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Din-Yu Hsieh

Division of Engineering and Applied Science
CALIFORNIA INSTITUTE OF TECHNOLOGY
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Abstract

A stability analysis is made for the laminar flow of a layer of a viscous and electrically conducting fluid down an inclined plane in a transverse magnetic field. It is found that the effect of the magnetic field, revealed through the Hartmann number, is to stabilize the flow. A simpler and physically clearer approximate treatment of the same problem based on the principle of local balance is also given. The results agree quite satisfactorily with the exact analysis.

1. Introduction

The stability of laminar flows of an electrically conducting fluid in a magnetic field has been studied fairly extensively. Among others, Chandrasekhar[1] has investigated the stability of flow between coaxial rotating cylinders with a magnetic field in the axial direction, Stuart[2] investigated the stability of pressure flow between parallel planes in a parallel magnetic field, while Lock[3] has studied the latter stability with a magnetic field perpendicular to the direction of motion and to the boundary planes. In all these cases, it is found that the presence of magnetic fields tends to stabilize the system. In the present paper, we shall investigate the effect of a magnetic field on the stability of the gravity flow of a conducting fluid down an inclined plane. The latter stability for non-conducting fluids has been studied by Yih[4][5], Benjamin[6] and Binnie[7], and also extended by the present author to the flow of superfluids[8]. It is found in general that the critical Reynolds number is quite low. From the study of the physical mechanism of this type of instability[9], it may be inferred that

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(1) S. Chandrasekhar, Proc. Roy. Soc. (London) A, 216, 293 (1953).

(2) J. T. Stuart, Proc. Roy. Soc. (London) A, 221, 189 (1954).

(3) R. C. Lock, Proc. Roy. Soc. (London) A, 233, 105 (1955).

(4) C.-S. Yih, Proc. 2nd U.S. Nat. Congr. Appl. Mech. (ASME, N.Y. 1955), 623.

(5) C.-S. Yih, Physics of Fluids, 6, 321 (1963).

(6) T. B. Benjamin, J. Fluid Mech. 2, 554 (1957).

(7) A. M. Binnie, J. Fluid Mech. 2, 551 (1957).

(8) D. Y. Hsieh, Physics of Fluids, 7, 1755(1964).

(9) M. S. Plesset and D. Y. Hsieh,(in press).

the crucial feature is the velocity profile of the undisturbed flow. Therefore we may expect, as indeed can be verified by analysis, that a magnetic field parallel to the inclined plane will have relatively slight effects on the stability, while a transverse magnetic field may have quite pronounced effects. This expectation is also in agreement with the results of Stuart[2] and Lock[3]. Therefore, in the following, we shall only consider the case in which the magnetic field is in the direction perpendicular to the inclined plane.

2. The Fundamental Equations

The hydromagnetic equations for a viscous, incompressible, conducting fluids are as follows.

Maxwell's equations:

$$\nabla \times \underset{\sim}{H} = \frac{4\pi}{c} \underset{\sim}{J} \quad , \tag{1}$$

$$\nabla \cdot \underset{\sim}{H} = 0 \quad , \tag{2}$$

$$\nabla \times \underset{\sim}{E} = - \frac{\mu}{c} \frac{\partial \underset{\sim}{H}}{\partial t} \quad , \tag{3}$$

$$\epsilon \nabla \cdot \underset{\sim}{E} = e \quad , \tag{4}$$

Ohm's law for a moving fluid:

$$\underset{\sim}{J} = \sigma (\underset{\sim}{E} + \frac{\mu}{c} \underset{\sim}{v} \times \underset{\sim}{H}) \quad , \tag{5}$$

