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in Burning Plasmas**

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Stability of Alfvén gap modes in burning plasmas

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A stability analysis is carried out for energetic particle-Alfvén gap modes. Three modes have been identified: the Toroidicity, Ellipticity and Noncircular Triangularity Induced Alfvén Eigenmodes (TAE, EAE and NAE). In highly elongated plasma cross sections with $\kappa - 1 \sim 1$, the EAE may be a more robust mode than the TAE and NAE. It is found that electron Landau damping in highly elongated plasmas has a strong stabilizing influence on the $n = 1$ EAE, while ion Landau damping stabilizes the $n = 1$ TAE in high density regimes. Furthermore, the NAE turns out to be stable for all currently proposed ignition experiments. The stability analysis of a typical burning plasma device, Burning Plasma Experiment (BPX) [Phys. Scr. **T16**, 89 (1987)] shows that $n > 1$ gap modes can pose a serious threat to the achievement of ignition conditions.

I Introduction

Since the original observation of “fishbone oscillations” in the PDX tokamak,¹ there has been considerable interest in the theory of resonant particle effects on MHD modes. The excitation of these modes by resonant interaction with an energetic particle species is expected to enhance loss of alpha particles in future burning plasma experiments and ignited devices such as the Burning Plasma Experiment (BPX),² Ignitor,³ and the International Thermonuclear Reactor (ITER).⁴ Specifically, it has been shown⁵ that MHD Alfvén waves whose frequency ω_A is lower than the alpha diamagnetic frequency $\omega_{*\alpha}$ can be driven unstable via transit resonance with alpha particles. Modes with toroidal wavenumber $n = 2$ and frequency of the order of the Alfvén frequency have been observed in TFTR during neutral beam injection.⁶ The current view is that Toroidal Alfvén Eigenmodes (TAE)⁷⁻¹¹ can pose the most serious threat. In the present paper we show that two other global Alfvén modes, the Ellipticity Induced Alfvén Eigenmode (EAE)¹² and the Noncircular Triangularity Induced Alfvén Eigenmode (NAE) can be excited via resonant interaction with alpha particles and that their stability threshold can be lower than that of the TAE in high density regimes and for toroidal wavenumber $n > 1$. The TAE⁷ results from the effects of toroidicity that couples the neighboring poloidal harmonics. Its frequency lies in the gaps generated by the toroidal coupling and in the limit of large aspect ratio (calculated at the gap position, $\epsilon_{gap} \ll 1$), the mode is localized in a narrow region of thickness $\sim \epsilon_{gap}a$ about the $q = (2m + 1)/2n$ surface.

The EAE¹² is a global mode resulting from two elliptically coupled harmonics. Its frequency lies in the Alfvén continuum gaps generated by the elliptical coupling. In the limit of small ellipticity ($\kappa - 1 \ll 1$) the mode is localized in a narrow region of thickness $\sim (\kappa - 1)a/2$ about the $q = (m + 1)/n$ surface.

The NAE results from the coupling induced by the triangularity of the plasma cross section. The mode consists primarily of two triangularly coupled harmonics, localized in a region of thickness $\sim \delta a/4$ about the $q = (2m + 3)/2n$ surface, and its real frequency lies in the Alfvén continuum gaps generated by the triangular coupling.

In particular since many tokamaks have finite ellipticity $(\kappa - 1)/2 \simeq 0.5$, as compared

to small toroidicity and triangularity ($\epsilon_{gap} \ll 1$, $\delta/4 \ll 1$) the EAE may be a potentially more dangerous mode than either the TAE or NAE.

II The Model

A The basic moment equations

Consider a plasma consisting of bulk electrons and ions and a species of energetic particles, each described by the Vlasov equation

$$\frac{\partial f_j}{\partial t} + \mathbf{v} \cdot \nabla f_j + \frac{q_j}{m_j} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_j = 0. \quad (1)$$

Here, j denotes particle species. Taking the zeroth order moment leads to the well known conservation of mass relation

$$\frac{\partial n_j}{\partial t} + \nabla \cdot n_j \mathbf{u}_j = 0. \quad (2)$$

Similarly, the first order moment yields the momentum equation

$$m_j \frac{\partial}{\partial t} (n_j \mathbf{u}_j) + \nabla \cdot \vec{\Pi}_j = q_j n_j (\mathbf{E} + \mathbf{u}_j \times \mathbf{B}) \quad (3)$$

where

$$\vec{\Pi}_j \equiv m_j \int \mathbf{v} \mathbf{v} f_j d\mathbf{v}. \quad (4)$$

Note that $\vec{\Pi}_j$ contains both thermal and inertial effects.

The desired form of the momentum equation is obtained by summing Eq. (3) over species and using the quasineutral condition. This gives

$$\mathbf{J} \times \mathbf{B} = \sum_j \left(\nabla \cdot \vec{\Pi}_j + m_j \frac{\partial}{\partial t} n_j \mathbf{u}_j \right). \quad (5)$$

Our approach is to solve the Vlasov equation and evaluate n_j , \mathbf{u}_j and $\vec{\Pi}_j$ in terms of the electric and magnetic fields. The model is closed by the addition of the low frequency Maxwell equations

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \end{aligned} \quad (6)$$

$$\nabla \cdot \mathbf{B} = 0.$$

Equations (2), (5) and (6) represent the basic moment description of the problem.

B Equilibrium

The equilibrium distribution function for a finite β , finite aspect ratio, noncircular axisymmetric Vlasov plasma is assumed to be an arbitrary function of the constants of motion ϵ and p_ϕ ; that is

$$f_{0j} = F_j(\epsilon, p_\phi) \quad (7)$$

where

$$\epsilon = \frac{m_j v^2}{2} + q_j \Phi \quad (8)$$

$$p_\phi = m_j R v_\phi + q_j \Psi$$

and $\Phi(R, Z)$, $\Psi(R, Z)$ are the equilibrium electric potential and poloidal flux function respectively. For low frequency phenomena f_0 could also be a function of the adiabatic invariants μ and $\sigma = v_{||}/|v_{||}|$.

We now make a number of simplifying assumptions that minimizes the volume of algebra required while maintaining the essential physics. First we assume that the ions and energetic species can carry only a toroidal current, so that Eq. (7) is the exact form of equilibrium distribution function. The electrons are allowed to carry both toroidal and poloidal current. Their distribution function is of the form $f_{0e} = F_e(\epsilon, p_\phi) + \tilde{F}_e(\epsilon, p_\phi, \mu, \sigma)$ where $\tilde{F}_e/F_e \sim r_{Le}/a$. The quantity \tilde{F}_e , which generates poloidal current, does not explicitly enter the evaluation of $\vec{\mathbf{u}}_e$ in the small gyroradius limit and makes inertial contributions of order m_e/m_i smaller than for the ions. Thus, \tilde{F}_e never explicitly enters the calculation and is hereafter suppressed from the analysis.

Second, observe that in the small gyroradius limit

$$F_j(\epsilon, p_\phi) \approx F_j(\epsilon, \Psi) + \frac{m_j R v_\phi}{q_j} \frac{\partial F_j}{\partial \Psi} + \dots \quad (9)$$

The dominant contribution to $\vec{\Pi}_j$ arises from the first term whose form implies that the equilibrium pressure is isotropic. This is a good assumption for most tokamaks. Anisotropic pressure would require finite μ dependence in F_j .

The third simplification arises from the assumption that the equilibrium electric field is zero: $\Phi(R, Z) = 0$. In practice small electric fields do exist, but as is shown in Appendix A, they do not alter the dispersion relation in any substantial manner.

Under these assumptions the equilibrium momentum balance equation reduces to

$$\mathbf{J} \times \mathbf{B} = \nabla p \quad (10)$$

where $\vec{\Pi}_j = p_j \vec{\mathbf{I}}$,

$$p(\Psi) = \sum_j p_j(\Psi) \quad (11)$$

and

$$p_j(\Psi) = \int \frac{m_j v^2}{3} F_j(\epsilon, \Psi) dv. \quad (12)$$

Equation (10) implies that any ideal MHD equilibrium solution is also a solution to the more general model considered here. Using Ampere's law and following standard procedures, it is possible to reduce Eq. (10) to a single Grad-Shafranov equation for Ψ given by

$$\Delta^* \Psi = -\mu_0 R^2 \frac{dp}{d\Psi} - F \frac{dF}{d\Psi} \quad (13)$$

$$\mathbf{B} = \frac{F}{R} \mathbf{e}_\phi + \frac{1}{R} \nabla \Psi \times \mathbf{e}_\phi$$

where $p(\Psi)$ and $F(\Psi)$ are free functions.

Finally, note that the assumption $\Phi = 0$ implies that the ions and energetic species are held in macroscopic equilibrium primarily by the $\mathbf{u}_j \times \mathbf{B}$ force. In the small gyroradius limit the toroidal fluid velocity for these species can be written as

$$u_{\phi j} = \frac{R}{q_j n_j} \frac{dp_j}{d\Psi}. \quad (14)$$

This completes the specification of the equilibrium problem. For the stability analysis it is assumed that a solution has been found to Eq. (10) or equivalently Eq. (13).

