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STABILITY OF BIFURCATING SOLUTIONS OF THE GIERER-MEINHARDT SYSTEM

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Introduction. In this paper we are concerned with stationary solutions of a system of semilinear parabolic partial differential equations arising in the biological pattern formation theory. A most fundamental problem in morphogenesis is to explain how the initially almost homogeneous state of cells (or tissues) gains spatial inhomogeneity or spatial patterns. In his pioneering paper [10], Turing considered this problem by introducing the model governed by a system of (linear) ordinary differential equations. Developing Turing's idea, Gierer and Meinhardt [4] proposed a reaction-diffusion system of the following type:

$$\text{(G-M)}\qquad \begin{cases} \dfrac{\partial a}{\partial s} = D_a \dfrac{\partial^2 a}{\partial y^2} - \tilde{\mu} a + c\tilde{\rho}\dfrac{a^2}{h} + \rho_0\tilde{\rho}\,, \\ \dfrac{\partial h}{\partial s} = D_h \dfrac{\partial^2 h}{\partial y^2} - \nu h + c'\rho' a^2\,, \end{cases} \qquad \text{for} \quad s > 0\,, \quad 0 < y < \tilde{L}\,.$$

Here, D_a, D_h, $\tilde{\mu}$, ν, c, c', $\tilde{\rho}$ and ρ' are all positive constants and ρ_0 is a nonnegative constant. The positive functions $a(s, y)$ and $h(s, y)$ represent the concentrations of an activator and an inhibitor, respectively.

From a biological point of view, a natural boundary condition is the zero flux condition, i.e.,

$$\partial a/\partial y = \partial h/\partial y = 0 \quad \text{at both end points of the interval}\,.$$

Note that the system (G-M) has a unique constant stationary solution subject to this boundary condition.

Under suitably chosen values of the constants D_a, D_h, $\cdots$, ρ_0, numerical analyses show that the solutions of (G-M) corresponding to almost constant initial values tend to the stationary solutions exhibiting spatial wavy patterns. (This suggests that the constant solution is unstable.) The place where the activator highly concentrates is regarded as the position at which cell differentiation or division begins.

It is also predicted numerically that the wave length of the stationary solutions remains almost constant when the spatial length $\tilde{L}$ is large; while the amplitude of the solutions becomes large when the ratio D_a/D_h is small enough (so-called "striking-patterns").

Recently, there have appeared some mathematical works related to the Gierer and Meinhardt models. Granero, Porati and Zanacca [3] applied the bifurcation theory to (G-M) (taking the spatial length $\tilde{L}$ as the bifurcation parameter) and showed the explicit forms of small non-homogeneous stationary solutions around the constant solution. On the other hand, as for the striking-patterns, Mimura [8] considered a system slightly different from (G-M), which is also proposed by Gierer and Meinhardt [4]. He studied stationary solutions with large amplitude with the aid of a singular perturbation technique. See also Mimura, Nishiura, and Yamaguti [9].

The purpose of this paper is to prove the existence of wave-like stationary solutions (G-M) in the vicinity of the constant solution and to investigate their stability. The main tools are the bifurcation theorem and the perturbation results on simple eigenvalues due to Crandall and Rabinowitz [1 and 2]. We shall adopt the ratio D_a/D_h as the bifurcation parameter.

The outline of this paper is as follows: In Section 1 we give an abstract formulation of (G-M). For convenience, in Section 2 we list the notations which will be used frequently in the following sections. The existence of wave-like stationary solutions is proved in Section 3 (Theorem 3.8); we also discuss there the relations between the number of waves and the length of the spatial interval (Proposition 3.12). The stability of the nontrivial solutions will be studied in Sections 4, 5 and 6. In Section 4 we reduce the stability problem to the investigation of the signs of a polynomial $P_\rho(\ell)$ at some special points (Proposition 4.2); and then in Section 5 we study $P_\rho(\ell)$ in detail. Lastly, in Section 6 we give the stability criterion for the bifurcating solutions around the constant solution (Theorems 6.1 and 6.2).

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1. Formulation in L^2 framework. In this section we give an abstract formulation of the Gierer-Meinhardt system (G-M) under the zero flux boundary condition and state some related fundamental facts without proof (see, e.g., [6]).

To simplify the notations, we normalize the system (G-M) by the change of independent variables and unknowns:

$$(1.1)\qquad \begin{aligned} &s = t/\nu\ , \quad y = \sqrt{D_h/\nu}x\ , \quad a(s, y) = [c\tilde{\rho}/(c'\rho')]u(t, x)\ , \\ &h(s, y) = [(c\tilde{\rho})^2/(c'\rho'\nu)]v(t, x)\ . \end{aligned}$$

Then we obtain the equations for new variables:

$$(1.2)\qquad \begin{cases} u_t = Du_{xx} - u/m + u^2/v + \rho\ , \\ v_t = v_{xx} - v + u^2\ , \end{cases} \qquad \text{for} \quad t > 0\ , \quad 0 < x < L\ ,$$

where

$$(1.3)\qquad D = D_a/D_h\ , \quad m = \nu/\tilde{\mu}\ , \quad \rho = (c'\rho'\rho_0)/(c\nu)\ , \quad L = \sqrt{\nu/D_h}\tilde{L}\ .$$

The boundary condition is

$$(1.4)\qquad u_x = v_x = 0 \quad \text{at} \quad x = 0\ , \quad L\ .$$

First, we rewrite (1.2), regarding the unknowns ${}^t(u, v)$ as a column vector. Let

$$H = L^2(0, L) \times L^2(0, L)$$

be the Hilbert space with the inner product

$$(U_1, U_2)_H = (u_1, u_2)_{L^2(0,L)} + (v_1, v_2)_{L^2(0,L)}$$

for $U_1 = {}^t(u_1, v_1)$, $U_2 = {}^t(u_2, v_2) \in H$. Furthermore, let

$$\mathscr{H} = \{U = {}^t(u, v);\, u, v \in H^2(0, L)\ , \quad du/dx = dv/dx = 0 \quad \text{at} \quad x = 0\ , \quad L\}\ .$$

Here, $H^2(0, L)$ denotes the usual Sobolev space of order 2 on the interval $(0, L)$. By the well-known imbedding theorem, if u_1 and u_2 belong to $H^2(0, L)$, the product u_1u_2 also belongs to $H^2(0, L)$, and if moreover u_1 and u_2 satisfy the zero flux boundary condition, u_1u_2 does likewise (see (1.8) below).

Now let $\tilde{U} = {}^t(\tilde{u}, \tilde{v}) \in \mathscr{H}$ be a stationary solution of the system (1.2) subject to (1.4). Next we divide the operations in (1.2) into the linear part $L(D, \tilde{U})$ and the nonlinear part $N[\tilde{U};\cdot]$ around $\tilde{U}$. Define a closed linear operator $L(D, \tilde{U})$ by

$$(1.5)\qquad L(D, \tilde{U}) = \begin{pmatrix} Dd^2/dx^2 - 1/m + 2\tilde{u}/\tilde{v} & -(\tilde{u}/\tilde{v})^2 \\ 2\tilde{u} & d^2/dx^2 - 1 \end{pmatrix}$$

with domain $\mathscr{H}$ and set

$$(1.6a)\qquad N[\tilde{U}; U] = \sum_{p=2}^{\infty} N^{(p)}[\tilde{U}; U]\ ,$$

where

$$(1.6b)\qquad N^{(p)}[\tilde{U}; U] = \begin{cases} {}^t((\tilde{u}^2/\tilde{v})(u/\tilde{u} - v/\tilde{v})^2,\ u^2) & \text{for} \quad p = 2\ , \\ {}^t((\tilde{u}^2/\tilde{v})(u/\tilde{u} - v/\tilde{v})^2(-v/\tilde{v})^{p-2},\ 0) & \text{for} \quad p > 2\ . \end{cases}$$

Then (1.2) becomes an evolution equation as in (1.9) below.

Now we remark two simple facts (1.7) and (1.8):

(1.7) *The closed linear operator $L(D, \tilde{U})$ has compact resolvents $(\lambda - L(D, \tilde{U}))^{-1}$ and generates the holomorphic semigroup $e^{tL(D,\tilde{U})}$ on H. Therefore, the spectrum of $L(D, \tilde{U})$ consists only of the eigenvalues.*

Furnishing $\mathscr{H}$ with the graph topology

$$(U, V)_{\mathscr{H}} = (U, V)_H + (L(D, \tilde{U})U, L(D, \tilde{U})V)_H ,$$

we have

(1.8) *The operator $N[\tilde{U}:\cdot]$ is an analytic mapping from a neighborhood of the origin of $\mathscr{H}$ into $\mathscr{H}$ (i.e., the series $\sum_p N^{(p)}[\tilde{U}; U]$ converges in $\mathscr{H}$ if $\|U\|_{\mathscr{H}}$ is sufficiently small).*

Consider the evolution problem

$$\frac{dU}{dt} = L(D, \tilde{U})U + N[\tilde{U}; U] , \quad U(0) = U_0 \in \mathscr{H} , \tag{1.9}$$

which is an interpretation of (1.2) in L^2 framework. This problem can be solved uniquely in $C^0([0, T]; \mathscr{H}) \cap C^1([0, T]; H)$ (the vector space of all continuous functions of $t \in [0, T]$ with values in $\mathscr{H}$ which have continuous derivative in H) for sufficiently small $\|U_0\|_{\mathscr{H}}$ by formulating it as an integral equation.