The equation of continuity:

$$\nabla \cdot \underset{\sim}{v} = 0 \tag{6}$$

The momentum equation:

$$\rho \frac{\partial \underset{\sim}{v}}{\partial t} + \rho(\underset{\sim}{v} \cdot \nabla)\underset{\sim}{v} = -\nabla p + \frac{\mu}{c}(\underset{\sim}{J} \times \underset{\sim}{H}) + \eta \nabla^2 \underset{\sim}{v} - \rho \nabla \Omega \qquad (7)$$

and the energy equation

$$\frac{\partial U}{\partial t} + (\underset{\sim}{v} \cdot \nabla) U = \frac{J^2}{\sigma} + \kappa \nabla^2 T + \frac{\eta}{2} \sum_{\alpha, \beta=1}^{3} \left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha} \right)^2 , \qquad (8)$$

where p, ρ, T and U are the pressure, density, temperature and internal energy of the fluid; ϵ, μ, σ, η and κ are the dielectric constant, permeability, coefficients of electrical conductivity, viscosity and thermal conductivity; Ω is the potential of the external force; and $\underset{\sim}{H}$, $\underset{\sim}{E}$, $\underset{\sim}{J}$, and e are magnetic field, electric field, current density and charge density. In arriving at the above set of equations, we have assumed that the constitutive and transport coefficients are constant scalars, that the displacement current can be neglected in Eq. (1), that the force due to the electric field may be neglected in Eq. (7) and that the Ohm's law as stated by Eq. (5), which neglects the convection current, may be justified. We should also have the equation of state which gives the expression of U in terms of ρ and T to complete the system. However, as it stands, Eq. (8) is not coupled with other equations; hence it can be disregarded if no information about U and T is desired. Likewise, Eq. (4), which is also uncoupled from the other equations, only serves to determine the charge density e.

3. The Primary Flow

The primary flow to be considered is a layer of fluid flowing in parallel flow on a plane making an angle θ with the horizontal direction. An external uniform magnetic field H is applied in the direction perpendicular to the plane. Let the thickness of the layer be h. A coordinate system is chosen with the origin at the free surface of the layer as shown in Fig. 1. Then Ω, the potential due to the gravitational force field, will be given by

$$\Omega = yg\cos\theta - xg\sin\theta \quad . \qquad (9)$$

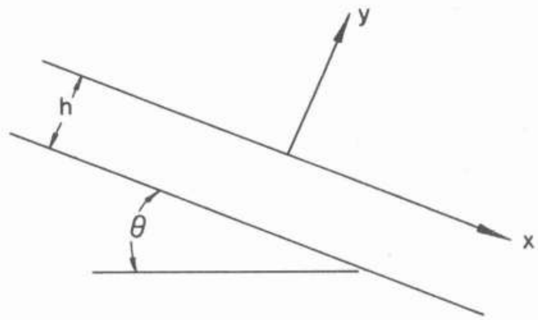

Figure 1

Now let us look for such an undisturbed system that

$$\underline{v} = \left(V(y), 0, 0\right) \;, \quad \underline{H} = \left(Hb(y), H, 0\right) \;, \quad \underline{J} = \left(0, 0, J(y)\right) \;, \quad \underline{E} = \left(0, 0, E\right) \;.$$

The electric field and current andadditional magnetic field are introduced in order to satisfy the governing equations.

Equation (5) yields

$$J(y) = \sigma\left(E + \frac{\mu}{c} HV(y)\right) \quad . \qquad (10)$$

Then Eq. (7) leads to $p = p(y)$, and

$$\eta V'' - \frac{\mu H\sigma}{c}\left(E + \frac{\mu H}{c} V\right) + \rho g \sin\theta = 0 \quad . \qquad (11)$$

With the introduction of the Hartmann number $M = \frac{\mu Hh}{c}\sqrt{\frac{\sigma}{\eta}}$, Eq. (11) can be written as

$$V'' - \frac{M^2}{h^2} V = \frac{\mu\sigma}{c\eta} HE - \frac{\rho g}{\eta} \sin\theta \quad . \qquad (12)$$

As $V(-h) = V'(o) = 0$, we obtain

$$V(y) = V_o \frac{\cosh M - \cosh\left(M \frac{y}{h}\right)}{\cosh M - 1} \;, \qquad (13)$$

where

$$V_o = \frac{(\cosh M-1)h^2}{M^2 \cosh M}\left(\frac{\rho g}{\eta} \sin\theta - \frac{\mu\sigma}{c\eta} HE\right) . \tag{14}$$