C Stability

The stability analysis proceeds in the standard manner. All perturbed quantities are written as

$$Q_1 = Q_1(R, Z) \exp(-i\omega t - in\phi). \quad (15)$$

The perturbed distribution function is found by the method of characteristics

$$f_{1j} = -\frac{q_j}{m_j} \int_{-\infty}^t (\mathbf{E}_1 + \mathbf{v} \times \mathbf{B}_1) \cdot \nabla_v F_j dt'. \quad (16)$$

Using the equilibrium relation $F_j = F_j(\epsilon, p_\phi)$, it can easily be shown that f_{1j} has the form

$$f_{1j} = -\frac{q_j}{\omega} \left[i \frac{\partial F_j}{\partial p_\phi} R E_{1\phi} + \left(\omega \frac{\partial F_j}{\partial \epsilon} - n \frac{\partial F_j}{\partial p_\phi} \right) \int_{-\infty}^t \mathbf{E}_1 \cdot \mathbf{v} dt' \right] \quad (17)$$

Equation (17) can be further simplified by making the assumption $E_{1\parallel} = 0$. This is a good approximation for MHD Alfvén waves although a small portion of the overall electron Landau damping is neglected. The condition $E_{1\parallel} = 0$ allows us to define

$$\mathbf{E}_{1\perp} \equiv i\omega \boldsymbol{\xi}_\perp \times \mathbf{B} \quad (18)$$

where $\boldsymbol{\xi}_\perp$ represents the perturbed $\mathbf{E} \times \mathbf{B}$ displacement. Using Eq. (18) we can rewrite the quantity $\mathbf{E}_1 \cdot \mathbf{v}$ (evaluated along the unperturbed orbit) as follows

$$\mathbf{E}_1 \cdot \mathbf{v} = -i\omega \boldsymbol{\xi}_\perp \cdot (\mathbf{v} \times \mathbf{B}) = -i\omega \frac{m_j}{q_j} \boldsymbol{\xi}_\perp \cdot \frac{d\mathbf{v}}{dt} = -i\omega \frac{m_j}{q_j} \left[\frac{d}{dt} (\boldsymbol{\xi}_\perp \cdot \mathbf{v}) - \mathbf{v} \cdot \frac{d\boldsymbol{\xi}_\perp}{dt} \right]. \quad (19)$$

The desired form of the perturbed distribution function becomes

$$f_{1j} = -q_j \frac{\partial F_j}{\partial p_\phi} (\boldsymbol{\xi}_\perp \cdot \nabla \Psi) + im_j \left(\omega \frac{\partial F_j}{\partial \epsilon} - n \frac{\partial F_j}{\partial p_\phi} \right) \left[\boldsymbol{\xi}_\perp \cdot \mathbf{v} - \int_{-\infty}^t \mathbf{v} \cdot \frac{d\boldsymbol{\xi}_\perp}{dt'} dt' \right]. \quad (20)$$

Equation (20) is exact. In principle we want to utilize this equation to calculate the perturbed n_j , u_j , and $\overleftrightarrow{\Pi}_j$ in terms of $\boldsymbol{\xi}_\perp$ by taking appropriate moments. Observe that most of the terms represent simple fluid-like contributions whose moments can be easily evaluated. The difficulty of course lies with the last term which involves a complicated integral along the unperturbed orbits. It is here that the substitution given by Eq. (19) proves useful. The form of the integral in Eq. (20) is smaller by r_{Lj}/a than that appearing

in Eq. (17). It can be easily evaluated in the small gyroradius limit using only the *zeroth* order guiding center orbits. This calculation is carried out in the next section in the interesting regime corresponding to finite wavelength, macroscopic modes.

D The drift kinetic expansion

The last step in the simplification of f_{1j} is the substitution of the small gyroradius, drift kinetic expansion into Eq. (20). The explicit transformation is given by

$$\mathbf{v} = v_{\parallel} \mathbf{b}(\mathbf{R}) + v_{\perp} [\mathbf{n}(\mathbf{R}) \cos \zeta - \boldsymbol{\tau}(\mathbf{R}) \sin \zeta] \quad (21)$$

$$\mathbf{x} = \mathbf{R} + \frac{v_{\perp}}{\Omega_j(\mathbf{R})} [\mathbf{n}(\mathbf{R}) \sin \zeta + \boldsymbol{\tau}(\mathbf{R}) \cos \zeta]$$

where $\Omega_j = q_j B / m_j$, $\mathbf{b} = \mathbf{B} / B$, \mathbf{R} is the guiding center of the particle, and $(\mathbf{n}, \boldsymbol{\tau}, \mathbf{b})$ are a right-handed set of locally orthogonal unit vectors in physical space. The quantity ζ is the gyrophase angle of the particle while v_{\parallel} and v_{\perp} are the parallel and perpendicular particle velocities. The variables v_{\parallel} and v_{\perp} actually represent an intermediate step in the transformation. In the final step v_{\parallel} and v_{\perp} are related to the energy and magnetic moment by the usual relations

$$\begin{aligned} \epsilon &= \frac{m_j}{2} (v_{\perp}^2 + v_{\parallel}^2) \\ \mu &= \frac{m_j v_{\perp}^2}{2B(\mathbf{R})}. \end{aligned} \quad (22)$$

Along the unperturbed orbits, the guiding center variables satisfy the following equations of motion

$$\begin{aligned} \frac{d\epsilon}{dt} &= 0 \\ \frac{d\mu}{dt} &\approx 0 \\ \frac{d\zeta}{dt} &\approx \Omega_j \\ \frac{d\mathbf{R}}{dt} &\approx v_{\parallel} \mathbf{b} + \mathbf{v}_D \end{aligned} \quad (23)$$

where

$$\begin{aligned} \mathbf{v}_D &= \mathbf{v}_{\nabla B} + \mathbf{v}_{\kappa} \\ \mathbf{v}_{\nabla B} &= \frac{v_{\perp}^2}{2\Omega_j} \mathbf{b} \times \nabla \ln B \\ \mathbf{v}_{\kappa} &= \frac{v_{\parallel}^2}{\Omega_j} \mathbf{b} \times \boldsymbol{\kappa} \end{aligned} \quad (24)$$

and $\boldsymbol{\kappa} = \mathbf{b} \cdot \nabla \mathbf{b}$. The equations for $\dot{\mu}$, $\dot{\zeta}$, and $\dot{\mathbf{R}}$ are correct to leading order in $\delta_j \equiv r_{Lj}/a$. The higher order corrections are not explicitly needed. Note also that the $\mathbf{E} \times \mathbf{B}$ drift velocity vanishes in the expression for \mathbf{v}_D since $\mathbf{E}_0 = 0$.

The next step is to substitute the drift kinetic expansion into the expression for $\mathbf{v} \cdot (d\boldsymbol{\xi}/dt)$ in Eq. (20). A simple calculation shows that $\mathbf{v} \cdot (d\boldsymbol{\xi}/dt)$ has the form

$$\mathbf{v} \cdot \frac{d\boldsymbol{\xi}}{dt} = a_0 + a_1 e^{i\zeta} + a_2 e^{2i\zeta} \quad (25)$$

with $a_j = a_j(\epsilon, \mu, \mathbf{R}, t')$. The last two terms are rapidly oscillating and nearly average to zero. It can be easily shown that their contributions to f_{1j} are smaller by δ_j than those already appearing. The quantity a_0 represents the resonant particle effects. It produces a larger contribution competitive with the fluid contribution and is dissipative in nature. This is the only term that need be maintained. A short calculation shows that

$$a_0(\epsilon, \mu, \mathbf{R}, t') = \frac{v_{\perp}^2}{2} \nabla \cdot \boldsymbol{\xi}_{\perp} + \left(\frac{v_{\perp}^2}{2} - v_{\parallel}^2 \right) \boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\kappa}. \quad (26)$$

The final form for f_{1j} is obtained by substituting Eq. (26) into Eq. (20) and introducing the drift kinetic expansion into the fluid-like terms. The result is

$$f_{1j} \approx -\frac{\partial F_j}{\partial \Psi} (\boldsymbol{\xi}_{\perp} \cdot \nabla \Psi) + im_j (\omega - \hat{\omega}_{*j}) \frac{\partial F_j}{\partial \epsilon} \left[\boldsymbol{\xi}_{\perp} \cdot \mathbf{v} - \int_{-\infty}^t a_0 dt' \right]. \quad (27)$$

In this expression $F_j = F_j(\epsilon, \Psi)$ and

$$\hat{\omega}_{*j}(\epsilon, \Psi) \equiv \frac{n}{q_j} \frac{\partial F_j / \partial \Psi}{\partial F_j / \partial \epsilon}. \quad (28)$$

It is important to notice that a_0 is a function of the invariants of motion ϵ, μ , the time t and the guiding center position R . Since the guiding center trajectories are much simpler than the full particle orbits, an analytic approximation can be used to obtain explicit evaluations.

