h_1, k_1\} + 1,$$

$$l_1 = \min\{h_1, k_1\} - h_2 + 1,$$

$$l_2 = k_2 - \max\{h_3, k_3\} + 1,$$

$$l_i = \min\{h_i, k_i\} - \max\{h_{i+1}, k_{i+1}\} + 1, \quad (3 \leq i \leq m-1),$$

$$l_m = \min\{h_m, k_m\} + \frac{1}{2}.$$

Since $s((l+1)\lambda_0)/s(\lambda_0) = \exp(l\lambda_0) + \exp((l-2)\lambda_0) + \cdots + \exp(-l\lambda_0)$, and $s((l+1/2)\lambda_0)/s((1/2)\lambda_0) = \exp(l\lambda_0) + \exp((l-1)\lambda_0) + \cdots + \exp(-l\lambda_0)$, we can conclude

$$\sum_{\substack{\max\{p_2, h_3\} \leq k_2 \leq p_1 \\ \max\{p_i, h_{i+1}\} \leq k_i \leq \max\{p_{i-1}, h_{i-1}\} \ (3 \leq i \leq m-1) \\ p_m \leq k_m \leq \max\{p_{m-1}, h_{m-1}\}}} \prod_{i=2}^{m-1} \frac{s(l_i\lambda_0)}{s(\lambda_0)} \cdot \frac{s(l_m\lambda_0)}{s(\frac{1}{2}\lambda_0)} = \sum_{0 \leq k \leq p_1} C_k c(k\lambda_0). \quad (1)$$

Notice that the coefficient C_k does not depend on h_0, h_1, nor h_2.

We shall compute the former part.

$$\sum_{h_2\le k_1\le h_0}\frac{s(l_0\lambda_0)s(l_1\lambda_0)}{(s(\lambda_0))^2}s\left(\frac{1}{2}\lambda_0\right)$$
$$=\frac{s((h_0-h_1+1)\lambda_0)}{s(\lambda_0)}\sum_{h_2\le k_1\le h_1}\frac{s((k_1-h_2+1)\lambda_0)}{s(\lambda_0)}s\left(\frac{1}{2}\lambda_0\right)$$
$$+\frac{s((h_1-h_2+1)\lambda_0)}{s(\lambda_0)}\sum_{h_1<k_1\le h_0}\frac{s((h_0-k_1+1)\lambda_0)}{s(\lambda_0)}s\left(\frac{1}{2}\lambda_0\right)$$
$$=\frac{s((h_0-h_1+1)\lambda_0)}{s(\lambda_0)}\sum_{h_2\le k_1\le h_1}\sum_{q=0}^{k_1-h_2}(-1)^q s\left(\left(k_1-h_2-q+\frac{1}{2}\right)\lambda_0\right)$$
$$+\frac{s((h_1-h_2+1)\lambda_0)}{s(\lambda_0)}\sum_{h_1<k_1\le h_0}\sum_{q=0}^{h_0-k_1}(-1)^q s\left(\left(h_0-k_1-q+\frac{1}{2}\right)\lambda_0\right)$$
$$=\frac{s((h_0-h_1+1)\lambda_0)}{s(\lambda_0)}\sum_{q=0}^{[(h_1-h_2)/2]}s\left(\left(h_1-h_2-2q+\frac{1}{2}\right)\lambda_0\right)$$
$$+\frac{s((h_1-h_2+1)\lambda_0)}{s(\lambda_0)}\sum_{q=0}^{[(h_0-h_1-1)/2]}s\left(\left(h_0-h_1-1-2q+\frac{1}{2}\right)\lambda_0\right). \tag{2}$$

We can calculate this from the following:

Lemma 4. *For $k\le h$, we have*

$$\frac{s((h+1)\lambda_0)}{s(\lambda_0)}s\left(\left(k+\frac{1}{2}\right)\lambda_0\right)=\sum_{p=0}^{h-k}(-1)^{h-k+p}s\left(\left(p+\frac{1}{2}\right)\lambda_0\right)$$
$$+\sum_{q=1}^{k}s\left(\left(h-k+2q+\frac{1}{2}\right)\lambda_0\right).$$

For $h\le k$, we have

$$\frac{s((h+1)\lambda_0)}{s(\lambda_0)}s\left(\left(k+\frac{1}{2}\right)\lambda_0\right)=\sum_{q=0}^{h}s\left(\left(k-h+2q+\frac{1}{2}\right)\lambda_0\right).$$

Proof. We use the following equality.

