Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **25:3** (2011), 43–54 DOI: 10.2298/FIL1103043G

STABILITY OF A MIXED TYPE ADDITIVE, QUADRATIC AND CUBIC FUNCTIONAL EQUATION IN RANDOM NORMED SPACES

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Abstract

In this paper, we obtain the general solution and the stability result for the following functional equation in random normed spaces (in the sense of Sherstnev) under arbitrary t-norms

$$f(x+3y)+f(x-3y)=9(f(x+y)+f(x-y))-16f(x).$$

1 Introduction

The stability problem of functional equations originated from a question of Ulam[31] in 1940, concerning the stability of group homomorphisms. Let $(G_1,.)$ be a group and let $(G_2,*,d)$ be a metric group with the metric $d(.,.)$. Given $\epsilon>0$, does there exist a $\delta>0$, such that if a mapping $h:G_1\to G_2$ satisfies the inequality $d(h(x.y),h(x)*h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\to G_2$ with $d(h(x),H(x))<\epsilon$ for all $x\in G_1$? In the other words, under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D. H. Hyers [16] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f:E\to E'$ be a mapping between Banach spaces such that

$$\|f(x+y)-f(x)-f(y)\|\le\delta$$

for all $x,y\in E$, and for some $\delta>0$. Then there exists a unique additive mapping $T:E\to E'$ such that

$$\|f(x)-T(x)\|\le\delta$$

for all $x\in E$. Moreover if $f(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in E$, then T is linear. In 1978, Th. M. Rassias [26] provided a generalization of Hyers'

2010 *Mathematics Subject Classifications.* 39B52,39B82,46S40,54E40.
Key words and Phrases. Additive functional equation; Quadratic functional equation; Cubic functional equation; Random normed space.
Received: December 16, 2009
Communicated by Dragan S. Djordjević

Theorem which allows the Cauchy difference to be unbounded. In 1991, Z. Gajda [10] answered the question for the case $p > 1$, which was rased by Rassias. This new concept is known as Hyers-Ulam-Rassias stability of functional equations (see [1, 2, 4],[11, 15, 17] and [24, 25]).
The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y). \tag{1.1}$$

is related to symmetric bi-additive function. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exits a unique symmetric bi-additive function B such that $f(x) = B(x,x)$ for all x (see [1, 19]). The bi-additive function B is given by

$$B(x,y)=\frac{1}{4}(f(x+y)-f(x-y)) \tag{1.2}$$

Hyers-Ulam-Rassias stability problem for the quadratic functional equation (1.1) was proved by Skof for functions $f : A \to B$, where A is normed space and B Banach space (see [30]). Cholewa[5] noticed that the Theorem of Skof is still true if relevant domain A is replaced an abelian group. In the paper [8], Czerwik proved the Hyers-Ulam-Rassias stability of the equation (1.1). Grabiec[12] has generalized these result mentioned above.
Jun and Kim [18] introduced the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.3}$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.3). The function $f(x) = x^3$ satisfies the functional equation (1.3), which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. Jun and Kim proved that a function f between real vector spaces X and Y is a solution of (1.3) if and only if there exits a unique function $C : X \times X \times X \longrightarrow Y$ such that $f(x) = C(x,x,x)$ for all $x \in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables.
In the sequel we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in [6, 28, 29]. Throughout this paper, Δ^+ is the space of distribution functions that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0,1]$, such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t)=\begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 1.1. *([28]). A mapping $T : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:*
(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a,1) = a$ for all $a \in [0,1]$;
(d) $T(a,b) \le T(c,d)$ whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous t-norms are $T_P(a,b) = ab$, $T_M(a,b) = \min(a,b)$ and $T_L(a,b) = \max(a+b-1,0)$ (the Lukasiewicz t-norm). Recall (see [13], [14]) that if T is a t-norm and $\{x_n\}$ is a given sequence of numbers in $[0,1]$, $T_{i=1}^{n}x_i$ is defined recurrently by $T_{i=1}^{1}x_i = x_1$ and $T_{i=1}^{n}x_i = T(T_{i=1}^{n-1}x_i, x_n)$ for $n \ge 2$. $T_{i=n}^{\infty}x_i$ is defined as $T_{i=1}^{\infty}x_{n+i}$. It is known([14]) that for the Lukasiewicz t-norm the following implication holds:

$$\lim_{n\to\infty} (T_L)_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty}(1-x_n) < \infty. \tag{1.4}$$

Definition 1.2. *([29]). A* random normed space *(briefly, RN-space) is a triple (X,μ,T), where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that, the following conditions hold:*
$(RN1)$ $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
$(RN2)$ $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $\alpha \neq 0$;
$(RN3)$ $\mu_{x+y}(t+s) \ge T(\mu_x(t), \mu_y(s))$ for all $x,y \in X$ and $t,s \ge 0$.

