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(*Invited paper*)

Stability of A Sincov Type Functional Equation

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Abstract. This paper aims to study the Hyers-Ulam stability of a Sincov type functional equation $f(x,y)+f(y,z)+f(x,z)=\ell(x+y+z)$ when the domain of the functions is an abelian group and the range is a Banach space.

1. Introduction

In 1903, the Russian mathematician D. M. Sincov (see [13] and [14]) studied the functional equation

$$f(x,y)+f(y,z)=f(x,z) \tag{1.1}$$

for all $x,y,z\in\mathbb{R}$ (the set of real numbers). Others like Moritz Cantor [2] and Gottlob Frege [5] (see also [6] and [7]) treated this functional equation before Sincov. The most general solution $f:\mathbb{R}^2\to\mathbb{R}$ of the functional equation (1.1) is given by $f(x,y)=\phi(y)-\phi(x)$, where $\phi:\mathbb{R}\to\mathbb{R}$ is an arbitrary function. The functional equation (1.1) is known as the Sincov functional equation.

The functional equation

$$f(x,y)+g(y,z)+h(x,z)=\ell(x+y+z) \tag{1.2}$$

for all $x,y,z\in G$, where G is an abelian group, is a generalization of the Sincov functional equation. The following theorem was proved by the author [12].

Theorem 1.1. *Suppose G and H are abelian groups written additively. Moreover, suppose the division by 2 is uniquely defined in H. The functions $f,g,h:G^2\to H$ and $\ell:G\to H$ satisfy the functional equation* (1.2) *for all $x,y,z\in G$ if and only if*

$$\ell(x)=2\,a(x)+2\,q(x)+c_1+c_2+c_3, \tag{1.3}$$

$$f(x,y)=\alpha(x)-\beta(y)+a(x)+a(y)+b(x,x)+4\,b(x,y)+b(y,y)+c_1, \tag{1.4}$$

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$$g(x,y) = \beta(x) - \gamma(y) + a(x) + a(y) + b(x,x) + 4\,b(x,y) + b(y,y) + c_2, \quad (1.5)$$

$$h(x,y) = \gamma(x) - \alpha(y) + a(x) + a(y) + b(x,x) + 4\,b(x,y) + b(y,y) + c_3, \quad (1.6)$$

where $\alpha, \beta, \gamma : G \to H$ are arbitrary functions, $b : G^2 \to H$ is a bihomomorphism, $a : G \to H$ is a homomorphism, and c_1, c_2, c_3 are arbitrary elements in H.

The following functional equation

$$f(x,y) + f(y,z) + f(x,z) = \ell(x+y+z) \quad (1.7)$$

is a special case of (1.2) and its solution can be obtained from Theorem 1.1. The general solution of (1.7) is given by

$$\ell(x) = 2\,a(x) + 2\,q(x) + 3\,c, \quad (1.8)$$

$$f(x,y) = \alpha(x) - \alpha(y) + a(x) + a(y) + b(x,x) + 4\,b(x,y) + b(y,y) + c \quad (1.9)$$

where $\alpha : G \to H$ are arbitrary function, $b : G^2 \to H$ is a bihomomorphism, $a : G \to H$ is a homomorphism, and c is an arbitrary element in H. The functional equation (1.7) was studied by J. A. Baker [1] in 1977.

A function $Q : G \to H$ satisfying $Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$ for all $x, y \in G$ is called a quadratic function. It is well known that a quadratic function $Q : G \to H$ can be represented as the diagonal of a symmetric bihomomorphism $b : G^2 \to H$, that is $Q(x) = b(x,x)$. Therefore, in (1.9), $b(x,x)+4b(x,y)+b(y,y)$ can be replaced by $2Q(x+y) - Q(x) - Q(y)$.