$$\frac{s((h+1)\lambda_0)}{s(\lambda_0)}s\left(\left(k+\frac{1}{2}\right)\lambda_0\right)=\frac{s((h+1)\lambda_0)}{s(\lambda_0)}s\left(\frac{1}{2}\lambda_0\right)\frac{s\left(\left(k+\frac{1}{2}\right)\lambda_0\right)}{s\left(\frac{1}{2}\lambda_0\right)}$$
$$=\sum_{p=0}^{h}(-1)^p s\left(\left(h-p+\frac{1}{2}\right)\lambda_0\right)$$
$$\times\sum_{q=0}^{2k}\exp((k-q)\lambda_0).$$

Notice that we have

$$s\left(\left(h+\frac{1}{2}\right)\lambda_0\right)\times\sum_{q=0}^{2k}\exp((k-q)\lambda_0)=\sum_{p=|h-k|}^{h+k}s\left(\left(p+\frac{1}{2}\right)\lambda_0\right).$$

The lemma follows from a straightforward computation. □

Using this lemma, we have:

Proposition 5. *For an integer p satisfying $0\le p\le\min\{h_0-h_1,h_1-h_2\}$, the coefficient D_p in the equation*

$$\sum_{h_2\le k_1\le h_0}\frac{s(l_0\lambda_0)s(l_1\lambda_0)}{(s(\lambda_0))^2}s\left(\frac{1}{2}\lambda_0\right)=\sum_{p=0}^{h_0-h_2}D_p s\left(\left(p+\frac{1}{2}\right)\lambda_0\right),$$

depends only on h_0-h_1 and h_1-h_2, and does not change when h_0-h_1 or h_1-h_2 are increased by even integers.

Proof. By carefully counting the appearance of $s((p+1/2)\lambda_0)$ in the formula (2), we can show that, if $p\equiv h_0-h_2 \pmod 2$, we have

$$D_p=\frac{p+(h_0-h_1)-(h_1-h_2)}{2}-\left[\frac{h_0-h_1-1}{2}\right]+\left[\frac{h_1-h_2}{2}\right],$$

and that, if $p\equiv h_0-h_2+1 \pmod 2$, we have

$$D_p=\frac{p+1-(h_0-h_1)+(h_1-h_2)}{2}+\left[\frac{h_0-h_1-1}{2}\right]-\left[\frac{h_1-h_2}{2}\right].$$ □

The coefficient m_p in Theorem 2 for $0 \le p \le \min\{h_0 - h_1, h_1 - h_2\} - p_1$ depends only on the coefficients C_k $(0 \le k \le p_1)$ in the formula (1) and the coefficients D_p $(0 \le p \le \min\{h_0 - h_1, h_1 - h_2\})$ in Proposition 5. By adding Λ_0 to Λ_G, the condition $h_2 \ge p_1$ does not alter and the values of $h_0 - h_1$ and $h_1 - h_2$ increase by even integers. Therefore m_p does not change. Thus the proof of Theorem 3 is completed. □

3. The Case n is Odd

We next treat the case $n = 2m + 1$ $(m \ge 2)$. We again recall the branching rule given in [4] for $G = SO(2m+4)$ and $K = SO(3) \times SO(2m+1)$, following the same notation for weights there.

The hightest weight Λ_G of an irreducible G-module is of the form $\Lambda_G = h_{-1}\lambda_{-1} + h_0\lambda_0 + h_1\lambda_1 + \cdots + h_{m-1}\lambda_{m-1} + \varepsilon h_m \lambda_m$, where $h_{-1}, h_0, h_1, \ldots, h_{m-1}$, h_m are integers satisfying $h_{-1} \ge h_0 \ge h_1 \ge \cdots \ge h_{m-1} \ge h_m \ge 0$ and ε is $+1$ or -1. The highest weight Λ_K of an irreducible K-module is of the form $\Lambda_K = p_{-1}\lambda_{-1} + p_1\lambda_1 + \cdots + p_m\lambda_m$, where $p_{-1}, p_1, \ldots, p_m$ are integers satisfying $p_{-1} \ge 0$ and $p_1 \ge \cdots \ge p_m \ge 0$.

