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Stability of a Cell at Optimum Point

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Abstract

In this paper we shall try to investigate the stability of a living cell at an optimum point, their biological significance with programming in C-language.

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1 Introduction

Many scientist Caterana Merla, Micaela Liberti, A. Martin Green, R. Stuart Wenham studied geometric structures of biological and solar cells by various methods. The cell theory, first developed in 1839 by Matthias Jakob Schleiden and Theodor Schwann, states that all organisms are composed of one or more cells. A living cell consists of cytoplasm, nucleus, mitochondrion, vacuole, lysosome and cell membrane etc. Cytoplasm is a clear gel containing proteins and amino acids, RNA and nucleotides, sugars and other carbohydrates, fatty acids, vitamins and ions.Nucleus which contains the genetic material, DNA, within its chromosomes; this is where DNA is stored and replicated and transcribed into RNA. Mitochondrion where respiration produces the major portion of cell waste products of glycolysis. Cell membrane is the lipid based sheath that envelops a cell, encloses the cytoplasm, and creates a selectively permeable barrier. The cell membrane (plasma membrane) act as a boundary that regulates what enters and leaves the cell.

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The thickness of cell membrane are thinner than the wave length of visible light (about 200nm). "The plasma membrane is the outer boundary of the cell; it is a continuous sheet of lipid molecules arranged as a molecular bilayer 4-5nm, thick" [6]. "Assuming that the potential changes linearly over the entire thickness of the membrane (3nm), the force needed to extract a transmembrane protein is 160pN, which is much larger than the measured rupture forces" [3]. "Membranes are typically 7.5-10nm in thickness with two layers of lipid molecules (a bilayer) containing various types of protein molecules" [2].

Here we are trying to study the stability of a cell at an optimum point when some force is applied at that point. For these we have used algorithmic techniques to prepare algorithms and got useful results for analysis. The results may be useful in Bio-mathematics, Biotechnology and precisely in cancer analysis.

2 Stability check up for autonomous system at optimum point

Let us consider that the section of a cell is like a convex hull and by introducing coordinate system we find the optimum point of that feasible region. At the optimum point a force is applied and then we check the stability at that point. Now any point on plane surface (here the point is an optimum point) can be expressed as

$$x = \alpha t + r$$

(2.1)

$$y = \beta t + s$$

where t denotes time and r, s, α, β are independent of drug. Here r, s are the point of application of drug. Therefore $\dot{x} = \alpha$ and $\dot{y} = \beta$. Here α is the horizontal velocity of cell membrane and β is the vertical velocity of cell membrane. After some perturbation (in the sense, when drug is to be applied at that point) these equations take the form

$$\dot{x} = \alpha + ax + by$$

(2.2)

$$\dot{y} = \beta + cx + dy$$

Where x and y are the horizontal displacement and vertical displacement of cell membrane respectively. Here a and c are concentration of drug, b and d are reactivity powers of drug. The above two equation are called homogeneous autonomous system. If the perturbation be such that the equations take the form

$$\dot{x} = \alpha + ax + by + k_1 x^p y, \quad p > 0$$

(2.3)

$$\dot{y} = \beta + cx + dy + k_2 x y^q, \quad q > 0$$

where k_1 and k_2 can be taken as horizontal inner pressure and vertical inner pressure of deformed cell membrane respectively and p, q are the degree of the displacement of cell membrane and movement of cytoplasm respectively, then two equations are said to be nonhomogeneous autonomous system.

In this section, we discuss the stability of nonhomogeneous autonomous system (2.3). For this first we state the Lyapunov theorem which indicates at which path the stability curve moves [1].