We call a stationary solution $\tilde{U}$ *stable* if the following condition is satisfied: For all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\|U_0\|_{\mathscr{H}} < \delta$ then the solution $U(t)$ of (1.9) exists in the whole interval $(0, \infty)$ and satisfies $\|U(t)\|_{\mathscr{H}} < \varepsilon$ for all $t > 0$. If moreover $U(t) \to 0$ in $\mathscr{H}$ as $t \to \infty$, then $\tilde{U}$ is called *asymptotically stable.*

A stationary solution is said to be *unstable* if it is not stable.

We here recall a criterion for the stability (see [6, 7]).

(1.10) *If all the eigenvalues of the operator $L(D, \tilde{U})$ have negative real parts, then the stationary solution $\tilde{U}$ is asymptotically stable.*

(1.11) *If $L(D, \tilde{U})$ has an eigenvalue with positive real part, then $\tilde{U}$ is unstable.*

2. List of notations. For convenience' sake, we list here the notations which will be defined and used repeatedly in the following sections.

(3.1), (3.2) $\quad \bar{u} = m(1 + \rho) = 2m/(1 + \mu) , \quad \bar{v} = \bar{u}^2 , \quad \bar{U} = {}^t(\bar{u}, \bar{v})$

(3.5) $$\varphi_j(x) = \begin{cases} 1/\sqrt{L} & \text{for} \quad j = 0 , \\ \sqrt{2/L}\cos(\pi jx/L) & \text{for} \quad j = 1, 2, 3, \cdots \end{cases}$$

(3.7) $$\mu = (1-\rho)/(1+\rho)\,, \quad \rho = (1-\mu)/(1+\mu)$$

(3.12) $$g(\ell) = (\mu\ell - 1)/[m\ell(\ell+1)]$$

(3.4) $$\ell_j = (\pi j/L)^2$$

(3.13) $$D_j = g(\ell_j)\,, \quad D_k = \max\{D_j\}$$

(3.15) $$\ell_j^* = (1+\ell_j)/(\mu\ell_j - 1)$$

(3.16) $$\hat{\ell} = (1+\sqrt{1+\mu})/\mu$$

(5.16) $$K(\ell) = |1-\sqrt{\ell^*/\ell}|^{-1} \quad \text{with} \quad \ell^* = (\ell+1)/(\mu\ell - 1)$$

(3.19) $$L_\rho(j) = \pi[\{(2j'+1)^2 + 4\mu j'^2\}^{1/2} - 2j' - 1]^{1/2}/\sqrt{2}$$

where $j' = j(j+1)$

(3.20) $$\Lambda_\rho = \pi/\sqrt{\hat{\ell}}$$

(3.29) $$h(\kappa) = [1+\kappa^2+\{(1+\kappa^2)^2+4\mu\kappa^2\}^{1/2}]/(2\mu)$$

(3.28) $$h_1(j) = h[j/(j+1)]\,, \quad h_2(j) = \begin{cases} +\infty & \text{for } j=1 \\ h[j/(j-1)] & \text{for } j>1 \end{cases}$$

(4.4) $$P_\rho(\ell) = 4(5-6\rho)\ell^4 - (-47+93\rho+96\mu)\ell^3 - 9(7-\rho)\ell^2 + 3(39+31\rho)\ell + 15(1+\rho)$$

(5.1) $\lambda_1(\rho), \lambda_2(\rho)$: positive zeros of $P_\rho(\ell)$ $(\lambda_1(\rho) \leqq \lambda_2(\rho))$

(5.12) $$Q(\ell) = -2\ell^4 + 23\ell^3 + 16\ell^2 - 153\ell - 36\,,$$

ξ_1, ξ_2: positive zeros of $Q(\ell)$ $(\xi_1 < \xi_2)$

(numerically, $\xi_1 = 2.6488\cdots, \xi_2 = 11.6100\cdots$)

(5.13) $$r_i = (\xi_i^2 - 2\xi_i - 1)/(\xi_i + 1)^2 \qquad (i = 1, 2)$$

(numerically, $r_1 = 0.0539\cdots, r_2 = 0.6953\cdots$)

(5.15) $$0 < r_1 < 5/27 < 16/65 < r_2 < 5/6 < 1$$

3. Existence of nonconstant stationary solutions. It is clear that the system (1.2) under the boundary condition (1.4) has a unique constant stationary solution

(3.1) $$\bar{U} = {}^t(\bar{u}, \bar{v})$$

for all $D > 0$, where

(3.2) $$\bar{u} = m(1+\rho) \quad \text{and} \quad \bar{v} = \bar{u}^2\,.$$

Our main concern here is the existence of nonconstant stationary solutions around $\bar{U}$ (their stability will be discussed in the following sections). For this purpose, we shall apply the theory of "bifurcation from simple eigenvalues" (see Theorem 3.6). We adopt D as the bifurcation parameter and keep the other constants m, ρ and L fixed.

Let $\tilde{U} = \bar{U} + U$ be a stationary solution, U being sufficiently small. Then U satisfies the equation

$$L(D)U + N[U] = 0\ , \tag{3.3}$$

where

$$L(D) = L(D, \bar{U}) \quad \text{and} \quad N[\cdot] = N[\bar{U};\cdot]\ .$$

If the operator $L(D_0)$ is invertible for some D_0, then the implicit function theorem yields that (3.3) has no nontrivial solution near $D = D_0$. Hence we assume that $L(D)$ satisfies the following condition:

(C) *The operator* $L(D)$ *has* 0 *as a simple eigenvalue, i.e.*, $\dim \mathrm{Ker}(L(D)) = 1$, *and moreover all the other eigenvalues remain in the left half plane* $\mathrm{Re}\,\lambda < 0$.

Note that the second part of (C) is necessary for nontrivial solutions around $\bar{U}$ to be stable, since the spectrum of $L(D, \tilde{U})$ is close to that of $L(D, \bar{U})$ when $\tilde{U}$ is in the vicinity of $\bar{U}$ (see also (1.10) and (1.11)). We must carefully define the multiplicity of the eigenvalues λ of $L(D)$ because $L(D)$ is not symmetric. We call $\dim \mathrm{Ker}(L(D) - \lambda)$ the *multiplicity* of λ; while $\dim \bigcup_{n=1}^{\infty} \mathrm{Ker}((L(D) - \lambda)^n)$ is called the *algebraic multiplicity* of λ.

First, we translate (C) into some conditions on D, m, ρ and L. This will be achieved in Lemma 3.5. Secondly, we prove the existence of wave-like stationary solutions around $\bar{U}$ in Theorem 3.8, and then we discuss the relations between the number of the waves of such solutions and L in Proposition 3.12.

Let us begin with some preliminaries necessary for studying the eigenvalues of $L(D)$. Put

$$\ell_j = (\pi j/L)^2 \quad \text{for} \quad j = 0, 1, 2, \cdots, \tag{3.4}$$

and

$$\varphi_j(x) = \begin{cases} 1/\sqrt{L} & \text{for} \quad j = 0\ , \\ \sqrt{2/L}\cos(\pi j x/L) & \text{for} \quad j = 1, 2, 3, \cdots\ . \end{cases} \tag{3.5}$$

Then ℓ_j and φ_j satisfy

$$\varphi_j'' = -\ell_j \varphi_j\ , \quad \varphi_j'(0) = \varphi_j'(L) = 0\ ,$$

where $'$ stands for d/dx. Furthermore, the family $\{\varphi_j\}$ forms a complete orthonormal system of $L^2(0, L)$.

The spectrum of $L(D)$ is characterized by the next lemma.

LEMMA 3.1 (Eigenvalues of $L(D)$). (a) *The spectrum of* $L(D)$ *consists of all the numbers* λ *satisfying the equation*

$$\lambda^2 + [(D+1)\ell_j + 1 - \mu/m]\lambda + D\ell_j^2 + (D - \mu/m)\ell_j + 1/m = 0 \tag{3.6}$$

for some $j \geqq 0$, *where*

$$\mu = (1-\rho)/(1+\rho) . \tag{3.7}$$

(b) *Let* λ *be an eigenvalue of* $L(D)$. *Then* λ *satisfies* (3.6) *for at most two* j*'s, say* n, n'; *moreover, the corresponding eigenspace* Ker $(L(D)-\lambda)$ *is spanned by*

$$^t(\varphi_j, [2\bar{u}/(\lambda + 1 + \ell_j)]\varphi_j) \quad \textit{with} \quad j = n, n' ; \tag{3.8}$$

and

(i) $\dim \operatorname{Ker}(L(D)-\lambda) = 1 , \quad$ *if* $\quad n = n'$,

(ii) $\dim \operatorname{Ker}(L(D)-\lambda) = 2 , \quad$ *if* $\quad n \neq n'$.