III Application to Energetic Particle-Alfvén Waves

A The moment equations

By focussing attention on energetic particle-Alfvén waves, the basic stability equation is obtained by linearizing the basic moment equation [Eq. (5)] and Maxwell's equations [Eq. (6)], and then substituting the expression for f_{1j} [Eq. (27)] to evaluate the perturbed $\vec{\Pi}_j$ and $n_j \mathbf{u}_j$. From the definition $\mathbf{E}_1 = i\omega \boldsymbol{\xi}_\perp \times \mathbf{B}$, Maxwell's equations yield

$$\mathbf{B}_1 = \nabla \times (\boldsymbol{\xi}_\perp \times \mathbf{B}) \quad (29)$$

$$\mu_0 \mathbf{J}_1 = \nabla \times \nabla \times (\boldsymbol{\xi}_\perp \times \mathbf{B}).$$

The linearized form of the momentum equation is given by

$$\mathbf{J}_1 \times \mathbf{B} + \mathbf{J} \times \mathbf{B}_1 = \sum_j \left[\nabla \cdot \vec{\Pi}_{1j} - i\omega m_j (n_{1j} \mathbf{u}_j + n_j \mathbf{u}_{1j}) \right]. \quad (30)$$

For simplicity, zero subscripts have been omitted from equilibrium quantities. Recall that at this point we have introduced the small gyroradius approximation in deriving the perturbed distribution function, but have made no assumptions regarding the size of β or ϵ . For Alfvén waves this is formally equivalent to introducing the gyroradius parameter $\delta \equiv r_{Li}/a$ and assuming that

$$\begin{aligned} \frac{\omega}{k_{\parallel} v_a} &\sim 1 & \frac{\omega}{\Omega_i} &\sim \delta \\ \frac{n_\alpha}{n_i} &\sim \delta & \frac{T_\alpha}{T_i} &\sim \frac{1}{\delta} \\ \frac{\omega_{*i}}{\omega} &\sim \frac{\omega_{*e}}{\omega} \sim \delta & \frac{\omega_{*\alpha}}{\omega} &\sim 1 \end{aligned} \quad (31)$$

where the subscripts e, i, α denote electrons, ions, and alphas, respectively.

On the basis of Eq. (39) it follows that the ion contribution dominates the inertial effects:

$$\sum_j m_j (n_{1j} \mathbf{u}_j + n_j \mathbf{u}_{1j}) = \sum_j m_j \int \mathbf{v} f_{1j} d\mathbf{v} \approx \omega^2 \rho \boldsymbol{\xi}_\perp. \quad (32)$$

For simplicity the subscript i has been omitted from ρ , the ion mass density.

Similarly, a straightforward calculation shows that the $\vec{\Pi}_{1j}$ tensor has the form

$$\begin{aligned} \vec{\Pi}_{1j} &= \int m_j \mathbf{v} \mathbf{v} f_{1j} d\mathbf{v} \\ &= \begin{bmatrix} p_{1\perp j} & 0 & 0 \\ 0 & p_{1\perp j} & 0 \\ 0 & 0 & p_{1\parallel j} \end{bmatrix} \end{aligned} \quad (33)$$

The elements $p_{1\perp j}$ and $p_{1\parallel j}$ are given by

$$p_{1\perp j} = -\boldsymbol{\xi}_\perp \cdot \nabla p_j - i \int (\omega - \hat{\omega}_{*j}) \frac{m_j v_\perp^2}{2} \frac{\partial F_j}{\partial \epsilon} s_j d\mathbf{v} \quad (34)$$

$$p_{1\parallel j} = -\boldsymbol{\xi}_\perp \cdot \nabla p_j - i \int (\omega - \hat{\omega}_{*j}) m_j v_\parallel^2 \frac{\partial F_j}{\partial \epsilon} s_j d\mathbf{v}$$

where

$$s_j = m_j \int_{-\infty}^t \left[\frac{v_\perp^2}{2} \nabla \cdot \boldsymbol{\xi}_\perp + \left(\frac{v_\perp^2}{2} - v_\parallel^2 \right) \boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa} \right] dt'. \quad (35)$$

Combining these results leads to a single vector equation for the unknown $\boldsymbol{\xi}_\perp$

$$-\rho \omega^2 \boldsymbol{\xi}_\perp = \mathbf{F}_\perp(\boldsymbol{\xi}_\perp) + i \mathbf{D}_\perp(\boldsymbol{\xi}_\perp). \quad (36)$$

Here

$$\mathbf{F}_\perp(\boldsymbol{\xi}_\perp) = \mathbf{J}_1 \times \mathbf{B} + \mathbf{J} \times \mathbf{B}_1 + \nabla(\boldsymbol{\xi}_\perp \cdot \nabla p) \quad (37)$$

is the ideal MHD force operator for incompressible displacements (and as implied, $\mathbf{F}_\perp \cdot \mathbf{b}$ can be easily shown to vanish). The operator \mathbf{D}_\perp contains the dissipative effects associated with resonant particles and can be written as

$$\mathbf{D}_\perp(\boldsymbol{\xi}_\perp) = \sum_j m_j \int \left[\frac{v_\perp^2}{2} \nabla_\perp + \left(v_\parallel^2 - \frac{v_\perp^2}{2} \right) \boldsymbol{\kappa} \right] \left[(\omega - \hat{\omega}_{*j}) \frac{\partial F_j}{\partial \epsilon} s_j \right] d\mathbf{v}. \quad (38)$$

Equations (36)–(38) describe the low frequency, finite wavenumber stability of energetic particle-Alfvén waves in an axisymmetric torus.

It is now convenient to introduce a subsidiary expansion in β . We assume $\delta \ll \beta \ll 1$, but note that no ordering is required for the inverse aspect ratio, the deviation from circularity, or the safety factor: $\epsilon \sim \kappa - 1 \sim q \sim 1$.

The low β assumption is useful because the dissipation operator becomes small compared to the inertial effects. This can be seen by recognizing that the orbit integral s_j scales as

$$s_j \sim \frac{m_j v_{\perp}^2 \xi_{\perp}}{R(\omega - k_{\parallel} v_{\parallel})}. \quad (39)$$

Equation (38) implies that

$$D_j \sim \frac{p_j \xi_{\perp}}{R^2} x_j Z(x_j) \quad (40)$$

where D_j is the j 'th particle contribution to $|\mathbf{D}_{\perp}|$, $x_j = \omega/k_{\parallel} v_{Tj}$, and $Z(x)$ is the plasma dispersion function. The function $xZ(x)$ is in general complex, but $|xZ(x)| \lesssim 1$ for all x assuming $\omega_i > 0$. By noting that $F_{\perp} \sim k_{\parallel}^2 B^2 \xi_{\perp} / \mu_0$ and $k_{\parallel} R \sim n/q \sim 1$ for Alfvén waves it follows that

$$|\mathbf{D}_{\perp} / \mathbf{F}_{\perp}| \lesssim \beta. \quad (41)$$

This scaling result is important because in the low β limit it implies that the kinetic effects make only a small modification to the ideal MHD wave. In particular, the real part of $i\mathbf{D}_{\perp}$ leads to a small β correction in the basic dispersion relation $\omega \approx k_{\parallel} v_a$. The imaginary part of $i\mathbf{D}_{\perp}$ produces the kinetic dissipation effects that determine the growth rate and stability boundaries. Since $Im(i\mathbf{D}_{\perp})$ is also small, it can be calculated perturbatively.

B General expression for the growth rate

In this section we make use of the low β ordering to derive a general expression for the growth rate of energetic particle Alfvén waves. The derivation is based on an energy relation obtained by multiplying Eq. (36) by $\xi_{\perp}^*/2$ and integrating over the plasma volume. We assume $\delta \ll 1$ but temporarily treat $\beta \sim 1$. Using Eq. (32) we obtain

$$\omega^2 K_M = \delta W_M + \delta W_K \quad (42)$$

where

$$K_M = \frac{1}{2} \int \rho |\xi_{\perp}|^2 dr \quad (43)$$

is the plasma kinetic energy normalization,

$$\delta W_M = -\frac{1}{2} \int \xi_{\perp}^* \cdot \mathbf{F}_{\perp}(\xi_{\perp}) dr \quad (44)$$

is the ideal MHD perpendicular potential energy and

$$\delta W_K = -\frac{i}{2} \int \xi_{\perp}^* \cdot \mathbf{D}_{\perp}(\xi_{\perp}) dr \quad (45)$$

is the kinetic contribution to the energy. After a simple integration by parts, δW_K can be rewritten as

$$\delta W_K = \frac{i}{2} \sum_j \int (\omega - \hat{\omega}_{*j}) \frac{\partial F_j}{\partial \epsilon} s_j \frac{ds_j^*}{dt} dv dr. \quad (46)$$

The next step is to recognize that s_j is an integral along the unperturbed particle orbits as a function of the guiding center coordinates. Therefore,

$$\frac{ds_j^*}{dt} = i\omega^* s_j^* + D s_j^*. \quad (47)$$

where

$$D \equiv \mathbf{v} \cdot \nabla + \frac{q_j}{m_j} \mathbf{v} \times B_0 \cdot \nabla_v \quad (48)$$

If we write the complex quantity s_j as $s_j = a_j + ic_j$ and make use of Eq. (47), we obtain

$$s_j \frac{ds_j^*}{dt} = i\omega^* |s_j|^2 + i(c_j D a_j - a_j D c_j) + \frac{1}{2} D (a_j^2 + c_j^2). \quad (49)$$

Since F and $\hat{\omega}_*$ are functions of the constants of motion ϵ and p_ϕ , the contribution to δW_K from the last term in Eq. (49) is an exact differential that integrates to zero over the phase space volume. The quantity δW_K can thus be rewritten as

$$\delta W_K = \frac{1}{2} \sum_j \int (\omega - \hat{\omega}_{*j}) \frac{\partial F_j}{\partial \epsilon} (\bar{R}_j + i\omega_i |s_j|^2) dv dr \quad (50)$$

where \bar{R}_j is a real quantity given by

$$\bar{R}_j = c_j D a_j - a_j d c_j - \omega_r |s_j|^2. \quad (51)$$

Consider now the limit $\omega_i \ll \omega_r$ corresponding to either marginal stability or the low β assumption. A straightforward dimensional analysis using Eq. (39) shows that

$$\int \frac{\partial F_j}{\partial \epsilon} |s_j|^2 dv dr \lesssim \frac{\epsilon p_j \xi_\perp^2}{k_{\parallel} v_{Tj} \omega_i} \sim \frac{1}{\omega_i}. \quad (52)$$

Thus, the term $i\omega_i |s_j|^2$ in Eq. (50) leads to a finite contribution in δW_K as $\omega_i \rightarrow 0$. More subtly, the \bar{R}_j term in Eq. (50) also leads to a finite contribution of order $\beta_j \delta W_M$. This is a direct consequence of Eq. (41) and the discussion therein. The subtlety is that each of the separate terms in \bar{R}_j diverge as $1/\omega_i$, but because of the general scaling argument associated with Eq. (41), these singularities must cancel identically.

