

## Research Article

# Stability of a Functional Equation Deriving from Cubic and Quartic Functions

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We obtain the general solution and the generalized Ulam-Hyers stability of the cubic and quartic functional equation  $4(f(3x+y) + f(3x-y)) = -12(f(x+y) + f(x-y)) + 12(f(2x+y) + f(2x-y)) - 8f(y) - 192f(x) + f(2y) + 30f(2x)$ .

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## 1. Introduction

The stability problem of functionalequations originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms. Let  $(G_1, \cdot)$  be a group and let  $(G_2, *)$  be a metric group with the metric  $d(\cdot, \cdot)$ . Given  $\epsilon > 0$ , does there exist a  $\delta > 0$ , such that if a mapping  $h : G_1 \rightarrow G_2$  satisfies the inequality  $d(h(x \cdot y), h(x) * h(y)) < \delta$  for all  $x, y \in G_1$ , then there exists a homomorphism  $H : G_1 \rightarrow G_2$  with  $d(h(x), H(x)) < \epsilon$  for all  $x \in G_1$ ? In the other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [2] gave the first affirmative answer to the question of Ulam for Banach spaces. Let  $f : E \rightarrow E'$  be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta \quad (1.1)$$

for all  $x, y \in E$ , and for some  $\delta > 0$ . Then there exists a unique additive mapping  $T : E \rightarrow E'$  such that

$$\|f(x) - T(x)\| \leq \delta \quad (1.2)$$

for all  $x \in E$ . Moreover, if  $f(tx)$  is continuous in  $t \in \mathbb{R}$  for each fixed  $x \in E$ , then  $T$  is linear. Finally, in 1978, Th. M. Rassias [3] proved the following theorem.

**Theorem 1.1.** *Let  $f : E \rightarrow E'$  be a mapping from a normed vector space  $E$  into a Banach space  $E'$  subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \quad (1.3)$$

for all  $x, y \in E$ , where  $\epsilon$  and  $p$  are constants with  $\epsilon > 0$  and  $p < 1$ . Then there exists a unique additive mapping  $T : E \rightarrow E'$  such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2-2^p} \|x\|^p \quad (1.4)$$

for all  $x \in E$ . If  $p < 0$ , then inequality (1.3) holds for all  $x, y \neq 0$ , and (1.4) for  $x \neq 0$ . Also, if the function  $t \mapsto f(tx)$  from  $\mathbb{R}$  into  $E'$  is continuous in real  $t$  for each fixed  $x \in E$ , then  $T$  is linear.

In 1991, Gajda [4] answered the question for the case  $p > 1$ , which was raised by Rassias. This new concept is known as Hyers-Ulam-Rassias stability of functional equations (see [2, 4–13]). On the other hand, J. M. Rassias [14–16] generalized the Hyers stability result by presenting a weaker condition controlled by a product of different powers of norms. According to J. M. Rassias theorem.

**Theorem 1.2.** *If it is assumed that there exist constants  $\Theta \geq 0$  and  $p_1, p_2 \in \mathbb{R}$  such that  $p = p_1 + p_2 \neq 1$ , and  $f : E \rightarrow E'$  is a map from a norm space  $E$  into a Banach space  $E'$  such that the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon \|x\|^{p_1} \|y\|^{p_2} \quad (1.3p)$$

for all  $x, y \in E$ , then there exists a unique additive mapping  $T : E \rightarrow E'$  such that

$$\|f(x) - T(x)\| \leq \frac{\Theta}{2-2^p} \|x\|^p, \quad (1.5)$$

for all  $x \in E$ . If in addition for every  $x \in E$ ,  $f(tx)$  is continuous in real  $t$  for each fixed  $x$ , then  $T$  is linear (see [14, 15, 17–22]).

The oldest cubic functional equation, and was introduced by J. M. Rassias [23, 24] is as follows:

$$f(x+2y) + 3f(x) = 3f(x+y) + f(x-y) + 6f(y). \quad (1.6)$$

Jun and Kim [25] introduced the following cubic functional equation:

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \quad (1.7)$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.7). The function  $f(x) = x^3$  satisfies the functional equation (1.7), which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. Jun and Kim proved that a function  $f$  between real vector spaces  $X$  and  $Y$  is a solution of (1.7) if and only if there exists a unique function  $C : X \times X \times X \rightarrow Y$  such that  $f(x) = C(x, x, x)$  for all  $x \in X$ , and  $C$  is symmetric for each fixed one variable and is additive for fixed two variables. The oldest quartic functional equation, and was introduced by J. M. Rassias [16, 26], and then was employed by Park and Bae [27], such that

$$f(x + 2y) + f(x - 2y) = 4(f(x + y) + f(x - y)) + 24f(y) - 6f(x). \quad (1.8)$$

In fact, they proved that a function  $f$  between real vector spaces  $X$  and  $Y$  is a solution of (1.8) if and only if there exists a unique symmetric multiadditive function  $Q : X \times X \times X \times X \rightarrow Y$  such that  $f(x) = Q(x, x, x, x)$  for all  $x$  (see also [27–33]). It is easy to show that the function  $f(x) = x^4$  satisfies the functional equation (1.8), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function.