Theorem 2.1. *Consider a system $\dot{x} = f(x, y)$, $x \in \mathbf{R^n}$. let $\bar{x}$ be an isolated critical point which we assume to be at the origin. if there exist a ć function $v(x)$ such that*
$v(x) : \mathbf{N_0} \longrightarrow \mathbf{R}$ defined on some neighborhood of $\mathbf{N_0} \subset \mathbf{R^n}$ of $\bar{x} = 0$, such that

1. $v(0) = 0$
2. $v(x) > 0$ *in* $\mathbf{N_0} - \{0\}$
3. $\dot{v(x)} = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i(x) \leq 0$ *in* $\mathbf{N_0} - \{0\}$ *Then* $\bar{x} = 0$ *is stable. Moreover, if*
4. $\dot{v(x)} < 0$ *in* $\mathbf{N_0} - \{0\}$*, then*
   $\bar{x} = 0$ *is asymptotically stable function satisfying properties (1) to (3) or (1) to (4) are called Lyapunov function.*

Therefore for nonhomogeneous autonomous system (2.3) the variational matrix is given by

(2.4)

$$A = \begin{pmatrix} a + k_1 p x^{p-1} y & b + k_1 x^p \\ c + k_2 y^q & d + k_2 q x y^{q-1} \end{pmatrix} = v(x, y)(say)$$

and the characteristic equation at the point (x,y) will be

(2.5)

$$\left| \; v(x, y) - \lambda I \; \right| = 0$$

So for optimum point (d_1, d_2), the characteristic equation is given by

(2.6)

$$\begin{vmatrix} a + k_1 p d_1^{p-1} d_2 - \lambda & b + k_1 d_1^p \\ c + k_2 d_2^q & d + k_2 q d_1 d_2^{q-1} - \lambda \end{vmatrix} = 0$$

Let λ_1=$a + k_1 p d_1^{p-1} d_2$, λ_2=$b + k_1 d_1^p$, λ_3=$c + k_2 d_2^q$, λ_4=$d + k_2 q d_1 d_2^{q-1}$. Now if $d_1 = 0$ and $p - 1 < 0$ or $d_2 = 0$ and $q - 1 < 0$ or $d_1 = 0$, $d_2 = 0$ and $p - 1 < 0$, $q - 1 < 0$ then λ_1 does not exist or λ_4 does not exist or λ_1 and λ_4 does not exist respectively.So for arbitrary point (d_1, d_2), if $d_1 = 0$ or $d_2 = 0$ or both $d_1 = 0$ and $d_2 = 0$ we must have $p - 1 > 0$ or $q - 1 > 0$ or both $p - 1 > 0$ and $q - 1 > 0$ respectively. Let τ=trace(A)=$(\lambda_1 + \lambda_4)$, δ=det(A)=$(\lambda_1\lambda_4 - \lambda_2\lambda_3)$, and Δ=τ^2-4δ.

Therefore the roots of the characteristic equation are

(2.7)

$$\lambda_1 = \frac{\tau + \sqrt{\Delta}}{2}$$

,

$$\lambda_2 = \frac{\tau + \sqrt{\Delta}}{2}$$

and we get different conditions for stability depending upon τ, δ and Δ.

- If $\tau > 0$ and $\delta < 0$ then the system is unstable and the point is a saddle.
- If $\tau < 0$ and $\delta < 0$ then the system is unstable and the point is a saddle.
- If $\tau > 0$ and $\delta > 0$ and $\Delta > 0$ then the system is unstable and the point is a node.
- If $\tau > 0$ and $\delta > 0$ and $\Delta < 0$ then the system is unstable and the point is a focus.
- If $\tau < 0$ and $\delta > 0$ and $\Delta > 0$ then the system is stable and the point is a node.
- If $\tau < 0$ and $\delta > 0$ and $\Delta < 0$ then the system is stable and the point is a focus.

- If $\tau = 0$ and $\delta > 0$ and $\Delta < 0$ then the system has a neutral stability and the point (0,0) is called center.
- If $\tau > 0$ and $\delta > 0$ and $\Delta = 0$ then the system is unstable and the point is a star.
- If $\tau < 0$ and $\delta > 0$ and $\Delta = 0$ then the system is stable and the point is a star.
- If $\tau = 0$ and $\delta = 0$ and $\Delta = 0$ then the equilibrium point is non hyperbolic and to check the stability of the system we have to use theorem (2.1).
- If $\tau = 0$ and $\delta < 0$ then the system is unstable and the point is saddle.
- If $\tau > 0$ and $\delta = 0$ then the equilibrium point is non hyperbolic and to check the stability of the system we have to use theorem (2.1).
- If $\tau < 0$ and $\delta = 0$ then the equilibrium point is non hyperbolic and to check the stability of the system we have to use theorem (2.1).