PROOF. Consider the eigenvalue problem $L(D)U = \lambda U$, i.e.,

$$(Dd^2/dx^2 + \mu/m)u - v/\bar{u}^2 = \lambda u , \tag{3.9a}$$

$$2\bar{u}u + (d^2/dx^2 - 1)v = \lambda v , \tag{3.9b}$$

under the zero flux boundary condition. Eliminating v, we get a single equation for u:

$$Dd^4u/dx^4 - [(1+\lambda)D + \lambda - \mu/m]d^2u/dx^2 + [\lambda^2 + (1-\mu/m)\lambda + 1/m]u = 0 . \tag{3.10}$$

The Fourier expansion of u with respect to φ_j's yields that (3.10) possesses a nontrivial solution if and only if

$$D\ell_j^2 + [(1+\lambda)D + \lambda - \mu/m]\ell_j + \lambda^2 + (1-\mu/m)\lambda + 1/m = 0 \tag{3.11}$$

is satisfied for some j. Thus (3.6) holds.

Conversely, let λ satisfy (3.6) for some j. Noting that (3.6) implies $\lambda + 1 + \ell_j \neq 0$, we set $u = \varphi_j$ and $v = [2\bar{u}/(\lambda + 1 + \ell_j)]\varphi_j$. Then it is easy to see that $^t(u, v)$ satisfies (3.10) and (3.9b), that is, (3.9a) and (3.9b). Therefore λ is an eigenvalues of $L(D)$. Thus assertion (a) is verified.

Regarding (3.11) as an equation for ℓ_j, we see that there are at most two of ℓ_j's satisfying (3.11). This is sufficient to prove assertion (b). q.e.d.

Now we can translate (C) into a condition on D, m, ρ and L by a study of equation (3.6) with $\lambda = 0$. First put

$$g(\ell) = (\mu\ell - 1)/[m\ell(\ell + 1)] . \tag{3.12}$$

Then equation (3.6) with $\lambda = 0$ may be written simply as $D = D_j$, where we define

$$D_j = g(\ell_j) \ . \tag{3.13}$$

For the existence of positive D_j, we require $0 \leqq \rho < 1$, that is, $0 < \mu \leqq 1$. Let $j \geqq 1$. Then the equation

$$g(\ell) = D_j \tag{3.14}$$

has two roots $\ell = \ell_j$ and $\ell = \ell_j^*$, where

$$\ell_j^* = (1 + \ell_j)/(\mu\ell_j - 1) \ . \tag{3.15}$$

Observe moreover that $g(\ell)$ is negative if $0 < \ell < 1/\mu$, strictly increasing in the interval $(0, \hat{\ell})$ and strictly decreasing in $(\hat{\ell}, +\infty)$, where

$$\hat{\ell} = (1 + \sqrt{1 + \mu})/\mu \ . \tag{3.16}$$

(If we consider the mapping $\ell \mapsto \ell^* = (1 + \ell)/(\mu\ell - 1)$, we see easily $(\ell^*)^* = \ell$, $g(\ell) = 1/(m\ell\ell^*) = \mu/\{m(\ell + \ell^* + 1)\} = g(\ell^*)$ and that $\hat{\ell}$ is the unique positive fixed point of this mapping.)

We are now ready to state the first translation of (C).

LEMMA 3.2. *The operator $L(D)$ satisfies condition* (C), *if and only if D, m, ρ and L satisfy the following* (i) *and* (ii):

(i) $\rho \in [0, 1)$ *and* $m > \mu$;

(ii) *There is only one natural number k attaining* $\max\{D_j\}$ *and* $D = D_k$.

PROOF. Let us first note that, by Lemma 3.1, 0 is an eigenvalue of $L(D)$ if and only if $D = D_j$ for some j. Let D_k be one of the values of D at which $L(D)$ satisfies (C). Note that both roots λ of the equation $\lambda^2 + p\lambda + q = 0$, with p and q real, have negative real parts, if and only if $p > 0$ and $q > 0$. This being applied to (3.6), condition (C) implies the following two inequalities (with $D = D_k$):

$$(D_k + 1)\ell_j + 1 - \mu/m > 0 \qquad \text{for all } j \ , \tag{3.17}$$

$$D_k\ell_j^2 + (D_k - \mu/m)\ell_j + 1/m > 0 \ , \qquad \text{if } j \neq k \ . \tag{3.18}$$

Conversely, if (3.17) and (3.18) hold, then $L(D_k)$ certainly satisfies (C).

Thus for the proof of this lemma, it is sufficient to show that (3.17) and (3.18) are equivalent to (i) and (ii), respectively. Clearly, (3.17) implies (i) (putting $j = 0$), and vice versa. Since the inequality in (3.18) holds automatically for $j = 0$, (3.18) is equivalent to the following condition:

$$D_k > D_j\,, \quad \text{if} \quad j \neq 0 \quad \text{and} \quad j \neq k\,.$$

This is obviously equivalent to (ii). q.e.d.

The stability of the constant solution $\bar{U}$ can be investigated by the use of (3.6) as in Lemma 3.2. Here we only state some results:

REMARK 3.3. In the case of $\rho \geqq 1$, $\bar{U}$ is stable for all $m > 0$ and $D > 0$.

In the case of $0 \leqq \rho < 1$, we have

(1) If $m > \mu$ and $D > D_k$, then $\bar{U}$ is stable;

(2) If $m > \mu$ and $D < D_k$, then $\bar{U}$ is unstable;

(3) If $m < \mu$, then $\bar{U}$ is unstable for all $D > 0$.

Next, we wish to study condition (ii) of Lemma 3.2 from a slightly different point of view. Let $D_k = \max\{D_j\}$. Then $L(D_k)$ has $\lambda = 0$ as an eigenvalue, the multiplicity of which is either one or two by Lemma 3.1. In applying the well-known theorem of Crandall and Rabinowitz, the simplicity of the eigenvalue 0 is indispensable (see Theorem 3.6 below). Hence we exclude the case $\dim \operatorname{Ker}(L(D_k)) = 2$ from our consideration. Thus we examine in what situation this case will take place. For this purpose, we introduce an important function (see Proposition 3.12 below):

$$L_\rho(j) = \pi[\{(2j' + 1)^2 + 4\mu j'^2\}^{1/2} - 2j' - 1]^{1/2}/\sqrt{2}\,, \tag{3.19}$$

where $j' = j(j + 1)$. Moreover, put

$$\Lambda_\rho = \pi/\sqrt{\hat{\ell}} \tag{3.20}$$

(see (3.16)). Then we have the following lemma:

LEMMA 3.4. *Assume that* $\rho \in [0, 1)$. *Let* $D_k = \max\{D_j\}$ *and* n *be the integral part of the positive number* L/Λ_ρ. *Then* $\dim \operatorname{Ker}(L(D_k)) = 2$ *holds if and only if* $L = L_\rho(n)$.

PROOF. From Lemma 3.1 we see that $\dim \operatorname{Ker}(L(D_k)) = 2$ holds if and only if $\max\{D_j\}$ is attained by two j's.

Note that, by the definition of n, ℓ_n is the largest among the ℓ_j's satisfying $\ell_j \leqq \hat{\ell}$. Thus $g(\ell_j) < g(\ell_n)$ for $j < n$, since $g(\ell)$ is increasing in the interval $(0, \hat{\ell})$. Similarly, we see that $g(\ell_{n+1}) > g(\ell_j)$ for $j > n + 1$. Therefore $\max\{D_j\}$ is attained by two j's, if and only if $g(\ell_n) = g(\ell_{n+1})$. However, if $g(\ell) = g(\ell_n)$ and $\ell \neq \ell_n$, then $\ell = \ell_n^*$. Consequently, dim $\operatorname{Ker}(L(D_k)) = 2$ holds if and only if $\ell_{n+1} = \ell_n^*$. It remains to show that $\ell_{n+1} = \ell_n^*$ is equivalent to $L = L_\rho(n)$. This is verified by a simple computation because $\ell_{n+1} = [(n + 1)/n]^2 \ell_n$. q.e.d.

Now we can state the translation of (C) in the final style.

LEMMA 3.5. *The operator $L(D)$ satisfies condition* (C), *if and only if D, m, ρ and L satisfy the following* (i), (ii) *and* (iii):

$$\text{(i)} \qquad \rho \in [0, 1) \quad \text{and} \quad m > \mu ;$$

$$\text{(ii)} \qquad L \neq L_\rho(n) ,$$

where n denotes the integral part of the number L/Λ_ρ;

$$\text{(iii)} \qquad D = D_k = \max \{D_j\} .$$

PROOF. This lemma is obvious by Lemmas 3.2 and 3.3. q.e.d.

To obtain nontrivial solutions of (3.3), we shall employ the following bifurcation theorem.