The total discharge per unit width, incidentally, is

$$Q = \int_{-h}^{o} V(y)dy = \frac{V_o}{\cosh M-1}\left(h \cosh M - \frac{h}{M} \sinh M\right) , \tag{15}$$

while $\frac{dQ}{dh}$, if we use Eq. (14) and note the linear dependence on h of M, is found to be

$$\frac{dQ}{dh} = V_o \frac{\cosh M+1}{\cosh M} . \tag{16}$$

We shall require that outside the fluid layer only the applied magnetic field (0, H, 0) is present, and that the total flux of current in the z-direction is zero. The latter requirement leads to the condition

$$\int_{-h}^{o} J(y)dy = 0 \tag{17}$$

while the former requires that

$$b(o) = b(-h) = 0 . \tag{18}$$

Equations (10), (13) and (17) then lead to

$$E = \frac{\mu H V_o}{c} \frac{\sinh M - M \cosh M}{M(\cosh M-1)} , \tag{19}$$

and then Eq. (5) yields

$$J(y) = \frac{\sigma\mu H V_o}{c} \frac{\sinh M - M \cosh\left(M \frac{y}{h}\right)}{M(\cosh M-1)} . \tag{20}$$

Equation (1) , with Eq. (18), then leads to

$$b(y) = -\frac{4\pi\sigma\mu V_o}{c^2} \frac{y \sinh M - h \sinh\left(\frac{My}{h}\right)}{M(\cosh M - 1)} .$$

4. The Stability Problem

Let us superimpose a two-dimensional disturbance on the primary flow so that the elevation of the free surface is given by

$$y = \zeta(x,t) = \zeta e^{ik(x-at)} . \quad (22)$$

Thus any perturbed quantity $g(x,y,t)$ may be written as

$$g(x,y,t) = g(y)e^{ik(x-at)} . \quad (23)$$

After the system is perturbed, we have $\underset{\sim}{v} = (v_x, V+v_y, v_z)$, $\underset{\sim}{H} = (Hb+H_x, H+H_y, H_z)$, $\underset{\sim}{J} = (J_x, J_y, J+J_z)$ and $\underset{\sim}{E} = (E_x, E_y, E+E_z)$. We shall assume that the disturbance is small and that the squares of the small quantities in all the equations can be neglected. Eliminating p from Eq. (7) and using Eqs. (1), (2) and (6), we obtain

$$v_y^{iv} - 2k^2 v_y'' + k^4 v_y - ik\frac{\rho}{\eta}\left[(V-a)(v_y'' - k^2 v_y) - V'' v_y\right]$$

$$= \frac{ik\mu H}{4\pi\eta}\left[\frac{i}{k}(H_y''' - k^2 H_y') - b(H_y'' - k^2 H_y) + b'' H_y\right] . \quad (24)$$

We can make the above equation non-dimensional by taking $v_y = V_o\psi$, $H_y = H\varphi$, $V = V_o\overline{V}$, $a = V_o\overline{a}$, $k = \frac{\overline{k}}{h}$ and $y = h\overline{y}$, and substitute them into the equation. Dropping bars of the newly introduced quantities to simplify writing, we find that Eq. (24) becomes

$$[\psi^{IV} - 2k^2\psi'' + k^4\psi] - ikR\,[(V-a)(\psi''-k^2\psi) - V''\psi]$$

$$= -\frac{M^2}{R_M}\,[\varphi''' - k^2\varphi' + ikb(\varphi' - k^2\varphi) + ikb''\varphi] \quad , \tag{25}$$

where

$$R = \frac{V_o \rho h}{\eta} \quad , \tag{26}$$

and

$$R_M = \frac{4\pi\mu\sigma V_o h}{c^2} \quad . \tag{27}$$

R and R_M are called the Reynolds number and magnetic Reynolds number respectively. Eliminating J_z from Eqs. (1) and (5), and using Eqs. (2) and (3), we obtain similarly:

$$\varphi'' - k^2\varphi = -R_M[\psi' + ikb\psi - ik(V-a)\varphi] \quad . \tag{28}$$

The electromagnetic fields in regions outside the fluid layer are governed by the Maxwell equations. With the space and time variation given by Eq. (23), the field quantities can be readily determined for a medium which is conducting or non-conducting.

The boundary conditions will contain the following features.

(i) The normal velocity of the fluid vanishes at the bottom:

$$\psi(-1) = 0 \quad . \tag{29}$$

(ii) The tangential velocity of the fluid vanishes at the bottom:

$$\psi'(-1) = 0 \quad , \tag{30}$$

after use of Eq. (6).