2. **Stability of functional equations**

Given an operator T and a solution class $\{u\}$ with the property that $T(u) = 0$, when does $\|T(v)\| \leq \varepsilon$ for an $\varepsilon > 0$ imply that $\|u - v\| \leq \delta(\varepsilon)$ for some u and for some $\delta > 0$? This problem is called the stability of the functional transformation. A great deal of work has been done in connection with the ordinary and partial differential equations. Let G be a group and H be a metric group with metric $d(\cdot,\cdot)$. S. M. Ulam [16, 17] asked given a number $\epsilon > 0$ does there exist a $\delta > 0$ such that if a function $f : G \to H$ satisfies the inequality $d(f(xy), f(x)f(y)) \leq \delta$ for all x, y in G, then there is a homomorphism $h : G \to H$ with $d(a(x), f(x)) \leq \epsilon$ for all x in G? If f is a function from a normed vector space into a Banach space and satisfies $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$, D. H. Hyers [8] proved that there exists an additive function A such that $\|f(x) - A(x)\| \leq \varepsilon$. If $f(x)$ is a real continuous function of x over $\mathbb{R}$, and $|f(x+y) - f(x) - f(y)| \leq \varepsilon$, it was shown by D. H. Hyers and S. M. Ulam [11] that there exists a constant k such that $|f(x) - kx| \leq 2\varepsilon$. Taking these results into account, we say that the additive Cauchy equation $f(x+y) = f(x) + f(y)$ is stable in the sense of Hyers and Ulam. The interested reader should refer to [9, 10] for an indepth account on the subject of stability of functional equations.

A careful examination of the proof of Hyers's theorem reveals that the proof remains true if one replaces the normed vector space by an abelian group.

Theorem 2.1. *Let G be an abelian group written additively and let B be a Banach space. If a function $f : G \to B$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \;\leq\; \varepsilon$$

for some $\varepsilon \geq 0$ and for all $x, y \in G$, then there exists a unique homomorphism $H : G \to B$ such that

$$\|f(x) - H(x)\| \;\leq\; \varepsilon$$

for all $x \in G$.

The Hyers-Ulam stability of the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{2.1}$$

was first proved by F. Skof [15] for functions from a normed space into a Banach space. P. W. Cholewa [3] demonstrated that Skof's theorem is also valid if the relevant domain is replaced by an abelian group. Later, S. Czerwik [4] proved the Hyers-Ulam-Rassias stability of the quadratic functional equation which includes the following theorem as a special case:

Theorem 2.2. *Let E be a normed space and let B be a Banach space. If a function $f : E \to B$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \;\leq\; \varepsilon$$

for some $\varepsilon \geq 0$ and for all $x, y \in E$, then there exists a unique quadratic function $Q : E \to B$ such that

$$\|f(x) - Q(x)\| \;\leq\; \frac{1}{3}(\varepsilon + \|f(0)\|)$$

for all $x \in E$.

The goal of this paper is to study the Hyers-Ulam stability of the functional equation (1.7) when the domain of the functions is an abelian group and the range is a Banach space.

3. Stability of the functional equation (1.7)

Theorem 3.1. *Let G be an abelian group written additively and B be a Banach space. If the functions $f : G^2 \to B$ and $\ell : G \to B$ satisfy the inequality*

$$\|f(x,y) + f(y,z) + f(z,x) - \ell(x+y+z)\| \leq \epsilon \tag{3.1}$$

for some $\epsilon \geq 0$ and for all $x, y, z \in G$, then there exist a unique homomorphism $A : G \to B$, a unique quadratic map $Q : G \to B$ and a function $\phi : G \to B$ such that

$$\|\ell(x) - 2A(x) - 2Q(x)\| \leq \frac{23}{3}\,\epsilon + 3\,\|f(0,0)\| \tag{3.2}$$

and

$$\|f(x,y) - f_o(x,y)\| \leq \frac{49}{3}\,\epsilon + \|f(0,0)\| \tag{3.3}$$

where

$$f_o(x,y) = \phi(x) - \phi(y) + A(x) + A(y) + 2Q(x+y) - Q(x) - Q(y). \quad (3.4)$$

Proof. Define

$$g(x,y) = \frac{f(x,y) - f(y,x)}{2} \quad (3.5)$$

and

$$h(x,y) = \frac{f(x,y) + f(y,x)}{2} - \gamma \quad (3.6)$$

where

$$\gamma = f(0,0). \quad (3.7)$$

From (3.5) and (3.6), we obtain

$$f(x,y) = g(x,y) + h(x,y) + \gamma. \quad (3.8)$$

Moreover, we see that

$$h(x,y) = h(y,x) \quad (3.9)$$

for all $x, y \in G$, that is h is symmetric in G. Interchanging x with y in (3.1), we have

$$\|f(y,x) + f(x,z) + f(z,y) - \ell(y+x+z)\| \leq \epsilon \quad (3.10)$$

for all $x, y, z \in G$. From (3.1), (3.5) and (3.10) we obtain

$$\begin{aligned} 2\,&\|g(x,y) + g(y,z) + g(z,x)\| \\ &= \|f(x,y) - f(y,x) + f(y,z) - f(z,y) + f(z,x) - f(x,z) \\ &\qquad - \ell(x+y+z) + \ell(y+x+z)\| \\ &\leq \|f(x,y) + f(y,z) + f(z,x) - \ell(x+y+z)\| \\ &\qquad + \|f(y,x) + f(x,z) + f(z,y) - \ell(y+x+z)\| \\ &\leq 2\epsilon. \end{aligned}$$