We deal with the following functional equation deriving from quartic and cubic functions:

$$\begin{aligned} 4(f(3x + y) + f(3x - y)) &= -12(f(x + y) + f(x - y)) + 12(f(2x + y) + f(2x - y)) \\ &\quad - 8f(y) - 192f(x) + f(2y) + 30f(2x). \end{aligned} \quad (1.9)$$

It is easy to see that the function  $f(x) = ax^4 + bx^3$  is a solution of the functional equation (1.9). In the present paper, we investigate the general solution and the generalized Hyers-Ulam-Rassias stability of the functional equation (1.9).

## 2. General solution

In this section, we establish the general solution of functional equation (1.9).

**Theorem 2.1.** *Let  $X, Y$  be vector spaces, and let  $f : X \rightarrow Y$  be a function. Then  $f$  satisfies (1.9) if and only if there exists a unique symmetric multiadditive function  $Q : X \times X \times X \times X \rightarrow Y$  and a unique function  $C : X \times X \times X \rightarrow Y$  such that  $C$  is symmetric for each fixed one variable and is additive for fixed two variables, and that  $f(x) = Q(x, x, x, x) + C(x, x, x)$  for all  $x \in X$ .*

*Proof.* Suppose there exists a symmetric multiadditive function  $Q : X \times X \times X \times X \rightarrow Y$  and a function  $C : X \times X \times X \rightarrow Y$  such that  $C$  is symmetric for each fixed one variable and is additive for fixed two variables, and that  $f(x) = Q(x, x, x, x) + C(x, x, x)$  for all  $x \in X$ . Then it is easy to see that  $f$  satisfies (1.9). For the convlet  $f$  satisfy (1.9). We decompose  $f$  into the even part and odd part by setting

$$f_e(x) = \frac{1}{2}(f(x) + f(-x)), \quad f_o(x) = \frac{1}{2}(f(x) - f(-x)) \quad (2.1)$$

for all  $x \in X$ . By (1.9), we have

$$\begin{aligned}
& 4f_e(3x + y) + 4f_e(3x - y) \\
&= \frac{1}{2} [4f(3x + y) + 4f(-3x - y) + 4f(3x - y) + 4f(-3x + y)] \\
&= \frac{1}{2} [4f(3x + y) + 4f(3x - y)] + \frac{1}{2} [4f((-3x) + (-y)) + 4f((-3x) - (-y))] \\
&= \frac{1}{2} [12f(2x + y) + 12f(2x - y) - 12f(x + y) - 12f(x - y) \\
&\quad - 8f(y) - 192f(x) + f(2y) + 30f(2x)] \\
&\quad + \frac{1}{2} [12f(-2x - y) + 12f((-2x) + y) - 12f(-x - y) - 12f(-x + y) \\
&\quad - 8f(-y) - 192f(-x) + f(-2y) + 30f(-2x)] \tag{2.2} \\
&= 12 \left[ \frac{1}{2} (f(2x + y) + f(-(2x + y))) \right] + 12 \left[ \frac{1}{2} (f(2x - y) + f(-(2x - y))) \right] \\
&\quad - 12 \left[ \frac{1}{2} (f(x + y) + f(-(x + y))) \right] - 12 \left[ \frac{1}{2} (f(x - y) + f(-(x - y))) \right] \\
&\quad - 8 \left[ \frac{1}{2} (f(y) + f(-y)) \right] - 192 \left[ \frac{1}{2} (f(x) + f(-x)) \right] \\
&\quad + \frac{1}{2} [f(2y) + f(-2y)] + 30 \left[ \frac{1}{2} (f(2x) + f(-2x)) \right] \\
&= 12(f_e(2x + y) + f_e(2x - y)) - 12(f_e(x + y) + f_e(x - y)) \\
&\quad - 8f_e(y) - 192f_e(x) + f_e(2y) + 30f_e(2x)
\end{aligned}$$

for all  $x, y \in X$ . This means that  $f_e$  satisfies (1.9), or

$$\begin{aligned}
4(f_e(3x + y) + f_e(3x - y)) &= -12(f_e(x + y) + f_e(x - y)) + 12(f_e(2x + y) + f_e(2x - y)) \\
&\quad - 8f_e(y) - 192f_e(x) + f_e(2y) + 30f_e(2x). \tag{1.9e}
\end{aligned}$$

Now, putting  $x = y = 0$  in (1.9e), we get  $f_e(0) = 0$ . Setting  $x = 0$  in (1.9e), by evenness of  $f_e$  we obtain

$$f_e(2y) = 16f_e(y) \tag{2.3}$$

for all  $y \in X$ . Hence, (1.9e) can be written as

$$\begin{aligned}
& f_e(3x + y) + f_e(3x - y) + 3(f_e(x + y) + f_e(x - y)) \\
&= 3(f_e(2x + y) + f_e(2x - y)) + 72f_e(x) + 2f_e(y) \tag{2.4}
\end{aligned}$$

for all  $x, y \in X$ . With the substitution  $y := 2y$  in (2.4), we have

$$\begin{aligned} f_e(3x+2y) + f_e(3x-2y) + 3f_e(x+2y) + 3f_e(x-2y) \\ = 48f_e(x+y) + 48f_e(x-y) + 72f_e(x) + 32f_e(y). \end{aligned} \quad (2.5)$$