(iii) The shearing stress vanishes at the free surface;

$$\frac{\partial}{\partial x}(v_y) + \frac{\partial}{\partial y}(V+v_x) = 0 \qquad \text{on} \qquad y = \zeta(x,t) \quad ;$$

or

$$\psi''(o) + k^2\psi(o) + \frac{ikM^2}{\cosh M - 1}\,\zeta = 0 \quad . \tag{31}$$

(iv) The normal stress is continuous across the free surface:

$$p = 2\eta\left(\frac{\partial v_y}{\partial y}\right) - \Gamma\rho\left(\frac{\partial^2\zeta}{\partial x^2}\right) + \rho g \cos\theta\,\zeta \quad , \qquad \text{at} \qquad y = 0 \quad ,$$

where $\rho\Gamma$ is the surface tension coefficient. Using Eq. (7), we then obtain

$$(\psi'''-k^2\psi') + \frac{M^2}{R_M}(\varphi''-k^2\varphi) - ik\left\{R[(V-a)\psi'-V'\psi] + \frac{M^2}{R_M}b'\varphi\right\}$$

$$= 2k^2\psi'+k^2\left(\Gamma k^2 R + \frac{M \sinh M}{\cosh M - 1}\cot\theta\right)\zeta \quad , \qquad \text{at} \qquad y = 0, \tag{32}$$

where the relation $V_o = \frac{\rho g \sin\theta h^2}{\eta}\,\frac{\cosh M - 1}{M \sinh M}$ which is derivable from Eqs. (14) and (19) has been used.

(v) The free surface will satisfy the kinematic surface condition:

$$\psi(o) - ik(V_o - a)\zeta = 0 \quad . \tag{33}$$

(vi) The electromagnetic fields in the fluid layer will be appropriately connected to those in the free space above and those in the material wall below.

5. Solution for Long Waves

In the analysis of the stability problem for non-conducting fluids[5], it is shown that the criterion for stability is essentially determined by the disturbances of long wavelength. We expect this feature to remain true so long as the magnetic field is not too strong. When the wavelength of the disturbance is long compared with the depth of the fluid layer, i.e., when $k \ll 1$, a scheme of successive approximation can be developed. For the first approximation, we shall neglect all quantities of order $O(k)$ in the system of differential equations and boundary conditions. Then Eqs. (25) and (28) become

$$\psi_o^{IV} + \frac{M^2}{R_M} \varphi_o''' = 0 \quad , \tag{34}$$

and

$$\varphi_o'' + R_M \psi_o' = 0 \quad , \tag{35}$$

and the boundary conditions become

(i) $\psi_o(-1) = 0 \quad ,$ (36)

(ii) $\psi_o'(-1) = 0 \quad ,$ (37)

(iii) $\psi_o''(o) + \frac{M^2}{\cosh M - 1} ik\zeta_o = 0 \quad ,$ (38)

(iv) $\psi_o'''(o) + \frac{M^2}{R_M} \varphi_o''(o) = 0 \quad ,$ (39)

and

(v) $\psi_o(o) - (V_o - a_o) ik\zeta_o = 0 \quad ,$ (40)

where the subscript zero signifies that it is the solution for the first approximation.

Equations (34) and (35) lead to

$$\psi_o^{IV} - M^2 \psi_o'' = 0 \quad . \tag{41}$$

Thus the general solution is

$$\psi_o = A_o \cosh My + B_o \sinh M_y + C_o y + D_o \quad . \tag{42}$$

Now Eq. (39), with application of Eq. (35), becomes

$$\psi_o'''(o) - M^2\psi'(o) = 0$$

which requires

$$C_o = 0 \quad .$$

Then Eqs. (36) and (37) lead to

$$\psi_o = A_o[\cosh M(1+y) - 1] \quad . \tag{43}$$

From Eq. (38), we get

$$ik\zeta_o = -A_o \cosh M(\cosh M - 1) \quad , \tag{44}$$

and from Eq. (40), as $V_o = 1$, we obtain

$$a_o = \frac{\cosh M + 1}{\cosh M} \quad . \tag{45}$$

It may be noted that a_o is identical with the expression of $\frac{dQ}{dh}$ given by Eq. (16).

