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Research Article

Stability of a Class of Fractional-Order Nonlinear Systems

Tianzeng Li and Yu Wang

School of Science, Sichuan University of Science and Engineering, Zigong 643000, China

Correspondence should be addressed to Yu Wang; wangyu_813@163.com

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In this letter stability analysis of fractional order nonlinear systems is studied. Some new sufficient conditions on the local (globally) asymptotic stability for a class of fractional order nonlinear systems with order $0 < \alpha < 2$ are proposed by using properties of Mittag-Leffler function and the Gronwall inequality. And the corresponding stabilization criteria are also given. The numerical simulations of two systems with order $0 < \alpha < 1$ and two systems with order $1 < \alpha < 2$ illustrate the effectiveness and universality of the proposed approach.

1. Introduction

During the last decade the fractional calculus has gained importance in both theoretical and applied aspects of several branches of science and engineering. There are two essential differences between integer order derivation and fractional order derivation. Firstly, the integer order derivative indicates a variation or certain attribute at particular time for a mechanical or physical process, while the fractional order derivative is concerned with the whole time domain. Secondly, the integer order derivative describes the local properties of a certain position, while the fractional order derivative is related to the whole space for a physical process. Then many physical systems are well characterized by the fractional order state equations [1–4], such as fractional order Lotka-Volterra equation [1] in biological systems, fractional order Schödinger equation [2] in quantum mechanics, fractional order Langevin equation [3] in anomalous diffusion, and fractional order oscillator equation [4] in damping vibration.

However there are several open problems in this area. Stability of fractional order systems is one of the most fundamental and important issues. On the other hand, because fractional differential operators are nonlocal and have weakly singular kernels, some methods in dealing with interorder systems cannot be simply extended to fractional-order methods. To the best of knowledge, the stability of fractional-order nonlinear systems is still relatively few. Reference [5–7] investigated the necessary and sufficient stability conditions for linear fractional order differential equations and linear time-delayed fractional differential equations. The stability of n-dimensional linear fractional order differential systems with order $1 < \alpha < 2$ has already been studied in [8]. However, only under some special circumstances or in certain cases, the practical problems may be regarded as linear systems. Therefore, stability of nonlinear system is of great significance, and it also has important value in application. In [9], the stability of fractional nonlinear time-delay systems for Caputo's derivative are investigated, and two theorems for Mittag-Leffler stability of the fractional order nonlinear time-delay systems are proved. In [10], the authors proposed the finite-time stabilization of a class of multistate time delay of fractional nonlinear systems. In [11, 12], the authors studied the stability of fractional nonlinear dynamic systems using Lyapunov direct method with the introductions of Mittag-Leffler stability and generalized Mittag-Leffler stability notions. In [13], the authors studied fractional order Lyapunov stability theorem and its applications in synchronization of complex dynamical networks. In [14], some new sufficient conditions ensuring asymptotical stability of fractional-order nonlinear system with delay are proposed firstly.

In this paper the stability of nonlinear fractional order nonlinear system is studied. And by using the Gronwall inequality and the properties of Mittag-Leffler function, we proposed some new sufficient conditions on the local (globally) asymptotic stability for a class of fractional order nonlinear systems with order $0 < \alpha < 2$. And the corresponding stabilization criteria are also given. Finally, four numerical simulation examples have illustrated the effectiveness and universality of the proposed methods.

2. Fractional Order Derivative and Mittag-Leffler Function

2.1. Definitions of Fractional Derivative and Mittag-Leffler Function. Fractional calculus plays an important role in modern science [15–17]. Some definitions for fractional derivatives are usually used, such as Grünwald-Letnikov (GL), Riemann-Liouville (RL), and Caputo definition. In this paper, we mainly use the Caputo definitions [15].

Definition 1 (see [15]). The fractional integral ${}_aD_t^{-\alpha}$ of function $f(t)$ is defined as follows:

$$ {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \tag{1}$$

where fractional order $\alpha > 0$, and $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ is the gamma function.

Definition 2 (see [15]). The Caputo derivative with order α of function $f(t)$ is given as

$$ {}_a^CD_t^{\alpha} f(t) = {}_aD_t^{-(n-\alpha)} \frac{d^n}{dt^n} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, \tag{2}$$

where $n-1 < \alpha < n$, $n \in Z^+$.

The formulas for Laplace transform of the Caputo fractional derivative ${}_a^CD_t^{\alpha} f(t)$ have the following form [16]:

$$ \mathfrak{L}\left\{ {}_a^CD_t^{\alpha} f(t); s \right\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \tag{3}$$

where $n-1 \le \alpha < n$, and $F(s) = \mathfrak{L}\{f(t); s\} = \int_0^\infty e^{-st} f(t) dt$.

As a generalization of the exponential function which is frequently used in the solutions of integer-order systems, the Mittag-Leffler function is frequently used in the solutions of fractional systems. The definition and properties are given in the following.

Definition 3 (see [17]). The Mittag-Leffler function is given as

$$ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \tag{4}$$

where $\alpha > 0$ and $z \in \mathbb{C}$.

The generalization of Mittag-Leffler function with two parameters is wildly used and defined as follows:

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \tag{5}$$

where $\alpha > 0$, $\beta > 0$, and $z \in \mathbb{C}$.

Remark 4. If $\beta = 1$, we have $E_{\alpha,1}(z) = E_\alpha(z)$, especially, $E_{1,1}(z) = E_1(z) = e^z$.

2.2. Properties of Mittag-Leffler Functions and the Gronwall Inequality. In this section, we give the Gronwall inequality and some important properties of the Mittag-Leffler functions which are used in the following.

Lemma 5 (see [15]). *Considering the Laplace transform of Mittag-Leffler function with two parameters, we have*

$$ \mathfrak{L}\left\{ t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha) \right\} = \frac{s^{\alpha-\beta}}{s^\alpha+\lambda}, \quad \left(\Re(s) > |\lambda|^{1/\alpha} \right), \tag{6}$$

where t and s are, respectively, the variables in the time domain and Laplace domain, $\Re(s)$ stands for the real part of s, $\lambda \in \mathbb{R}$, and $\mathfrak{L}\{\cdot\}$ denotes the Laplace transform.

Proof. The proof of this Lemma can be found in [15]. □

Lemma 6 (see [18, 19]). *If $0 < \alpha < 2$, $\beta \in \mathbb{R}$, and μ satisfies $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$, there exist $C_1 > 0$ and $C_2 > 0$ such that*

$$ \left| E_{\alpha,\beta}(z) \right| \le C_1 (1+|z|)^{(1-\beta)/\alpha} \exp\left(\mathrm{Re}\left(z^{1/\alpha} \right) \right) + \frac{C_2}{1+|z|}, \tag{7}$$

where $|\arg(z)| < \mu$, $|z| \ge 0$.

Lemma 7 (see [18, 19]). *For the Mittage-Leffler function $E_{\alpha,\beta}(\mathbf{A}t^\alpha)$, there exist finite real constants $K_{E_{\alpha,1}} \ge 1$, $K_{E_{\alpha,\alpha}} \ge 1$, and $K_{E_{\alpha,\beta}} \ge 1$ such that*

$$ E_{\alpha,1}(\mathbf{A}t^\alpha) \le K_{E_{\alpha,1}} \left\| e^{\mathbf{A}t^\alpha} \right\|, \qquad \left\| E_{\alpha,\alpha}(\mathbf{A}t^\alpha) \right\| \le K_{E_{\alpha,\alpha}} \left\| e^{\mathbf{A}t^\alpha} \right\|, \quad \text{for any } 0 < \alpha < 1,$$
$$ E_{\alpha,\beta}(\mathbf{A}t^\alpha) \le K_{E_{\alpha,\beta}} \left\| e^{\mathbf{A}t^\alpha} \right\|, \quad \text{for any } \alpha > 1,\ \beta = 1, 2, \alpha, \tag{8}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$.

Proof. The proof of this Lemma can be found in [18]. □

Lemma 8 (Gronwall inequality [19, 20]). *Let $\alpha > 0$, $u(t)$ is a nonnegative function locally integrable on $[0, T)$ and $a(t)$ is a nonnegative, nondecreasing continuous function defined on $[0, T)$, $a(t) < M$ (constant), and suppose $z(t)$ is nonnegative and locally integrable on $[0, T)$ with*

$$ z(t) \le u(t) + a(t) \int_0^t (t-\tau)^{\alpha-1} z(\tau)\, d\tau, \tag{9}$$

on this interval. Then

$$ z(t) \le u(t) + \int_0^t \left[\sum_{k=1}^{\infty} \frac{(\Gamma(\alpha) a(t))^k}{\Gamma(k\alpha)} (t-\tau)^{k\alpha-1} u(\tau) \right] d\tau. \tag{10}$$

Moreover, if $u(t)$ is a nondecreasing function on $[0, T)$, we have

$$ z(t) \le u(t) E_\alpha\left(\Gamma(\alpha) a(t) t^\alpha \right). \tag{11}$$

3. Stability and Stabilization of Fractional Order Nonlinear System

3.1. Stability and Stabilization of Fractional Order Nonlinear System with Order $0 < \alpha < 1$. Firstly, we consider the Caputo fractional nonlinear systems [16, 21]

$$ {}_0^C D_t^{\alpha} \mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t)) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(\mathbf{x}(t)) \tag{12} $$

with the initial condition $\mathbf{x_0} = \mathbf{x}(0)$, where $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$ denotes the state vector of the system, $\alpha \in (0,1)$ is the order of the fractional-order derivative, $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ defines a nonlinear vector field in the n-dimensional vector space, and $\mathbf{A}\mathbf{x}(t)$ and $\mathbf{g}(\mathbf{x}(t))$ denote the linear and nonlinear parts of $\mathbf{f}(\mathbf{x}(t))$, respectively. If $\mathbf{f}(\mathbf{x}^*) = 0$, the constant $\mathbf{x}^*$ is called the equilibrium point of Caputo fractional nonlinear system (12). Without loss of generality, we suppose the equilibrium point is $\mathbf{x} = 0$.

Theorem 9. *The fractional order nonlinear system* (12) *is local asymptotically stable, if it satisfies the following conditions:* (1) $\mathrm{Re}(\mathrm{eig}(\mathbf{A})) < 0$ *and* $\omega = -\max \mathrm{Re}(\mathrm{eig}(\mathbf{A})) > \Gamma(\alpha)$, *where* $\Gamma(\cdot)$ *is the gamma function;* (2) $\mathbf{g}(\mathbf{x}(t))$ *satisfies* $\|\mathbf{g}(\mathbf{x}(t))\| = o\|\mathbf{x}(t)\|$ *as* $\|\mathbf{x}\| \to 0$.