Therefore

$$\|g(x,y) + g(y,z) + g(z,x)\| \leq \epsilon \quad (3.11)$$

for all $x, y, z \in G$. From (3.5) we see that

$$g(x,y) = -g(y,x) \quad (3.12)$$

for all $x, y \in G$, that is g is anti-symmetric in G. Using (3.12) in (3.11) we have

$$\|g(x,y) - g(x,z) + g(y,z)\| \leq \epsilon \quad (3.13)$$

for all $x, y, z \in G$. Letting $z = 0$ in (3.13) we have

$$\|g(x,y) - g(x,0) + g(y,0)\| \leq \epsilon. \tag{3.14}$$

Define $\phi : G \to B$ by

$$\phi(x) = g(x,0). \tag{3.15}$$

Then (3.14) yields

$$\|g(x,y) - \phi(x) + \phi(y)\| \leq \epsilon. \tag{3.16}$$

for all $x, y \in G$. Since, by (3.6), (3.1) and (3.10)

$$\begin{aligned}
2\,&\|h(x,y) + h(y,z) + h(z,x) - \ell(x+y+z) + 3\gamma\| \\
&= \|f(x,y) + f(y,x) + f(y,z) + f(z,y) + f(z,x) + f(x,z) \\
&\qquad - \ell(x+y+z) - \ell(y+x+z)\| \\
&\leq \|f(x,y) + f(y,z) + f(z,x) - \ell(x+y+z)\| \\
&\qquad + \|f(y,x) + f(x,z) + f(z,y) - \ell(y+x+z)\| \\
&\leq 2\,\epsilon
\end{aligned}$$

hence we have

$$\|h(x,y) + h(y,z) + h(z,x) - \ell(x+y+z) + 3\gamma\| \leq \epsilon \tag{3.17}$$

for all $x, y, z \in G$. Next we define $F : G \to B$ by

$$F(x) = \ell(x) - 3\gamma. \tag{3.18}$$

Then by (3.18), the equation (3.17) reduces to

$$\|h(x,y) + h(y,z) + h(z,x) - F(x+y+z)\| \leq \epsilon \tag{3.19}$$

for all $x, y, z \in G$. Letting $z = 0$ in (3.19) we obtain

$$\|h(x,y) + h(y,0) + h(0,x) - F(x+y)\| \leq \epsilon. \tag{3.20}$$

Let us define $\psi : G \to B$ by

$$\psi(x) = h(x,0). \tag{3.21}$$

Then using the fact that h is symmetric, (3.20) can be written as

$$\|h(x,y) + \psi(y) + \psi(x) - F(x+y)\| \leq \epsilon \tag{3.22}$$

for all $x, y, z \in G$. Let $z = y = 0$ in (3.20), we obtain

$$\|h(x,0) + h(0,0) + h(0,x) - F(x)\| \leq \epsilon. \tag{3.23}$$

Since from (3.6) and (3.7) we see that

$$h(0,0) = 0 \tag{3.24}$$

the inequality (3.23) can be written as

$$\|2\psi(x) - F(x)\| \le \epsilon \tag{3.25}$$

for all $x \in G$. Since

$$\begin{aligned}
&\|h(x,y) + \psi(x) + \psi(y) - 2\psi(x+y)\| \\
&\quad = \|h(x,y) + \psi(x) + \psi(y) - F(x+y) + F(x+y) - 2\psi(x+y)\| \\
&\quad \le \|h(x,y) + \psi(x) + \psi(y) - F(x+y)\| + \|2\psi(x+y) - F(x+y)\| \\
&\quad \le 2\epsilon
\end{aligned}$$