The general expression for φ_o can be written down immediately from Eq. (35):

$$\varphi_o = -R_M \left[\frac{A_o}{M} \sinh M(1+y) + C_o'y + D_o' \right] \quad . \tag{46}$$

The coefficients C_o' and D_o' can be readily determined, by the application of the boundary conditions (vi). Now for almost all conducting liquid of interest, the ratio $\frac{R_M}{R}$ is extremely small. For mercury, it is about 1.5×10^{-7}, while for liquid sodium, it is 7.5×10^{-6}. Therefore in the range of Reynolds number usually encountered in laminar flow, R_M is very small. The value of φ_o, as may be seen easily, is in general of the order $O(R_M)$. Hence, as far as the stability of the flow is concerned, there is practically no need to obtain the explicit expression for φ_o.

For the next approximation, we shall take $\psi = \psi_o + \psi_1$, where $\psi_1 = O(k)$, etc., and neglect all terms that are quadratic in k in the differential equations and boundary conditions. Keeping in mind that $\varphi_o = O(R_M)$ and $b = O(R_M)$, we obtain from Eqs. (25) and (28):

$$\psi_1^{IV} - M^2\psi_1'' = ik\{R[(V-a_o)\psi_o'' - V''\psi_o] + O(R_M)\} \quad . \tag{47}$$

As

$$V = \frac{\cosh M - \cosh My}{\cosh M - 1} \quad , \qquad V'' = -\frac{M^2 \cosh My}{\cosh M - 1} \quad ,$$

thus with $A_o = 1$, Eq. (47) becomes

$$\psi_1^{IV} - M^2\psi_1'' = ikR \frac{M^2 \tanh M \sinh My}{\cosh M - 1} \quad . \tag{48}$$

The general solution of the above equation is:

$$\psi_1 = A_1 \cosh My + B_1 \sinh My + C_1 y + D_1 + \frac{ikR}{2M} \frac{\tanh M}{\cosh M - 1} y \cosh My \quad . \tag{49}$$

Now, from Eq. (28), we obtain:

$$\varphi_1'' + R_M\psi_1' = O(R_M^2) \quad . \tag{50}$$

Thus in particular,

$$\varphi_1''(o) = -R_M\left(B_1 M + C_1 + \frac{ikR}{2M} \frac{\tanh M}{\cosh M - 1}\right) \quad . \tag{51}$$

The boundary condition (iv) becomes

$$\psi_1'''(o) + \frac{M^2}{R_M}\varphi_1''(o) = ik\Big\{R[(V_o - a_o)\psi_o'(o) - V'(o)\psi_o(o)]$$

$$-ik\zeta_o\left[\Gamma k^2 R + \frac{M\sinh M}{\cosh M - 1}\cot\theta\right] + O(R_M)\Big\} \quad . \quad (52)$$

Then Eqs. (49) and (51) together with the first order solutions lead to

$$C_1 = ik\left\{R\frac{\sinh M}{M(\cosh M - 1)} - \frac{\cosh M}{M^2}\left[M\sinh M\cot\theta + \Gamma k^2 R(\cosh M - 1)\right]\right\}. \quad (53)$$

Boundary conditions (i) and (ii) yield

$$A_1\cosh M - B_1\sinh M - C_1 + D_1 = \frac{ikR}{2M}\frac{\sinh M}{\cosh M - 1} \quad , \quad (54)$$

and

$$-A_1 M\sinh M + B_1 M\cosh M + C_1 = -\frac{ikR}{2M}\frac{\tanh M(\cosh M + M\sinh M)}{\cosh M - 1} \quad , (55)$$

while boundary conditions (iii) and (v) are

$$\psi_1''(o) + \frac{M^2}{\cosh M - 1} ik\zeta_1 = 0 \quad , \quad (56)$$

and

$$\psi_1(o) - ik\zeta_1(V_o - a_o) + ik\zeta_o a_1 = 0 \quad , \quad (57)$$

and they combine to give

$$a_1 = \frac{A_1 + D_1\cosh M}{\cosh^2 M(\cosh M - 1)} \quad . \quad (58)$$

Multiply Eq. (54) by $\cosh M$, Eq. (55) by $\frac{\sinh M}{M}$, and then add to obtain

$$A_1 + D_1\cosh M = C_1\left(\cosh M - \frac{\sinh M}{M}\right) + \frac{ikR\sinh M}{2M(\cosh M - 1)}\left[\cosh M - \frac{\sinh M}{M} - \frac{\sinh^2 M}{\cosh M}\right] \quad (59)$$