Replacing  $y$  by  $x+2y$  in (2.4), we obtain

$$\begin{aligned} 16f_e(2x+y) + 16f_e(x-y) + 48f_e(x+y) + 48f_e(y) \\ = 3f_e(3x+2y) + 3f_e(x-2y) + 2f_e(x+2y) + 72f_e(x). \end{aligned} \quad (2.6)$$

Substituting  $-y$  for  $y$  in (2.6) gives

$$\begin{aligned} 16f_e(2x-y) + 16f_e(x+y) + 48f_e(x-y) + 48f_e(y) \\ = 3f_e(3x-2y) + 3f_e(x+2y) + 2f_e(x-2y) + 72f_e(x). \end{aligned} \quad (2.7)$$

By utilizing (2.5), (2.6), and (2.7), we obtain

$$4f_e(2x+y) + 4f_e(2x-y) + f_e(x+2y) + f_e(x-2y) = 20f_e(x+y) + 20f_e(x-y) + 90f_e(x). \quad (2.8)$$

Interchanging  $x$  and  $y$  in (2.5), we get

$$\begin{aligned} f_e(2x+3y) + f_e(2x-3y) + 3f_e(2x+y) + 3f_e(2x-y) \\ = 48f_e(x+y) + 48f_e(x-y) + 32f_e(x) + 72f_e(y). \end{aligned} \quad (2.9)$$

If we add (2.5) to (2.9), we have

$$\begin{aligned} f_e(2x+3y) + f_e(3x+2y) + f_e(2x-3y) + f_e(3x-2y) + 3f_e(2x+y) \\ + 3f_e(x+2y) + 3f_e(2x-y) + 3f_e(x-2y) \\ = 96f_e(x+y) + 96f_e(x-y) + 104f_e(x) + 104f_e(y). \end{aligned} \quad (2.10)$$

And by utilizing (2.6), (2.7), and (2.10), we arrive at

$$\begin{aligned} 3f_e(2x+3y) + 3f_e(2x-3y) \\ = -25f_e(2x+y) - 25f_e(2x-y) - 4f_e(x-2y) - 4f_e(x+2y) \\ + 224f_e(x+y) + 224f_e(x-y) + 456f_e(x) + 216f_e(y). \end{aligned} \quad (2.11)$$

Let us interchange  $x$  and  $y$  in (2.11). Then we see that

$$\begin{aligned} & 3f_e(3x+2y) + 3f_e(3x-2y) \\ &= -25f_e(x+2y) - 25f_e(x-2y) - 4f_e(2x-y) - 4f_e(2x+y) \\ &+ 224f_e(x+y) + 224f_e(x-y) + 456f_e(y) + 216f_e(x). \end{aligned} \quad (2.12)$$

Comparing (2.12) with (2.5), we get

$$\begin{aligned} 4f_e(2x-y) + 4f_e(2x+y) &= -16f_e(x+2y) - 16f_e(x-2y) + 80f_e(x+y) \\ &+ 80f_e(x-y) + 360f_e(y). \end{aligned} \quad (2.13)$$

If we compare (2.13) and (2.8), we conclude that

$$f_e(x+2y) + f_e(x-2y) + 6f_e(x) = 4f_e(x+y) + 4f_e(x-y) + 24f_e(y). \quad (2.14)$$

This means that  $f_e$  is quartic function. Thus, there exists a unique symmetric multiadditive function  $Q : X \times X \times X \times X \rightarrow Y$  such that  $f_e(x) = Q(x, x, x, x)$  for all  $x \in X$ . On the other hand, we can show that  $f_o$  satisfies (1.9), or

$$\begin{aligned} 4(f_o(3x+y) + f_o(3x-y)) &= -12(f_o(x+y) + f_o(x-y)) + 12(f_o(2x+y) + f_o(2x-y)) \\ &- 8f_o(y) - 192f_o(x) + f_o(2y) + 30f_o(2x). \end{aligned} \quad (1.9o)$$

Now setting  $x = y = 0$  in (1.9o) gives  $f_o(0) = 0$ . Putting  $x = 0$  in (1.9o), then by oddness of  $f_o$ , we have

$$f_o(2y) = 8f_o(y). \quad (2.15)$$

Hence, (1.9o) can be written as

$$f_o(3x+y) + f_o(3x-y) + 3f_o(x+y) + 3f_o(x-y) = 3f_o(2x+y) + 3f_o(2x-y) + 12f_o(x) \quad (2.16)$$

for all  $x, y \in X$ . Replacing  $x$  by  $x+y$ , and  $y$  by  $x-y$  in (2.16) we have

$$8f_o(2x+y) + 8f_o(x+2y) + 24f_o(x) + 24f_o(y) = 3f_o(3x+y) + 3f_o(x+3y) + 12f_o(x+y) \quad (2.17)$$

and interchanging  $x$  and  $y$  in (2.16) yields

$$f_o(x+3y) - f_o(x-3y) + 3f_o(x+y) - 3f_o(x-y) = 3f_o(x+2y) - 3f_o(x-2y) + 12f_o(y). \quad (2.18)$$

Which on substitution of  $-y$  for  $y$  in (2.16) gives

$$f_o(3x - y) + f_o(3x + y) + 3f_o(x - y) + 3f_o(x + y) = 3f_o(2x - y) + 3f_o(2x + y) + 12f_o(x). \quad (2.19)$$