therefore

$$\|h(x,y) + \psi(x) + \psi(y) - 2\psi(x+y)\| \le 2\epsilon \tag{3.26}$$

for all $x, y \in G$. Since

$$\begin{aligned}
&2\,\|\psi(x+y) + \psi(y+z) + \psi(z+x) - \psi(x) - \psi(y) - \psi(z) - \psi(x+y+z)\| \\
&\quad = \|2\psi(x+y) - \psi(x) - \psi(y) - h(x,y) + 2\psi(y+z) - \psi(y) - \psi(z) \\
&\qquad\quad - h(y,z) + 2\psi(z+x) - \psi(z) - \psi(x) - h(z,x) - \psi(x+y+z) \\
&\qquad\quad + F(x+y+z) + h(x,y) + h(y,z) + h(z,x) - F(x+y+z)\| \\
&\quad \le \|h(x,y) + \psi(x) + \psi(y) - 2\psi(x+y)\| \\
&\qquad\quad + \|h(y,z) + \psi(y) + \psi(z) - 2\psi(y+z)\| \\
&\qquad\quad + \|h(z,x) + \psi(z) + \psi(x) - 2\psi(z+x)\| \\
&\qquad\quad + \|2\psi(x+y+z) - F(x+y+z)\| \\
&\qquad\quad + \|h(x,y) + h(y,z) + h(z,x) - F(x+y+z)\| \\
&\quad \le 8\epsilon
\end{aligned}$$

therefore

$$\begin{aligned}
&\|\psi(x+y) + \psi(y+z) + \psi(z+x) - \psi(x) - \psi(y) - \psi(z) - \psi(x+y+z)\| \\
&\quad \le 4\epsilon
\end{aligned} \tag{3.27}$$

for all $x, y, z \in G$. Defining the functions $a, q : G \to B$ as

$$a(x) = \frac{\psi(x) - \psi(-x)}{2} \tag{3.28}$$

and

$$q(x) = \frac{\psi(x) + \psi(-x)}{2} \tag{3.29}$$

we see that

$$\psi(x) = a(x) + q(x). \tag{3.30}$$

Since $h(0,0) = 0$, from (3.21) we see that

$$\psi(0) = 0. \tag{3.31}$$

Letting $z = -x - y$ in (3.27) and using (3.31) we obtain

$$\|\psi(x+y) + \psi(-x) + \psi(-y) - \psi(x) - \psi(y) - \psi(-x-y)\| \leq 4\epsilon. \tag{3.32}$$

Using (3.28) in (3.32) we have

$$2\,\|a(x+y) - a(x) - a(y)\| \leq 4\epsilon. \tag{3.33}$$

Therefore

$$\|a(x+y) - a(x) - a(y)\| \leq 2\epsilon \tag{3.34}$$

for all $x, y \in G$. Thus by Theorem 2.1, there exists a unique homomorphism $A : G \to B$ such that

$$\|a(x) - A(x)\| \leq 2\epsilon \tag{3.35}$$

for all $x \in G$.

Next, letting $z = -y$ in (3.27) we have

$$\|\psi(x+y) + \psi(0) + \psi(x-y) - 2\psi(x) - \psi(y) - \psi(-y)\| \leq 4\epsilon.$$

Since $\psi(0) = 0$, the last inequality yields

$$\|\psi(x+y) + \psi(x-y) - 2\psi(x) - \psi(y) - \psi(-y)\| \leq 4\epsilon \tag{3.36}$$

for all $x, y \in G$. Replacing y by $-y$ and x by $-x$ in (3.36), we obtain

$$\|\psi(-x-y) + \psi(y-x) - 2\psi(-x) - \psi(-y) - \psi(y)\| \leq 4\epsilon. \tag{3.37}$$

Using (3.36) and (3.37), we see that

$$\begin{aligned}
&2\,\|q(x+y) + q(x-y) - 2q(x) - 2q(y)\| \\
&\quad = \|\psi(x+y) + \psi(-x-y) + \psi(x-y) + \psi(-x+y) - 2\psi(x) \\
&\qquad\quad - 2\psi(-x) - \psi(y) - \psi(-y) - \psi(-y) - \psi(y)\| \\
&\quad = \|\psi(x+y) + \psi(x-y) - 2\psi(x) - \psi(y) - \psi(-y)\| \\
&\qquad\quad + \|\psi(-x-y) + \psi(-x+y) - 2\psi(-x) - \psi(-y) - \psi(y)\| \\
&\quad \leq 8\epsilon.
\end{aligned}$$