The desired expression for the growth rate is obtained by setting the real and imaginary parts of Eq. (42) to zero and introducing the low β expansion. The real part yields

$$\omega_r^2 = \frac{\delta W_M}{K_M} + O(\beta). \quad (53)$$

The real frequency and eigenfunction correspond to the ideal MHD Alfvén wave.

In the limit $\omega_i \ll \omega_r$ consistent with low β , the imaginary part of Eq. (42) yields

$$\omega_i = \frac{W_K}{K_M + K_K} \quad (54a)$$

$$\approx \frac{W_K}{K_M} \quad (54b)$$

where

$$W_K = \frac{-L}{\omega_i \rightarrow 0} \left[\frac{1}{4\omega_r} \sum_j \int (\hat{\omega}_{*j} - \omega_r) \frac{\partial F_j}{\partial \epsilon} \omega_i |s_j|^2 \right] dv dr$$

$$K_K = -\frac{1}{4\omega_r} \sum_j \int \frac{\partial F_j}{\partial \epsilon} \bar{R}_j dv dr.$$
(55)

Note that K_K is neglected in the second expression for ω_i since $K_K \sim \beta K_M$. Also, for rigid rotor distribution functions $\hat{\omega}_{*j} = \omega_{*j} = \text{const.}$

Equation (54b) is a general expression for the growth rate. It is valid for arbitrary aspect ratio and noncircularity but requires $\delta \ll \beta \ll 1$. Its evaluation requires a knowledge of the trajectory integral s_j along the unperturbed guiding center orbits. This can be carried out “exactly” numerically for realistic tokamak geometries, or evaluated approximately analytically using the additional assumption of large aspect ratio. For the usual case of $\partial F_j / \partial \epsilon < 0$, we see that a given species produces a stabilizing contribution to the dispersion relation when its diamagnetic frequency is less than the real frequency of the mode: $\hat{\omega}_{*j} < \omega_r$. The converse situation applies when $\hat{\omega}_{*j} > \omega_r$. The point of equality $\hat{\omega}_{*j} = \omega_r$ represents the transition from the positive to negative dissipation for the j 'th species.

C The TAE growth rate

The general expression for the growth rate given by Eq. (54b) is now applied to the TAE instability. To proceed analytically we assume large aspect ratio as well as low β . Specifically, we assume a circular cross section tokamak satisfying the ohmic scaling expansion $\beta \sim \epsilon^2$, $q \sim 1$.

The analysis begins with the evaluation of the unperturbed particle orbits and the trajectory integral s_j . Consistent with the ohmic expansion, the leading order guiding center orbits are given by

$$r(t') = r(t)$$

$$\begin{aligned}\theta(t') &= \frac{v_{\parallel} B_{\theta}}{r B_0} (t' - t) \\ \phi(t') &= \frac{v_{\parallel}}{R_0} (t' - t).\end{aligned}\tag{56}$$

The particles lie on circular flux surfaces and move parallel to \mathbf{B} .

Next, note that within the tokamak expansion $\nabla \cdot \boldsymbol{\xi}_{\perp} + 2\boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\kappa} \sim O(\epsilon^2)$ and $\boldsymbol{\kappa} = \mathbf{e}_R/R_0 + O(\epsilon^2/a)$ for Alfvén waves. Thus, to leading order ϵ , s_j reduces to

$$s_j = -m_j \int_{-\infty}^t \left(v_{\parallel}^2 + \frac{v_{\perp}^2}{2} \right) \frac{\xi_R}{R_0} dt'.\tag{57}$$

Also note that both v_{\parallel} and v_{\perp} are constant to leading order in ϵ .

The integral s_j is evaluated by expanding the eigenfunction $\boldsymbol{\xi}_r \equiv \boldsymbol{\xi}$ as a general Fourier series

$$\begin{aligned}\xi &= \sum_m \xi_m(r) e^{im\theta} \\ \xi_{\theta} &= i \sum_m \frac{(r\xi_m)'}{m} e^{im\theta}\end{aligned}\tag{58}$$

where the second expression follows from the relation $\nabla \cdot \boldsymbol{\xi}_{\perp} \sim O(\epsilon)$. Substituting into Eq. (58) and focussing on circulating particles we obtain

$$s_j = \frac{im_j}{2R_0} \left(v_{\parallel}^2 + \frac{v_{\perp}^2}{2} \right) \sum_m \left[\xi_{m-1} + \xi_{m+1} - \frac{(r\xi_{m-1})'}{m-1} + \frac{(r\xi_{m+1})'}{m+1} \right] \frac{e^{-i\omega t - in\phi + im\theta}}{\omega - \omega_m}\tag{59}$$

where

$$\omega_m = \left(\frac{m}{q} - n \right) \frac{v_{\parallel}}{R_0}.\tag{60}$$

Equation (59) is valid for both the TAE and EAE instabilities.

Consider now the TAE instability. This perturbation consists primarily of two toroidally coupled harmonics ξ_m and ξ_{m+1} . All other harmonics are essentially zero. Furthermore, the strongest coupling occurs in a narrow region of thickness $\sim \epsilon a$ localized about the surface $r = r_0$ corresponding to $q_0 \equiv q(r_0) = (2m+1)/2n$. The mode localization implies that the $\xi'_{m\pm 1}$ terms dominate in Eq. (59). Substituting these results into the expression

for s_j and maintaining only those terms which do not average to zero in θ , leads to the following expression for $|s_j|^2$:

$$|s_j|^2 = \frac{m_j^2 r_0^2}{4R_0^2} \left(v_{\parallel}^2 + \frac{v_{\perp}^2}{2} \right)^2 \left[\frac{|\xi'_m|^2}{m^2} + \frac{|\xi'_{m+1}|^2}{(m+1)^2} \right] \left[\frac{1}{|\omega - \omega_{m-1}|^2} + \frac{1}{|\omega - \omega_m|^2} \right]. \quad (61)$$

In this expression all equilibrium quantities are evaluated on the surface $r = r_0$.

At this point the general expression for the growth rate [Eq. (54b)] can be simplified significantly by noting that for localized perturbations, the kinetic energy normalization reduces to

$$K_M \approx \frac{r_0^2 \rho_0}{2} \int \left[\frac{|\xi'_m|^2}{m^2} + \frac{|\xi'_{m+1}|^2}{(m+1)^2} \right] dr \quad (62)$$

where $\rho_0 = \rho_i(r_0)$. Observe that K_M and $|s_j|^2$ have the same combination of ξ'_m and ξ'_{m+1} terms which thus cancel when evaluating ω_i . Using the fact that $\omega_r \approx k_{\parallel} v_a$ with $k_{\parallel} = 1/2R_0 q_0$ for the TAE instability, we obtain the following expression for the growth rate

$$\frac{\omega_i}{k_{\parallel} v_a} = L \sum_j \frac{\mu_0 m_j^2 q_0^2}{2B_0^2} \int \left(v_{\parallel}^2 + \frac{v_{\perp}^2}{2} \right)^2 \left(\omega_r \frac{\partial F_j}{\partial \epsilon} - \frac{n}{q_j} \frac{\partial F_j}{\partial \Psi} \right) \left[\frac{\omega_i}{|\omega - \omega_{m-1}|^2} + \frac{\omega_i}{|\omega - \omega_m|^2} \right] dv. \quad (63)$$

In the limit of small ω_i , the v_{\parallel} integral can be carried out analytically. A short calculation yields

$$\frac{\omega_i}{k_{\parallel} v_a} = \sum_j \left[\frac{2\pi^2 \mu_0 m_j^2 R_0 q_0^3}{B_0^2} \int \left(v_{\parallel}^2 + \frac{v_{\perp}^2}{2} \right)^2 \left(\omega_r \frac{\partial F_j}{\partial \epsilon} - \frac{n}{q_j} \frac{\partial F_j}{\partial \Psi} \right) v_{\perp} dv_{\perp} \right] \Big|_{v_{\parallel}=v_a} + \Big|_{v_{\parallel}=v_a/3}. \quad (64)$$

Equation (64) gives the TAE growth rate for arbitrary distribution functions $F_j(\epsilon, \Psi)$.