Replace  $y$  by  $x + 2y$  in (2.16). Then we have

$$8f_o(2x + y) + 8f_o(x - y) + 24f_o(x + y) - 24f_o(y) = 3f_o(3x + 2y) + 3f_o(x - 2y) + 12f_o(x). \quad (2.20)$$

From the substitution  $y := -y$  in (2.20) it follows that

$$8f_o(2x - y) + 8f_o(x + y) + 24f_o(x - y) + 24f_o(y) = 3f_o(3x - 2y) + 3f_o(x + 2y) + 12f_o(x). \quad (2.21)$$

If we add (2.20) to (2.21), we have

$$\begin{aligned} 3f_o(3x - 2y) + 3f_o(3x + 2y) &= 8f_o(2x + y) + 8f_o(2x - y) - 3f_o(x + 2y) - 3f_o(x - 2y) \\ &\quad + 32f_o(x - y) + 32f_o(x + y) - 24f_o(x). \end{aligned} \quad (2.22)$$

Let us interchange  $x$  and  $y$  in (2.22). Then we see that

$$\begin{aligned} 3f_o(2x + 3y) - 3f_o(2x - 3y) &= 8f_o(x + 2y) - 8f_o(x - 2y) - 3f_o(2x + y) + 3f_o(2x - y) \\ &\quad + 32f_o(x + y) - 32f_o(x - y) - 24f_o(y). \end{aligned} \quad (2.23)$$

With the substitution  $y := x + y$  in (2.16), we have

$$f_o(4x + y) + f_o(2x - y) + 3f_o(2x + y) - 3f_o(y) = 3f_o(3x + y) + 3f_o(x - y) + 12f_o(x), \quad (2.24)$$

and replacing  $-y$  by  $y$  gives

$$f_o(4x - y) + f_o(2x + y) + 3f_o(2x - y) + 3f_o(y) = 3f_o(3x - y) + 3f_o(x + y) + 12f_o(x). \quad (2.25)$$

If we add (2.24) to (2.25), we have

$$\begin{aligned} f_o(4x + y) + f_o(4x - y) &= 3f_o(3x + y) + 3f_o(3x - y) - 4f_o(2x - y) - 4f_o(2x + y) \\ &\quad + 3f_o(x - y) + 3f_o(x + y) + 24f_o(x). \end{aligned} \quad (2.26)$$

By comparing (2.19) with (2.26), we arrive at

$$f_o(4x + y) + f_o(4x - y) = 5f_o(2x + y) + 5f_o(2x - y) - 6f_o(x + y) - 6f_o(x - y) + 60f_o(x) \quad (2.27)$$

and replacing  $y$  by  $2y$  in (2.16) gives

$$f_o(3x + 2y) + f_o(3x - 2y) = 24f_o(x + y) + 24f_o(x - y) - 3f_o(x + 2y) - 3f_o(x - 2y) + 12f_o(x). \quad (2.28)$$

By comparing (2.28) with (2.22), we arrive at

$$3f_o(x + 2y) + 3f_o(x - 2y) = 20f_o(x + y) + 20f_o(x - y) - 4f_o(2x + y) - 4f_o(2x - y) + 30f_o(x). \quad (2.29)$$

Let us interchange  $x$  and  $y$  in (2.28). Then we see that

$$f_o(2x + 3y) - f_o(2x - 3y) = 24f_o(x + y) - 24f_o(x - y) - 3f_o(2x + y) + 3f_o(2x - y) + 12f_o(y). \quad (2.30)$$

Thus combining (2.30) with (2.23) yields

$$4f_o(x + 2y) - 4f_o(x - 2y) = 3f_o(2x - y) - 3f_o(2x + y) + 20f_o(x + y) - 20f_o(x - y) + 30f_o(y). \quad (2.31)$$

By comparing (2.31) with (2.18), we arrive at

$$4f_o(x + 3y) - 4f_o(x - 3y) = 9f_o(2x - y) - 9f_o(2x + y) + 48f_o(x + y) - 48f_o(x - y) + 138f_o(y). \quad (2.32)$$

Which, by putting  $y := 2y$  in (2.17), leads to

$$64f_o(x + y) + 8f_o(x + 4y) + 24f_o(x) + 192f_o(y) = 3f_o(3x + 2y) + 3f_o(x + 6y) + 12f_o(x + 2y). \quad (2.33)$$

Replacing  $y$  by  $-y$  in (2.33) gives

$$64f_o(x - y) + 8f_o(x - 4y) + 24f_o(x) - 192f_o(y) = 3f_o(3x - 2y) + 3f_o(x - 6y) + 12f_o(x - 2y). \quad (2.34)$$

If we subtract (2.33) from (2.34), we obtain

$$\begin{aligned} 8f_o(x + 4y) - 8f_o(x - 4y) &= 3f_o(3x + 2y) - 3f_o(3x - 2y) + 3f_o(x + 6y) \\ &\quad - 3f_o(x - 6y) + 12f_o(x + 2y) - 12f_o(x - 2y) \\ &\quad + 64f_o(x - y) - 64f_o(x + y) - 384f_o(y). \end{aligned} \quad (2.35)$$



Setting  $x$  instead of  $y$  and  $y$  instead of  $x$  in (2.27), we get

$$f_o(x + 4y) - f_o(x - 4y) = 5f_o(x + 2y) - 5f_o(x - 2y) + 6f_o(x - y) - 6f_o(x + y) + 60f_o(y). \quad (2.36)$$