THEOREM 3.6 (Crandall-Rabinowitz [1, Theorem 2.4]). *Let L_0 and L_1 be closed linear operators on H with $\mathrm{Dom}(L_1) \supset \mathrm{Dom}(L_0) = \mathscr{H}$. Let M be an analytic mapping from a neighborhood of the origin of $\mathscr{H}$ into $\mathscr{H}$ with $M(0) = M'(0) = 0$, where $M'(0)$ denotes the Fréchet derivative of M at $U = 0$.*

Suppose that the following (1) *and* (2) *are satisfied*:

(1) $\dim \mathrm{Ker}(L_0) = \mathrm{codim}\, \mathrm{Range}(L_0) = 1$ *and* $\mathrm{Ker}(L_0)$ *is spanned by* U_0;

$$(2) \qquad L_1 U_0 \notin \mathrm{Range}(L_0) .$$

Then the equation

$$L_0 U + \tau L_1 U + M(U) = 0 , \quad (\tau, U) \in \boldsymbol{R}^1 \times \mathscr{H} \tag{3.21}$$

has a one-parameter family of nontrivial solutions $(\tau(\varepsilon), U(\varepsilon))$ such that

$$\tau(0) = 0 , \quad U(\varepsilon) = \varepsilon U_0 + V(\varepsilon) \quad \text{and} \quad V(\varepsilon) = O(\varepsilon^2) .$$

Moreover, the functions $\varepsilon \mapsto \tau(\varepsilon) \in \boldsymbol{R}^1$ and $\varepsilon \mapsto U(\varepsilon) \in \mathscr{H}$ are analytic. The solution set near $\tau = 0$, $U = 0$ consists of the trivial solution $(\tau, 0)$ and bifurcating solution $(\tau(\varepsilon), U(\varepsilon))$.

REMARK 3.7. Although the analyticity result is not explicitly mentioned in [1], this follows at once from the regularity of the implicit function theorem (see, for example, [6]).

Now we wish to apply Theorem 3.6 to our case with $L_0 = L(D_k)$, $M = N$ and

$$L_1 = \begin{pmatrix} d^2/dx^2 & 0 \\ 0 & 0 \end{pmatrix} . \tag{3.22}$$

Then our equation (3.3) takes the form of (3.21) with $D = D_k + \tau$. Moreover, let

$$U_0 = {}^t(\varphi_k, [2\bar{u}/(1 + \lambda_k)]\varphi_k) . \tag{3.23}$$

By Lemma 3.5, $\mathrm{Ker}(L(D_k))$ is of dimension one and spanned by U_0, if and only if m, ρ and L satisfy the following two conditions:

(H-1) $\rho \in [0, 1)$ and $m > \mu$;

(H-2) $L \neq L_\rho(n)$, where n is the integral part of L/Λ_ρ.

Our first result is stated as follows:

THEOREM 3.8 (Existence). *Let* $D_k = \max\{D_j\}$. *Suppose that* (H-1) *and* (H-2) *hold. Then there is a one-parameter family of nontrivial stationary solutions* $(D(\varepsilon), \tilde{U}(\varepsilon))$ *of* (1.2) *subject to the boundary condition* (1.4) *for* $|\varepsilon| < \varepsilon_0$ *such that*

$$\begin{cases} D(\varepsilon) = D_k + \tau(\varepsilon) \quad \text{with} \quad \tau(0) = 0, \\ \tilde{u}(\varepsilon) = \bar{u} + \varepsilon\cdot\sqrt{2/L}\cdot\cos(\pi kx/L) + O(\varepsilon^2) , \\ \tilde{v}(\varepsilon) = \bar{v} + \varepsilon\cdot[2\bar{u}\sqrt{2/L}/(1 + \lambda_k)]\cos(\pi kx/L) + O(\varepsilon^2) . \end{cases} \tag{3.24}$$

Moreover, the functions $\varepsilon \mapsto \tau(\varepsilon) \in \boldsymbol{R}^1$, $\varepsilon \mapsto U(\varepsilon) = \tilde{U}(\varepsilon) - \bar{U} \in \mathscr{H}$ *are analytic for* $|\varepsilon| < \varepsilon_0$. *The solution set of* (3.3) *consists of two curves* $(D, 0)$ *and* $(D(\varepsilon), U(\varepsilon))$ *in a neighborhood of the bifurcation point* $(D_k, 0)$.

PROOF. Set $L_0 = L(D_k)$, $M = N$ and define L_1 by (3.22). As stated above, by (H-1) and (H-2), $\mathrm{Ker}(L(D_k))$ is of dimension one and spanned by U_0.

To characterize $\mathrm{Range}(L(D_k))$, we consider the adjoint operator $L(D_k)^*$ of $L(D_k)$, which is formally given by transposing the matrix $L(D_k)$. The eigenfunction U_0^* of $L(D_k)^*$ corresponding to the eigenvalue 0 can be found by the same method as in Lemma 3.1 and is given by

$$U_0^* = {}^t(\varphi_k, -[\bar{u}^2(1 + \lambda_k)]^{-1}\varphi_k) . \tag{3.25}$$

Then the Fredholm alternative implies that $\mathrm{Range}(L(D_k))$ is of codimension one and

$$\mathrm{Range}(L(D_k)) = \{U \in H;\ (U, U_0^*)_H = 0\} . \tag{3.26}$$

Hence the condition (1) of Theorem 3.6 is satisfied.

The condition (2) can be verified as follows: Using (3.22), (3.23) and (3.25), we have

$$(L_1 U_0, U_0^*)_H = (\varphi_k'', \varphi_k)_{L^2} = -\lambda_k \neq 0 , \tag{3.27}$$

whence, by (3.26), $L_1 U_0 \notin \mathrm{Range}(L(D_k))$. q.e.d.

The number k is related to the shape of the bifurcating solutions. In view of the leading terms of the nontrivial solutions (3.24), we see that k expresses the number of the waves of such solutions. Hence we give the following definition.

DEFINITION 3.9. The natural number k uniquely determined by (H-1), (H-2) and (iii) of Lemma 3.5 is called the *mode* of the bifurcating solutions.

Now we discuss the relations between the mode k of $\tilde{U}(\varepsilon)$ given in Theorem 3.8 and the length L of the interval.

Let us start with a study of the location of ℓ_k. First we introduce the following functions:

$$(3.28)\qquad h_1(j) = h[j/(j+1)] \quad \text{and} \quad h_2(j) = \begin{cases} +\infty & \text{for} \quad j = 1\,, \\ h[j/(j-1)] & \text{for} \quad j > 1\,, \end{cases}$$

where

$$(3.29)\qquad h(\kappa) = [1 + \kappa^2 + \{(1+\kappa^2)^2 + 4\mu\kappa^2\}^{1/2}]/(2\mu)\,.$$

Then it is easily seen that $h_1(j)$ is an increasing function of j and $h_2(j)$ is a decreasing function of j. As $j \to \infty$, $h_1(j)$ and $h_2(j)$ tend to the common limit

$$h_1(\infty) = h_2(\infty) = h(1) = \hat{\ell}\,.$$

Hence we have

$$(3.30)\qquad h_1(j) < \hat{\ell} < h_2(j) \quad \text{for all} \quad j \geqq 1\,.$$

We also see that

$$(3.31)\qquad h_1(j) = [\pi j/L_\rho(j)]^2 \quad \text{and} \quad h_2(j) = [\pi j/L_\rho(j-1)]^2\,.$$

Now we can determine the location of ℓ_k as follows:

LEMMA 3.10. *Assume that* (H-1) *and* (H-2) *hold. Then* $\max\{D_j\}$ *is attained by* ℓ_k *satisfying*

$$(3.32)\qquad h_1(k) < \ell_k < h_2(k)\,.$$

PROOF. Let us first observe that $\ell/\ell^* = \kappa^2$ holds if and only if $\ell = h(\kappa)$. Hence we have for $\kappa \neq 1$

(a) $\ell \leqq \hat{\ell}$ and $\ell^* < \kappa^2\ell$ hold if and only if $\ell > h(\kappa^{-1})$;

(b) $\ell > \hat{\ell}$ and $\ell^* > \kappa^2\ell$ hold if and only if $\ell < h(\kappa^{-1})$.

Now suppose that $\max\{D_j\}$ is attained by $j = k$. Then ℓ_k satisfies either (i) $\ell_k \leqq \hat{\ell} \leqq \ell_k^* < \ell_{k+1}$ or (ii) $\ell_{k-1} < \ell_k^* < \hat{\ell} < \ell_k$. From (i) we have $\ell_k > h_1(k)$, since $\ell_{k+1} = [(k+1)/k]^2\ell_k$. Thus, noting that $h_2(k) > \hat{\ell}$, we see that (i) implies (3.32). Similarly, we find that (ii) yields (3.32).

Conversely, if ℓ_k satisfies (3.32), then (i) or (ii) holds. Thus ℓ_k attains $\max\{D_j\}$. q.e.d.

The next lemma gives us an important information on $L_\rho(j)$.

LEMMA 3.11. *Let* $L_\rho(j)$ *and* Λ_ρ *be defined by* (3.19) *and* (3.20), *respectively. Then* $L_\rho(j)$ *is strictly increasing in* j, *more precisely,* $L_\rho(j)$ *satisfies*

$$L_\rho(j-1) < \Lambda_\rho \cdot j < L_\rho(j) \quad \text{for all} \quad j \geqq 1 . \tag{3.33}$$

PROOF. We have only to combine (3.30) with (3.31). q.e.d.

Using the following proposition, the mode k of bifurcating solutions can be obtained as a function of ρ and L; conversely, we can also find the bifurcating solutions with the preassigned mode k.

PROPOSITION 3.12 (Relations between k and L). (a) *Let* (H-1) *and* (H-2) *hold. Then the mode* k *of the bifurcating solutions* $\tilde{U}(\varepsilon)$ *given in Theorem* 3.8 *can be specified as*

$$k = \begin{cases} n \quad , & \text{if} \quad L < L_\rho(n) , \\ n+1 , & \text{if} \quad L > L_\rho(n) , \end{cases} \tag{3.34}$$

where n *is the integral part of the number* L/Λ_ρ.