The growth rate can be easily evaluated for a Maxwellian distribution function

$$F_j = n_j \left(\frac{m_j}{2\pi T_j} \right)^{3/2} \exp(-m_j v^2 / 2T_j). \quad (65)$$

Here $n_j = n_j(\Psi)$ and $T_j = T_j(\Psi)$. Some straightforward algebra leads to

$$\left(\frac{\omega_i}{k_{\parallel} v_a} \right)_j = -q_0^2 \beta_j \left[G_{mj}^T - n q_0 \delta_j \frac{(H_{mj}^T + \eta_j J_{mj}^T)}{1 + \eta_j} \right] \quad (66)$$

where $\beta_j = 2\mu_0 n_j T_j / B_0^2$, $\delta_j = -r_{L\theta} (dp_j/dr) / p_j$, $r_{L\theta} = v_{Tj} / \Omega_{\theta j}$, $\Omega_{\theta j} = q_j B_p / m_j$ and $\eta_j = d \ln T_j / d \ln n_j$. Each of these quantities is evaluated at $r = r_0$. The functions G_j^T , H_j^T and J_j^T are functions of the single parameter $\lambda_j = v_a / v_{Tj}$ and are given by

$$\begin{aligned} G_{mj}^T &= g_m(\lambda_j) + g_m(\lambda_j/3) & g_m(\lambda_j) &= \frac{\pi^{1/2}}{2} \lambda_j (1 + 2\lambda_j^2 + 2\lambda_j^4) e^{-\lambda_j^2} \\ H_{mj}^T &= h_m(\lambda_j) + \frac{1}{3} h_m(\lambda_j/3) & h_m(\lambda_j) &= \frac{\pi^{1/2}}{2} (1 + 2\lambda_j^2 + 2\lambda_j^4) e^{-\lambda_j^2} \\ J_{mj}^T &= j_m(\lambda_j) + \frac{1}{3} j_m(\lambda_j/3) & j_m(\lambda_j) &= \frac{\pi^{1/2}}{2} (3/2 + 2\lambda_j^2 + \lambda_j^4 + 2\lambda_j^6) e^{-\lambda_j^2}. \end{aligned} \quad (67)$$

The Maxwellian distribution function is a good approximation for electrons and ions. For these species the parameter δ_j (the ratio of poloidal gyroradius to pressure scale length) satisfies $\delta_j \ll 1$. Consequently only the G_j^T term need be maintained.

For the alpha particles it is more reasonable to assume a slowing down distribution. A simple model that has the correct asymptotic behavior and allows a simple evaluation of the integrals is as follows

$$F_\alpha = \frac{A}{(v^2 + v_0^2)^{3/2}} \quad 0 < v < v_\alpha \quad (68)$$

where $A = A(\Psi)$ and $v_0^2 = v_0^2(\Psi)$ are two free functions that are easily related to the density and temperature. The quantity v_0^2 represents the low velocity transition to the bulk plasma. Typically $m_\alpha v_0^2 / 2 \approx 2T(\Psi)$ where T is the bulk plasma temperature. The cutoff velocity v_α is defined by $m_\alpha v_\alpha^2 / 2 \equiv E_\alpha = 3.5$ MeV. Clearly $v_\alpha^2 \gg v_0^2$. After another straightforward calculation we obtain an analogous expression for the alpha particle contribution to the growth rate

$$\left(\frac{\omega_i}{k_{\parallel} v_a} \right)_\alpha \approx -q_0^2 \beta_\alpha (G_{s\alpha}^T - n q_0 \delta_\alpha H_{s\alpha}^T) \quad (69)$$

with $\beta_\alpha = 2\mu_0 n_\alpha T_\alpha / B_0^2$, $\delta_\alpha = -(2/3) r_{L\theta} (dp_\alpha/dr) / p_\alpha$, $r_{L\theta} = v_\alpha / \Omega_{\theta\alpha}$, and $\Omega_{\theta\alpha} = q_\alpha B_p / m_\alpha$. The functions G and H are functions of the parameter $\lambda_\alpha = v_a / v_\alpha$ and can be written as

$$\begin{aligned} G_{s\alpha}^T &= g_s(\lambda_\alpha) + g_s(\lambda_\alpha/3) & g_s(\lambda_\alpha) &= \left(\frac{3\pi}{16} \right) \lambda_\alpha (3 + 4\lambda_\alpha - 6\lambda_\alpha^2 - \lambda_\alpha^4) H(1 - \lambda_\alpha) \\ H_{s\alpha}^T &= h_s(\lambda_\alpha) + \frac{1}{3} h_s(\lambda_\alpha/3) & h_s(\lambda_\alpha) &= \left(\frac{3\pi}{16} \right) (1 + 6\lambda_\alpha^2 - 4\lambda_\alpha^3 - 3\lambda_\alpha^4) H(1 - \lambda_\alpha). \end{aligned} \quad (70)$$

Here H is the Heaviside step function.

The final form of the growth rate is obtained by combining these expressions and assuming $n_i \approx n_e \equiv n$, $T_e \approx T_i \equiv T$

$$\frac{\omega_i}{k_{\parallel} v_a} = -q_0^2 \left[\frac{\beta_c}{2} (G_{mi}^T + G_{me}^T) + \beta_{\alpha} (G_{s\alpha}^T - nq_0 \delta_{\alpha} H_{s\alpha}^T) \right]. \quad (71)$$

where $\beta_c = 4\mu_0 n T / B_0^2$ is the core beta. Equation (71) is a more accurate expression of the TAE growth rate given in the original work of Fu and Van Dam,⁹ where the effect of the ion Landau damping was not included. Applications of Eq. (71) and comparisons with other expressions of the TAE growth rate are discussed shortly.

D The EAE growth rate

We consider the EAE in the limit of small ellipticity ($\kappa - 1 \ll 1$). The strongest coupling occurs about the $q \simeq q_0 = (m + 1)/n$ surface and the perturbation consists primarily of two elliptically coupled harmonics m and $m+2$. Following the same procedures used for the TAE, the expression for $|s_j|^2$ becomes

$$|s_j|^2 = \frac{m_j^2 r_0^2}{4R_0^2} \left(v_{\parallel}^2 + \frac{v_{\perp}^2}{2} \right)^2 \left\{ \left[\frac{|\xi'_m|^2}{m^2} + \frac{|\xi'_{m+2}|^2}{(m+2)^2} \right] \frac{1}{|\omega - \omega_{m-1}|^2} + \left| \frac{\xi'_{m+2}}{m+2} - \frac{\xi'_m}{m} \right|^2 \frac{1}{|\omega - \omega_{m+1}|^2} \right\}.$$

Notice that, unlike the TAE, K_M and $|s_j|^2$ do not have the same combination of ξ'_m and ξ'_{m+2} . Thus, the lowest order eigenfunctions given in Ref. [12] are needed. As for the TAE, the largest contribution comes from the resonant layer about the $q = q_0$ surface. In the limit of small ω_i and using a slowing down distribution function for the alphas, the growth rate has the following form

$$\frac{\omega_i}{k_{\parallel} v_a} = - \left(\frac{q_0}{2} \right)^2 \left[\frac{\beta_c}{2} (G_{mi}^E + G_{me}^E) + \beta_{\alpha} \left(G_{s\alpha}^E - \frac{nq_0}{2} \delta_{\alpha} H_{s\alpha}^E \right) \right] \quad (72)$$

where

$$G_{mj}^E = g_m(\lambda_j/2) + \int dx g_m(u_{nj}/|y|) F(\xi')$$

$$G_{s\alpha}^E = g_s(\lambda_\alpha/2) + \int dx g_s(u_{n\alpha}/|y|) F(\xi')$$

$$H_{s\alpha}^E = \frac{1}{2} h_s(\lambda_\alpha/2) + \int dx \frac{u_{n\alpha}/|y|}{\lambda_\alpha} h_s(u_{n\alpha}/|y|) F(\xi')$$

$$F(\xi') = \left| \frac{\xi'_{m+2}}{m+2} - \frac{\xi'_m}{m} \right|^2 / \int \left(\left| \frac{\xi'_{m+2}}{m+2} \right|^2 + \left| \frac{\xi'_m}{m} \right|^2 \right) dx$$

$$u_{nj} = \frac{2n}{q_0 \Delta'} \lambda_j \quad u_{n\alpha} = \frac{2n}{q_0 \Delta'} \lambda_\alpha \quad dx = 2\pi^2 \frac{R_0 r_0^2 \Delta'}{s_0} dy$$

$$y = \frac{2s_0}{\Delta'} \left(\frac{r - r_0}{r_0} \right) \quad s_0 = \frac{r_0 q'_0}{q_0} \quad \Delta' = \left(r \frac{\kappa(r) - 1}{2} \right)' \Big|_{r_0}$$

Applications of Eq. (72) are also discussed shortly.

E The NAE growth rate

The Noncircular Triangularity Induced Alfvén Eigenmode (NAE) which results from triangular coupling is localized about the $q \simeq q_0 = (2m + 3)/2n$ surface and consists primarily of the two harmonics m and $m + 3$. The procedure to calculate the growth rate is identical to that used for the TAE and in the limit of small triangularity, the mode is highly localized so that the growth rate is independent of the eigenfunction (as for the TAE). It is easy to show that using a slowing down distribution function for the alphas, the NAE growth rate has the form

$$\frac{\omega_i}{k_{\parallel} v_a} = - \left(\frac{q_0}{3} \right)^2 \left[\frac{\beta_c}{2} \left(G_{mi}^N + G_{me}^N \right) + \beta_\alpha \left(G_{s\alpha}^N - \frac{nq_0}{3} \delta_\alpha H_{s\alpha}^N \right) \right] \quad (73)$$

where

$$G_{mj}^N = g_m(3\lambda_j) + g_m(3\lambda_j/5)$$

$$G_{s\alpha}^N = g_s(3\lambda_\alpha) + g_s(3\lambda_\alpha/5)$$

$$H_{s\alpha}^N = 3h_s(3\lambda_\alpha) + (3/5)h_s(3\lambda_\alpha/5)$$

IV Discussion

Marginal stability conditions can be easily evaluated by setting $\omega_i = 0$ in Eqs. (71), (72) and (74). In order to find the parameter $\beta_\alpha = 2\mu_0 n_\alpha T_\alpha / B_0^2$, we first calculate T_α from the α equilibrium distribution function