Every normed spaces $(X, \|.\|)$ defines a random noremed space (X,μ,T_M) where

$$\mu_x(t) = \frac{t}{t+\|x\|},$$

for all $t > 0$, and T_M is the minimum t-norm. This space is called the induced random normed space.

Definition 1.3. *Let (X,μ,T) be a RN-space.*
(1) A sequence $\{x_n\}$ in X is said to be convergent *to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $\mu_{x_n - x}(\epsilon) > 1-\lambda$ whenever $n \ge N$.*
(2) A sequence $\{x_n\}$ in X is called Cauchy sequence *if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $\mu_{x_n - x_m}(\epsilon) > 1-\lambda$ whenever $n \ge m \ge N$.*
(3) A RN-space (X,μ,T) is said to be complete *if and only if every Cauchy sequence in X is convergent to a point in X.*

Theorem 1.4. *([28]). If (X,μ,T) is a RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty}\mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

The generalized Hyers-Ulam-Rassias stability of different functional equations in random normed spaces has been recently studied in [3, 20, 21, 22, 23, 27]. Recently, Eshaghi Gordji, Bavand Savadkouhi and Zolfaghari [9] established the stability of mixed type additive, quadratic and cubic functional equation in quasi-Banach

spaces.
In this paper we deal with the following functional equation:

$$f(x+3y)+f(x-3y)=9(f(x+y)+f(x-y))-16f(x) \tag{1.5}$$

in random normed spaces.

2 Main results

From now on, we suppose that X is a real linear space, (Y,μ,T) is a complete RN-space and $f:X\to Y$ is a function with $f(0)=0$ for which there is $\rho: X\times X\to D^+$ ($\rho(x,y)$ is denoted by $\rho_{x,y}$) with the property:

$$\mu_{f(x+3y)+f(x-3y)-9[f(x+y)+f(x-y)]+16f(x)}(t)\geq \rho_{x,y}(t) \tag{2.1}$$

for all $x,y\in X$ and all $t>0$.

Theorem 2.1. *Let f be even and let*

$$\lim_{n\to\infty} T_{i=1}^{\infty}(\rho_{0,3^{n+i-1}x}(3^{i+2n}t))=1=\lim_{n\to\infty}\rho_{3^n x,3^n y}(3^{2n}t) \tag{2.2}$$

for all $x,y\in X$ and all $t>0$, then there exist a unique quadratic mapping $Q:X\to Y$ such that

$$\mu_{Q(x)-f(x)}(t)\geq T_{i=1}^{\infty}(\rho_{0,3^{i-1}x}(3^i t)), \tag{2.3}$$

for all $x\in X$ and all $t>0$.

Proof. Setting $x=0$ in (2.1), we get

$$\mu_{2f(3y)-18f(y)}(t)\geq \rho_{0,y}(t) \tag{2.4}$$

for all $y\in X$. If we replace y in (2.4) by x, we get

$$\mu_{f(3x)-9f(x)}(t)\geq \rho_{0,x}(2t)\geq \rho_{0,x}(t)$$

for all $x\in X$ and all $t>0$. Thus we have

$$\mu_{\frac{f(3x)}{3^2}-f(x)}(t)\geq \rho_{0,x}(3^2 t)$$

for all $x\in X$ and all $t>0$. Therefore,

$$\mu_{\frac{f(3^{k+1}x)}{3^{2(k+1)}}-\frac{f(3^k x)}{3^{2k}}}(t)\geq \rho_{0,3^k x}(3^{2(k+1)}t)$$