THEOREM 6. *The irreducible K-module $V_K(\Lambda_K)$ with the highest weight $\Lambda_K = p_{-1}\lambda_{-1} + p_1\Lambda_1 + \cdots + p_m\lambda_m$ appears in the decomposition of the irreducible G-module $V_G(\Lambda_G)$ with the highest weight $\Lambda_G = h_{-1}\lambda_{-1} + h_0\lambda_0 + h_1\lambda_1 + \cdots + h_{m-1}\lambda_{m-1} + \varepsilon h_m\lambda_m$ if and only if the following conditions are satisfied.*

1. $p_m \le h_{m-2}$, $h_{i+1} \le p_i \le h_{i-2}$ $(1 \le i \le m-1)$.
2. *In the following expression, in which we calculate the left hand side and arrange them as in the right hand side, $m_{p_{-1}}$ does not vanish*:

$$\sum_{(q_0, q_1, \ldots, q_m)} \frac{\prod_{i=0}^{m} s(r_i\lambda_{-1})}{(s(\lambda_{-1}))^m} = \sum_{p \ge 0} m_p s\left(\left(p + \frac{1}{2}\right)\lambda_{-1}\right),$$

where the sum in the left hand side is taken over all the sequences of integers $q_0, q_1, \ldots, q_m$ satisfying

$$q_0 \ge q_1 \ge \cdots \ge q_m \ge 0,$$

$$h_m \le q_m \le \min\{p_{m-1}, h_{m-1}\},$$

$$\max\{p_{i+1}, h_i\} \le q_i \le \min\{p_{i-1}, h_{i-1}\} \quad (2 \le i \le m-1),$$

$$\max\{p_2, h_1\} \le q_1 \le h_0, \quad \max\{p_1, h_0\} \le q_0 \le h_{-1},$$

and $r_0, r_1, \dots, r_m$ *are given by*

$$r_0 = q_0 - \max\{q_1, p_1\} + 1,$$

$$r_i = \min\{q_i, p_i\} - \max\{q_{i+1}, p_{i+1}\} + 1 \quad (1 \le i \le m-1),$$

$$r_m = \min\{q_m, p_m\} + \frac{1}{2}.$$

We have $m(\Lambda_G, \Lambda_K) = m_{p_{-1}}$.

The fundamental weights for the pair (G, K) are given by

$$\Lambda_1 = 2\lambda_{-1},$$

$$\Lambda_2 = 2\lambda_{-1} + 2\lambda_0,$$

$$\Lambda_3 = \lambda_{-1} + \lambda_0 + \lambda_1.$$

Let Λ_0 be any linear combination of Λ_1, Λ_2, and Λ_3 with non-negative integral coefficients. Our main theorem on the stability bound is given as follows:

Theorem 7. *Assume that* $h_{-1} - h_0 \ge p_{-1} + p_1$, $h_0 - h_1 \ge p_{-1} + p_1$, *and* $h_1 \ge p_1$. *Then we have* $m(\Lambda_G + \Lambda_0, \Lambda_K) = m(\Lambda_G, \Lambda_K)$.

For the proof, we assume $h_1 \ge p_1$. Then the summation over $(q_0, q_1, \dots, q_m)$ in Theorem 6 splits into the product of two parts.

$$\sum_{(q_0, q_1, \dots, q_m)} \frac{\prod_{i=0}^{m} s(r_i\lambda_{-1})}{(s(\lambda_{-1}))^m}$$

$$= \sum_{\substack{h_0 \le q_0 \le h_{-1} \\ h_1 \le q_1 \le h_0}} \frac{s(r_0\lambda_{-1})}{s(\lambda_{-1})} s\left(\frac{1}{2}\lambda_{-1}\right)$$

$$\times \sum_{\substack{\max\{p_3, h_2\} \le q_2 \le p_1 \\ \max\{p_{i+1}, h_i\} \le q_i \le \max\{p_{i-1}, h_{i-1}\} \ (3 \le i \le m-1) \\ h_m \le q_m \le \max\{p_{m-1}, h_{m-1}\}}} \prod_{i=1}^{m-1} \frac{s(r_i\lambda_{-1})}{s(\lambda_{-1})} \cdot \frac{s(r_m\lambda_{-1})}{s\left(\frac{1}{2}\lambda_{-1}\right)},$$

where we have

$$r_0 = q_0 - q_1 + 1,$$

$$r_1 = p_1 - \max\{q_2, p_2\},$$

$$r_i = \min\{q_i, p_i\} - \max\{q_{i+1}, p_{i+1}\} + 1, \quad (2 \le i \le m-1),$$

$$r_m = \min\{q_m, p_m\} + \frac{1}{2}.$$

The latter part is represented as

$$\sum_{\substack{\max\{p_3, h_2\} \le q_2 \le p_1 \\ \max\{p_{i+1}, h_i\} \le q_i \le \max\{p_{i-1}, h_{i-1}\} \ (3 \le i \le m-1) \\ h_m \le q_m \le \max\{p_{m-1}, h_{m-1}\}}} \prod_{i=1}^{m-1} \frac{s(r_i\lambda_{-1})}{s(\lambda_{-1})} \cdot \frac{s(r_m\lambda_{-1})}{s(\frac{1}{2}\lambda_{-1})} = \sum_{0 \le k \le p_1} C_k c(k\lambda_{-1}). \quad (3)$$

Notice that the coefficient C_k does not depend on h_{-1}, h_0, nor h_1.