Combining (2.35) and (2.36) yields

$$\begin{aligned} 3f_o(3x + 2y) - 3f_o(3x - 2y) &= 28f_o(x + 2y) - 28f_o(x - 2y) + 3f_o(x - 6y) - 3f_o(x + 6y) \\ &\quad + 16f_o(x + y) - 16f_o(x - y) + 864f_o(y) \end{aligned} \quad (2.37)$$

and subtracting (2.21) from (2.20), we obtain

$$\begin{aligned} 3f_o(3x + 2y) - 3f_o(3x - 2y) &= 3f_o(x + 2y) - 3f_o(x - 2y) + 8f_o(2x + y) - 8f_o(2x - y) \\ &\quad + 16f_o(x + y) - 16f_o(x - y) - 48f_o(y). \end{aligned} \quad (2.38)$$

By comparing (2.37) with (2.38), we arrive at

$$\begin{aligned} 3f_o(x + 6y) - 3f_o(x - 6y) &= 25f_o(x + 2y) - 25f_o(x - 2y) + 8f_o(2x - y) \\ &\quad - 8f_o(2x + y) + 912f_o(y). \end{aligned} \quad (2.39)$$

Interchanging  $y$  with  $2y$  in (2.32) gives the equation

$$\begin{aligned} 4f_o(x + 6y) - 4f_o(x - 6y) &= 48f_o(x + 2y) - 48f_o(x - 2y) + 72f_o(x - y) \\ &\quad - 72f_o(x + y) + 1104f_o(y). \end{aligned} \quad (2.40)$$

We obtain from (2.39) and (2.40)

$$\begin{aligned} 44f_o(x + 2y) - 44f_o(x - 2y) &= 32f_o(2x - y) - 32f_o(2x + y) + 216f_o(x + y) \\ &\quad - 216f_o(x - y) + 336f_o(y). \end{aligned} \quad (2.41)$$

By using (2.31) and (2.41), we lead to

$$f_o(2x + y) - f_o(2x - y) = 4f_o(x + y) - 4f_o(x - y) - 6f_o(y). \quad (2.42)$$

And interchanging  $x$  with  $y$  in (2.42) gives

$$f_o(x + 2y) + f_o(x - 2y) = 4f_o(x + y) + 4f_o(x - y) - 6f_o(x). \quad (2.43)$$

If we compare (2.43) and (2.29), we conclude that

$$8f_o(x+y) + 8f_o(x-y) + 48f_o(x) = 4f_o(2x+y) + 4f_o(2x-y). \quad (2.44)$$

This means that  $f_o$  is cubic function and that there exists a unique function  $C : X \times X \times X \rightarrow Y$  such that  $f_o(x) = C(x, x, x)$  for all  $x \in X$ , and  $C$  is symmetric for each fixed one variable and is additive for fixed two variables. Thus for all  $x \in X$ , we have

$$f(x) = f_e(x) + f_o(x) = C(x, x, x) + Q(x, x, x, x). \quad (2.45)$$

This completes the proof of theorem.  $\square$

The following corollary is an alternative result of Theorem 2.1.

**Corollary 2.2.** *Let  $X, Y$  be vector spaces, and let  $f : X \rightarrow Y$  be a function satisfying (1.9). Then the following assertions hold.*

- (a) *If  $f$  is even function, then  $f$  is quartic.*
- (b) *If  $f$  is odd function, then  $f$  is cubic.*

### 3. Stability

We now investigate the generalized Hyers-Ulam-Rassias stability problem for functional equation (1.9). From now on, let  $X$  be a real vector space and let  $Y$  be a Banach space. Now before taking up the main subject, given  $f : X \rightarrow Y$ , we define the difference operator  $D_f : X \times X \rightarrow Y$  by

$$\begin{aligned} D_f(x, y) = & 4[f(3x+y) + f(3x-y)] - 12[f(2x+y) + f(2x-y)] + 12[f(x+y) + f(x-y)] \\ & - f(2y) + 8f(y) - 30f(2x) + 192f(x) \end{aligned} \quad (3.1)$$

for all  $x, y \in X$ . We consider the following functional inequality:

$$\|D_f(x, y)\| \leq \phi(x, y) \quad (3.2)$$

for an upper bound  $\phi : X \times X \rightarrow [0, \infty)$ .

**Theorem 3.1.** *Let  $s \in \{1, -1\}$  be fixed. Suppose that an even mapping  $f : X \rightarrow Y$  satisfies  $f(0) = 0$ , and*

$$\|D_f(x, y)\| \leq \phi(x, y) \quad (3.3)$$

*for all  $x, y \in X$ . If the upper bound  $\phi : X \times X \rightarrow [0, \infty)$  is a mapping such that the series  $\sum_{i=0}^{\infty} 2^{4si} \phi(0, x/2^{si})$  converges, and that  $\lim_{n \rightarrow \infty} 2^{4sn} \phi(x/2^{sn}, y/2^{sn}) = 0$  for all  $x, y \in X$ , then*

the limit  $Q(x) = \lim_{n \rightarrow \infty} 2^{4sn} f(x/2^{sn})$  exists for all  $x \in X$ , and  $Q : X \rightarrow Y$  is a unique quartic function satisfying (1.9), and

$$\|f(x) - Q(x)\| \leq \frac{1}{16} \sum_{i=(s-1)/2}^{\infty} 2^{4s(i+1)} \phi\left(0, \frac{x}{2^{s(i+1)}}\right) \quad (3.4)$$

for all  $x \in X$ .