for all $x\in X$ and all $k\in\mathbb{N}$. Therefore we have

$$\mu_{\frac{f(3^{k+1}x)}{3^{2(k+1)}}-\frac{f(3^k x)}{3^{2k}}}(\frac{t}{3^{k+1}})\geq \rho_{0,3^k x}(3^{k+1}t)$$

for all $x \in X$, $t > 0$ and all $k \in \mathbb{N}$. Thus we have

$$\mu_{\frac{f(3^n x)}{3^{2n}} - f(x)}(t) \geq T_{k=0}^{n-1}(\mu_{\frac{f(3^{k+1}x)}{3^{2(k+1)}} - \frac{f(3^k x)}{3^{2k}}}(\frac{t}{3^{k+1}})) \geq T_{k=0}^{n-1}(\rho_{0,3^k x}(3^{k+1}t))$$
$$= T_{i=1}^{n}(\rho_{0,3^{i-1}x}(3^i t)) \tag{2.5}$$

for all $x \in X$ and $t > 0$. In order to prove the convergence of the sequence $\{\frac{f(3^n x)}{3^{2n}}\}$, we replace x with $3^m x$ in (2.5) to find that

$$\mu_{\frac{f(3^{n+m}x)}{3^{2(n+m)}} - \frac{f(3^m x)}{3^{2m}}}(t) \geq T_{i=1}^{n}(\rho_{0,3^{i+m-1}x}(3^{i+2m}t))$$

Since the right hand side of the inequality tends to 1 as m and n tend to infinity, the sequence $\{\frac{f(3^n x)}{3^{2n}}\}$ is a Cauchy sequence. Therefore, we may define $Q(x) = \lim_{n\to\infty} \frac{f(3^n x)}{3^{2n}}$ for all $x \in X$. Now, we show that Q is a quadratic map. Replacing x, y with $3^n x$ and $3^n y$ respectively in (2.1), it follows that

$$\mu_{\frac{f(3^n x+3^{n+1}y)}{3^{2n}} + \frac{f(3^n x-3^{n+1}y)}{3^{2n}} - 9[\frac{f(3^n x+3^n y)}{3^{2n}} + \frac{f(3^n x-3^n y)}{3^{2n}}] + 16\frac{f(3^n x)}{3^{2n}}}(t) \geq \rho_{3^n x,3^n y}(3^{2n}t).$$

Taking the limit as $n \to \infty$, we find that Q satisfies (1.5) for all $x, y \in X$. Therefore the mapping $Q : X \to Y$ is quadratic.
To prove (2.3), take the limit as $n \to \infty$ in (2.5). Finally, to prove the uniqueness of the quadratic function Q subject to (2.3), let us assume that there exist a quadratic function Q' which satisfies (2.3). Since $Q(3^n x) = 3^{2n}Q(x)$ and $Q'(3^n x) = 3^{2n}Q'(x)$ for all $x \in X$ and $n \in \mathbb{N}$, from (2.3) it follows that

$$\mu_{Q(x)-Q'(x)}(2t) = \mu_{\frac{Q(3^n x)}{3^{2n}} - \frac{Q'(3^n x)}{3^{2n}}}(2t) = \mu_{Q(3^n x)-Q'(3^n x)}(2.3^{2n}t)$$
$$\geq T(\mu_{Q(3^n x)-f(3^n x)}(3^{2n}t), \mu_{f(3^n x)-Q'(3^n x)}(3^{2n}t))$$
$$\geq T(T_{i=1}^{\infty}(\rho_{0,3^{i+n-1}x}(3^{i+2n}t)), T_{i=1}^{\infty}(\rho_{0,3^{i+n-1}x}(3^{i+2n}t))) \tag{2.6}$$

for all $x \in X$ and all $t > 0$. By letting $n \to \infty$ in above inequality, we find that $Q = Q'$. □

Theorem 2.2. *Let f be odd and let*

$$\lim_{n\to\infty} T_{i=1}^{\infty}(\rho_{2^{i+n-1},2^{i+n-1}x}(2^n t)) = 1 = \lim_{n\to\infty} \rho_{2^n x,2^n y}(2^n t) \tag{2.7}$$

for all $x, y \in X$ and all $t > 0$, then there exist a unique additive mapping $A : X \to Y$ such that

$$\mu_{f(2x)-8f(x)-A(x)}(t) \geq T_{i=1}^{\infty}(\rho_{2^{i-1}x,2^{i-1}x}(t)), \tag{2.8}$$

for all $x \in X$ and all $t > 0$.