Proof. Applying the Laplace transform on (12), we have

$$ S^{\alpha}\mathbf{X}(S) - S^{\alpha-1}\mathbf{x}_0 = \mathbf{A}\mathbf{X}(S) + \mathfrak{L}\{\mathbf{g}(\mathbf{x}(t))\}; \tag{13} $$

that is,

$$ \mathbf{X}(S) = (\mathbf{I}S^{\alpha} - \mathbf{A})^{-1}\left(S^{\alpha-1}\mathbf{x}_0 + \mathfrak{L}\{\mathbf{g}(\mathbf{x}(t))\}\right), \tag{14} $$

where $\mathbf{X}(S)$ is the Laplace transform of $\mathbf{x}(t)$, $\mathbf{I}$ is an $n \times n$ identity matrix, and $\mathfrak{L}\{\cdot\}$ denotes the Laplace transform. By using the Laplace inverse transform, we obtain the solution of (16),

$$ \begin{aligned} \mathbf{x}(t) &= E_{\alpha,1}(\mathbf{A}t^{\alpha})\mathbf{x}_0 \\ &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\mathbf{A}(t-\tau)^{\alpha})\mathbf{g}(\mathbf{x}(\tau))\, d\tau. \end{aligned} \tag{15} $$

It follows from Lemma 7 that there exist constants $M_1 > 0$ and $M_2 > 0$ such that

$$ \begin{aligned} \|\mathbf{x}(t)\| &\le M_1 \left\| e^{\mathbf{A}t^{\alpha}} \right\| \|\mathbf{x}_0\| \\ &\quad + M_2 \int_0^t (t-\tau)^{\alpha-1} \left\| e^{\mathbf{A}(t-\tau)^{\alpha}} \right\| \|\mathbf{g}(\mathbf{x}(\tau))\|\, d\tau. \end{aligned} \tag{16} $$

Since matrix $\mathbf{A}$ is stable, there is a constant $M_3 > 0$ such that $\|e^{\mathbf{A}t^{\alpha}}\| \le M_3 e^{-\omega t^{\alpha}}$. Substituting it into (16), one has

$$ \begin{aligned} \|\mathbf{x}(t)\| &\le M_1 M_3 \|\mathbf{x}_0\| e^{-\omega t^{\alpha}} \\ &\quad + M_2 M_3 \int_0^t (t-\tau)^{\alpha-1} e^{-\omega(t-\tau)^{\alpha}} \|\mathbf{g}(\mathbf{x}(\tau))\|\, d\tau. \end{aligned} \tag{17} $$

Based on the condition (2) $\|\mathbf{g}(\mathbf{x}(t))\| = o\|\mathbf{x}(t)\|$, that is, $\lim_{\|\mathbf{x}\| \to 0}(\|\mathbf{g}(\mathbf{x}(t))\| / \|\mathbf{x}(t)\|) = 0$, there is a constant $\delta > 0$, such that

$$ \|\mathbf{g}(\mathbf{x}(t))\| \le \frac{1}{M_2 M_3} \|\mathbf{x}(t)\|, \tag{18} $$

where $\|\mathbf{x}(t)\| < \delta$. And $(t-\tau)^{\alpha} < t^{\alpha} - \tau^{\alpha}$ when $0 < \alpha < 1$ and $t > \tau$, then

$$ \begin{aligned} \|\mathbf{x}(t)\| &\le M_1 M_3 \|\mathbf{x}_0\| e^{-\omega t^{\alpha}} + \int_0^t (t-\tau)^{\alpha-1} e^{-\omega(t-\tau)^{\alpha}} \|\mathbf{x}(\tau)\| \\ &\le M_1 M_3 \|\mathbf{x}_0\| e^{-\omega t^{\alpha}} + \int_0^t (t-\tau)^{\alpha-1} e^{-\omega(t^{\alpha}-\tau^{\alpha})} \|\mathbf{x}(\tau)\|\, d\tau. \end{aligned} \tag{19} $$

Multiplying the inequality by $e^{\omega t^{\alpha}}$, we will get

$$ e^{\omega t^{\alpha}} \|\mathbf{x}(t)\| \le M_1 M_3 \|\mathbf{x}_0\| + \int_0^t (t-\tau)^{\alpha-1} e^{\omega \tau^{\alpha}} \|\mathbf{x}(\tau)\|\, d\tau. \tag{20} $$

Applying Lemma 8 (Gronwall inequality) to (20), we have

$$ \begin{aligned} &e^{\omega t^{\alpha}} \|\mathbf{x}(t)\| \\ &\quad \le M_1 M_3 \|\mathbf{x}_0\| E_{\alpha,1}(\Gamma(\alpha) t^{\alpha}) \le M_1 M_3 M_4 \|\mathbf{x}_0\| e^{\Gamma(\alpha) t^{\alpha}}. \end{aligned} \tag{21} $$

Then

$$ \begin{aligned} \|\mathbf{x}(t)\| &\le M_1 M_3 \|\mathbf{x}_0\| E_{\alpha,1}(\Gamma(\alpha) t^{\alpha}) e^{-\omega t^{\alpha}} \\ &\le M_1 M_3 M_4 \|\mathbf{x}_0\| e^{-(\omega - \Gamma(\alpha)) t^{\alpha}}. \end{aligned} \tag{22} $$

Therefore, when $t \to \infty$, $\|\mathbf{x}(t)\| \to 0$ for $\omega > \Gamma(\alpha)$, which implies that the system (12) is asymptotically stable. □

Theorem 10. *The fractional order nonlinear system* (12) *is globally asymptotically stable, if it satisfies the following conditions:* (1) $\mathbf{g}(\mathbf{x}(t))$ *satisfies* $\mathbf{g}(0) = 0$ *and the Lipschitz condition with respect to* $\mathbf{x}$, *that is,* $\|\mathbf{g}(\mathbf{x}_1) - \mathbf{g}(\mathbf{x}_2)\| \le L\|\mathbf{x}_1 - \mathbf{x}_2\|$; (2) $\mathrm{Re}(\mathrm{eig}(\mathbf{A})) < 0$ *and* $\omega = -\max \mathrm{Re}(\mathrm{eig}(\mathbf{A})) > LM_3M_4\Gamma(\alpha)$, *where* M_3 *and* M_4 *satisfy* $\|e^{\mathbf{A}t}\| \le M_3 e^{-\omega t}$ *and* $E_{\alpha,1}(kt^{\alpha}) < M_4 e^{kt^{\alpha}}$.

Proof. Applying the Laplace transform and Laplace inverse transform on (12), we obtain the solution of (12),

$$ \begin{aligned} \mathbf{x}(t) &= E_{\alpha,1}(\mathbf{A}t^{\alpha})\mathbf{x}_0 \\ &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\mathbf{A}(t-\tau)^{\alpha})\mathbf{g}(\mathbf{x}(\tau))\, d\tau. \end{aligned} \tag{23} $$

It follows from Lemma 7 that there exist constants $M_1 > 0$, $M_2 > 0$, and $M_3 > 0$ such that

$$\begin{aligned}\|\mathbf{x}(t)\| &\le M_1 \left\|e^{\mathbf{A}t^{\alpha}}\right\| \|\mathbf{x}_0\| \\ &\quad + M_2 \int_0^t (t-\tau)^{\alpha-1} \left\|e^{\mathbf{A}(t-\tau)^{\alpha}}\right\| \|\mathbf{g}(\mathbf{x}(\tau))\| \, d\tau \\ &\le M_1 M_3 \|\mathbf{x}_0\| e^{-\omega t^{\alpha}} \\ &\quad + M_2 M_3 \int_0^t (t-\tau)^{\alpha-1} e^{-\omega(t-\tau)^{\alpha}} \|\mathbf{g}(\mathbf{x}(\tau))\| \, d\tau \\ &\le M_1 M_3 \|\mathbf{x}_0\| e^{-\omega t^{\alpha}} \\ &\quad + L M_2 M_3 \int_0^t (t-\tau)^{\alpha-1} e^{-\omega(t-\tau)^{\alpha}} \|\mathbf{x}(\tau)\| \, d\tau \\ &\le M_1 M_3 \|\mathbf{x}_0\| e^{-\omega t^{\alpha}} \\ &\quad + L M_2 M_3 \int_0^t (t-\tau)^{\alpha-1} e^{-\omega(t^{\alpha}-\tau^{\alpha})} \|\mathbf{x}(\tau)\| \, d\tau.\end{aligned} \tag{24}$$

Multiplying the inequality by $e^{\omega t^{\alpha}}$, we will get

$$\begin{aligned} e^{\omega t^{\alpha}} & \|\mathbf{x}(t)\| \\ &\le M_1 M_3 \|\mathbf{x}_0\| + L M_2 M_3 \int_0^t (t-\tau)^{\alpha-1} e^{\omega \tau^{\alpha}} \|\mathbf{x}(\tau)\| \, d\tau.\end{aligned} \tag{25}$$

Applying Lemma 8 (Gronwall inequality) to (25), we have

$$\begin{aligned} e^{\omega t^{\alpha}} \|\mathbf{x}(t)\| &\le M_1 M_3 \|\mathbf{x}_0\| E_{\alpha,1}\left(L M_2 M_3 \Gamma(\alpha) t^{\alpha}\right) \\ &\le M_1 M_3 M_4 \|\mathbf{x}_0\| e^{L M_2 M_3 \Gamma(\alpha) t^{\alpha}}.\end{aligned} \tag{26}$$

Then

$$\begin{aligned} \|\mathbf{x}(t)\| &\le M_1 M_3 M_4 \|\mathbf{x}_0\| E_{\alpha,1}\left(L M_2 M_3 \Gamma(\alpha) t^{\alpha}\right) e^{\omega t^{\alpha}} \\ &\le M_1 M_3 M_4 \|\mathbf{x}_0\| e^{-(\omega - L M_2 M_3 \Gamma(\alpha)) t^{\alpha}}.\end{aligned} \tag{27}$$

Therefore, when $t \to \infty$, $\|\mathbf{x}(t)\| \to 0$ for $\omega > L M_2 M_3 \Gamma(\alpha)$, which implies that the system (12) is globally asymptotically stable. □

The controlled fractional order nonlinear system with linear feedback control input is given as

$$\begin{aligned} {}_0^C D_t^{\alpha} \mathbf{x}(t) &= \mathbf{f}(\mathbf{x}(t)) + \mathbf{u}(t) \\ &= (\mathbf{A} + \mathbf{K}) \mathbf{x}(t) + \mathbf{g}(\mathbf{x}(t)) = \overline{\mathbf{A}} \mathbf{x}(t) + \mathbf{g}(\mathbf{x}(t)),\end{aligned} \tag{28}$$

where $\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t)$ is the linear feedback control input, $\overline{\mathbf{A}} = \mathbf{A} + \mathbf{K}$, and the feedback gain matrix $\mathbf{K} \in \mathbb{R}^{n \times n}$ needs to be determined.