*Proof.* Let  $s = 1$ . Putting  $x = 0$  in (3.3), we get

$$\|f(2y) - 16f(y)\| \leq \phi(0, y). \quad (3.5)$$

Replacing  $y$  by  $x/2$  in (3.5), yields

$$\left\|f(x) - 16f\left(\frac{x}{2}\right)\right\| \leq \phi\left(0, \frac{x}{2}\right). \quad (3.6)$$

Interchanging  $x$  with  $x/2$  in (3.6), and multiplying by 16 it follows that

$$\left\|16f\left(\frac{x}{2}\right) - 16^2f\left(\frac{x}{4}\right)\right\| \leq 16\phi\left(0, \frac{x}{4}\right). \quad (3.7)$$

Combining (3.6) and (3.7), we lead to

$$\left\|16^2f\left(\frac{x}{4}\right) - f(x)\right\| \leq \phi\left(0, \frac{x}{2}\right) + 16\phi\left(0, \frac{x}{4}\right). \quad (3.8)$$

From the inequality (3.6) we use iterative methods and induction on  $n$  to prove our next relation:

$$\left\|16^n f\left(\frac{x}{2^n}\right) - f(x)\right\| \leq \frac{1}{16} \sum_{i=0}^{n-1} 16^{i+1} \phi\left(0, \frac{x}{2^{i+1}}\right). \quad (3.9)$$

We multiply (3.9) by  $16^m$  and replace  $x$  by  $x/2^m$  to obtain that

$$\left\|16^{m+n} f\left(\frac{x}{2^{m+n}}\right) - 16^m f\left(\frac{x}{2^m}\right)\right\| \leq \sum_{i=0}^{n-1} 16^{m+i} \phi\left(0, \frac{x}{2^{i+m+1}}\right). \quad (3.10)$$

This shows that  $\{16^n f(x/2^n)\}$  is a Cauchy sequence in  $Y$  by taking the limit  $m \rightarrow \infty$ . Since  $Y$  is a Banach space, it follows that the sequence  $\{16^n f(x/2^n)\}$  converges. We define  $Q : X \rightarrow Y$  by  $Q(x) = \lim_{n \rightarrow \infty} 2^{4n} f(x/2^n)$  for all  $x \in X$ . It is clear that  $Q(-x) = Q(x)$  for all  $x \in X$ , and it follows from (3.3) that

$$\|D_Q(x, y)\| = \lim_{n \rightarrow \infty} 16^n \left\|D_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| \leq \lim_{n \rightarrow \infty} 16^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \quad (3.11)$$

for all  $x, y \in X$ . Hence, by Corollary 2.2,  $Q$  is quartic. It remains to show that  $Q$  is unique. Suppose that there exists another quartic function  $Q' : X \rightarrow Y$  which satisfies (1.9) and (3.4). Since  $Q(2^n x) = 16^n Q(x)$ , and  $Q'(2^n x) = 16^n Q'(x)$  for all  $x \in X$ , we conclude that

$$\begin{aligned} \|Q(x) - Q'(x)\| &= 16^n \left\| Q\left(\frac{x}{2^n}\right) - Q'\left(\frac{x}{2^n}\right) \right\| \\ &\leq 16^n \left\| Q\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| + 16^n \left\| Q'\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| \\ &\leq 2 \sum_{i=0}^{\infty} 16^{n+i} \phi\left(0, \frac{x}{2^{n+i+1}}\right) \end{aligned} \quad (3.12)$$

for all  $x \in X$ . By letting  $n \rightarrow \infty$  in this inequality, it follows that  $Q(x) = Q'(x)$  for all  $x \in X$ , which gives the conclusion. For  $s = -1$ , we obtain

$$\left\| \frac{f(2^m x)}{16^m} - f(x) \right\| \leq \frac{1}{16} \sum_{i=1}^{m-2} \frac{\phi(0, 2^{i+1} x)}{16^{i+1}}, \quad (3.13)$$

from which one can prove the result by a similar technique.  $\square$

**Theorem 3.2.** Let  $s \in \{1, -1\}$  be fixed. Suppose that an odd mapping  $f : X \rightarrow Y$  satisfies

$$\|D_f(x, y)\| \leq \phi(x, y) \quad (3.14)$$

for all  $x, y \in X$ . If the upper bound  $\phi : X \times X \rightarrow [0, \infty)$  is a mapping such that  $\sum_{i=0}^{\infty} 2^{3si} \phi(0, x/2^{si})$  converges, and that  $\lim_{n \rightarrow \infty} 2^{3si} \phi(x/2^{si}, y/2^{si}) = 0$  for all  $x, y \in X$ , then the limit  $C(x) = \lim_{n \rightarrow \infty} 2^{3sn} f(x/2^{sn})$  exists for all  $x \in X$ , and  $C : X \rightarrow Y$  is a unique cubic function satisfying (1.9), and

$$\|f(x) - C(x)\| \leq \frac{1}{8} \sum_{i=(s-1)/2}^{\infty} 2^{3s(i+1)} \phi\left(0, \frac{x}{2^{s(i+1)}}\right) \quad (3.15)$$

for all  $x \in X$ .