Thus we can state

Theorem 2.2. *For a system defined by (2.3), if $\bar{x}$ be an optimum point (which not necessarily origen) then the stability of the system follows,*

1. *If $\tau > 0$ and $\delta < 0$ then the system is unstable and the point is a saddle.*
2. *If $\tau < 0$ and $\delta < 0$ then the system is unstable and the point is a saddle.*
3. *If $\tau > 0$ and $\delta > 0$ and $\Delta > 0$ then the system is unstable and the point is a node.*
4. *If $\tau > 0$ and $\delta > 0$ and $\Delta < 0$ then the system is unstable and the point is a focus.*
5. *If $\tau < 0$ and $\delta > 0$ and $\Delta > 0$ then the system is stable and the point is a node.*
6. *If $\tau < 0$ and $\delta > 0$ and $\Delta < 0$ then the system is stable and the point is a focus.*
7. *If $\tau = 0$ and $\delta > 0$ and $\Delta < 0$ then the system has a neutral stability and the point (0,0) is called center.*
8. *If $\tau > 0$ and $\delta > 0$ and $\Delta = 0$ then the system is unstable and the point is a star.*

9. *If $\tau < 0$ and $\delta > 0$ and $\Delta = 0$ then the system is stable and the point is a star.*

10. *If $\tau = 0$ and $\delta = 0$ and $\Delta = 0$ then the equilibrium point is non hyperbolic and to check the stability of the system we have to use theorem (2.1).*

11. *If $\tau = 0$ and $\delta < 0$ then the system is unstable and the point is saddle.*

12. *$\tau > 0$ and $\delta = 0$ then the equilibrium point is non hyperbolic and to check the stability of the system we have to use theorem (2.1).*

13. *If $\tau < 0$ and $\delta = 0$ then the equilibrium point is non hyperbolic and to check the stability of the system we have to use theorem (2.1).*

Now if we put $k_1 = 0$ and $k_2 = 0$ in equation (2.3) then we get the homogeneous autonomous system. That is

$$f(x, y) = \alpha + ax + by$$

(2.8)

$$g(x, y) = \beta + cx + dy$$

and the Jacobian at (x, y) of the given system is given by

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = v_1(x, y)(say)$$

Let τ_1=trace(B)=(a+d), δ_1=det(B)=(ad-bc), Δ_1=τ_1^2-4δ_1.

The Characteristic equation is given

$$\mid v(x, y) - \bar{\lambda} I \mid = 0$$

and therefore

$$\bar{\lambda_1} = \frac{\tau + \sqrt{\Delta}}{2}$$

,

$$\bar{\lambda_2} = \frac{\tau - \sqrt{\Delta}}{2}.$$

So we get the different stability conditions as a corollary of theorem (2.2)

Corollary 2.2. For a homogeneous system defined by (2.8), if $\bar{x}$ be an optimum point (which not necessarily origen) then the stability of the system follows,

1. If $\tau > 0$ and $\delta < 0$ then the system is unstable and the point is a saddle.
2. If $\tau < 0$ and $\delta < 0$ then the system is unstable and the point is a saddle.
3. If $\tau > 0$ and $\delta > 0$ and $\Delta > 0$ then the system is unstable and the point is a node.
4. If $\tau > 0$ and $\delta > 0$ and $\Delta < 0$ then the system is unstable and the point is a focus.
5. If $\tau < 0$ and $\delta > 0$ and $\Delta > 0$ then the system is stable and the point is a node.
6. If $\tau < 0$ and $\delta > 0$ and $\Delta < 0$ then the system is stable and the point is a focus.
7. If $\tau = 0$ and $\delta > 0$ and $\Delta < 0$ then the system has a neutral stability and the point (0,0) is called center.
8. If $\tau > 0$ and $\delta > 0$ and $\Delta = 0$ then the system is unstable and the point is a star.
9. If $\tau < 0$ and $\delta > 0$ and $\Delta = 0$ then the system is stable and the point is a star.
10. If $\tau = 0$ and $\delta = 0$ and $\Delta = 0$ then the equilibrium point is non hyperbolic and to check the stability of the system we have to use theorem (2.1).
11. If $\tau = 0$ and $\delta < 0$ then the system is unstable and the point is saddle.
12. $\tau > 0$ and $\delta = 0$ then the equilibrium point is non hyperbolic and to check the stability of the system we have to use theorem (2.1).
13. If $\tau < 0$ and $\delta = 0$ then the equilibrium point is non hyperbolic and to check the stability of the system we have to use theorem (2.1).