Therefore, our aim is to design a suitable feedback gain matrix $\mathbf{K}$ such that the controlled system is local (globally) asymptotically stable.

Theorem 11. *The controlled fractional order nonlinear system* (28) *is local asymptotically stable, if it satisfies the following conditions:* (1) $\mathrm{Re}(\mathrm{eig}\,(\overline{\mathbf{A}})) < 0$ *and* $\omega = -\max \mathrm{Re}(\mathrm{eig}\,(\overline{\mathbf{A}})) > \Gamma(\alpha)$*;* (2) $\mathbf{g}(\mathbf{x}(t))$ *satisfies* $\|\mathbf{g}(\mathbf{x}(t))\| = o\|\mathbf{x}(t)\|$ *as* $\|\mathbf{x}\| \to 0$.

Proof. The proof is similar to that of Theorem 9. □

Theorem 12. *The controlled fractional order nonlinear system* (28) *is globally asymptotically stable, if it satisfies the following conditions:* (1) $\mathbf{g}(\mathbf{x}(t))$ *satisfies* $\mathbf{g}(0) = 0$ *and the Lipschitz condition with respect to* $\mathbf{x}$*, that is,* $\|\mathbf{g}(\mathbf{x}_1) - \mathbf{g}(\mathbf{x}_2)\| \le L\|\mathbf{x}_1 - \mathbf{x}_2\|$*;* (2) $\mathrm{Re}(\mathrm{eig}\,(\overline{\mathbf{A}})) < 0$ *and* $\omega = -\max \mathrm{Re}(\mathrm{eig}\,(\overline{\mathbf{A}})) > L M_3 M_4 \Gamma(\alpha)$*, where* M_3 *and* M_4 *satisfy* $\|e^{\overline{\mathbf{A}}t}\| \le M_3 e^{-\omega t}$ *and* $E_{\alpha,1}(kt^{\alpha}) < M_4 e^{kt^{\alpha}}$.

Proof. The proof is similar to that of Theorem 10. □

3.2. Stability and Stabilization of Fractional Order Nonlinear System with Order $1 < \alpha < 2$.

Firstly, we consider the Caputo fractional nonlinear systems [16, 21]

$${}_0^C D_t^{\alpha} \mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t)) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(\mathbf{x}(t)) \tag{29}$$

with the initial conditions $\mathbf{x}_0 = \mathbf{x}(0)$ and $\mathbf{x}_1 = \mathbf{x}^{(1)}(0)$, where $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$ denotes the state vector of the system, $\alpha \in (1,2)$ is the order of the fractional order derivative, $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ defines a nonlinear vector field in the n-dimensional vector space, and $\mathbf{A}\mathbf{x}(t)$ and $\mathbf{g}(\mathbf{x}(t))$ denote the linear and nonlinear parts of $\mathbf{f}(\mathbf{x}(t))$, respectively.

Theorem 13. *The fractional order nonlinear system* (29) *is local asymptotically stable, if it satisfies the following conditions:* (1) $\mathrm{Re}(\mathrm{eig}\,(\mathbf{A})) < 0$ *and* $\omega = -\max \mathrm{Re}(\mathrm{eig}\,(\mathbf{A})) > \Gamma(\alpha)^{1/\alpha}$*;* (2) $\mathbf{g}(\mathbf{x}(t))$ *satisfies* $\|\mathbf{g}(\mathbf{x}(t))\| = o\|\mathbf{x}(t)\|$ *as* $\|\mathbf{x}\| \to 0$.

Proof. Applying the Laplace transform on (29), we have

$$S^{\alpha} \mathbf{X}(S) - S^{\alpha-1} \mathbf{x}_0 - S^{\alpha-2} \mathbf{x}_1 = \mathbf{A}\mathbf{X}(S) + \mathfrak{L}\{\mathbf{g}(\mathbf{x}(t))\}; \tag{30}$$

that is,

$$\mathbf{X}(S) = \left(\mathbf{I}S^{\alpha} - \mathbf{A}\right)^{-1} \left(S^{\alpha-1}\mathbf{x}_0 + S^{\alpha-2}\mathbf{x}_1 + \mathfrak{L}\{\mathbf{g}(\mathbf{x}(t))\}\right), \tag{31}$$

where $\mathbf{X}(S)$ is the Laplace transform of $\mathbf{x}(t)$, and $\mathbf{I}$ is an $n \times n$ identity matrix. By using the Laplace inverse transform, we obtain the solution of (29),

$$\begin{aligned} \mathbf{x}(t) &= E_{\alpha,1}\left(\mathbf{A}t^{\alpha}\right) \mathbf{x}_0 + t E_{\alpha,2}\left(\mathbf{A}t^{\alpha}\right) \mathbf{x}_1 \\ &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\mathbf{A}(t-\tau)^{\alpha}\right) \mathbf{g}(\mathbf{x}(\tau)) \, d\tau.\end{aligned} \tag{32}$$

It follows from Lemma 7 that there exist constants $M_1 > 0$ and $M_2 > 0$ such that

$$\begin{aligned} \|\mathbf{x}(t)\| &\le M_1 \left\|e^{\mathbf{A}t^{\alpha}}\right\| \|\mathbf{x}_0\| + M_2 \left\|e^{\mathbf{A}t^{\alpha}}\right\| \|\mathbf{x}_1\| t \\ &\quad + M_3 \int_0^t (t-\tau)^{\alpha-1} \left\|e^{\mathbf{A}(t-\tau)^{\alpha}}\right\| \|\mathbf{g}(\mathbf{x}(\tau))\| \, d\tau.\end{aligned} \tag{33}$$

Since matrix $\mathbf{A}$ is stable, there is a constant $M_4 > 0$ such that $\|e^{\mathbf{A}t^{\alpha}}\| \le M_4 e^{-\omega t^{\alpha}}$. Substituting it into (33), one has

$$\|\mathbf{x}(t)\| \le M_1 M_4 \|\mathbf{x}_0\| e^{-\omega t^{\alpha}} + M_2 M_4 \|\mathbf{x}_1\| e^{-\omega t^{\alpha}} t + M_3 M_4 \int_0^t (t-\tau)^{\alpha-1} e^{-\omega (t-\tau)^{\alpha}} \|\mathbf{g}(\mathbf{x}(\tau))\| \, d\tau. \tag{34}$$

Based on the condition (2) $\|\mathbf{g}(\mathbf{x}(t))\| = o\|\mathbf{x}(t)\|$, that is, $\lim_{\|\mathbf{x}\|\to 0}(\|\mathbf{g}(\mathbf{x}(t))\|/\|\mathbf{x}(t)\|) = 0$, there is a constant $\delta > 0$, such that

$$\|\mathbf{g}(\mathbf{x}(t))\| \le \frac{1}{M_3 M_4} \|\mathbf{x}(t)\|, \tag{35}$$

where $\|\mathbf{x}(t)\| < \delta$. And $(t-\tau)^{\alpha} > t-\tau$ when $1 < \alpha < 2$, and then

$$\|\mathbf{x}(t)\| \le M_1 M_4 \|\mathbf{x}_0\| e^{-\omega t} + M_2 M_4 \|\mathbf{x}_1\| e^{-\omega t} t + \int_0^t (t-\tau)^{\alpha-1} e^{-\omega (t-\tau)} \|\mathbf{x}(\tau)\| \, d\tau. \tag{36}$$

Multiplying the inequality by $e^{\omega t}$, we will get

$$e^{\omega t}\|\mathbf{x}(t)\| \le M_1 M_4 \|\mathbf{x}_0\| + M_2 M_4 \|\mathbf{x}_1\| t + \int_0^t (t-\tau)^{\alpha-1} e^{\omega \tau} \|\mathbf{x}(\tau)\| \, d\tau. \tag{37}$$

Applying Lemma 8 (Gronwall inequality) and Lemma 6 to (37), we have

$$\begin{aligned} e^{\omega t}\|\mathbf{x}(t)\| &\le (M_1 M_4 \|\mathbf{x}_0\| + M_2 M_4 \|\mathbf{x}_1\| t) E_{\alpha,1}(\Gamma(\alpha) t^{\alpha}) \\ &\le (M_1 M_4 \|\mathbf{x}_0\| + M_2 M_4 \|\mathbf{x}_1\| t) \times \left(C_1 e^{\Gamma(\alpha)^{1/\alpha} t} + \frac{C_2}{1+\Gamma(\alpha) t^{\alpha}} \right). \end{aligned} \tag{38}$$

Then

$$\|\mathbf{x}(t)\| \le C_1 (M_1 M_4 \|\mathbf{x}_0\| + M_2 M_4 \|\mathbf{x}_1\| t) e^{-(\omega - \Gamma(\alpha)^{1/\alpha}) t} + \frac{C_2 (M_1 M_4 \|\mathbf{x}_0\| + M_2 M_4 \|\mathbf{x}_1\| t)}{1+\Gamma(\alpha) t^{\alpha}} e^{-\omega t}. \tag{39}$$

Therefore, when $t \to \infty$, $\|\mathbf{x}(t)\| \to 0$ for $\omega > \Gamma(\alpha)^{1/\alpha}$, which implies that the system (29) is asymptotically stable. □

Theorem 14. *The fractional order nonlinear system* (29) *is globally asymptotically stable, if it satisfies the following conditions:* (1) $\mathbf{g}(\mathbf{x}(t))$ *satisfies* $\mathbf{g}(0) = 0$ *and the Lipschitz condition with respect to* $\mathbf{x}$*, that is,* $\|\mathbf{g}(\mathbf{x}_1) - \mathbf{g}(\mathbf{x}_2)\| \le L\|\mathbf{x}_1 - \mathbf{x}_2\|$*;* (2) $\mathrm{Re}(\mathrm{eig}(\mathbf{A})) < 0$ *and* $\omega = -\max \mathrm{Re}(\mathrm{eig}(\mathbf{A})) > LM_3 M_4 \Gamma(\alpha)$*, where* M_3 *and* M_4 *satisfy* $\|e^{\mathbf{A}t}\| \le M_3 e^{-\omega t}$ *and* $E_{\alpha,1}(kt^{\alpha}) < M_4 e^{kt^{\alpha}}$.