$$n_\alpha T_\alpha = \frac{1}{3} m_\alpha \int v^2 f_\alpha d^3 v.$$

For an isotropic slowing down distribution function with a cutoff energy E_α , the temperature is approximately constant and it is given by $T_\alpha \simeq E_\alpha/6$. The α particle density can be obtained by balancing the production with the slowing down rate

$$\langle \sigma v \rangle \frac{n_e^2}{4} = \frac{n_\alpha}{\tau_s}.$$

where $\tau_s = 1.17 \cdot 10^{12} [T_e(\text{keV})]^{3/2} / n_e (\text{cm}^{-3})$ is the slowing down time. If we assume that $T_e \simeq T_i$, then the marginal stability condition for the TAE and NAE can be written in the form

$$n_e^{T,N} = n_e^{T,N}(T_e, B_0, q(r), a, R_0, m, n). \quad (75)$$

Since the growth rate of the EAE depends on the mode eigenfunctions, the dependence on the plasma elongation must be included and the EAE marginal stability condition becomes

$$n_e^E = n_e^E(T_e, B_0, q(r), \kappa(r), a, R_0, m, n). \quad (76)$$

Stability domains in the (n_e, T) space, for typical ignition experiment parameters ($B_0 = 8$ T, $q_0 \simeq 1$, $q_a \simeq 3.5$, $\kappa(a) = 1.8$, $a = .78$, $R_0 = 2.4$) are illustrated in Fig. 1. Instability occurs in a well defined range $n_{min} < n < n_{max}$. The lower boundary occurs when the Alfvén velocity becomes so high that there are no resonant α particles, while the higher limit occurs when the ion or the electron Landau damping becomes dominant. For BPX operational density of $n_0 \simeq 5 \times 10^{20} \text{ m}^{-3}$ and temperature of $T_0 \simeq 20 \text{ keV}$ the $m = 1, n = 1$ gap modes seem to be stable. Modes with $m > 1, n > 1$ show a lower instability threshold and therefore are more likely to be destabilized in a burning plasma. In particular the EAE $m = 3, n = 3$ seems to pose a potential threat to a BPX-like device.

Several interesting conclusions can be extracted from Eqs. (71), (72) and (73). First, we observe that for large β , the resonance $v_{\parallel} = v_a/3$ of the TAE strongly enhances the ion Landau damping. This explains the stability domain of the TAE at high plasma densities as shown in Fig. 2.

The next point to note is that the electron Landau damping in highly elongated plasmas strongly stabilizes low- n EAEs. Figure 3 shows the EAE ($m = 1, n = 1$) stability boundaries for different values of the plasma elongation.

We also observe that for all three gap modes the destabilizing term is proportional to the product nq_0 . Therefore, we expect the high- m modes to be the most unstable as shown in Fig. 1.

Other expressions for the TAE growth rate have been compared with Eq. (71). The most recent one is given in Ref. [13] where a slowing down distribution function for the alphas has been used, but the effects of the poloidal sideband coupling has not been included.

According to Ref. [13] and in the limit of $v_{\alpha}/v_A \gg 1$, the marginal stability condition for the TAE ($n = 1, m = 1, 3$) can be written in the following form

$$\beta_{\alpha} \geq \frac{8}{3\pi} \sum_j \beta_j g(\lambda_j) / (\delta_{\alpha} - 2\lambda_{\alpha}) \quad (77)$$

where $\delta_{\alpha}, \lambda_{\alpha}, \lambda_j$ and $g(\lambda_j)$ have the same meaning as in Eqs. (67-71).

For the same conditions and neglecting the sideband coupling, Eq. (71) yields

$$\beta_{\alpha} \geq \frac{32}{9\pi} \sum_j \beta_j g(\lambda_j) / (\delta_{\alpha} - 2\lambda_{\alpha}) \quad (78)$$

which is in general agreement with the result of Ref. [13]. However, we emphasize the importance of the sideband coupling on the stabilizing effect of the ion Landau damping, as included in Eq. (71).

In the derivation of the gap mode growth rates, the effects of continuum damping and trapped particle resonance have not been included. The latter are expected to be important

for high- n modes. Finally, because of the limits of validity $\epsilon \ll 1, \kappa - 1 \ll 1, \delta \ll 1$, low (m, n) the expressions (71), (72) and (73) can only be considered as an approximate criterion to test stability against Alfvén gap modes.

V Conclusions

We have demonstrated that two new global Alfvén modes, the Ellipticity and Triangularity Induced Alfvén Eigenmodes, can be destabilized via transit resonance with energetic alpha particles. The growth or damping of these modes depends upon a competition between the alpha particle driver, electron and ion Landau damping. The electron Landau damping turns out to be particularly important for the $m = 1, n = 1$ EAE in highly elongated plasmas ($\kappa \sim 2$). We have also shown that ion Landau damping can stabilize the TAE at high density regimes. For sufficiently high densities and $n > 1$ the EAE can have a stability threshold lower than that of the TAE. Both the EAE and TAE need further investigation to determine how detrimental their effects can be on alpha particle confinement in ignited tokamaks.

In particular, the behavior of higher n modes which have lower thresholds, but greater localization needs to be addressed.

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Appendix A

Effects of an Equilibrium Electric Field

In this appendix we show that the existence of an equilibrium electric field $\mathbf{E}_0 \equiv -\nabla\phi_0$ does not fundamentally alter Eq. (27) upon which our stability analysis is based. To lowest order in r_{Le}/a the electrons are adiabatic and therefore neglecting the electron inertia, the parallel electron momentum equation can be written as

$$\mathbf{E}_{\parallel} = -\nabla_{\parallel} \frac{1}{e} \int \frac{dp_e}{n_e}. \quad (A1)$$

Linearizing Eq. (A1) gives the following expression of the perturbed parallel electric field

$$E_{\parallel 1} = -\mathbf{b} \cdot \nabla(\boldsymbol{\xi}_{\perp} \cdot \mathbf{E}_0). \quad (A2)$$

The definition of the $\mathbf{E} \times \mathbf{B}$ displacement can be rewritten as follows:

$$\mathbf{E}_1 \equiv i\omega \boldsymbol{\xi}_{\perp} \times B_0 - \nabla(\boldsymbol{\xi}_{\perp} \cdot \mathbf{E}_0) \quad (A3)$$

an Eq. (19) becomes

$$\mathbf{E}_1 \cdot \mathbf{v} = -i\omega \frac{m_j}{q_j} \left[\frac{d}{dt}(\boldsymbol{\xi}_{\perp} \cdot \mathbf{v}) - \mathbf{v} \cdot \frac{d\boldsymbol{\xi}_{\perp}}{dt} \right] - \frac{d}{dt}(\boldsymbol{\xi}_{\perp} \cdot \mathbf{E}_0). \quad (A4)$$

Substituting these results into Eq. (17) leads to the following form of Eq. (20)

$$f_{1j} = -q_j \left[\frac{\partial F_j}{\partial p_{\phi}} \boldsymbol{\xi}_{\perp} \cdot \nabla\psi + \frac{\partial F_j}{\partial \epsilon} \boldsymbol{\xi}_{\perp} \cdot \nabla\phi_0 \right] + im_j \left(\omega \frac{\partial F_j}{\partial \epsilon} - n \frac{\partial F_j}{\partial p_{\phi}} \right) \left[(\boldsymbol{\xi}_{\perp} \cdot \mathbf{v}) - \int_{-\infty}^t \mathbf{v} \cdot \frac{d\boldsymbol{\xi}_{\perp}}{dt'} dt' \right]. \quad (A5)$$

Following the averaging procedure described in section IID leads to an unchanged form of Eq. (27)

$$f_{1j} \approx -\boldsymbol{\xi}_{\perp} \cdot \nabla F + im_j(\omega - \hat{\omega}_j^*) \frac{\partial F}{\partial \epsilon} \left[\boldsymbol{\xi}_{\perp} \cdot \mathbf{v} - \int_{-\infty}^t a_0 dt' \right].$$

However, in the calculation of the divergence of the pressure tensor, the equilibrium electric field contributes through an inertial term $\simeq \sum_j \rho_j \omega \omega_{Ej}$; where ρ_j is the density of the species j , and $\omega_{Ej} \equiv \mathbf{k}_{\perp} \cdot (\mathbf{E}_0 \times \mathbf{B}_0)/B_0^2$. For Alfvén waves ($\omega \sim \omega_A$) and for small α particle population ($\rho_i \gg \rho_{\alpha}$), the ion inertia dominates ($\rho_i \omega^2 \gg \rho_j \omega \omega_{Ej}$) and the equilibrium electric field contribution can be neglected.

Figure Captions

Fig. 1 Plot of the stability domain in the (n_e, T) for BPX-like parameters ($B_0 = 8.1$ T, $a = .79$ m, $R_0 = 2.59$ m, $q(0) = 1$, $q(a) = 3.5$): (a) $m = 1, n = 1$; (b) $m = 3, n = 3$; (c) $m = 5, n = 4$.

Fig. 2 Plot of the stability domain in the (n_e, T) space of the TAE $m = 1, n = 1$ for BPX, including ion Landau damping (solid line) and neglecting it (dashed line).

Fig. 3 Plot of the stability domain in the (n_e, T) space of the EAE $m = 1, n = 1$ for BPX, for different values of the plasma elongation and with $q(a) = 4$.