3 C-programming for stability of equations (2.3) and (2.8)

- For nonhomogeneous autonomus system the C-programme be:

```
#include< stdio.h >
#include< math.h >
#define f1(x,y) a11*x+a12*y+a13*pow(x,t)*y
#define f2(x,y) a21*x+a22*y+a23*x*pow(y,n)
#define g11(x,y) a13*y*t*pow(x,t-1)
#define g12(x,y) a13*pow(x,t)
#define g21(x,y) a23*pow(y,t)
#define g1(x,y) a23*x*n*pow(y,n-1)
main()
{
int i,j;
float s[10][10],x1,x2,m,n,det,u[10][10],
a11,a12,a21,a22,r1,r2,p1,q1,p,q,trace,t,a23.a13,b[10][10];
printf(“Give the value of a13,a23:”);
scanf(“%f%f”,&a13,&a23);
printf(“ Give the valu of t & n:”);
scanf(“%f%f”,&t,&n);
printf(“ Give the valu of x & y:”);
scanf(“%f%f”,&x1,&x2);
s[1][1]=g11(x1,x2):
s[1][2]=g12(x1,x2):
s[2][1]=g21(x1,x2):
s[2][2]=g22(x1,x2):
u[1][1]=s[1][1];
u[1][2]=s[1][2];
u[2][1]=s[2][1];
u[2][2]=s[2][2];
printf(“Give the value of a11,a12,a21,a22:”);
scanf(“%f%f%f%f”,&a11,&a12,&a21,&a22);
b[1][1]=u[1][1]+a11;
b[1][2]=u[1][2]+a12;
b[2][1]=u[2][1]+a21;
b[2][2]=u[2][2]+a22;
det=(b[1][1]*b[2][2])-(b[1][2]*b[2][1]);
p=b[1][1]+b[2][2];
q=b[1][1]*b[2][2]-b[1][2]*b[2][1];
printf(“The characterestic equation is:”);
if(q >= 0)
```

$printf($"$m^2 - \%fm + \%f = 0$",p,q);
else
$printf($"$m^2 - \%fm\ \%f = 0$",p,q);
trace=p;
$q_1 = p * p - 4 * q$;
if($trace > 0$ && $q < 0$)
$printf($"The equilibrium point is saddle.Therefore the system be unstable.");
$elseif(trace < 0$ && $q < 0)$
$printf($"The equilibrium point is saddle and the system be unstable.");
$elseif(trace > 0$ && $q > 0$ && $q_1 > 0)$
$printf($"The equilibrium point is node and the system be unstable.");
$elseif(trace > 0$ && $q > 0$ && $q_1 < 0)$
$printf($"The equilibrium point is focus and the system be unstable.");
$elseif(trace < 0$ && $q > 0$ && $q_1 > 0)$
$printf($"The equilibrium point is node and the system be stable.");
$elseif(trace < 0$ && $q > 0$ && $q_1 < 0)$
$printf($"The equilibrium point is focus and the system be stable.");
$elseif(trace == 0$ && $q > 0$ && $q_1 < 0)$
$printf($"The equilibrium point is called center and the system has a neutral stability.");
$elseif(trace > 0$ && $q > 0$ && $q_1 == 0)$
$printf($"The equilibrium point is star and the system be unstable.");
$elseif(trace < 0$ && $q > 0$ && $q_1 == 0)$
$printf($"The equilibrium point is star and the system be stable.");
$elseif(trace == 0$ && $q == 0$ && $q_1 == 0)$
$printf($"The equilibrium point be nonhyperbolic .Therefore to check the stability we have to use Lyapunov theorem .");
$elseif(trace == 0$ && $q < 0)$
$printf($"The equilibrium point is saddle and the system be unstable.");
$elseif(trace == 0$ && $q == 0)$
$printf($"The equilibrium point be nonhyperbolic .Therefore to check the stability we have to use Lyapunov theorem .");
$elseif(trace < 0$ && $q == 0)$
$printf($"The equilibrium point be nonhyperbolic .Therefore to check the stability we have to use Lyapunov theorem .");
}