Proof. By putting $y = x$ in (2.1), we obtain

$$\mu_{f(4x)-10f(2x)+16f(x)}(t) \geq \rho_{x,x}(t) \tag{2.9}$$

for all $x \in X$ and $t > 0$. Let $g : X \to Y$ be a mapping defined by $g(x) := f(2x) - 8f(x)$. Then we conclude that

$$\mu_{g(2x)-2g(x)}(t) \geq \rho_{x,x}(t) \tag{2.10}$$

for all $x \in X$. Thus we have

$$\mu_{\frac{g(2x)}{2}-g(x)}(t) \geq \rho_{x,x}(2t) \tag{2.11}$$

for all $x \in X$ and all $t > 0$. Hence,

$$\mu_{\frac{g(2^{k+1}x)}{2^{k+1}}-\frac{g(2^k x)}{2^k}}(t) \geq \rho_{2^k x,2^k x}(2^{k+1}t)$$

for all $x \in X$ and all $k \in \mathbb{N}$. So we have

$$\mu_{\frac{g(2^{k+1}x)}{2^{k+1}}-\frac{g(2^k x)}{2^k}}(\frac{t}{2^{k+1}}) \geq \rho_{2^k x,2^k x}(t)$$

for all $x \in X$, $t > 0$ and all $k \in \mathbb{N}$. As $1 > \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^n}$, by the triangle inequality it follows that

$$\begin{aligned}\mu_{\frac{g(2^n x)}{2^n}-g(x)}(t) \geq T_{k=0}^{n-1}(\mu_{\frac{g(2^{k+1}x)}{2^{k+1}}-\frac{g(2^k x)}{2^k}}(\frac{t}{2^{k+1}})) &\geq T_{k=0}^{n-1}(\rho_{2^k x,2^k x}(t))\\ &= T_{i=1}^{n}(\rho_{2^{i-1}x,2^{i-1}x}(t))\end{aligned} \tag{2.12}$$

for all $x \in X$ and $t > 0$. In order to prove the convergence of the sequence $\{\frac{g(2^n x)}{2^n}\}$, we replace x with $2^m x$ in (2.12) to obtain

$$\mu_{\frac{g(2^{n+m}x)}{2^{n+m}}-\frac{g(2^m x)}{2^m}}(t) \geq T_{i=1}^{n}(\rho_{2^{i+m-1}x,2^{i+m-1}x}(2^m t)).$$

Since the right hand side of the inequality tends to 1 as m and n tend to infinity, the sequence $\{\frac{g(2^n x)}{2^n}\}$ is a Cauchy sequence. Therefore, we may define $A(x) = \lim_{n\to\infty} \frac{g(2^n x)}{2^n}$ for all $x \in X$. Now, we show that A is an additive mapping. Replacing x, y with $2^n x$ and $2^n y$ respectively, in (2.1), it follows that

$$\mu_{\frac{f(2^n x+3.2^n y)}{2^n}+\frac{f(2^n x-3.2^n y)}{2^n}-9[\frac{f(2^n x+2^n y)}{2^n}+\frac{f(2^n x-2^n y)}{2^n}]+16\frac{f(2^n x)}{2^n}}(t) \geq \rho_{2^n x,2^n y}(2^n t).$$

Taking the limit as $n \to \infty$, the mapping $A : X \to Y$ is additive.
To prove (2.8), take the limit as $n \to \infty$ in (2.12). Finally, to prove the uniqueness property of A subject to (2.8), let us assume that there exist an additive function A' which satisfies (2.8). Since $A(2^n x) = 2^n A(x)$ and $A'(2^n x) = 2^n A'(x)$ for all $x \in X$ and $n \in \mathbb{N}$, from (2.8) it follows that

$$\begin{aligned}\mu_{A(x)-A'(x)}(2t) &= \mu_{A(2^n x)-A'(2^n x)}(2^{n+1}t)\\ &\geq T(\mu_{A(2^n x)-g(2^n x)}(2^n t), \mu_{g(2^n x)-A'(2^n x)}(2^n t))\\ &\geq T(T_{i=1}^{\infty}(\rho_{2^{i+n-1}x,2^{i+n-1}x}(2^n t)), T_{i=1}^{\infty}(\rho_{2^{i+n-1}x,2^{i+n-1}x}(2^n t)))\end{aligned}$$

for all $x \in X$ and all $t > 0$. Taking the limit as $n \to \infty$, we find that $A = A'$. □