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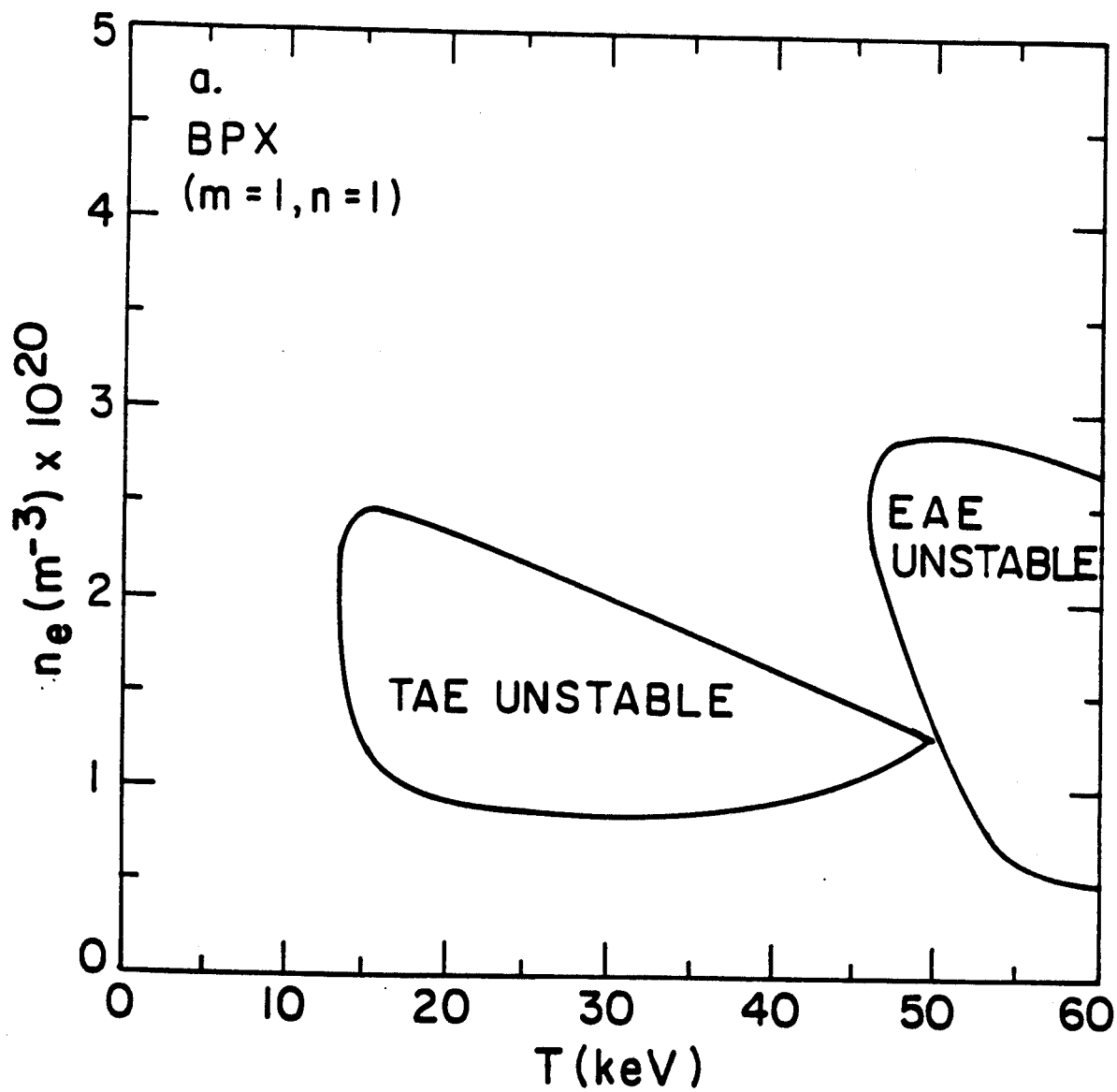