We shall compute the former part.

$$\begin{aligned}
\sum_{\substack{h_0 \le q_0 \le h_{-1} \\ h_1 \le q_1 \le h_0}} \frac{s(r_0\lambda_{-1})}{s(\lambda_{-1})} s\left(\frac{1}{2}\lambda_{-1}\right) &= \sum_{\substack{h_0 \le q_0 \le h_{-1} \\ h_1 \le q_1 \le h_0}} \frac{s((q_0 - q_1 + 1)\lambda_{-1})}{s(\lambda_{-1})} s\left(\frac{1}{2}\lambda_{-1}\right) \\
&= \sum_{\substack{h_0 \le q_0 \le h_{-1} \\ h_1 \le q_1 \le h_0}} \sum_{q=0}^{q_0-q_1} (-1)^q s\left(\left(q_0 - q_1 - q + \frac{1}{2}\right)\lambda_{-1}\right) \\
&= \sum_{0 \le p \le h_{-1} - h_1} D_p s\left(\left(p + \frac{1}{2}\right)\lambda_{-1}\right),
\end{aligned}$$

where D_p is given by

$$D_p = \sum_{\substack{h_0 \le q_0 \le h_{-1} \\ h_1 \le q_1 \le h_0 \\ p \le q_0 - q_1}} (-1)^{q_0 - q_1 - p}. \quad (4)$$

PROPOSITION 8. *For an integer p satisfying $0 \le p \le \min\{h_{-1} - h_0, h_0 - h_1\}$, the coefficient D_p in* (4) *depends only on $h_{-1} - h_0$ and $h_0 - h_1$, and does not change when $h_{-1} - h_0$ or $h_0 - h_1$ are increased by even integers.*

PROOF. We notice that

$$D_0 = \sum_{\substack{h_0 \le q_0 \le h_{-1} \\ h_1 \le q_1 \le h_0}} (-1)^{q_0 - q_1} = \begin{cases} 1, & \text{when both } h_{-1} - h_0 \text{ and } h_0 - h_1 \text{ are even,} \\ 0, & \text{otherwise.} \end{cases}$$

For p satisfying $0 < p \le \min\{h_{-1}-h_0, h_0-h_1\}$, we have

$$(-1)^p D_p = D_0 - \sum_{\substack{0\le p_0,p_1\\ p_0+p_1<p}} (-1)^{p_0+p_1},$$

from which Proposition 8 is obvious. □

The coefficient m_p in Theorem 6 for $0\le p\le \min\{h_{-1}-h_0, h_0-h_1\}-p_1$ depends only on the coefficients C_k $(0\le k\le p_1)$ in the formula (3) and the coefficients D_p $(0\le p\le \min\{h_{-1}-h_0,h_0-h_1\})$ in the formula (4). By adding Λ_0 to Λ_G, the condition $h_1\ge p_1$ does not alter and the values of $h_{-1}-h_0$ and h_0-h_1 increase by even integers. Therefore m_p does not change. Thus the proof of Theorem 7 is completed. □

4. The Case $G = SO(6)$ and $K = SO(3)\times SO(3)$

For the sake of completeness, we state the result for $n=3$, which is omitted in the section 3. We follow the notation in [4].

The hightest weight Λ_G of an irreducible G-module is of the form $\Lambda_G = h_{-1}\lambda_{-1}+h_0\lambda_0+\varepsilon h_1\lambda_1$, where h_{-1}, h_0, h_1 are integers satisfying $h_{-1}\ge h_0\ge h_1 \ge 0$ and ε is $+1$ or -1. The highest weight Λ_K of an irreducible K-module is of the form $\Lambda_K = p_{-1}\lambda_{-1}+p_1\lambda_1$, where p_{-1}, p_1 are integers satisfying $p_{-1}\ge 0$ and $p_1\ge 0$. We give the branching rule in the different but equivalent manner.