Proof. Applying the Laplace transform and Laplace inverse transform on (29), we obtain the solution of (29),

$$\mathbf{x}(t) = E_{\alpha,1}(\mathbf{A}t^{\alpha}) \mathbf{x}_0 + t E_{\alpha,2}(\mathbf{A}t^{\alpha}) \mathbf{x}_1 + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\mathbf{A}(t-\tau)^{\alpha}) \mathbf{g}(\mathbf{x}(\tau)) \, d\tau. \tag{40}$$

It follows from Lemma 7 that there exist constants $M_1 > 0$, $M_2 > 0$, and $M_3 > 0$ such that

$$\begin{aligned} \|\mathbf{x}(t)\| &\le M_1 \left\| e^{\mathbf{A}t^{\alpha}} \right\| \|\mathbf{x}_0\| + M_2 \left\| e^{\mathbf{A}t^{\alpha}} \right\| \|\mathbf{x}_1\| t + M_3 \int_0^t (t-\tau)^{\alpha-1} \left\| e^{\mathbf{A}(t-\tau)^{\alpha}} \right\| \|\mathbf{g}(\mathbf{x}(\tau))\| \, d\tau \\ &\le M_1 M_4 e^{-\omega t^{\alpha}} \|\mathbf{x}_0\| + M_2 M_4 e^{-\omega t^{\alpha}} \|\mathbf{x}_1\| t + M_3 M_4 \int_0^t (t-\tau)^{\alpha-1} e^{-\omega (t-\tau)^{\alpha}} \|\mathbf{g}(\mathbf{x}(\tau))\| \, d\tau \\ &\le M_1 M_4 e^{-\omega t^{\alpha}} \|\mathbf{x}_0\| + M_2 M_4 e^{-\omega t^{\alpha}} \|\mathbf{x}_1\| t + LM_3 M_4 \int_0^t (t-\tau)^{\alpha-1} e^{-\omega (t-\tau)^{\alpha}} \|\mathbf{x}(\tau)\| \, d\tau \\ &\le M_1 M_4 e^{-\omega t} \|\mathbf{x}_0\| + M_2 M_4 e^{-\omega t} \|\mathbf{x}_1\| t + LM_3 M_4 \int_0^t (t-\tau)^{\alpha-1} e^{-\omega (t-\tau)} \|\mathbf{x}(\tau)\| \, d\tau. \end{aligned} \tag{41}$$

Multiplying the inequality by $e^{\omega t}$, we will get

$$e^{\omega t}\|\mathbf{x}(t)\| \le M_1 M_4 \|\mathbf{x}_0\| + M_2 M_4 \|\mathbf{x}_1\| t + LM_3 M_4 \int_0^t (t-\tau)^{\alpha-1} e^{\omega \tau} \|\mathbf{x}(\tau)\| \, d\tau. \tag{42}$$

Applying Lemma 8 (Gronwall inequality) and Lemma 6 to (42), we have

$$\begin{aligned} e^{\omega t}\|\mathbf{x}(t)\| &\le (M_1 M_4 \|\mathbf{x}_0\| + M_2 M_4 \|\mathbf{x}_1\| t) E_{\alpha,1}(LM_3 M_4 \Gamma(\alpha) t^{\alpha}) \\ &\le C_1 (M_1 M_4 \|\mathbf{x}_0\| + M_2 M_4 \|\mathbf{x}_1\| t) e^{(LM_3 M_4)^{1/\alpha} t} + \frac{C_2 (M_1 M_4 \|\mathbf{x}_0\| + M_2 M_4 \|\mathbf{x}_1\| t)}{1 + LM_3 M_4 \Gamma(\alpha) t^{\alpha}} e^{-\omega t}. \end{aligned} \tag{43}$$

Then

$$\|\mathbf{x}(t)\| \le C_1 (M_1 M_4 \|\mathbf{x}_0\| + M_2 M_4 \|\mathbf{x}_1\| t) e^{-(\omega - (LM_3 M_4)^{1/\alpha}) t} + \frac{C_2 (M_1 M_4 \|\mathbf{x}_0\| + M_2 M_4 \|\mathbf{x}_1\| t)}{1 + LM_3 M_4 \Gamma(\alpha) t^{\alpha}} e^{-\omega t}. \tag{44}$$

Therefore, when $t \to \infty$, $\|\mathbf{x}(t)\| \to 0$ for $\omega > (LM_3 M_4 \Gamma(\alpha))^{1/\alpha}$, which implies that the system (29) is globally asymptotically stable. □

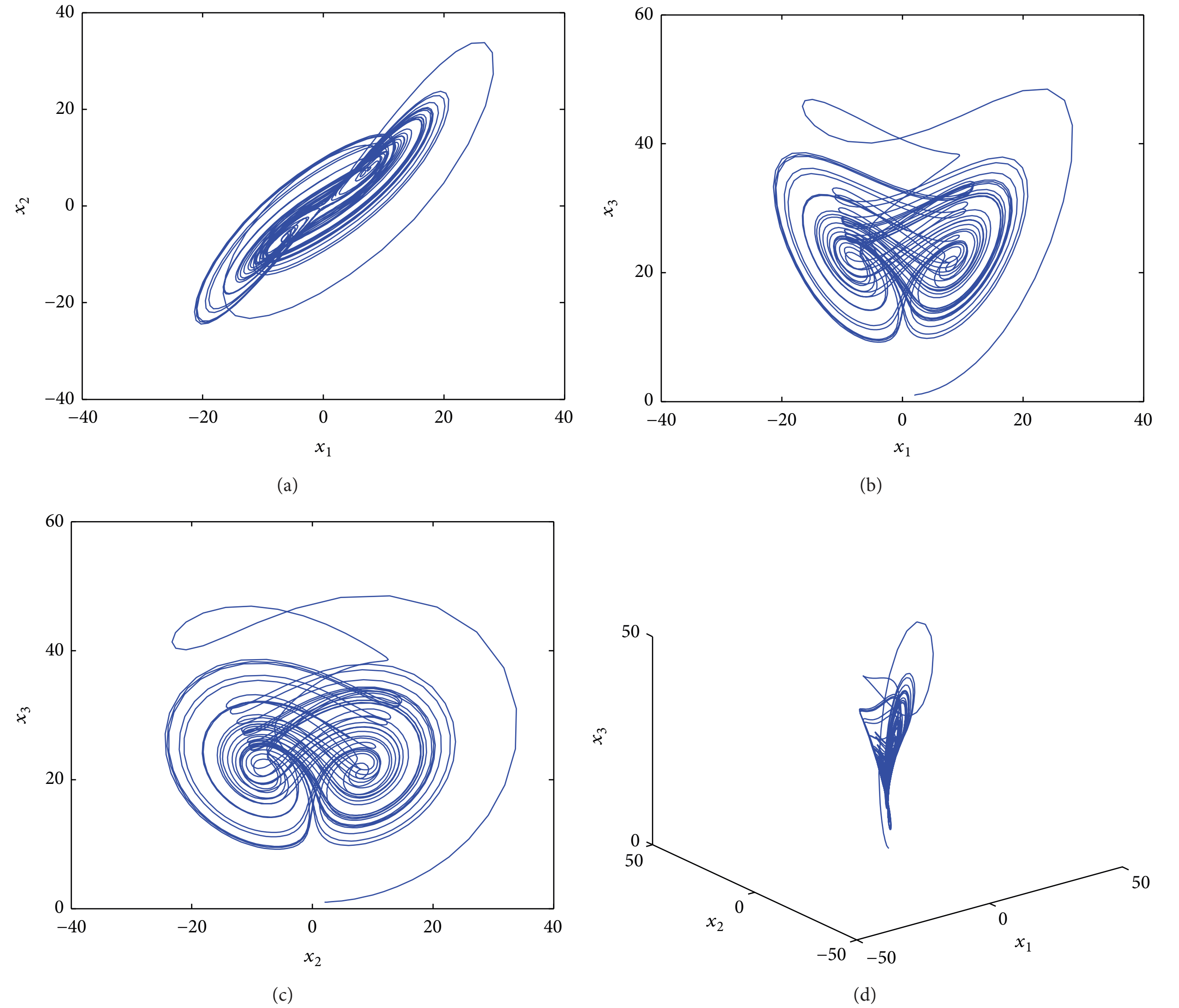

Figure 1: Chaotic attractors in the fractional order Chen system with $\alpha = 0.95$. The panels (a), (b), (c), and (d) show the $x_2 - x_1$, $x_3 - x_1$, $x_3 - x_2$, and 3D views, respectively.

The controlled fractional order nonlinear system with linear feedback control input is given as

$$\begin{aligned} {}_{0}^{C}D_t^{\alpha}\mathbf{x}(t) &= \mathbf{f}(\mathbf{x}(t)) + \mathbf{u}(t) \\ &= (\mathbf{A}+\mathbf{K})\,\mathbf{x}(t) + \mathbf{g}(\mathbf{x}(t)) = \overline{\mathbf{A}}\mathbf{x}(t) + \mathbf{g}(\mathbf{x}(t)), \end{aligned} \tag{45}$$

where $\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t)$ is the linear feedback control input, $\overline{\mathbf{A}} = \mathbf{A} + \mathbf{K}$, and the feedback gain $\mathbf{K} \in \mathbb{R}^{n\times n}$ needs to be determined.

Therefore, our aim is to design a suitable feedback gain matrix $\mathbf{K}$ such that the controlled system is local (globally) asymptotically stable.

Theorem 15. *The controlled fractional order nonlinear system* (45) *is local asymptotically stable, if it satisfies the following conditions:* (1) $\mathrm{Re}(\mathrm{eig}(\overline{\mathbf{A}})) < 0$ *and* $\omega = -\max \mathrm{Re}(\mathrm{eig}(\overline{\mathbf{A}})) > \Gamma(\alpha)$*;* (2) $\mathbf{g}(\mathbf{x}(t))$ *satisfies* $\|\mathbf{g}(\mathbf{x}(t))\| = o\|\mathbf{x}(t)\|$ *as* $\|\mathbf{x}\| \to 0$.