Theorem 2.3. *Let f be odd and let*

$$\lim_{n\to\infty} T_{i=1}^{\infty}(\rho_{2^{n+i-1},2^{n+i-1}x}(2^{3n+2i}t)) = 1 = \lim_{n\to\infty} \rho_{2^n x,2^n y}(2^{3n}t) \tag{2.13}$$

for all $x, y \in X$ and all $t > 0$, then there exist a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(2x)-2f(x)-C(x)}(t) \geq T_{i=1}^{\infty}(\rho_{2^{i-1}x,2^{i-1}x}(2^{2i}t)), \tag{2.14}$$

for all $x \in X$ and all $t > 0$.

Proof. By putting $y = x$ in (2.1), we obtain

$$\mu_{f(4x)-10f(2x)+16f(x)}(t) \geq \rho_{x,x}(t) \tag{2.15}$$

for all $x \in X$ and $t > 0$. Let $h : X \to Y$ be a mapping defined by $h(x) := f(2x) - 2f(x)$. Then we conclude that

$$\mu_{h(2x)-8h(x)}(t) \geq \rho_{x,x}(t) \tag{2.16}$$

for all $x \in X$. Thus we have

$$\mu_{\frac{h(2x)}{2^3}-h(x)}(t) \geq \rho_{x,x}(2^3 t) \tag{2.17}$$

for all $x \in X$ and all $t > 0$. Hence,

$$\mu_{\frac{h(2^{k+1}x)}{2^{3(k+1)}}-\frac{h(2^k x)}{2^{3k}}}(t) \geq \rho_{2^k x,2^k x}(2^{3(k+1)}t)$$

for all $x \in X$ and all $k \in \mathbb{N}$. So we have

$$\mu_{\frac{h(2^{k+1}x)}{2^{3(k+1)}}-\frac{h(2^k x)}{2^{3k}}}(\frac{t}{2^{k+1}}) \geq \rho_{2^k x,2^k x}(2^{2(k+1)}t)$$

for all $x \in X$, $t > 0$ and all $k \in \mathbb{N}$. As $1 > \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^n}$, by the triangle inequality it follows that

$$\begin{aligned}\mu_{\frac{h(2^n x)}{2^{3n}}-h(x)}(t) &\geq T_{k=0}^{n-1}(\mu_{\frac{h(2^{k+1}x)}{2^{3(k+1)}}-\frac{h(2^k x)}{2^{3k}}}(\frac{t}{2^{k+1}})) \geq T_{k=0}^{n-1}(\rho_{2^k x,2^k x}(2^{2(k+1)}t)) \\ &= T_{i=1}^{n}(\rho_{2^{i-1}x,2^{i-1}x}(2^{2i}t))\end{aligned} \tag{2.18}$$

for all $x \in X$ and $t > 0$. In order to prove the convergence of the sequence $\{\frac{h(2^n x)}{2^{3n}}\}$, we replace x with $2^m x$ in (2.18) to obtain

$$\mu_{\frac{h(2^{n+m}x)}{2^{3(n+m)}}-\frac{h(2^m x)}{2^{3m}}}(t) \geq T_{i=1}^{n}(\rho_{2^{i+m-1}x,2^{i+m-1}x}(2^{2i+3m}t)).$$

Since the right hand side of the inequality tends to 1 as m and n tend to infinity, the sequence $\{\frac{h(2^n x)}{2^{3n}}\}$ is a Cauchy sequence. Therefore, we may define $C(x) =$

$\lim_{n\to\infty} \frac{h(2^n x)}{2^{3n}}$ for all $x \in X$. Now, we show that C is a cubic mapping. Replacing x, y with $2^n x$ and $2^n y$ respectively, in (2.1), it follows that

$$\mu_{\frac{f(2^n x+3.2^n y)}{2^{3n}}+\frac{f(2^n x-3.2^n y)}{2^{3n}}-9[\frac{f(2^n x+2^n y)}{2^{3n}}+\frac{f(2^n x-2^n y)}{2^{3n}}]+16\frac{f(2^n x)}{2^{3n}}}(t) \geq \rho_{2^n x, 2^n y}(2^{3n} t).$$