(b) *Assume that* k *is given beforehand. Let* (H-1) *hold. Then there exist bifurcating solutions of mode* k, *if and only if* L *satisfies*

$$L_\rho(k-1) < L < L_\rho(k) . \tag{3.35}$$

PROOF. (a) First, suppose that $L < L_\rho(n)$. Then from (3.31) we see that $\ell_n < h_2(n)$. Then by Lemma 3.10 we have $k = n$. Next, suppose that $L > L_\rho(n)$. Then by the definition of n we see that $L < (n+1)\Lambda_\rho$. Hence it follows from (3.33) that $L < L_\rho(n+1)$. Then the above argument yields that $k = n+1$.

(b) From (3.31) we see that (3.35) is equivalent to (3.32). Thus by Lemma 3.10 we obtain assertion (b). q.e.d.

REMARK 3.13. Note that the quantity L/k corresponds to the wave length of the bifurcating solutions. From (3.33), $L_\rho(j)/j$ tends to the constant Λ_ρ as $j \to +\infty$. This means that the wave length of the bifurcating solutions tends to Λ_ρ as $L \to +\infty$.

4. Preliminaries for stability analysis (Calculations). In order to study the stability of the bifurcating solutions, we apply the results on perturbation of simple eigenvalues due to Crandall and Rabinowitz

[2] to our case. In fact, using their results, we can show the following stability criterion (for the proof, see Appendix).

PROPOSITION 4.1 (Stability criterion). *Assume that* (H-1) *and* (H-2) *are satisfied. Let* $(D(\varepsilon), \tilde{U}(\varepsilon))$ *be the bifurcating solutions given in Theorem* 3.8. *Then if* $D(\varepsilon) < D_k$ *(i.e.,* $\tau(\varepsilon) < 0$*), then the corresponding solution* $\tilde{U}(\varepsilon)$ *is stable; and if* $D(\varepsilon) > D_k$ *(i.e.,* $\tau(\varepsilon) > 0$*), then the corresponding solution is unstable.*

Now let us expand $\tau(\varepsilon)$ and $U(\varepsilon)$ as the power series in ε (so-called Poincaré-Lindstedt series):

$$U(\varepsilon) = \varepsilon U_0 + \varepsilon^2 U^{(2)} + \varepsilon^3 U^{(3)} + \cdots , \tag{4.1}$$

$$\tau(\varepsilon) = \varepsilon\tau^{(1)} + \varepsilon^2\tau^{(2)} + \varepsilon^3\tau^{(3)} + \cdots . \tag{4.2}$$

Then we can show the following expressions of $\tau^{(1)}$ and $\tau^{(2)}$, which are the object of this section and will be crucial to our stability analysis.

PROPOSITION 4.2. *Assume that* (H-1) *and* (H-2) *hold. Let* $\tau(\varepsilon) = \varepsilon\tau^{(1)} + \varepsilon^2\tau^{(2)} + \cdots$ *be the function given by Theorem* 3.8. *Then we have*

$$\begin{aligned} &\tau^{(1)} = 0 , \\ &\tau^{(2)} = -\frac{P_\rho(\ell_k)}{4L(D_k - D_{2k})\bar{u}^4(1 + \ell_k)^4(1 + 4\ell_k)\ell_k} , \end{aligned} \tag{4.3}$$

where $P_\rho(\ell)$ *is a polynomial in* ℓ *with parameter* ρ *defined by*

$$\begin{aligned} P_\rho(\ell) = 4(5 - 6\rho)\ell^4 - (-47 + 93\rho + 96\mu)\ell^3 - 9(7 - \rho)\ell^2 \\ + 3(39 + 31\rho)\ell + 15(1 + \rho) . \end{aligned} \tag{4.4}$$

Since $D_k - D_{2k}$ and the other factors in the right hand side of (4.3) are positive, we see that

$$\operatorname{sign} \tau^{(2)} = -\operatorname{sign} P_\rho(\ell_k) .$$

Combining this observation with Proposition 4.1, we are led to the following assertion.

COROLLARY 4.3. *The bifurcating solutions of mode* k, $\tilde{U}(\varepsilon)$, *are stable if* $P_\rho(\ell_k) > 0$; $\tilde{U}(\varepsilon)$ *are unstable if* $P_\rho(\ell_k) < 0$.

Under the assumption

$$P_\rho(\ell_k) \neq 0 , \tag{H-3}$$

the stability analysis of bifurcating solutions has been completely reduced to the investigation of the signs of $P_\rho(\ell_k)$. In the next section we shall study the polynomial $P_\rho(\ell)$ in greater detail and in Section 6 we shall state the stability criterion in terms of ρ, L and k. (If $P_\rho(\ell_k) = 0$, then we need to compute $\tau^{(3)}$, which is fairly tedious. We exclude this case from our considerations.)

The remainder part of this section is devoted to the proof of Proposition 4.2. The next lemma is essential for computing the coefficients $U^{(n)}$, $\tau^{(n)}$.

LEMMA 4.4. *Let* $F = {}^t(\sum_j f_j\varphi_j, \sum_j g_j\varphi_j) \in H$ *be given. Then the equation*

$$L(D_k)U = F \quad \text{and} \quad (U, U_0^*)_H = 0 \tag{4.5}$$

has a unique solution $U = KF \in \mathscr{H}$, *if and only if* $(F, U_0^*)_H = 0$. *Furthemore, if we expand* $U = {}^t(\sum_j u_j\varphi_j, \sum_j v_j\varphi_j)$, *then*

$$u_j = \begin{cases} [f_j - g_j/\{\bar{u}^2(1 + \ell_j)\}]/d_j & \text{for } j \neq k\,, \\ g_k/\bar{u}[\{2 - \bar{u}(1 + \ell_k)\}] & \text{for } j = k\,, \end{cases}$$

$$v_j = (2\bar{u}u_j - g_j)/(1 + \ell_j) \qquad \text{for } j > 0\,,$$

where

$$d_j = -\ell_j D_k + \mu/m - 2/[\bar{u}(1 + \ell_j)]\,.$$

PROOF. By the Riesz-Schauder theory, (4.5) is solvable if and only if $F \in \mathrm{Range}(L(D_k))$, i.e., $(F, U_0^*)_H = 0$. The Fourier coefficients are obtained by direct computations. q.e.d.

In what follows, the solution U of (4.5) is denoted by KF.

REMARK 4.5. For $j \neq 0$, it follows from the definition of D_k that

$$d_j = \ell_j(D_j - D_k) < 0 \quad \text{if} \quad j \neq k\,.$$

Now we show the algorithm to determine the coefficients $\tau^{(n)}$, $U^{(n)}$. Substituting (4.1) and (4.2) into the equation

$$L(D_k)U + \tau L_1 U + N[U] = 0\,,$$

we have the following system of equations:

$$\begin{cases} L(D_k)U_0 = 0\,, \\ L(D_k)U^{(2)} + \tau^{(1)}L_1U_0 + N^{(2)}[U_0] = 0\,, \\ L(D_k)U^{(3)} + \tau^{(2)}L_1U_0 + \tau^{(1)}L_1U^{(2)} \\ \qquad\qquad + DN^{(2)}[U_0]U^{(2)} + N^{(3)}[U_0] = 0\,, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\,, \end{cases} \tag{4.6}$$

where $DN^{(2)}[U_0]$ denotes the Fréchet derivative of $N^{(2)}[\cdot]$ at U_0.

The first equation of (4.3) holds automatically. To obtain $U^{(2)}$, we must impose the following condition in view of Lemma 4.4:

$$(\tau^{(1)}L_1U_0 + N^{(2)}[U_0],\ U_0^*)_H = 0\ ,$$

whence we can determine $\tau^{(1)}$ by (3.27) as follows:

$$\tau^{(1)} = (N^{(2)}[U_0],\ U_0^*)_H/\ell_k\ . \tag{4.7}$$

With this $\tau^{(1)}$ we obtain

$$U^{(2)} = -K(\tau^{(1)}L_1U_0 + N^{(2)}[U_0])\ . \tag{4.8}$$

From the third equation of (4.6), we have

$$\tau^{(2)} = (\tau^{(1)}L_1U_0 + DN^{(2)}[U_0]U^{(2)} + N^{(3)}[U_0],\ U_0^*)/\ell_k \tag{4.9}$$

and

$$U^{(3)} = -K(\tau^{(2)}L_1U_0 + \tau^{(1)}L_1U^{(2)} + DN^{(2)}[U_0]U^{(2)} + N^{(3)}[U_0])\ .$$

In the same way, we can determine $\tau^{(n)}$ and $U^{(n)}$ successively and uniquely.