*Proof.* Let  $s = 1$ . Set  $x = 0$  in (3.14). We obtain

$$\|8f(y) - f(2y)\| \leq \phi(0, y). \quad (3.16)$$

Replacing  $y$  by  $x/2$  in (3.16) to get

$$\left\| 8f\left(\frac{x}{2}\right) - f(x) \right\| \leq \phi\left(0, \frac{x}{2}\right). \quad (3.17)$$

An induction argument now implies

$$\left\| 8^n f\left(\frac{x}{2^n}\right) - f(x) \right\| \leq \frac{1}{8} \sum_{i=0}^{n-1} 8^{i+1} \phi\left(0, \frac{x}{2^{i+1}}\right). \quad (3.18)$$

Multiply (3.18) by  $8^m$  and replace  $x$  by  $x/2^m$ , we obtain that

$$\left\| 8^{m+n} f\left(\frac{x}{2^{m+n}}\right) - 8^m f\left(\frac{x}{2^m}\right) \right\| \leq \sum_{i=0}^{n-1} 8^{m+i} \phi\left(0, \frac{x}{2^{m+i+1}}\right). \quad (3.19)$$

The right hand side of the inequality (3.19) tends to 0 as  $m \rightarrow \infty$  because of

$$\sum_{i=0}^{\infty} 8^i \phi\left(0, \frac{x}{2^{i+1}}\right) < \infty \quad (3.20)$$

by assumption, and thus the sequence  $\{2^{3n} f(x/2^n)\}$  is Cauchy in  $Y$ , as desired. Therefore we may define a mapping  $C : X \rightarrow Y$  as  $C(x) = \lim_{n \rightarrow \infty} 2^{3n} f(x/2^n)$ . The rest of proof is similar to the proof of Theorem 3.1.  $\square$

**Theorem 3.3.** *Let  $s \in \{1, -1\}$  be fixed. Suppose a mapping  $f : X \rightarrow Y$  satisfies  $f(0) = 0$ , and  $\|D_f(x, y)\| \leq \phi(x, y)$  for all  $x, y \in X$ . If the upper bound  $\phi : X \times X \rightarrow [0, \infty)$  is a mapping such that*

$$\begin{aligned} & \sum_{i=0}^{\infty} \left[ (|s| + s) 2^{4si} \phi\left(0, \frac{x}{2^{si-1}}\right) + (|s| - s) 2^{3si} \phi\left(0, \frac{x}{2^{si-1}}\right) \right] < \infty, \\ & \lim_{n \rightarrow \infty} \left[ (|s| + s) 2^{(4sn-1)} \phi\left(\frac{x}{2^{sn}}, \frac{y}{2^{sn}}\right) + (|s| - s) 2^{3sn} \phi\left(\frac{x}{2^{sn}}, \frac{y}{2^{sn}}\right) \right] = 0 \end{aligned} \quad (3.21)$$

for all  $x, y \in X$ . Then there exists a unique quartic function  $Q : X \rightarrow Y$  and a unique cubic function  $C : X \rightarrow Y$  satisfying

$$\|f(x) - Q(x) - C(x)\| \leq \sum_{i=(s-1)/2}^{\infty} \left\{ \left( \frac{2^{4s(i+1)}}{32} + \frac{2^{3s(i+1)}}{16} \right) \left[ \phi\left(0, \frac{x}{2^{s(i+1)}}\right) + \phi\left(0, \frac{-x}{2^{s(i+1)}}\right) \right] \right\} \quad (3.22)$$

for all  $x \in X$ .

*Proof.* Let  $f_e(x) = (1/2)(f(x) + f(-x))$  for all  $x \in X$ . Then  $f_e(0) = 0$  and  $f_e$  is even function satisfying  $\|D_{f_e}(x, y)\| \leq (1/2)[\phi(x, y) + \phi(-x, -y)]$  for all  $x, y \in X$ . From Theorem 3.1, it follows that there exists a unique quartic function  $Q : X \rightarrow Y$  satisfies

$$\|f_e(x) - Q(x)\| \leq \frac{1}{32} \sum_{i=(s-1)/2}^{\infty} \left\{ 2^{4s(i+1)} \phi\left(0, \frac{x}{2^{s(i+1)}}\right) + 2^{4s(i+1)} \phi\left(0, \frac{-x}{2^{s(i+1)}}\right) \right\} \quad (3.23)$$

for all  $x \in X$ . Let now  $f_o(x) = (1/2)(f(x) - f(-x))$  for all  $x \in X$ . Then  $f_o$  is odd function satisfying

$$\|D_{f_o}(x, y)\| \leq \frac{1}{2} [\phi(x, y) + \phi(-x, -y)] \quad (3.24)$$

for all  $x, y \in X$ . Hence, in view of Theorem 3.2, it follows that there exists a unique cubic function  $C : X \rightarrow Y$  such that

$$\|f_o(x) - C(x)\| \leq \frac{1}{16} \sum_{i=(s-1)/2}^{\infty} \left\{ 2^{3s(i+1)} \phi\left(0, \frac{x}{2^{s(i+1)}}\right) + 2^{3s(i+1)} \phi\left(0, \frac{-x}{2^{s(i+1)}}\right) \right\} \quad (3.25)$$