Taking the limit as $n \to \infty$, we find that C satisfies (1.5) for all $x, y \in X$. Hence, the mapping $C : X \to Y$ is cubic.
To prove (2.14), take the limit as $n \to \infty$ in (2.18). Finally, to prove the uniqueness property of C subject to (2.14), let us assume that there exist a cubic function C' which satisfies (2.8). Since $C(2^n x) = 2^{3n} C(x)$ and $C'(2^n x) = 2^{3n} C'(x)$ for all $x \in X$ and $n \in \mathbb{N}$, from (2.14) it follows that

$$\begin{aligned}
\mu_{C(x)-C'(x)}(2t) &= \mu_{C(2^n x)-C'(2^n x)}(2^{3n+1} t) \\
&\geq T(\mu_{C(2^n x)-h(2^n x)}(2^{3n} t), \mu_{h(2^n x)-C'(2^n x)}(2^{3n} t)) \\
&\geq T(T_{i=1}^{\infty}(\rho_{2^{i+n-1}x, 2^{i+n-1}x}(2^{2i+3n} t)), T_{i=1}^{\infty}(\rho_{2^{i+n-1}x, 2^{i+n-1}x}(2^{2i+3n} t)))
\end{aligned}$$

for all $x \in X$ and all $t > 0$. Taking the limit as $n \to \infty$, we find that $C = C'$. □

Theorem 2.4. *Let f be odd and let*

$$\lim_{n\to\infty} T_{i=1}^{\infty}(\rho_{2^{n+i-1}x, 2^{n+i-1}x}(2^{3n+2i} t)) = 1 = \lim_{n\to\infty} T_{i=1}^{\infty}(\rho_{2^{i+n-1}, 2^{i+n-1}x}(2^n t)) \quad (2.19)$$

and

$$\lim_{n\to\infty} \rho_{2^n x, 2^n y}(2^{3n} t) = 1 = \lim_{n\to\infty} \rho_{2^n x, 2^n y}(2^n t) \quad (2.20)$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(x)-A(x)-C(x)}(t) \geq T(T_{i=1}^{\infty}(\rho_{2^{i-1}x, 2^{i-1}x}(6t)), T_{i=1}^{\infty}(\rho_{2^{i-1}x, 2^{i-1}x}(6.2^{2i} t))) \quad (2.21)$$

for all $x \in X$ and all $t > 0$.

Proof. By Theorems 2.2 and 2.3, there exists a unique additive mapping $A' : X \to Y$ and a unique cubic mapping $C' : X \to Y$ such that

$$\mu_{f(2x)-8f(x)-A'(x)}(t) \geq T_{i=1}^{\infty}(\rho_{2^{i-1}x, 2^{i-1}x}(t)),$$

and

$$\mu_{f(2x)-2f(x)-C'(x)}(t) \geq T_{i=1}^{\infty}(\rho_{2^{i-1}x, 2^{i-1}x}(2^{2i} t))$$

for all $x \in X$ and all $t > 0$. So it follows from the last inequalities that

$$\mu_{f(x)+\frac{1}{6}A'(x)-\frac{1}{6}C'(x)}(t) \geq T(T_{i=1}^{\infty}(\rho_{2^{i-1}x, 2^{i-1}x}(6t)), T_{i=1}^{\infty}(\rho_{2^{i-1}x, 2^{i-1}x}(6.2^{2i} t)))$$

for all $x \in X$ and all $t > 0$. Hence we obtain (2.21) by letting $A(x) = \frac{-1}{6} A'(x)$ and $C(x) = \frac{1}{6} C'(x)$ for all $x \in X$. The uniqueness property of A and C, are trivial. □