PROOF OF PROPOSITION 4.2. We divide the computations into three steps;

Step 1: Substitute

$$N^{(2)}[U_0] = {}^t([\bar{u}(1+\ell_k)]^{-2}(1-\ell_k)^2\varphi_k^2,\ \varphi_k^2) \tag{4.10}$$

together with (3.25) into (4.7). Then we have

$$\tau^{(1)} = \frac{\ell_k - 3}{\bar{u}^2(1+\ell_k)^3}\int_0^L \varphi_k(x)^3 dx = 0\ . \tag{4.11}$$

Step 2: Note that $2\cos^2\theta = 1 + \cos 2\theta$, so that

$$\varphi_k^2 = (1/\sqrt{2L})\cdot(\sqrt{2}\,\varphi_0 + \varphi_{2k})\ . \tag{4.12}$$

Substituting (4.10), (4.11) and (4.12) into (4.8) and then applying Lemma 4.4, we obtain

$$U^{(2)} = {}^t(u_0^{(2)}\varphi_0 + u_{2k}^{(2)}\varphi_{2k},\ v_0^{(2)}\varphi_0 + v_{2k}^{(2)}\varphi_{2k})\ , \tag{4.13}$$

where

$$\begin{cases} u_0^{(2)} = 4\ell_k/\{\sqrt{L}\,\bar{u}^2 d_0(1+\ell_k)^2\}\ , \\ u_{2k}^{(2)} = \dfrac{1}{\sqrt{2L}\,\bar{u}^2 d_{2k}}\left[\dfrac{1}{1+\ell_{2k}} - \left[\dfrac{1-\ell_k}{1+\ell_k}\right]^2\right], \\ v_0^{(2)} = 2\bar{u}u_0^{(2)} + 1/\sqrt{L}\ , \\ v_{2k}^{(2)} = (2\bar{u}u_{2k}^{(2)} + 1/\sqrt{2L})/(1+\ell_{2k})\ . \end{cases} \tag{4.14}$$

Setp 3: Set

$$U_0 = {}^t(u^{(0)},\ v^{(0)}) \quad \text{and} \quad U^{(2)} = {}^t(u^{(2)},\ v^{(2)})\ .$$

Then

$$DN^{(2)}[U_0]U^{(2)} = {}^t((2/\bar{u}^4)\cdot(\bar{u}u^{(0)} - v^{(0)})(\bar{u}u^{(2)} - v^{(2)}),\ 2u^{(0)}v^{(2)})\ ,$$
$$N^{(3)}[U_0] = {}^t(-(1/\bar{u}^6)\cdot(\bar{u}u^{(0)} - v^{(0)})^2 v^{(0)},\ 0)\ .$$

Substituting these representations together with (3.23), (3.25) and (4.13) into (4.9), we get

$$\tau^{(2)} = \frac{2}{\bar{u}^3\ell_k(1+\ell_k)}[(\{(\ell_k - 2)\bar{u}u_0^{(2)} - (\ell_k - 1)v_0^{(2)}\}\varphi_0,\ \varphi_k^2)_{L^2}$$
$$+(\{(\ell_k - 2)\bar{u}u_{2k}^{(2)} - (\ell_k - 1)v_{2k}^{(2)}\}\varphi_{2k},\ \varphi_k^2)_{L^2}$$
$$-[(1-\ell_k)/(1+\ell_k)]^2(\varphi_k^2,\ \varphi_k^2)_{L^2}]\ .$$

Use (4.12), (4.14) and $\ell_{2k} = 4\ell_k$, then

$$\tau^{(2)} = \frac{2}{\bar{u}^3(1+\ell_k)}\left[-\frac{\bar{u}u_0^{(2)}}{\sqrt{L}} + \frac{(4\ell_k - 9)\bar{u}u_{2k}^{(2)}}{\sqrt{2L}(4\ell_k+1)} + \frac{5 + 26\ell_k - 23\ell_k^2 - 8\ell_k^3}{2L(1+4\ell_k)(1+\ell_k)^2}\right]$$
$$= \frac{2}{\bar{u}^3(1+\ell_k)}\left[-\frac{4\ell_k}{L\bar{u}d_0(1+\ell_k)^2} + \frac{2\ell_k^2(2-\ell_k)(4\ell_k - 9)}{L\bar{u}d_{2k}(1+4\ell_k)^2(1+\ell_k)^2}\right.$$
$$\left. + \frac{5 + 26\ell_k - 23\ell_k^2 - 8\ell_k^3}{2L(1+4\ell_k)(1+\ell_k)^2}\right].$$

Observing that

$$\bar{u}d_0 = -(1+\rho)\ ,$$
$$\bar{u}d_{2k} = \bar{u}\ell_{2k}(D_{2k} - D_k) = \frac{3(1+\rho) + 15(1+\rho)\ell_k - 12(1-\rho)\ell_k^2}{(1+\ell_k)(1+4\ell_k)}\ ,$$

we finally find (4.3). q.e.d.

5. Preliminaries for stability analysis (Studies of $P_\rho(\ell)$). We have assumed (H-1) and (H-2) in Section 3 for the existence of nontrivial solutions $\tilde{U}(\varepsilon)$ of mode k. Moreover, in Section 4 we have added (H-3) to simplify our situation. Under these assumptions, the question of the stability of $\tilde{U}(\varepsilon)$ has been reduced to the positivity of $P_\rho(\ell_k)$ (see Corollary 4.3).

Now our task in this section is to examine whether this will really take place or not. First, recall that ℓ_k satisfies (3.32) of Lemma 3.10. Next, as will be shown in Lemma 5.1 below, $P_\rho(\ell)$ has one or two positive zeros $\lambda_1(\rho)$ and $\lambda_2(\rho)$. Therefore, it is important to classify the values of ρ and k to arrange the four quantities $h_1(k)$, $h_2(k)$, $\lambda_1(\rho)$ and $\lambda_2(\rho)$ in order of magnitude. This will be done in Lemma 5.3. Then we can sort the values of ρ, k and ℓ_k according to the signs of $P_\rho(\ell_k)$ as in Lemmas 5.4 and 5.5 below. (The conditions on ρ, k and ℓ_k appearing in these lemmas are easily converted into those on ρ, k and L, which will be given in the next section. Since k is determined by ρ and L, this will suffice to establish our stability criterion for the bifurcating solutions.)

In Lemma 5.6, we shall see in what situation (H-3) is violated.

First of all, we shall point out that the following five special values of ρ between 0 and 1:

$$\rho = r_1\,,\quad 5/27\,,\quad 16/65\,,\quad r_2 \quad \text{and} \quad 5/6$$

will be critical, where r_1 and r_2 will be defined in (5.13) below.

We begin with a study of positive zeros of $P_\rho(\ell)$. We first note that

$$\ell_k > h_1(1)\,,$$

since $h_1(j)$ is increasing in j. The next lemma is a clue to our analysis.

LEMMA 5.1 (Positive zeros of $P_\rho(\ell)$). *Let $P_\rho(\ell)$ be the polynomial in ℓ defined by* (4.4). *Then we have*

(a) *If $\rho \in [0, 5/6)$, then $P_\rho(\ell)$ has exactly two positive simple zeros*

$$\ell = \lambda_1(\rho)\,,\quad \lambda_2(\rho) \qquad (\lambda_1(\rho) < \lambda_2(\rho))\,. \tag{5.1a}$$

Moreover,

(a-i) *If $\rho \in [0, 5/27] \cup [16/65, 5/6)$, then $\lambda_1(\rho) \leqq h_1(1) < \lambda_2(\rho)$, where the equality holds if and only if $\rho = 5/27$ or $16/65$.*

(a-ii) *If $\rho \in (5/27, 16/65)$, then $h_1(1) < \lambda_1(\rho) < \lambda_2(\rho)$;*

(b) *If $\rho \in [5/6, 1)$, then $P_\rho(\ell)$ has exactly one positive simple zero*

$$\ell = \lambda_1(\rho)\,, \tag{5.1b}$$

which satisfies $\lambda_1(\rho) < h_1(1)$.

PROOF. Let us first estimate the number of positive zeros of $P_\rho(\ell)$ by the sign law of Descartes. Since the coefficient of ℓ^3 in $P_\rho(\ell)$ is negative for $\rho \in [0, 1)$, the signs of the coefficients are as follows:

$$\begin{array}{cccccl} + & - & - & + & + & \text{for } \rho \in [0, 5/6)\,, \\ 0 \text{ or } - & - & - & + & + & \text{for } \rho \in [5/6, 1)\,. \end{array}$$

Hence, if $\rho \in [5/6, 1)$, then $P_\rho(\ell)$ has exactly one positive simple zero; while, if $\rho \in [0, 5/6)$, there are three possibilities: two positive simple zeros, one positive double zero or no positive zero.

Next, we examine the sign of the value $P_\rho(\alpha)$, where

$$\alpha = h_1(1) = (5 + \sqrt{25 + 16\mu})/(8\mu)\,. \tag{5.2}$$

Inverting this relation, we have

$$(5.3) \qquad \rho = (4\alpha^2 - 5\alpha - 1)/(4\alpha^2 + 5\alpha + 1) .$$

Using this expression, we obtain

$$(5.4) \qquad P_\rho(\alpha) = -4(1+\alpha)\alpha^2(\alpha - 2)(4\alpha - 9)/(4\alpha + 1) .$$

It is also seen from (5.2) that α is an increasing function of ρ; and that

$$\text{if} \quad \rho = 0, \text{ then } \alpha = (5 + \sqrt{41})/8 ,$$
$$\text{if} \quad \rho = 5/27, \text{ then } \alpha = 2, \text{ and}$$
$$\text{if} \quad \rho = 16/65, \text{ then } \alpha = 9/4 .$$

Thus, by (5.4), $P_\rho(\alpha) < 0$ if $\rho \in [0, 5/27) \cup (16/65, 1)$. Consequently, the corresponding assertions ((b) and part of (a-i)) are verified, because $P_\rho(0) > 0$.