Proof. The proof is similar to that of Theorem 13. □

Theorem 16. *The controlled fractional-order nonlinear system* (45) *is globally asymptotically stable, if it satisfies the following conditions:* (1) $\mathbf{g}(\mathbf{x}(t))$ *satisfies* $\mathbf{g}(0) = 0$ *and the Lipschitz condition with respect to* $\mathbf{x}$*, that is,* $\|\mathbf{g}(\mathbf{x}_1) - \mathbf{g}(\mathbf{x}_2)\| \le L\|\mathbf{x}_1 - \mathbf{x}_2\|$*;* (2) $\mathrm{Re}(\mathrm{eig}(\overline{\mathbf{A}})) < 0$ *and* $\omega = -\max \mathrm{Re}(\mathrm{eig}(\overline{\mathbf{A}})) > LM_3M_4\Gamma(\alpha)$*, where* M_3 *and* M_4 *satisfy* $\|e^{\overline{\mathbf{A}}t}\| \le M_3e^{-\omega t}$ *and* $E_{\alpha,1}(kt^{\alpha}) < M_4e^{kt^{\alpha}}$.

Proof. The proof is similar to that of Theorem 14. □

Remark 17. There are many fractional order chaotic (hyperchaotic) systems which satisfy $\|g(x(t))\| = o\|x(t)\|$ as $\|x\| \to 0$ or the Lipschitz condition, such as fractional order Lorenz

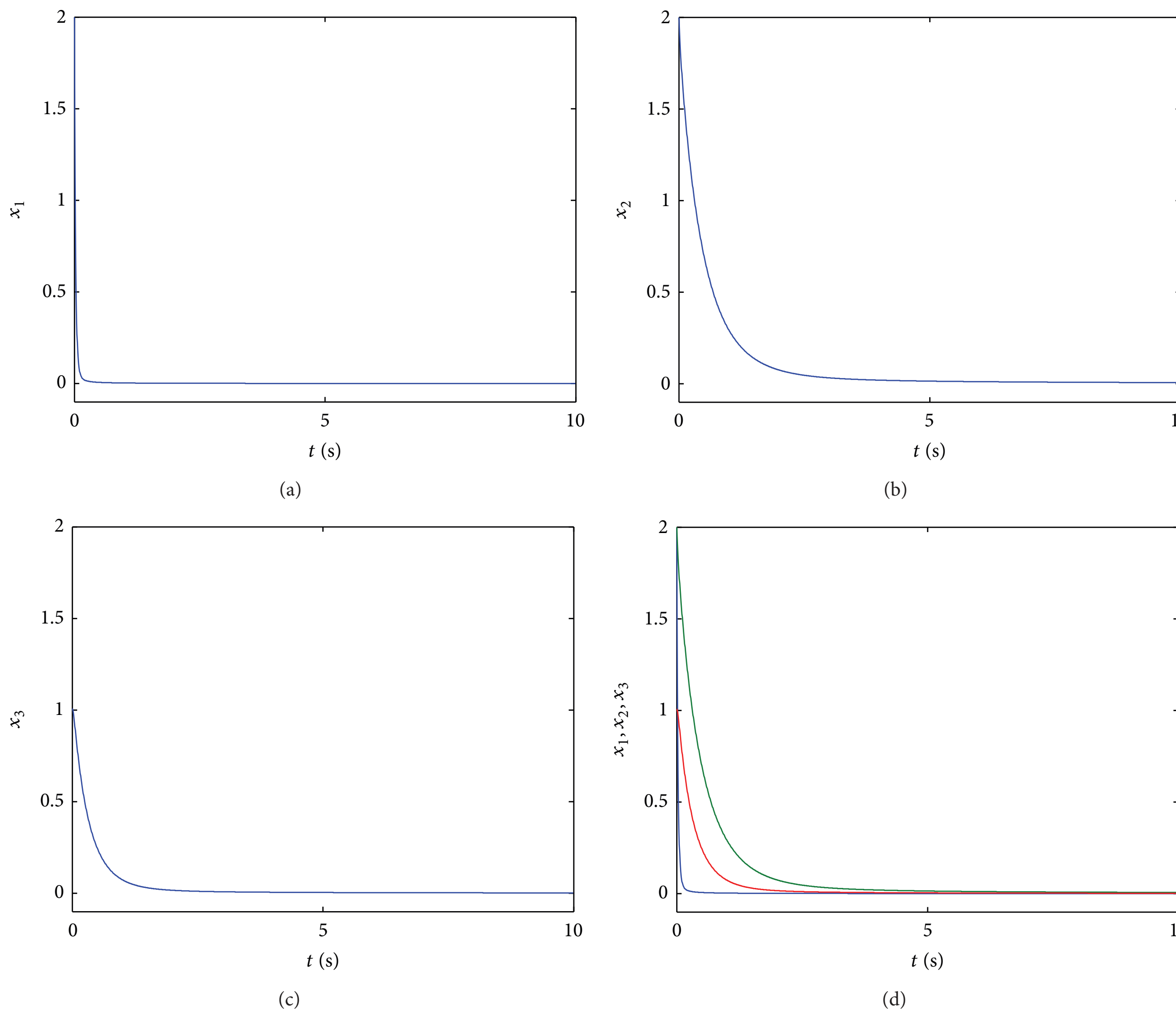

Figure 2: Time waveforms of state variables $x_1(a)$, $x_2(b)$, and $x_3(c)$ of the controlled fractional order Chen system.

system, fractional order Chen system, fractional order Lü system, fractional order Liu system, and so forth [22]. Therefore, Theorems 9–16 can be used as the criteria to control chaos in a class of fractional-order systems. Compared with nonlinear control methods, the advantage of linear control lies in reducing control cost and is easy to implement.

Remark 18. The obtained sufficient conditions could be applied to a class of fractional order hyperchaotic systems [23–25]. On the one hand, complex multiscroll chaotic systems have garnered much attention in recent years. J. H. Lü has done a large amount of remarkable work. In fact, the sufficient conditions could be applied to a class of complex multiscroll chaotic systems, which could also generate a complex four-scroll chaotic attractor.

4. Four Illustrative Examples

In this section, we apply the proposed method in stabilizing a fractional order Chen system, Chua system, Lü system, and Liu system to verify its effectiveness and universality.

4.1. Stabilization of Fractional Order Chaotic Chen System. The fractional order Chen system [21, 22] with order $\alpha = 0.95$ can be de described by

$$\begin{aligned} {}_0^C D_t^\alpha x_1 &= a\left(x_2 - x_1\right), \\ {}_0^C D_t^\alpha x_2 &= (c-a)\, x_1 + c x_2 - x_1 x_3, \\ {}_0^C D_t^\alpha x_3 &= x_1 x_2 - b x_3. \end{aligned} \tag{46}$$

When the parameters are chosen as $a = 35$, $b = 3$, $c = 28$, and $\alpha = 0.95$, system (46) exhibits the chaotic behavior, as shown in Figure 1. We consider system (46) as form (12)

$${}_0^C D_t^\alpha \mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(\mathbf{x}(t)), \tag{47}$$

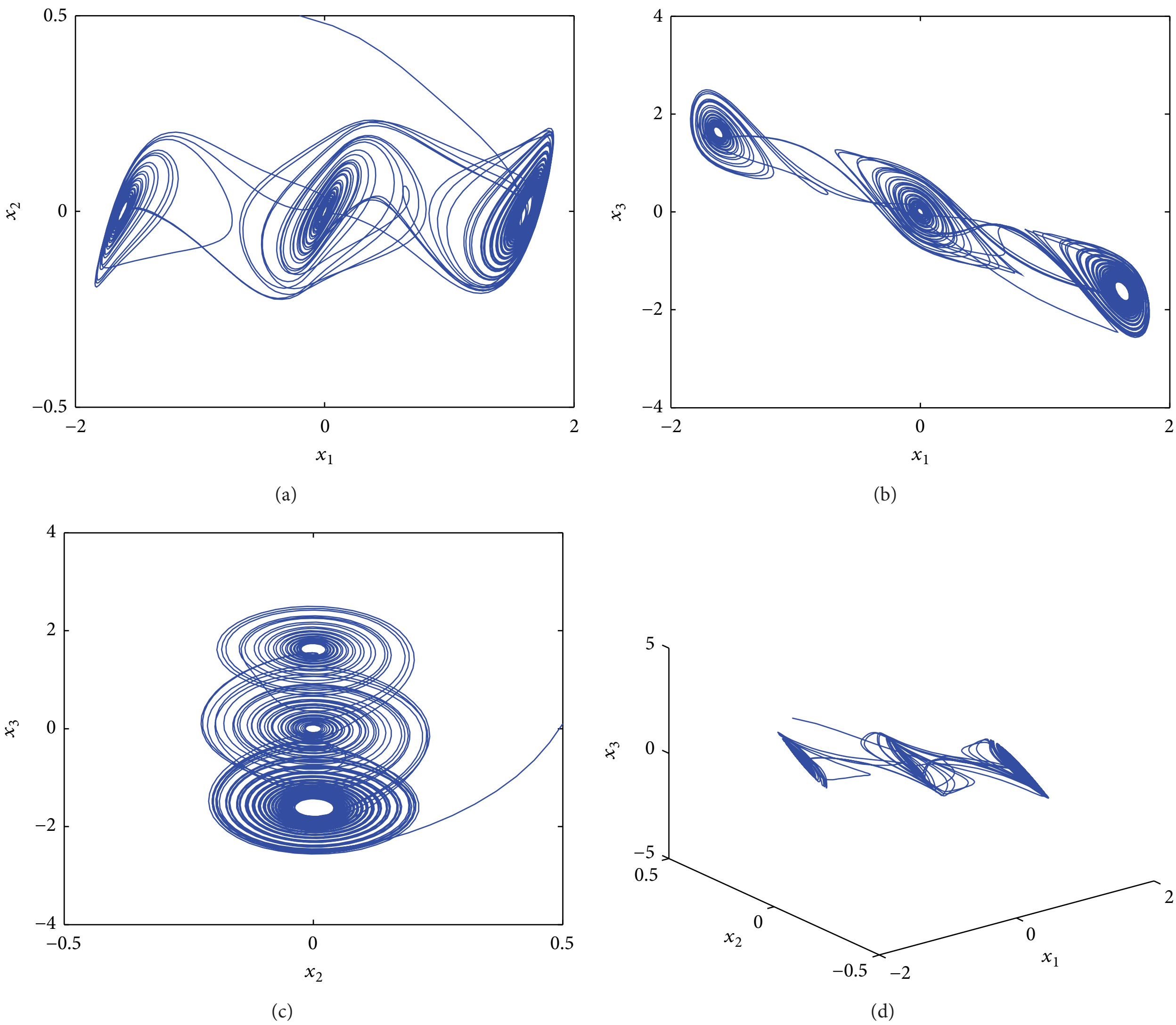

FIGURE 3: Chaotic attractors in the fractional order Chua system with $\alpha = 0.99$. The panels (a), (b), (c), and (d) show the $x_2 - x_1$, $x_3 - x_1$, $x_3 - x_2$, and 3D views, respectively.