Theorem 2.5. *Let*

$$\begin{aligned}
&\lim_{n\to\infty} T_{i=1}^{\infty}[T(\rho_{0,3^{n+i-1}x}(3^{i+2n}t), \rho_{0,-3^{n+i-1}x}(3^{i+2n}t))]\\
&= \lim_{n\to\infty} T_{i=1}^{\infty}[T(\rho_{2^{n+i-1}x,2^{n+i-1}x}(2^{3n+2i}t), \rho_{-2^{n+i-1}x,-2^{n+i-1}x}(2^{3n+2i}t))]\\
&= \lim_{n\to\infty} T_{i=1}^{\infty}[T(\rho_{2^{i+n-1}x,2^{i+n-1}x}(2^{n}t), \rho_{-2^{i+n-1}x,-2^{i+n-1}x}(2^{n}t))] = 1 \qquad (2.22)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} T(\rho_{3^n x,3^n y}(3^{2n}t), \rho_{-3^n x,-3^n y}(3^{2n}t)) &= \lim_{n\to\infty} T(\rho_{2^n x,2^n y}(2^{3n}t), \rho_{-2^n x,-2^n y}(2^{3n}t))\\
&= \lim_{n\to\infty} T(\rho_{2^n x,2^n y}(2^{n}t), \rho_{-2^n x,-2^n y}(2^{n}t))\\
&= 1 \qquad (2.23)
\end{aligned}$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique additive mapping $A : X \to Y$, a unique quadratic mapping $Q : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\begin{aligned}
\mu_{f(x)-A(x)-Q(x)-C(x)}(t) \geq T\{&T_{i=1}^{\infty}[T(\rho_{0,3^{i-1}x}(\frac{3^i t}{2}), \rho_{0,-3^{i-1}x}(\frac{3^i t}{2}))]\\
&, T_{i=1}^{\infty}\{T[T(T_{i=1}^{\infty}(\rho_{2^{i+n-1}x,2^{i+n-1}x}(3t))\\
&, T_{i=1}^{\infty}(\rho_{2^{i+n-1}x,2^{i+n-1}x}(3.2^{2i}t)))\\
&, T(T_{i=1}^{\infty}(\rho_{-2^{i+n-1}x,-2^{i+n-1}x}(3t))\\
&, T_{i=1}^{\infty}(\rho_{-2^{i+n-1}x,-2^{i+n-1}x}(3.2^{2i}t)))]\}\} \qquad (2.24)
\end{aligned}$$

for all $x \in X$ and all $t > 0$.

Proof. Let

$$f_e(x) = \frac{1}{2}[f(x) + f(-x)]$$

for all $x \in X$. Then $f_e(0) = 0$, $f_e(-x) = f_e(x)$, and

$$\mu_{f_e(x+3y)+f_e(x-3y)-9[f_e(x+y)+f_e(x-y)]+16f_e(x)}(t) \geq T(\rho_{x,y}(t), \rho_{-x,-y}(t))$$

for all $x, y \in X$. Hence, in view of Theorem 2.1, there exist a unique quadratic function $Q : X \to Y$ such that

$$\mu_{f_e(x)-Q(x)}(t) \geq T_{i=1}^{\infty}[T(\rho_{0,3^{i-1}x}(3^i t), \rho_{0,-3^{i-1}x}(3^i t))]. \qquad (2.25)$$

Let

$$f_o(x) = \frac{1}{2}[f(x) - f(-x)]$$

for all $x \in X$. Then $f_o(0) = 0$, $f_o(-x) = -f_o(x)$, and

$$\mu_{f_o(x+3y)+f_o(x-3y)-9[f_o(x+y)+f_o(x-y)]+16f_o(x)}(t) \geq T(\rho_{x,y}(t), \rho_{-x,-y}(t))$$

for all $x, y \in X$. From Theorem 2.4, it follows that there exists a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f_o(x)-A(x)-C(x)}(t) \geq T_{i=1}^{\infty}\{T[T(T_{i=1}^{\infty}(\rho_{2^{i+n-1}x,2^{i+n-1}x}(6t)), T_{i=1}^{\infty}(\rho_{2^{i+n-1}x,2^{i+n-1}x}(6.2^{2i}t))) \\ , T(T_{i=1}^{\infty}(\rho_{-2^{i+n-1}x,-2^{i+n-1}x}(6t)), T_{i=1}^{\infty}(\rho_{-2^{i+n-1}x,-2^{i+n-1}x}(6.2^{2i}t)))]\} \tag{2.26}$$

Obviously, (2.24) follows from (2.25) and (2.26). □

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