We now pass to the case $\rho \in (5/27, 16/65)$. We have seen above that $P_\rho(\alpha) > 0$. Let us evaluate $P_\rho(\ell)$ at $\ell = 9/4$:

$$P_\rho(9/4) = -3(65\rho - 16)(461\rho - 115)/[64(1+\rho)] .$$

This is negative, because $16/65 < 115/461$. Hence, we see that

$$(5.5) \qquad \alpha < \lambda_1(\rho) < 9/4 < \lambda_2(\rho) \quad \text{for} \quad \rho \in (5/27, 16/65) .$$

If $\rho = 5/27$, then $\lambda_1(\rho) = 2 < \lambda_2(\rho)$ by (5.5) and the continuity of $\lambda_i(\rho)$ in ρ. In the case of $\rho = 16/65$, $\lambda_1(\rho) = 9/4$ can not be a double zero, since a direct computation shows that $\partial P_\rho(9/4)/\partial\ell < 0$ at $\rho = 16/65$.

q.e.d.

From this lemma, we can immediately see that (i) $P_\rho(\ell_k)$ is positive if and only if one of the following two conditions holds:

$$(5.6) \qquad \rho \in [0, 5/6) \quad \text{and} \quad \ell_k > \lambda_2(\rho) ,$$

$$(5.7) \qquad \rho \in (5/27, 16/65) \quad \text{and} \quad \ell_k < \lambda_1(\rho) ;$$

and (ii) $P_\rho(\ell_k)$ is negative if and only if one of the following three conditions holds:

$$(5.8) \qquad \rho \in [0, 5/27] \cup [16/65, 5/6) \quad \text{and} \quad \ell_k < \lambda_2(\rho) ,$$

$$(5.9) \qquad \rho \in (5/27, 16/65) \quad \text{and} \quad \lambda_1(\rho) < \ell_k < \lambda_2(\rho) ,$$

$$(5.10) \qquad \rho \in [5/6, 1) .$$

Noting that $h_1(k)$ and $h_2(k)$ tend to $\hat{\ell}$ as $k \to \infty$, we see that ℓ_k tends to $\hat{\ell}$ as $k \to \infty$. Therefore, if $\hat{\ell} < \lambda_2(\rho)$, then condition (5.6) is violated for large k. This means that in this case there exist no stable bifurcating solutions for large L. Contrarily, if $\hat{\ell} > \lambda_2(\rho)$, then condition (5.6) always holds for sufficiently large k; hence, the bifurcating solutions are always stable whenever L is large enough.

Motivated by these observations, we arrange $\hat{\ell}$ and $\lambda_2(\rho)$ in order of magnitude. This is achieved by investigating the value $P_\rho(\hat{\ell})$: First, inverting the relation

$$\hat{\ell} = (1 + \sqrt{1+\mu})/\mu \ ,$$

we have

$$\rho = (\hat{\ell}^2 - 2\hat{\ell} - 1)/(\hat{\ell} + 1)^2 \ .$$

Using this expression, we find that

$$P_\rho(\hat{\ell}) = 2\hat{\ell} Q(\hat{\ell})/(1 + \hat{\ell}) \ , \tag{5.11}$$

where

$$Q(\ell) = -2\ell^4 + 23\ell^3 + 16\ell^2 - 153\ell - 36 \ . \tag{5.12}$$

It is easily seen that $Q(\ell)$ has exactly two positive simple zeros ξ_1 and ξ_2 ($\xi_1 < \xi_2$). Further, we set

$$r_i = (\xi_i^2 - 2\xi_i - 1)/(\xi_i + 1)^2 \quad \text{for} \quad i = 1, 2 \ . \tag{5.13}$$

These quantities are approximately given by

$$\begin{aligned} \xi_1 &= 2.6488\cdots \ , \quad \xi_2 = 11.6100\cdots \ , \\ r_1 &= 0.0539\cdots \ , \quad r_2 = 0.6953\cdots \ . \end{aligned} \tag{5.14}$$

Thus we see that

$$0 < r_1 < 5/27 < 16/65 < r_2 < 5/6 < 1 \ . \tag{5.15}$$

With the aid of these preliminaries, we can state as follows:

LEMMA 5.2. (a) *If $\rho \in [0, r_1) \cup (r_2, 5/6)$, then $\hat{\ell} < \lambda_2(\rho)$.*

(b) *If $\rho \in [r_1, r_2]$, then $\lambda_2(\rho) \leqq \hat{\ell}$, where the equality is valid if and only if $\rho = r_1$ or r_2.*

PROOF. If $Q(\hat{\ell}) < 0$ (i.e., $1 + \sqrt{2} < \hat{\ell} < \xi_1$ or $\hat{\ell} > \xi_2$), then from (5.11) we have $P_\rho(\hat{\ell}) < 0$. This implies $\lambda_1(\rho) < \hat{\ell} < \lambda_2(\rho)$, which lead us to assertion (a).

Next, consider the case $Q(\hat{\ell}) \geqq 0$, i.e., $\xi_1 \leqq \hat{\ell} \leqq \xi_2$. First suppose that $\lambda_1(\rho) \leqq h_1(1) < \lambda_2(\rho)$ holds. Then we have $P_\rho(\hat{\ell}) \geqq 0$ and $h_1(1) < \hat{\ell}$. Therefore, $\hat{\ell} \geqq \lambda_2(\rho)$ holds. Secondly, suppose $h_1(1) < \lambda_1(\rho)$. In this case, we see that $\rho \in (5/27, 16/65)$ by Lemma 5.1; and that $\lambda_1(\rho) < 9/4$ by (5.5).

On the other hand, we find that $\hat{\ell} > 3.34$ for $\rho \in (5/27, 16/65)$. Thus, we have $\hat{\ell} > \lambda_2(\rho)$. q.e.d.

Now we wish to consider the four quantities $\lambda_1(\rho)$, $\lambda_2(\rho)$, $h_1(k)$ and $h_2(k)$.

Observe that $h(\kappa) = \lambda$ if and only if $\kappa = \sqrt{\lambda/\lambda^*}$, where $h(\kappa)$ is defined by (3.29) and $\lambda^* = (\lambda + 1)/(\mu\lambda - 1)$. Thus we introduce the following function $K(\ell)$ for $\ell \neq \hat{\ell}$:

$$K(\ell) = |1 - \sqrt{\ell^*/\ell}|^{-1} \quad \text{with} \quad \ell^* = (\ell + 1)/(\mu\ell - 1) \,. \tag{5.16}$$

Note that $K(\ell) > 1$.

Then we can classify the possible cases as follows:

LEMMA 5.3. (a) *If* $\rho \in [0, r_1) \cup (r_2, 5/6)$, *then*

$$\lambda_1(\rho) < h_1(k) < \lambda_2(\rho) < h_2(k) \quad \textit{for} \quad k < K(\lambda_2(\rho)) \, ;$$
$$\lambda_1(\rho) < h_1(k) < h_2(k) \leqq \lambda_2(\rho) \quad \textit{for} \quad k \geqq K(\lambda_2(\rho)) \, ,$$

where the equality holds if and only if $k = K(\lambda_2(\rho))$.

(b) *If* $\rho \in (r_1, 5/27] \cup [16/65, r_2)$, *then*

$$\lambda_1(\rho) \leqq h_1(k) < \lambda_2(\rho) < h_2(k) \quad \textit{for} \quad k < K(\lambda_2(\rho)) \, ,$$

where the equality holds if and only if $\rho = 5/27$ *or* $16/65$;

$$\lambda_1(\rho) < \lambda_2(\rho) \leqq h_1(k) < h_2(k) \quad \textit{for} \quad k \geqq K(\lambda_2(\rho)) \, ,$$

where the equality holds if and only if $k = K(\lambda_2(\rho))$.

(c) *If* $\rho = r_1$ *or* r_2, *then*

$$\lambda_1(\rho) < h_1(k) < \lambda_2(\rho) = \hat{\ell} < h_2(k) \quad \textit{for} \quad k \geqq 1 \, .$$

(d) *If* $\rho \in (5/27, 16/65)$, *then*

$$h_1(k) < \lambda_1(\rho) < \lambda_2(\rho) < h_2(k) \quad \textit{for} \quad k = 1 \, ;$$
$$\lambda_1(\rho) < \lambda_2(\rho) < h_1(k) < h_2(k) \quad \textit{for} \quad k > 1 \, .$$

PROOF. Let us commence by proving (a). From Lemma 5.2, we see that $\hat{\ell} < \lambda_2(\rho)$ for $\rho \in [0, r_1) \cup (r_2, 5/6)$. Thus, $h_1(k) < \lambda_2(\rho)$ for all k, since $h_1(k) < \hat{\ell}$. Moreover, by Lemma 5.1 (a-i), we have $\lambda_1(\rho) < h_1(1)$. Therefore, we find $\lambda_1(\rho) < h_1(k)$ for all k, because $h_1(k)$ is increasing in k. A simple computation shows that $\lambda_2(\rho) < h_2(k)$ if and only if $k < K(\lambda_2(\rho))$. Thus we obtain assertion (a).