for all  $x \in X$ . On the other hand, we have  $f(x) = f_e(x) + f_o(x)$  for all  $x \in X$ . Then by combining (3.23) and (3.25), it follows that

$$\begin{aligned} \|f(x) - C(x) - Q(x)\| &\leq \|f_e(x) - Q(x)\| + \|f_o(x) - C(x)\| \\ &\leq \sum_{i=(s-1)/2}^{\infty} \left\{ \left( \frac{2^{4s(i+1)}}{32} + \frac{2^{3s(i+1)}}{16} \right) \left[ \phi\left(0, \frac{x}{2^{s(i+1)}}\right) + \phi\left(0, \frac{-x}{2^{s(i+1)}}\right) \right] \right\} \end{aligned} \quad (3.26)$$

for all  $x \in X$ , and the proof of theorem is complete.  $\square$

We are going to investigate the Hyers-Ulam-Rassias stability problem for functional equation (1.9).

**Corollary 3.4.** *Let  $p \in (-\infty, 3) \cup (4, +\infty)$ ,  $\theta > 0$ . Suppose  $f : X \rightarrow Y$  satisfies  $f(0) = 0$ , and inequality*

$$\|D_f(x, y)\| \leq \theta(\|x\|^p + \|y\|^p), \quad (3.27)$$

for all  $x, y \in X$ . Then there exists a unique quartic function  $Q : X \rightarrow Y$ , and a unique cubic function  $C : X \rightarrow Y$  satisfying

$$\|f(x) - Q(x) - C(x)\| \leq \begin{cases} \theta \|x\|^p \left( \frac{1}{2^p - 2^4} + \frac{1}{2^p - 2^3} \right), & p > 4, \\ \theta \|x\|^p \left( \frac{1}{2^4 - 2^p} + \frac{1}{2^3 - 2^p} \right), & p < 3 \end{cases} \quad (3.28)$$

for all  $x \in X$ .

*Proof.* Let  $s = 1$  in Theorem 3.3. Then by taking  $\phi(x, y) = \theta(\|x\|^p + \|y\|^p)$  for all  $x, y \in X$ , the relations (3.21) hold for  $p > 4$ . Then there exists a unique quartic function  $Q : X \rightarrow Y$  and a unique cubic function  $C : X \rightarrow Y$  satisfying

$$\|f(x) - Q(x) - C(x)\| \leq \theta \left\| \frac{x}{2} \right\|^p \left( \frac{1}{1 - 2^{4-p}} + \frac{1}{1 - 2^{3-p}} \right) \quad (3.29)$$

for all  $x \in X$ . Let now  $s = -1$  in Theorem 3.3 and put  $\phi(x, y) = \theta(\|x\|^p + \|y\|^p)$  for all  $x, y \in X$ . Then the relations (3.21) hold for  $p < 3$ . Then there exists a unique quartic function  $Q : X \rightarrow Y$  and a unique cubic function  $C : X \rightarrow Y$  satisfying

$$\|f(x) - Q(x) - C(x)\| \leq \theta \|x\|^p \left( \frac{1}{2^4 - 2^p} + \frac{1}{2^3 - 2^p} \right) \quad (3.30)$$

for all  $x \in X$ . □

Similarly, we can prove the following Ulam stability problem for functional equation (1.9) controlled by the mixed type product-sum function

$$(x, y) \mapsto \theta(\|x\|_X^u \|y\|_X^v + \|x\|^p + \|y\|^p) \quad (3.31)$$

introduced by J. M. Rassias (e.g., [34]).

**Corollary 3.5.** *Let  $u, v, p$  be real numbers such that  $u + v, p \in (-\infty, 3) \cup (4, +\infty)$ , and let  $\theta > 0$ . Suppose  $f : X \rightarrow Y$  satisfies  $f(0) = 0$ , and inequality*

$$\|D_f(x, y)\| \leq \theta(\|x\|_X^u \|y\|_X^v + \|x\|^p + \|y\|^p), \quad (3.32)$$

for all  $x, y \in X$ . Then there exists a unique quartic function  $Q : X \rightarrow Y$ , and a unique cubic function  $C : X \rightarrow Y$  satisfying

$$\|f(x) - Q(x) - C(x)\| \leq \begin{cases} \theta \|x\|^p \left( \frac{1}{2^p - 2^4} + \frac{1}{2^p - 2^3} \right), & p > 4, \\ \theta \|x\|^p \left( \frac{1}{2^4 - 2^p} + \frac{1}{2^3 - 2^p} \right), & p < 3 \end{cases} \quad (3.33)$$

for all  $x \in X$ .

By Corollary 3.4, we solve the following Hyers-Ulam stability problem for functional equation (1.9).

**Corollary 3.6.** *Let  $\epsilon$  be a positive real number. Suppose  $f : X \rightarrow Y$  satisfies  $f(0) = 0$ , and  $\|D_f(x, y)\| \leq \epsilon$ , for all  $x, y \in X$ . Then there exists a unique quartic function  $Q : X \rightarrow Y$ , and a unique cubic function  $C : X \rightarrow Y$  satisfying*

$$\|f(x) - Q(x) - C(x)\| \leq \frac{22}{105} \epsilon \quad (3.34)$$

for all  $x \in X$ .

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