where

$$\mathbf{A} = \begin{pmatrix} -a & a & 0 \\ c-a & c & 0 \\ 0 & 0 & -b \end{pmatrix}, \qquad \mathbf{g}(\mathbf{x}(t)) = \begin{pmatrix} 0 \\ -x_1 x_3 \\ x_1 x_2 \end{pmatrix},$$
$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}. \tag{48}$$

Adding control input $\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t)$ to system (47), the controlled system can be rewritten as ${}_{0}^{C}D_t^{\alpha}\mathbf{x}(t) = \overline{\mathbf{A}}\mathbf{x}(t) + \mathbf{g}(\mathbf{x}(t))$. It is easy to demonstrate that $\mathbf{g}(\mathbf{x}(t))$ satisfies

$$\lim_{\|\mathbf{x}\|\to 0}\left(\frac{\|\mathbf{g}(\mathbf{x}(t))\|}{\|\mathbf{x}(t)\|}\right) \le \lim_{\|\mathbf{x}\|\to 0}\frac{\sqrt{x_1^2x_3^2 + x_1^2x_2^2}}{\sqrt{x_1^2}} = \lim_{\|\mathbf{x}\|\to 0}\sqrt{x_2^2 + x_3^2} = 0; \tag{49}$$

that is, $\|\mathbf{g}(\mathbf{x}(t))\| = o\|\mathbf{x}(t)\|$. The feedback gain matrix is selected as

$$K = \begin{pmatrix} 0 & -35 & 0 \\ 7 & -30 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{50}$$

which satisfies the conditions $\mathrm{Re}(\mathrm{eig}(\overline{\mathbf{A}})) < 0$ and $\omega = -\max \mathrm{Re}(\mathrm{eig}(\overline{\mathbf{A}})) = 2 > \Gamma(\alpha) = 1.0315$ in Theorem 11. The simulation result is shown in Figure 2, which shows that the zero solution of the controlled system is asymptotically stable.

4.2. Stabilization of Fractional Order Chaotic Chua System. The fractional order Chua system [26] with order $\alpha = 0.99$ can be described by

$$\begin{aligned} {}_{0}^{C}D_t^{\alpha}x_1 &= a\left(x_2 - x_1 - h(x_1)\right), \\ {}_{0}^{C}D_t^{\alpha}x_2 &= x_1 - x_2 + x_3, \\ {}_{0}^{C}D_t^{\alpha}x_3 &= -bx_2, \end{aligned} \tag{51}$$

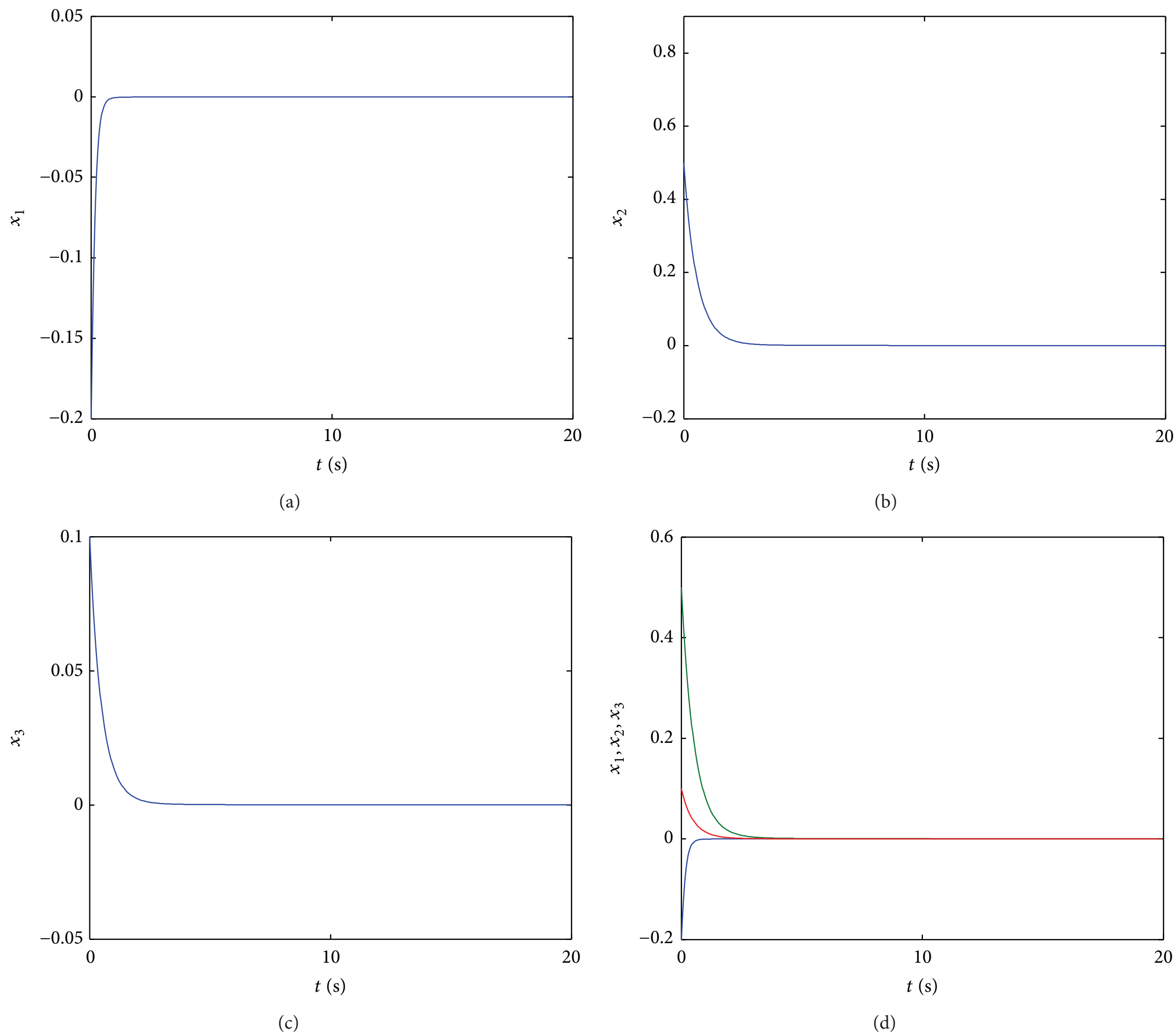

FIGURE 4: Time waveforms of state variables $x_1(a)$, $x_2(b)$, and $x_3(c)$ of the controlled fractional order Chua system.

where $h(x_1) = cx_1 + dx_1|x_1| + ex_1^3$. When the parameters are chosen as $a = 12.8, b = 19.1, c = -0.4, d = -1.1, e = 0.45$, and $\alpha = 0.99$, system (55) exhibits the chaotic behavior, as shown in Figure 3.

We consider system (51) as form (12)

$$ {}_0^C D_t^{\alpha} \mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(\mathbf{x}(t)), \quad (52)$$

where

$$\mathbf{A} = \begin{pmatrix} -a(1+c) & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{pmatrix},$$

$$\mathbf{g}(\mathbf{x}(t)) = \begin{pmatrix} -adx_1|x_1| - aex_1^3 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}. \quad (53)$$

Adding control input $\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t)$ to system (52), the controlled system can be rewritten as ${}_0^C D_t^{\alpha}\mathbf{x}(t) = \overline{\mathbf{A}}\mathbf{x}(t) + \mathbf{g}(\mathbf{x}(t))$. It is easy to demonstrate that $\mathbf{g}(\mathbf{x}(t))$ satisfies

$$\lim_{\|\mathbf{x}\|\to 0}\left(\frac{\|\mathbf{g}(\mathbf{x}(t))\|}{\|\mathbf{x}(t)\|}\right) \le \lim_{\|\mathbf{x}\|\to 0}\frac{\sqrt{a^2d^2x_1^4 + a^2e^2x_1^6}}{\sqrt{x_1^2}} = \lim_{\|\mathbf{x}\|\to 0}\sqrt{a^2d^2x_1^2 + a^2e^2x_1^4} = 0; \quad (54)$$

that is, $\mathbf{g}(\mathbf{x}(t)) = o\|\mathbf{x}(t)\|$. The feedback gain matrix is selected as

$$\mathbf{K} = \begin{pmatrix} 0 & -12.8 & 0 \\ -1 & -1 & 0 \\ 0 & 19.1 & -2 \end{pmatrix} \quad (55)$$