Using Eq. (53), we then arrive at:

$$a_1 = \frac{ik}{\cosh^2 M(\cosh M - 1)} \left\{ R \frac{\tanh M}{2M(\cosh M - 1)} \left(3\cosh^2 M - \frac{3\cosh M \sinh M}{M} - \sinh^2 M \right) - \frac{\cosh M}{M^2} \left(\cosh M - \frac{\sinh M}{M} \right) [M \sinh M \cot \theta + (\cosh M - 1)\Gamma k^2 R] \right\} . \tag{60}$$

6. Stability Criterions

The flow system is stable or unstable according as the imaginary part of $a = a_o + a_1$ is positive or negative. For this problem, a_o is real, while a_1 is imaginary; therefore, the stability criterion is determined by the sign of a_1.

It may be seen from the expression of a_1, that coefficients associated with Γ and $\cot\theta$ are all negative; this means that surface tension always tends to stabilize the flow while gravity will tend to stabilize or distabilize the system according to whether the fluid is flowing down the upperside or the underside of the plane.

If we neglect the effect of surface tension, we may conclude that the flow is stable if

$$R < F(M) \cot\theta \quad , \tag{61}$$

where

$$F(M) = \frac{2\cosh^2 M \left(\cosh M - \frac{\sinh M}{M} \right) (\cosh M - 1)}{3\cosh^2 M - \frac{3\cosh M \sinh M}{M} - \sinh^2 M} \quad . \tag{62}$$

$F(M)$ is a monotonically increasing function and as a function of M is shown in Fig. 2. Thus we may conclude that the magnetic field tends to stabilize the flow system. As $M \to 0$, it may be readily verified that $F(M) \to \frac{5}{4}$, thus the results agree with those of Yih[5] and Benjamin[6]. For large M,

$$F(M) \approx \frac{M-1}{2(2M-3)} \; e^{2M} \quad . \tag{63}$$

The above result will not hold for very large M, since in that case, the stability will be most likely controlled by the shear wave disturbance rather than these soft waves[5], while Lock[3] has shown that the critical Reynolds number for the former type of disturbance in his problem is only linearly proportional to M for large M.

7. Simple Description of the Stability Problem

The principle of local balance[10] has been extended to the stability problem of laminar flow down an inclined plane[9]. That approach offers a much simpler analysis and clearer physical picture while it retains satisfying accuracy. We shall also apply that principle to this problem.

Once the primary flow is obtained, the speed of propagation of disturbances of long wavelength, a_o, may be immediately written down from the total discharge as expressed in Eq. (16)[11]. Thus

$$a_o = \frac{dQ}{dh} = V_o \; \frac{\cosh M+1}{\cosh M} \quad . \tag{64}$$

Let us now transform the coordinate system from the original (x, y) to a new coordinate system (x', y) by the Galilean transformation:

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(10) M.S. Plesset and D.Y. Hsieh, Physics of Fluids, 7, 1099 (1964).

(11) M.J. Lighthill and G.B. Whitham, Proc. Roy. Soc. A, 229, 281 (1955).

$$x' = x - a_o t \quad , \qquad y' = y \quad , \tag{65}$$

which brings the wave disturbance of the free surface to rest. In this new frame of reference, the disturbance wave is given by

$$y = \zeta e^{ikx'} \quad , \tag{66}$$

or, taking the imaginary part:

$$y = \zeta \sin kx' \quad . \tag{66'}$$

Also in the new frame, the fluid in the layer is moving with velocity:

$$U(y) = V(y) - a_o = V_o \frac{1 - \cosh M \cosh(My/h)}{\cosh M(\cosh M - 1)} \quad . \tag{67}$$