Therefore

$$\|q(x+y) + q(x-y) - 2q(x) - 2q(y)\| \leq 4\epsilon \tag{3.38}$$

for all $x, y \in G$. Therefore by Theorem 2.2, there exists a unique quadratic map $Q : G \to B$ such that

$$\|q(x) - Q(x)\| \leq \frac{4\epsilon + \|q(0)\|}{3} \tag{3.39}$$

for all $x \in G$. Since $q(0) = 0$, the equation (3.39) implies that

$$\|q(x) - Q(x)\| \le \frac{4\epsilon}{3}. \tag{3.40}$$

Using (3.40) and (3.29), we obtain

$$\left\| \frac{\psi(x) + \psi(-x)}{2} - Q(x) \right\| \le \frac{4\epsilon}{3}. \tag{3.41}$$

Similarly, from (3.28) and (3.35), we have

$$\left\| \frac{\psi(x) - \psi(-x)}{2} - A(x) \right\| \le 2\epsilon. \tag{3.42}$$

Hence from (3.41) and (3.42) we obtain

$$\begin{aligned}
&\|\psi(x) - A(x) - Q(x)\| \\
&\quad = \left\| \frac{\psi(x) - \psi(-x)}{2} - A(x) + \frac{\psi(x) + \psi(-x)}{2} - Q(x) \right\| \\
&\quad \le \left\| \frac{\psi(x) - \psi(-x)}{2} - A(x) \right\| + \left\| \frac{\psi(x) + \psi(-x)}{2} - Q(x) \right\| \\
&\quad \le \frac{10}{3}\,\epsilon.
\end{aligned}$$

Therefore

$$\|\psi(x) - A(x) - Q(x)\| \le \frac{10}{3}\,\epsilon. \tag{3.43}$$

Next, using (3.26) and (3.43), we compute

$$\begin{aligned}
&\|h(x,y) - 2Q(x+y) + Q(x) + Q(y) - 2A(x+y) + A(x) + A(y)\| \\
&\quad = \|h(x,y) + 2\psi(x+y) - \psi(x) - \psi(y) - 2\psi(x+y) + \psi(x) + \psi(y) \\
&\qquad\quad - 2Q(x+y) + Q(x) + Q(y) - 2A(x+y) + A(x) + A(y)\| \\
&\quad \le \|h(x,y) + \psi(x) + \psi(y) - 2\psi(x+y)\| \\
&\qquad\quad + 2\,\|\psi(x+y) - Q(x+y) - A(x+y)\| \\
&\qquad\quad + \|\psi(x) - Q(x) - A(x)\| + \|\psi(y) - Q(y) - A(y)\| \\
&\quad \le \frac{46}{3}\,\epsilon.
\end{aligned}$$

Therefore

$$\|h(x,y) - 2Q(x+y) + Q(x) + Q(y) - A(x) - A(y)\| \le \frac{46}{3}\,\epsilon \tag{3.44}$$

for all $x, y \in G$. Finally, by (3.17) and (3.44), we get

$$\|g(x,y) - \phi(x) + \phi(y) + h(x,y) - 2Q(x+y) + Q(x) + Q(y) - A(x) - A(y) + \gamma\|$$

$$\leq \|g(x,y) - \phi(x) + \phi(y)\| \\ + \|h(x,y) - 2Q(x+y) + Q(x) + Q(y) - A(x) - A(y)\| + \|\gamma\| \\ \leq \frac{49}{3}\,\epsilon + \|\gamma\|.$$

Hence we have

$$\|g(x,y) + h(x,y) + \gamma - f_o(x,y)\| \leq \frac{49}{3}\,\epsilon + \|\gamma\| \tag{3.45}$$

where

$$f_o(x,y) = \phi(x) - \phi(y) + A(x) + A(y) + 2Q(x+y) - Q(x) - Q(y). \tag{3.46}$$

Since

$$f(x,y) = g(x,y) + h(x,y) + \gamma \tag{3.47}$$

the inequality (3.45) reduces to

$$\|f(x,y) - f_o(x,y)\| \leq \frac{49}{3}\,\epsilon + \|\gamma\|. \tag{3.48}$$

Similarly, using (3.25) and (3.43), we see that

$$\|\ell(x) - 2A(x) - 2Q(x)\| \\ = \|\ell(x) - 2A(x) - 2Q(x) - 3\gamma + 3\gamma\| \\ = \|F(x) - 2A(x) - 2Q(x) + 3\gamma\| \\ \leq \|2\psi(x) - F(x)\| + 2\|\psi(x) - A(x) - Q(x)\| + 3\|\gamma\| \\ \leq \frac{23}{3}\,\epsilon + 3\|\gamma\|$$

and the proof of the theorem is now complete. □

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