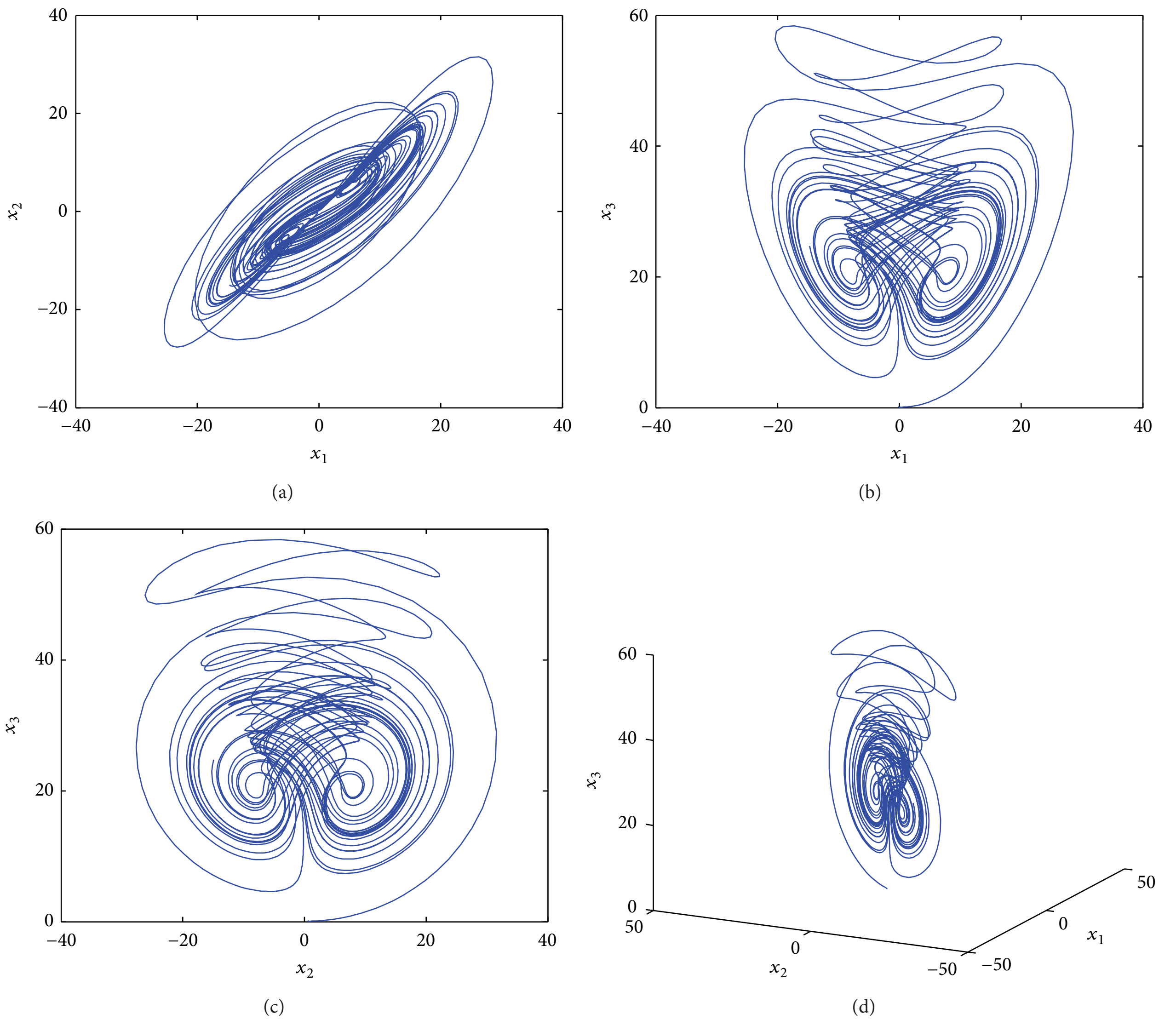

FIGURE 5: Chaotic attractors in the fractional order Lü system with $\alpha = 1.09$. The panels (a), (b), (c), and (d) show the $x_2 - x_1$, $x_3 - x_1$, $x_3 - x_2$, and 3D views, respectively.

which satisfies the conditions $\operatorname{Re}(\operatorname{eig}(\overline{\mathbf{A}})) < 0$ and $\omega = -\max \operatorname{Re}(\operatorname{eig}(\overline{\mathbf{A}})) = 2 > \Gamma(\alpha) = 1.0059$ in Theorem 11. The simulation result is shown in Figure 4, which shows that the zero solution of the controlled system is asymptotically stable.

4.3. Stabilization of Fractional Order Chaotic Lü System. The fractional order Lü [27] system with order $\alpha = 1.09$ can be de described by

$$\begin{aligned} {}_0^C D_t^\alpha x_1 &= a\left(x_2 - x_1\right), \\ {}_0^C D_t^\alpha x_2 &= -x_1 x_3 + c x_2, \\ {}_0^C D_t^\alpha x_3 &= x_1 x_2 - b x_3. \end{aligned} \tag{56}$$

When the parameters are chosen as $a = 36$, $b = 3$, $c = 20$, and $\alpha = 1.09$, system (56) exhibits the chaotic behavior, as shown in Figure 5.

We consider system (56) as form (33)

$${}_0^C D_t^\alpha \mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(\mathbf{x}(t)), \tag{57}$$

where

$$\mathbf{A} = \begin{pmatrix} -a & a & 0 \\ 0 & c & 0 \\ 0 & 0 & -b \end{pmatrix}, \qquad \mathbf{g}(\mathbf{x}(t)) = \begin{pmatrix} 0 \\ -x_1 x_3 \\ x_1 x_2 \end{pmatrix}, \qquad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}. \tag{58}$$

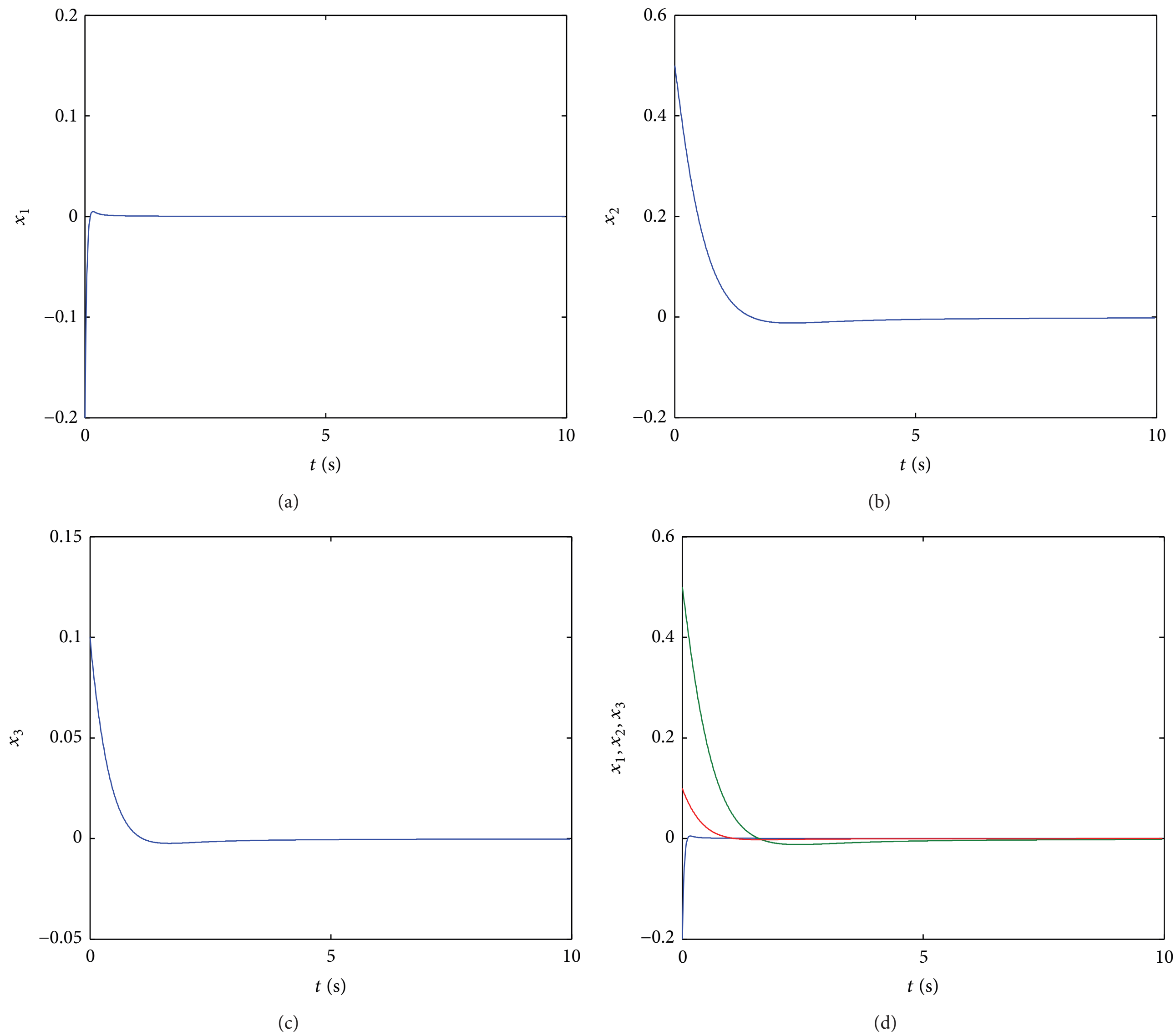

FIGURE 6: Time waveforms of state variables $x_1(a)$, $x_2(b)$, and $x_3(c)$ of the controlled fractional order Lü system.

Adding control input $\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t)$ to system (57), the controlled system can be rewritten as ${}_0^C D_t^{\alpha}\mathbf{x}(t) = \overline{\mathbf{A}}\mathbf{x}(t) + \mathbf{g}(\mathbf{x}(t))$. It is easy to demonstrate that $\mathbf{g}(\mathbf{x}(t))$ satisfies

$$\begin{aligned} &\lim_{\|\mathbf{x}\|\to 0}\left(\frac{\|\mathbf{g}(\mathbf{x}(t))\|}{\|\mathbf{x}(t)\|}\right) \\ &\quad \le \lim_{\|\mathbf{x}\|\to 0}\frac{\sqrt{x_1^2x_3^2 + x_1^2x_2^2}}{\sqrt{x_1^2}} = \lim_{\|\mathbf{x}\|\to 0}\sqrt{x_3^2 + x_2^2} = 0; \end{aligned} \tag{59}$$

that is, $\mathbf{g}(\mathbf{x}(t)) = o\|\mathbf{x}(t)\|$. The feedback gain matrix is selected as

$$\mathbf{K} = \begin{pmatrix} 0 & -36 & 0 \\ 0 & -22 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{60}$$

which satisfies the conditions $\mathrm{Re}(\mathrm{eig}(\overline{\mathbf{A}})) < 0$ and $\omega = -\max \mathrm{Re}(\mathrm{eig}(\overline{\mathbf{A}})) = 2 > \Gamma(\alpha) = 0.9555$ in Theorem 15. The simulation result is shown in Figure 6, which shows that the zero solution of the controlled system is asymptotically stable.

4.4. Stabilization of Fractional Order Chaotic Liu System. The fractional order Liu system [28, 29] with order $\alpha = 1.05$ can be de described by

$$\begin{aligned} {}_0^C D_t^{\alpha} x_1 &= a(x_2 - x_1), \\ {}_0^C D_t^{\alpha} x_2 &= bx_1 - cx_1x_3, \\ {}_0^C D_t^{\alpha} x_3 &= -dx_3 + hx_1^2. \end{aligned} \tag{61}$$

When the parameters are chosen as $a = 10$, $b = 40$, $c = 10$, $d = 2.8$, $h = 4$, and $\alpha = 1.05$, system (65) exhibits the chaotic behavior, as shown in Figure 7.