We now compute the pressure exerted on the "wavy wall" (Eq. (66')) by a layer of incompressible inviscid fluid bounded by a rigid wall at $y = -h$, with primary velocity given by Eq. (67). Let the fluid velocity in the presence of a wavy wall be given by $(U+u, v)$, then the momentum and continuity equation for steady flow are

$$(U + u) \frac{\partial u}{\partial x} + v \frac{\partial}{\partial y} (U + u) = - \frac{1}{\rho} \frac{\partial p}{\partial x} \quad , \tag{68}$$

$$(U + u) \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial y} \quad , \tag{69}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad . \tag{70}$$

We eliminate p and u and neglect quadratic terms in u and v, and get

$$\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} - \frac{U''}{U} v = 0 \quad . \tag{71}$$

Now let

$$v(x,y) = f(y) \cos kx \quad . \tag{72}$$

Then Eq. (71) becomes

$$f'' - \left(\frac{U''}{U} + k^2 \right) f = 0 \quad . \tag{73}$$

For long wavelength disturbance, we have $kh \ll 1$, thus we may approximate Eq. (73) by

$$f'' - \frac{U''}{U} f = 0 \quad . \tag{74}$$

One solution of the last equation is U, then by reduction of order, we obtain the general solution of Eq. (74):

$$f = A\,U(y) + B\,U(y) \int_o^y \frac{dy'}{U^2(y')} \quad . \tag{75}$$

The boundary conditions at the wavy surface and the bottom require that:

$$f = 0 \qquad \text{at} \qquad y = -h \quad , \tag{76}$$

and

$$f = U_o k\zeta \qquad \text{at} \qquad y = 0 \quad , \tag{77}$$

where

$$U_o = U(o) \quad .$$

Thus

$$f = k\zeta \left[U(y) + \frac{U(y)}{G} \int_o^y \frac{dy'}{U^2(y')} \right] \quad . \tag{78}$$

where

$$G = \int_o^h \frac{dy}{U^2(y)} \quad . \tag{79}$$

For this case, as $U'(o) = 0$, the pressure applied to the wavy wall is readily seen from Eqs. (68) and (70) to be:

$$p(o) = \frac{\rho U_o f'(o)}{k} \sin kx \quad . \tag{80}$$

As

$$f'(y) = k\zeta \left[U'(y) + \frac{U'(y)}{G} \int_o^y \frac{dy'}{U^2(y')} + \frac{1}{GU(y)} \right] ,$$

we obtain

$$p(o) = \frac{\rho\zeta}{G} \sin kx \quad . \tag{81}$$

This pressure is counterbalanced by the gravitational restoring force $\rho g \cos \theta \zeta \sin kx$ in the stability problem, (cf. Fig.1), hence the critical condition for stability may be expressed by

$$g \cos \theta = \frac{1}{G} \quad . \tag{82}$$

Now

$$G = \int_o^h \frac{dy}{U^2(y)} = \frac{\cosh^2 M (\cosh M - 1)^2}{V_o^2} \int_o^h \frac{dy}{\left(\cosh M \cosh M \frac{y}{h} - 1\right)^2}$$

$$= \frac{\cosh^2 M (\cosh M - 1)^2 h}{V_o^2 M} \int_o^M \frac{dz}{(\cosh M \cosh z - 1)^2} \quad .$$

The integral is readily integrated and we obtain

$$\frac{1}{G} = \frac{V_o^2 M}{\cosh^2 M (\cosh M - 1)^2 h} \frac{\sinh^3 M}{\cosh M + \pi/2} \quad . \tag{83}$$

As in the derivation of Eq. (32) we can now express

$$g \cos \theta = \frac{V_o \eta \cot \theta}{\rho h^2} \frac{M \sinh M}{\cosh M - 1} \quad . \tag{84}$$

Then Eq. (82) may be expressed as

$$\frac{V_o \eta \cot \theta}{\rho h^2} \frac{M \sinh M}{\cosh M - 1} = \frac{V_o^2 M \sinh^3 M}{h \cosh^2 M (\cosh M - 1)^2 (\cosh M + \pi/2)} \quad ,$$

or

$$F'(M) \cot \theta = R \quad , \tag{85}$$

where

$$F'(M) = \frac{\cosh^2 M (\cosh M - 1)(\cosh M + \pi/2)}{\sinh^2 M} \quad . \tag{86}$$

The comparison between F(M) and F'(M) is shown in Fig. 2. The general agreement between F and F' over all values of M shows that the above simple description reveals the essential physical mechanism of the instability.