Theorem 9. *The irreducible K-module $V_K(\Lambda_K)$ with the highest weight $\Lambda_K = p_{-1}\lambda_{-1}+p_1\Lambda_1$ appears in the decomposition of the irreducible G-module $V_G(\Lambda_G)$ with the highest weight $\Lambda_G = h_{-1}\lambda_{-1}+h_0\lambda_0+\varepsilon h_1\lambda_1$ if and only if, when we calculate*

$$\sum_{\substack{h_0\le q_0\le h_{-1}\\ h_1\le q_1\le h_0}} \left(\sum_{p=0}^{q_0-q_1}\sum_{q=0}^{q_1} s\left(\left(q_0-q_1-p+q+\frac{1}{2}\right)\lambda_{-1}\right)s\left(\left(p+q+\frac{1}{2}\right)\lambda_1\right)\right.$$
$$\left. + \sum_{q=0}^{q_0-q_1-1}\sum_{p=0}^{q}(-1)^{q_0-q_1-q} s\left(\left(q-p+\frac{1}{2}\right)\lambda_{-1}\right)s\left(\left(p+\frac{1}{2}\right)\lambda_1\right)\right),$$

the coefficient of $s((p_{-1}+1/2)\lambda_{-1})s((p_1+1/2)\lambda_1)$ does not vanish. Then the coefficient is equal to $m(\Lambda_G,\Lambda_K)$.

The fundamental weights for the pair (G, K) are given by

$$\Lambda_1 = 2\lambda_{-1},$$
$$\Lambda_2 = \lambda_{-1} + \lambda_0 + \lambda_1,$$
$$\Lambda_3 = \lambda_{-1} + \lambda_0 - \lambda_1.$$

Let Λ_0 be any linear combination of Λ_1, Λ_2, and Λ_3 with non-negative integral coefficients. Our main theorem on the stability bound is given as follows:

Theorem 10. *Assume that* $h_{-1} - h_0 \geq p_{-1} + p_1$, $h_0 - h_1 \geq p_{-1} + p_1$, *and* $h_0 \geq (3/2)(p_{-1} + p_1)$. *Then we have* $m(\Lambda_G + \Lambda_0, \Lambda_K) = m(\Lambda_G, \Lambda_K)$.

Proof. We consider the set S of the sequence (ℓ_0, ℓ_1, p, q) of non-negative integers.

$$S = \left\{ (\ell_0, \ell_1, p, q) \left| \begin{array}{l} 0 \leq \ell_0 \leq h_{-1} - h_0, \quad 0 \leq \ell_1 \leq h_0 - h_1, \\ 0 \leq p \leq \ell_0 + \ell_1, \quad 0 \leq q \leq h_0 - \ell_1, \\ p_{-1} = \ell_0 + \ell_1 - p + q, \quad p_1 = p + q \end{array} \right. \right\}.$$

Then we have

$$m(\Lambda_G, \Lambda_K) = \#S + \sum_{\substack{0 \leq \ell_0 \leq h_{-1} - h_0 \\ 0 \leq \ell_1 \leq h_0 - h_1 \\ p_{-1} + p_1 < \ell_0 + \ell_1}} (-1)^{(\ell_0 + \ell_1) - (p_{-1} + p_1)}$$

If (ℓ_0, ℓ_1, p, q) satisfies $p_{-1} = \ell_0 + \ell_1 - p + q$ and $p_1 = p + q$, we have $\ell_0 + \ell_1 + 2q = p_{-1} + p_1$. Under the assumption of Theorem 10, we can conclude

$$S = \left\{ (\ell_0, \ell_1, p, q) \left| \begin{array}{l} 0 \leq \ell_0, \quad 0 \leq \ell_1, \\ 0 \leq p \leq \ell_0 + \ell_1, \quad 0 \leq q, \\ p_{-1} = \ell_0 + \ell_1 - p + q, \quad p_1 = p + q \end{array} \right. \right\},$$

and $\#S$ does not depend on h_{-1}, h_0, nor h_1. We also have

$$\sum_{\substack{0 \leq \ell_0 \leq h_{-1} - h_0 \\ 0 \leq \ell_1 \leq h_0 - h_1 \\ p_{-1} + p_1 < \ell_0 + \ell_1}} (-1)^{(\ell_0 + \ell_1) - (p_{-1} + p_1)} = (-1)^{p_{-1} + p_1} \left(D_0 - \sum_{\substack{0 \leq \ell_0, \ell_1 \\ \ell_0 + \ell_1 \leq p_{-1} + p_1}} (-1)^{\ell_0 + \ell_1} \right),$$

where D_0 is the same number in the section 3. When we add Λ_0 to Λ_G, the value of h_0 increases and the values of $h_{-1} - h_0$ and $h_0 - h_1$ increase by even integers.

Therefore the assumption of Theorem 10 remains to hold, and, since the value D_0 does not change, the equality $m(\Lambda_G + \Lambda_0, \Lambda_K) = m(\Lambda_G, \Lambda_K)$ holds. □

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