FIGURE 1a

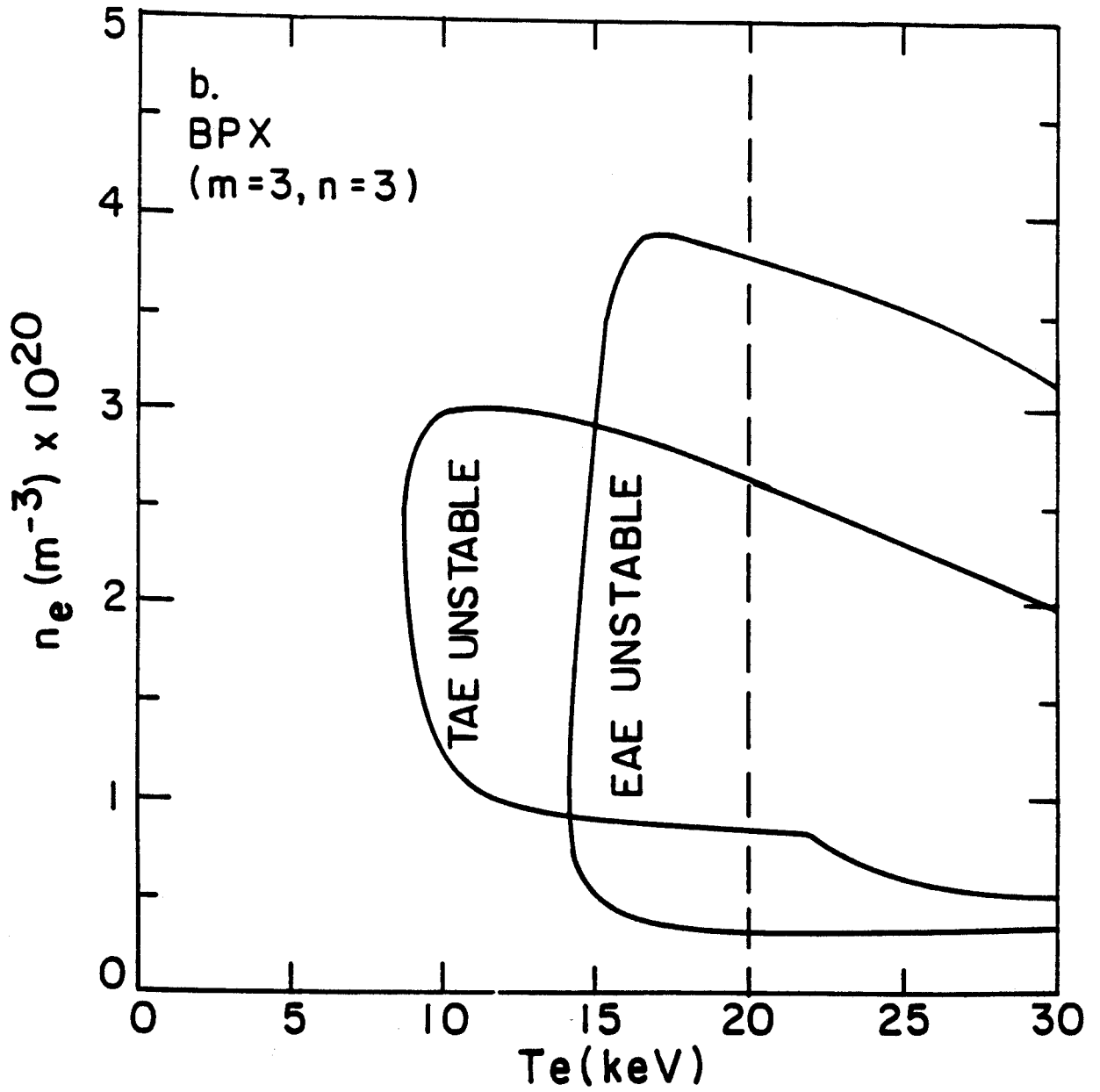


FIGURE 1b

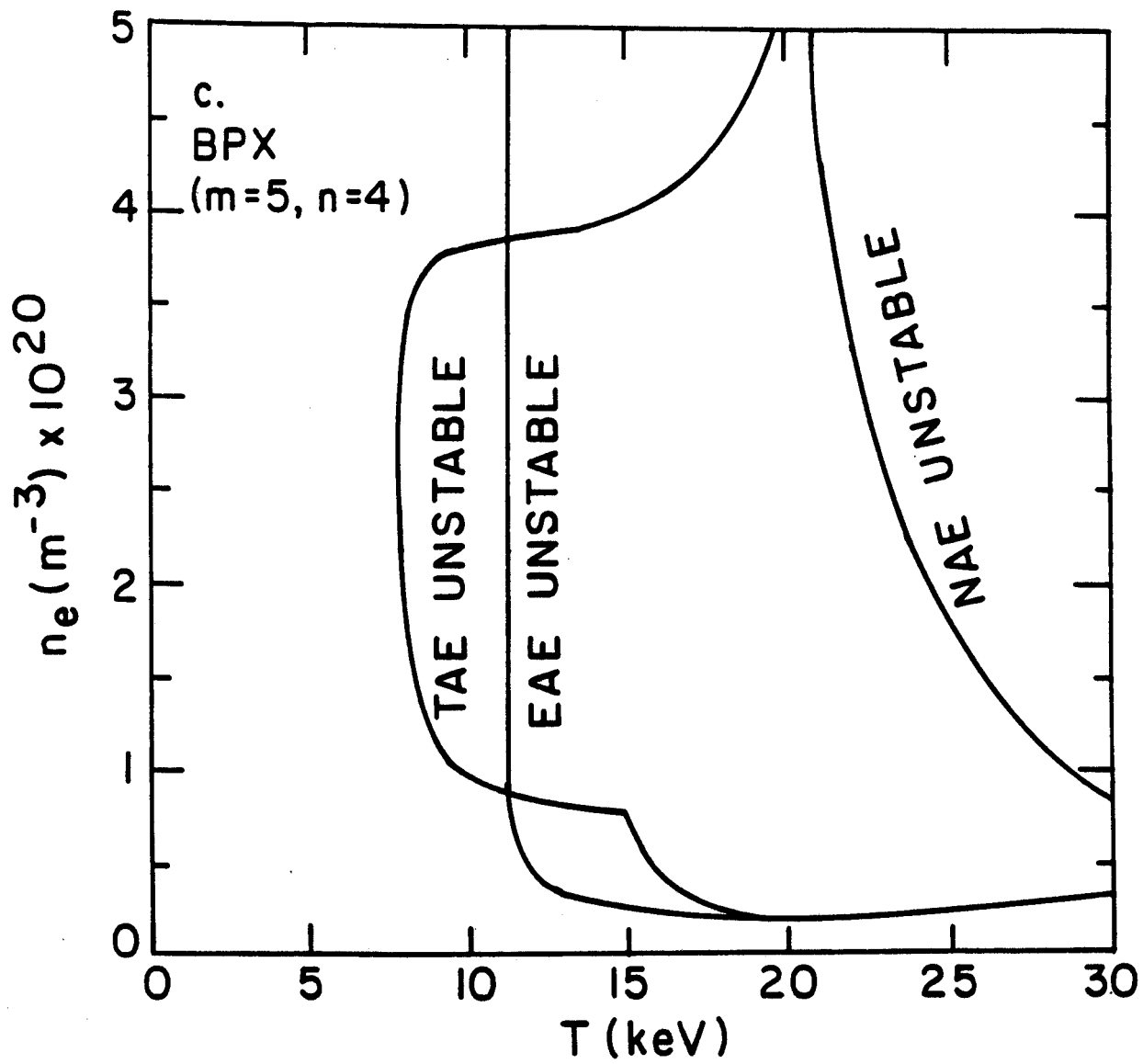


FIGURE 1c

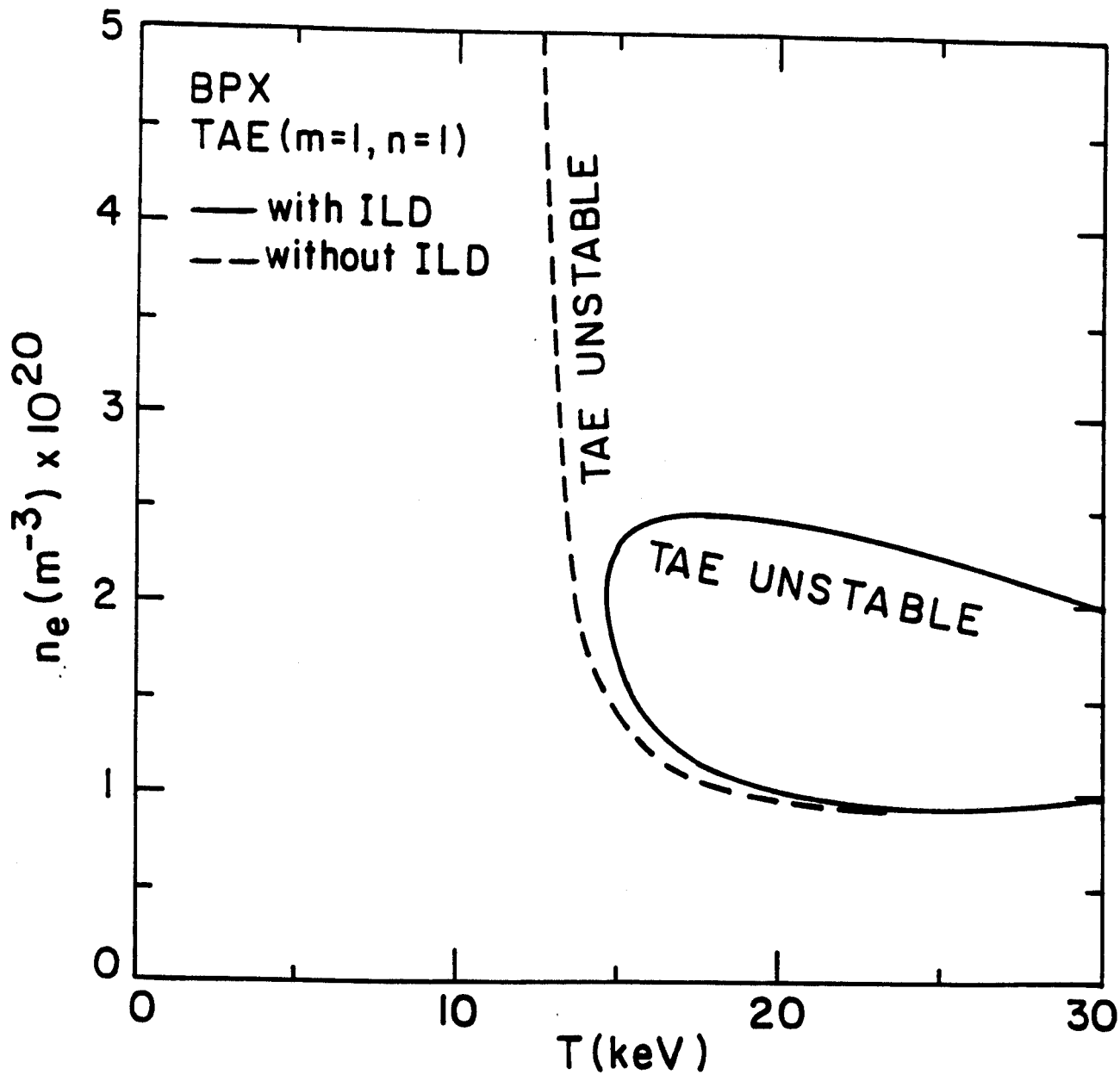


FIGURE 2

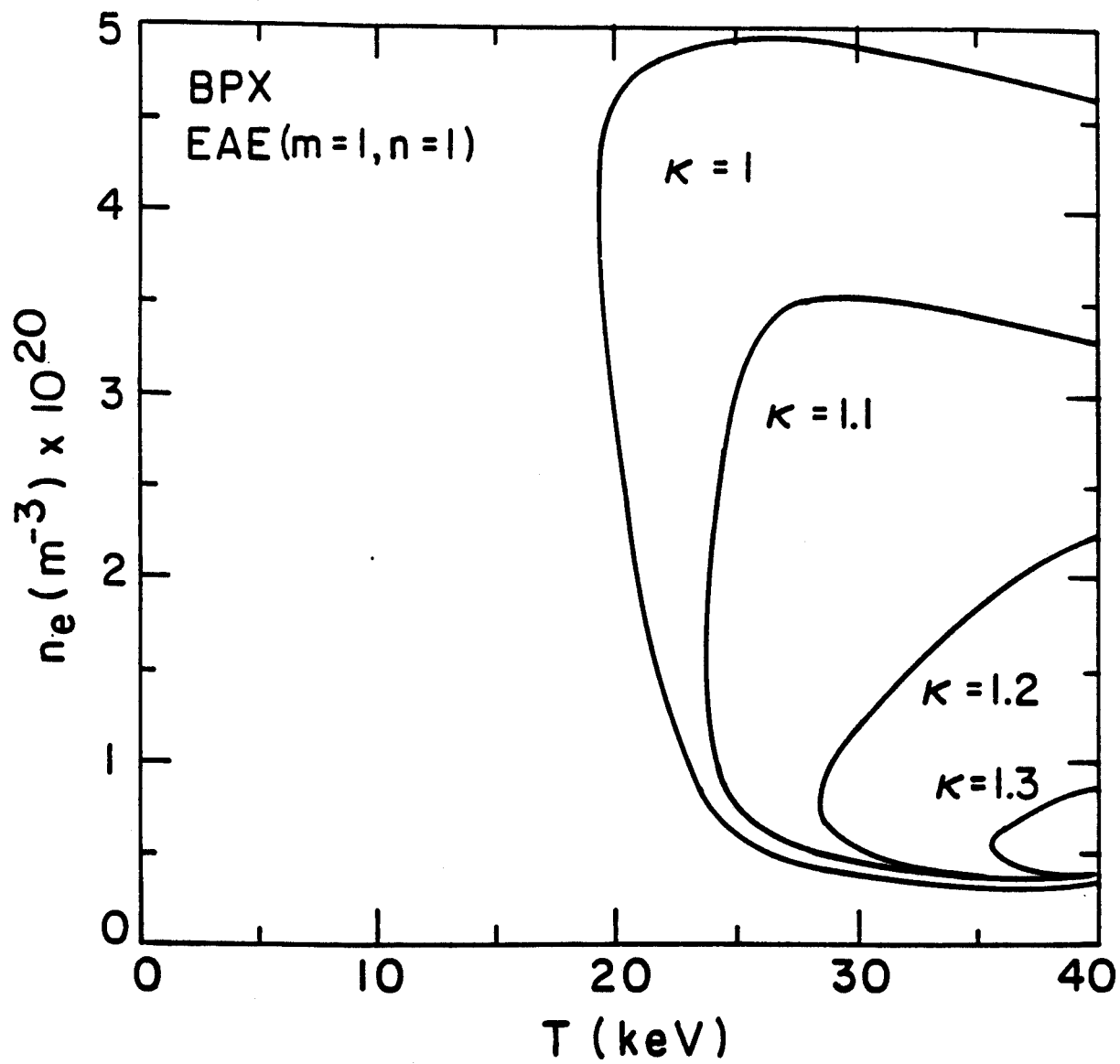


FIGURE 3