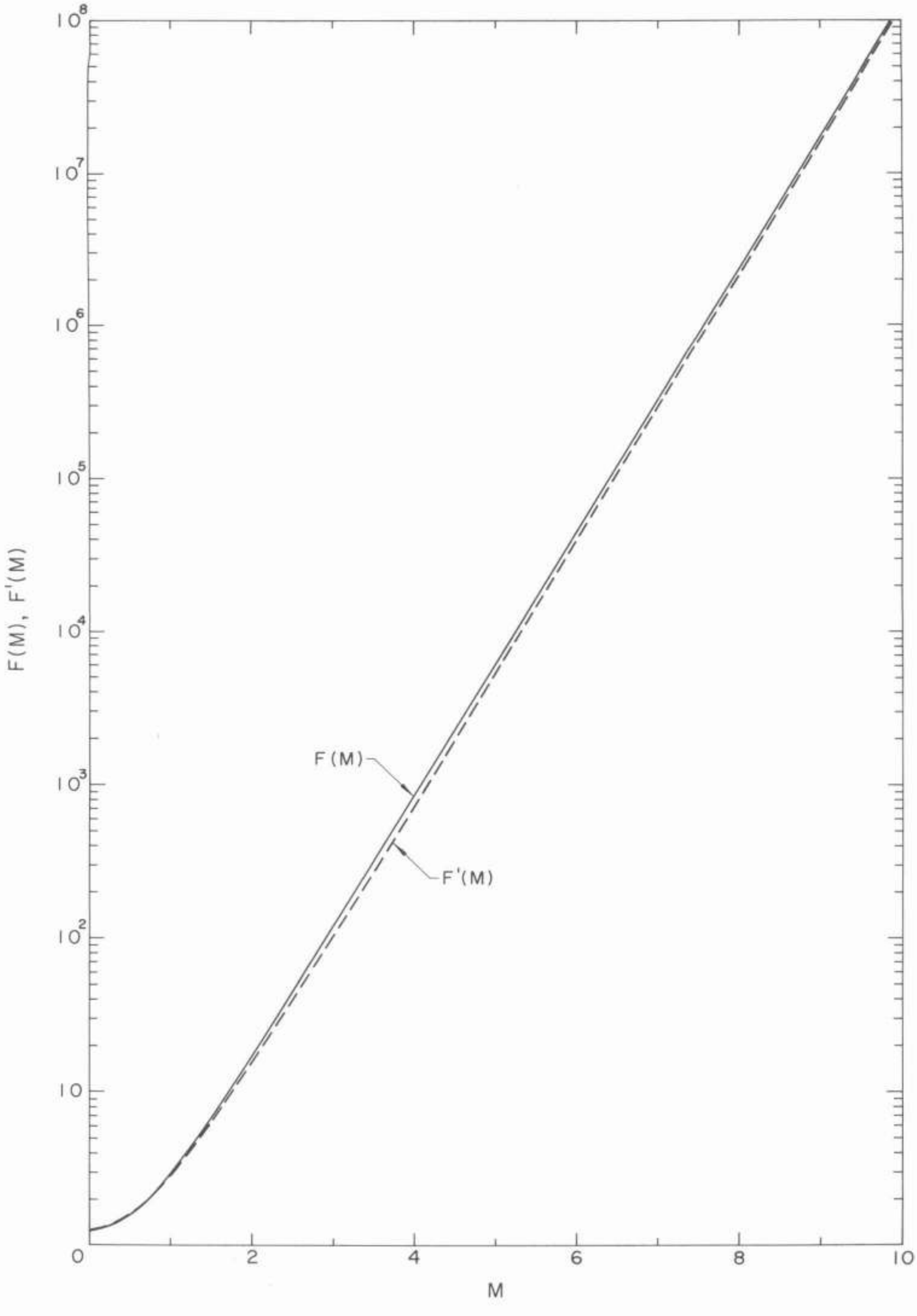

Figure 2. Comparison of the critical values of $R/\cot\theta$, as a function of the Hartmann number M, where R is the Reynolds number, and θ is the angle of inclination of the plane; F is the value for this ratio obtained from the complete theory, and F' is that obtained from the approximate theory of local balance.

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