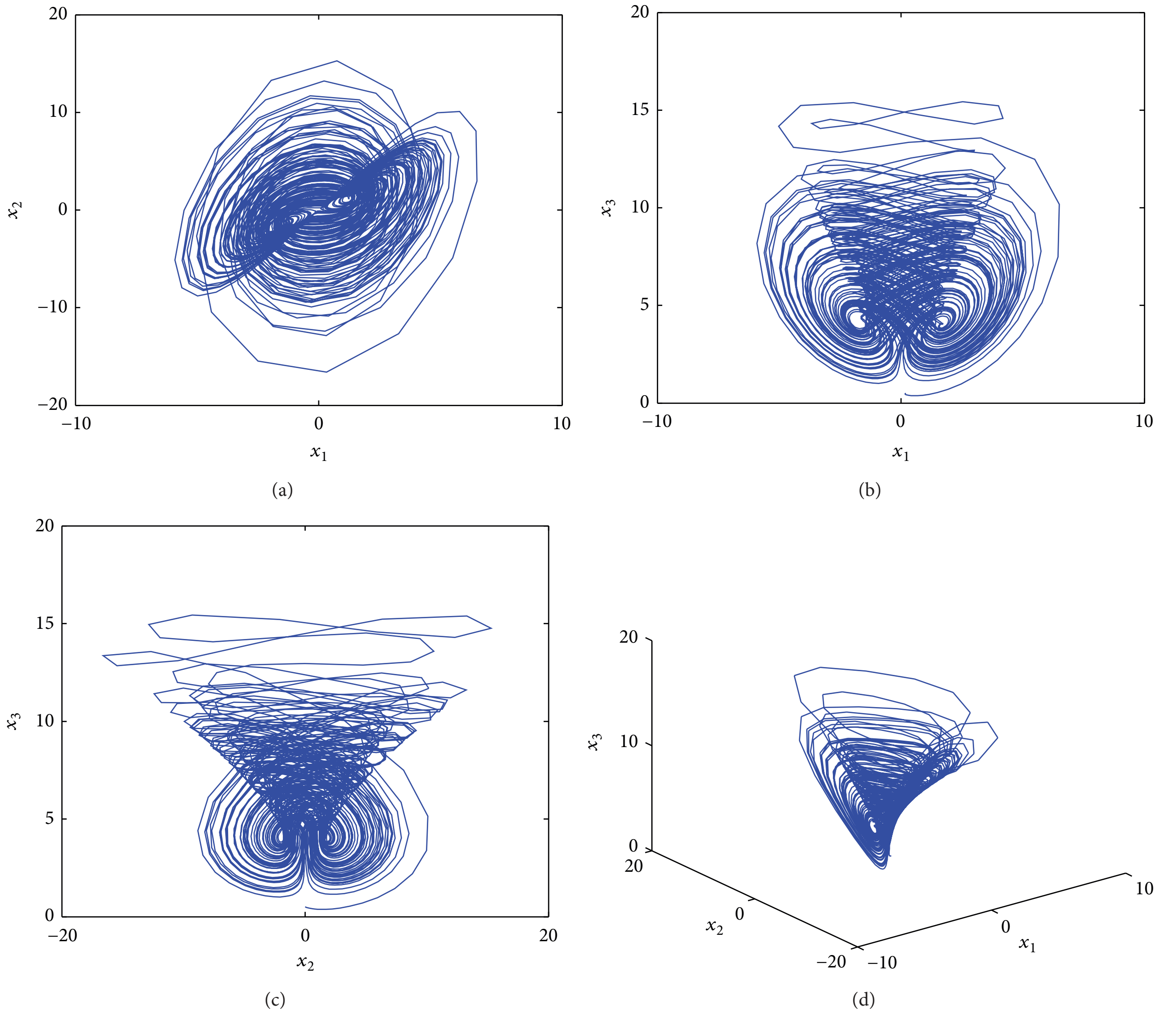

FIGURE 7: Chaotic attractors in the fractional order Liu system with $\alpha = 1.05$. The panels (a), (b), (c), and (d) show the $x_2 - x_1$, $x_3 - x_1$, $x_3 - x_2$, and 3D views, respectively.

We consider system (61) as form (29)

$$ {}_0^C D_t^{\alpha} \mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(\mathbf{x}(t)), \tag{62} $$

where

$$ \mathbf{A} = \begin{pmatrix} -a & a & 0 \\ b & 0 & 0 \\ 0 & 0 & -d \end{pmatrix}, \qquad \mathbf{g}(\mathbf{x}(t)) = \begin{pmatrix} 0 \\ -c x_1 x_3 \\ h x_1^2 \end{pmatrix}, \tag{63} $$

$$ \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}. $$

Adding control input $\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t)$ to system (62), the controlled system can be rewritten as ${}_0^C D_t^{\alpha} \mathbf{x}(t) = \overline{\mathbf{A}}\mathbf{x}(t) + \mathbf{g}(\mathbf{x}(t))$. It is easy to demonstrate that $\mathbf{g}(\mathbf{x}(t))$ satisfies

$$ \lim_{\|\mathbf{x}\| \to 0} \left(\frac{\|\mathbf{g}(\mathbf{x}(t))\|}{\|\mathbf{x}(t)\|} \right) \leq \lim_{\|\mathbf{x}\| \to 0} \frac{\sqrt{c^2 x_1^2 x_3^2 + h^2 x_1^4}}{\sqrt{x_1^2}} = \lim_{\|\mathbf{x}\| \to 0} \sqrt{c^2 x_3^2 + h^2 x_1^2} = 0; \tag{64} $$

that is, $\mathbf{g}(\mathbf{x}(t)) = o\|\mathbf{x}(t)\|$. The feedback gain matrix is selected as

$$ \mathbf{K} = \begin{pmatrix} 0 & -10 & 0 \\ -40 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{65} $$

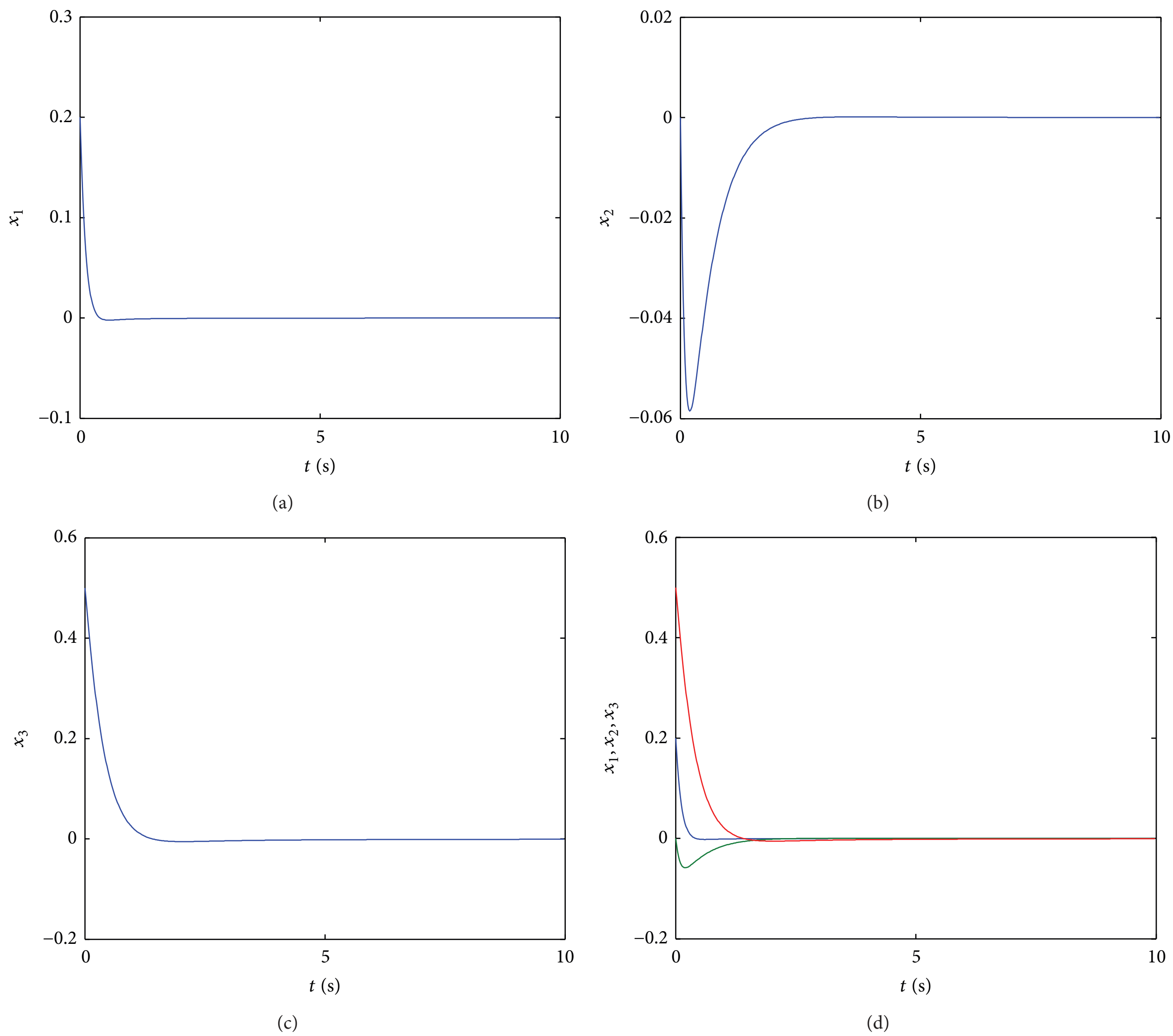

Figure 8: Time waveforms of state variables $x_1(a)$, $x_2(b)$, and $x_3(c)$ of the controlled fractional order Liu system.

which satisfies the conditions $\mathrm{Re}(\mathrm{eig}(\overline{\mathbf{A}})) < 0$ and $\omega = -\max \mathrm{Re}(\mathrm{eig}(\overline{\mathbf{A}})) = 2 > \Gamma(\alpha) = 0.9735$ in Theorem 15. The simulation result is shown in Figure 8, which shows that the zero solution of the controlled system is asymptotically stable.

5. Conclusion

Stability of the nonlinear dynamical systems is important for scientists and engineers. Fractional dynamic systems were used intensively during the last decade in order to describe the behavior of complex systems in physical and engineering. In this paper the stabilization of nonlinear fractional order dynamic system is studied. And by using the Gronwall inequality and the properties of Mittag-Leffler function, we proposed some new sufficient conditions on the local (globally) asymptotic stability for a class of fractional order nonlinear systems. Finally the corresponding stabilization criteria are also given. Four numerical simulation examples have illustrated the effectiveness and universality of the proposed methods.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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