- For homogeneous autonomus system the C-programme be:
  #include$< stdio.h >$
  #include$< math.h >$
  #define $f_1(x, y)$ a_{11}*x+a_{12}*y
  #define $f_2(x, y)$ a_{21}*x+a_{22}*y

```
#define d1f1(x,y) a11
#define d1f2(x,y) a12
#define d2f1(x,y) a21
#define d2f2(x,y) a22
main()
{
int i,j;
float s[10][10],x1,x2,m,n,det,u[10][10], a11, a12, a21, a22, r1, r2, p1, q1,p,q,trace;
printf("Give the value of a11, a12, a21, a22:");
scanf("%f%f%f%f",&a11,&a12,&a21,&a22);
printf(" Give the valu of x & y:");
scanf("%f%f",&x1, &x2);
s[1][1]=d1f1(x1, x2);
s[1][2]=d1f2(x1, x2);
s[2][1]=d2f1(x1, x2);
s[2][2]=d2f2(x1, x2);
u[1][1]=s[1][1];
u[1][2]=s[1][2];
u[2][1]=s[2][1];
u[2][2]=s[2][2];
det=(u[1][1]*u[2][2])-(u[1][2]*u[2][1]);
p = a11 + a22;
q = a11 * a22 - a12 * a21;
printf("The characterestic equation is:");
if(q >= 0)
printf("m^2 - %fm + %f = 0",p,q);
else
printf("m^2 - %fm %f = 0",p,q);
trace=p;
q1 = p * p - 4 * q;
if(trace > 0 && q < 0)
printf("The equilibrium point is saddle.Therefore the system be unsta-
ble.");
elseif(trace < 0 && q < 0)
printf("The equilibrium point is saddle and the system be unstable.");
elseif(trace > 0 && q > 0 && q1 > 0)
printf("The equilibrium point is node and the system be unstable.");
elseif(trace > 0 && q > 0 && q1 < 0)
printf("The equilibrium point is focus and the system be unstable.");
elseif(trace < 0 && q > 0 && q1 > 0)
printf("The equilibrium point is node and the system be stable.");
elseif(trace < 0 && q > 0 && q1 < 0)
```

$printf$(“The equilibrium point is focus and the system be stable.”);
$elseif(trace == 0 \&\& q > 0 \&\& q_1 < 0)$
$printf$(“The equilibrium point is called center and the system has a neutral stability.”);
$elseif(trace > 0 \&\& q > 0 \&\& q_1 == 0)$
$printf$(“The equilibrium point is star and the system be unstable.”);
$elseif(trace < 0 \&\& q > 0 \&\& q_1 == 0)$
$printf$(“The equilibrium point is star and the system be stable.”);
$elseif(trace == 0 \&\& q == 0 \&\& q_1 == 0)$
$printf$(“The equilibrium point be nonhyperbolic .Therefore to check the stability we have to use Lyapunov theorem .”);
$elseif(trace == 0 \&\& q < 0)$
$printf$(“The equilibrium point is saddle and the system be unstable.”);
$elseif(trace == 0 \&\& q == 0)$
$printf$(“The equilibrium point be nonhyperbolic .Therefore to check the stability we have to use Lyapunov theorem .”);
$elseif(trace < 0 \&\& q == 0)$
$printf$(“The equilibrium point be nonhyperbolic .Therefore to check the stability we have to use Lyapunov theorem .”);
}

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