Assertions (b) and (c) can be verified similarly.

Now let us pass to (d). By Lemma 5.1 (a-ii), we see that $h_1(1) < \lambda_1(\rho) < \lambda_2(\rho)$. Since $h_1(2) \leqq h_1(k)$ for $k \geqq 2$, it is sufficient for the proof to show $\lambda_2(\rho) < h_1(2)$. For this purpose, we put $\beta = h_1(2)$, that is,

$$\beta = (13 + \sqrt{169 + 144\mu})/(18\mu) .$$

Then we have

$$\rho = (9\beta^2 - 13\beta - 4)/(9\beta^2 + 13\beta + 4)$$

and

$$P_\rho(\beta) = 2\beta R(\beta)/[3(9\beta + 4)] ,$$

where

$$R(\ell) = -54\ell^4 + 291\ell^3 + 102\ell^2 - 1111\ell - 112 .$$

Using these expressions, the argument analogous to that in the proof of Lemma 5.2 gives us that if $\rho \in (0.1847\cdots, 0.5228\cdots)$, then $P_\rho(\beta) > 0$. In view of $0.1847\cdots < 5/27$ and $16/65 < 0.5228\cdots$, we have $\lambda_1(\rho) < 9/4 < \lambda_2(\rho)$ and $9/4 < \beta$ for $\rho \in (5/27, 16/65)$. Therefore we see that $\lambda_2(\rho) < \beta = h_1(2)$. q.e.d.

We are now ready to classify the values of ρ, k and ℓ_k according to the signs of $P_\rho(\ell_k)$. We note that the following conditions stated in Lemmas 5.4 and 5.5 are compatible with (3.32). Thus, by a suitable choice of ρ and L, we can find the values of k and ℓ_k satisfying (3.32) and one of these conditions (recall that k and ℓ_k are determined by ρ and L).

LEMMA 5.4 (Positive case). *The value $P_\rho(\ell_k)$ is positive, if and only if ρ, k and ℓ_k satisfy one of the following conditions*:

(5.17) $$\rho \in [0, 5/6) , \quad k < K(\lambda_2(\rho)) \quad \text{and} \quad \ell_k > \lambda_2(\rho) ;$$

(5.18) $$\rho \in (r_1, r_2) \quad \text{and} \quad k \geqq K(\lambda_2(\rho)) ;$$

(5.19) $$\rho \in (5/27, 16/65) , \quad k = 1 \quad \text{and} \quad \ell_k < \lambda_1(\rho) .$$

LEMMA 5.5 (Negative case). *The value $P_\rho(\ell_k)$ is negative, if and only if ρ, k and ℓ_k satisfy one of the following conditions*:

(5.20) $$\rho \in [0, 5/27] \cup [16/65, 5/6) , \quad k < K(\lambda_2(\rho)) \quad \text{and} \quad \ell_k < \lambda_2(\rho) ;$$

(5.21) $$\rho \in [0, r_1) \cup (r_2, 5/6) \quad \text{and} \quad k \geqq K(\lambda_2(\rho)) ;$$

(5.22) $$\rho \in (5/27, 16/65) , \quad k = 1 \quad \text{and} \quad \lambda_1(\rho) < \ell_k < \lambda_2(\rho) ;$$

(5.23) $$\rho \in [5/6, 1) .$$

LEMMA 5.6 (The case $P_\rho(\ell_k) = 0$). *The value $P_\rho(\ell_k)$ vanishes, if and only if one of the following conditions is satisfied*:

(5.24) $$\rho \in [0, 5/6) \quad \text{and} \quad \ell_k = \lambda_2(\rho) ;$$

$$\rho \in (5/27, 16/65)\ , \quad k = 1 \quad \textit{and} \quad \ell_k = \lambda_1(\rho)\ . \tag{5.25}$$

REMARK 5.7. In conditions (5.17) and (5.20), if $\rho = r_1$ or r_2 then $K(\lambda_2(\rho))$ is understood as $+\infty$. Thus in this case there are no restrictions on k.

PROOFS OF LEMMAS 5.4, 5.5 AND 5.6. These lemmas can be obtained easily by combining (5.6), (5.7), (5.8), (5.9) and (5.10) with Lemma 5.3. q.e.d.

6. Stability criterion for nonconstant bifurcating solutions. Assume that (H-1), (H-2) and (H-3) are satisfied. Then there exist the bifurcating solutions $\tilde{U}(\varepsilon)$ of mode k by Theorem 3.8.

Let $L_\rho(j)$ be the sequence defined by (3.19) and $K(\ell)$ be the function defined by (5.16). Moreover, let $\lambda_1(\rho)$ and $\lambda_2(\rho)$ be the positive zeros of the polynomial $P_\rho(\ell)$ defined by (4.4), r_1 and r_2 be the numbers given by (5.13).

Note that ρ, k and L satisfy (3.34) and (3.35).

Then our stability criterion is as follows:

THEOREM 6.1 (Stable case). *The k-mode bifurcating solutions $\tilde{U}(\varepsilon)$ are stable, if ρ, k and L satisfy one of the following conditions*:

(S.1) $\rho \in [0, 5/6)\ , \quad k < K(\lambda_2(\rho)) \quad \textit{and} \quad L_\rho(k-1) < L < \pi k/\sqrt{\lambda_2(\rho)}\ ;$

(S.2) $\rho \in (r_1, r_2) \quad \textit{and} \quad k \geqq K(\lambda_2(\rho))\ ;$

(S.3) $\rho \in (5/27, 16/65)\ , \quad k = 1 \quad \textit{and} \quad \pi/\sqrt{\lambda_1(\rho)} < L < L_\rho(1)\ .$

THEOREM 6.2 (Unstable case). *The k-mode bifurcating solutions $\tilde{U}(\varepsilon)$ are unstable, if one of the following conditions is satisfied*:

(US.1) $\rho \in [0, 5/27] \cup [16/65, 5/6)\ , \quad k < K(\lambda_2(\rho)) \quad \textit{and} \quad \pi k/\sqrt{\lambda_2(\rho)} < L < L_\rho(k)\ ;$

(US.2) $\rho \in [0, r_1) \cup (r_2, 5/6) \quad \textit{and} \quad k \geqq K(\lambda_2(\rho))\ ;$

(US.3) $\rho \in (5/27, 16/65)\ , \quad k = 1 \quad \textit{and} \quad \pi/\sqrt{\lambda_2(\rho)} < L < \pi/\sqrt{\lambda_1(\rho)}\ ;$

(US.4) $\rho \in [5/6, 1)\ .$

REMARK 6.3. In (S.1) and (US.1), if $\rho = r_1$ or r_2, then we have $K(\lambda_2(\rho)) = +\infty$. Thus in this case there are no restrictions on k.

PROOFS OF THEOREMS 6.1 AND 6.2. These theorems follow at once from Lemmas 5.4 and 5.5, observing that

$$h_1(k) = [\pi k/L_\rho(k)]^2\ , \quad h_2(k) = [\pi k/L_\rho(k-1)]^2 \quad \text{and} \quad \ell_k = (\pi k/L)^2\ . \quad \text{q.e.d.}$$

REMARK 6.4. Note that Theorems 6.1 and 6.2 are always applicable unless ρ and L satisfy one of the following conditions:

(E.1) $0 \leqq \rho < 1$ and $L \in \{L_\rho(j);\ j = 1, 2, 3, \cdots\}$;

(E.2) $0 \leqq \rho < 5/6$ and $L \in \{\pi j/\sqrt{\lambda_2(\rho)};\ j = 1, 2, 3, \cdots\}$;

(E.3) $5/27 < \rho < 16/65$ and $L = \pi/\sqrt{\lambda_1(\rho)}$.

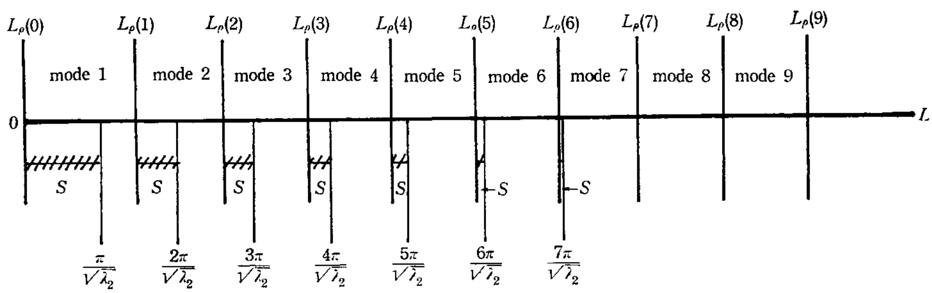
Here, if (E.1) holds, then (H-2) is violated; while (H-3) is violated if (E.2) or (E.3) holds. (See Lemma 5.6.)

REMARK 6.5. Recall that k is determined by ρ and L (see (3.34)). Conversely, assume that the mode k is prescribed. Then we can find stable bifurcating solutions of this mode k, by a suitable choice of ρ and L satisfying one of (S.1), (S.2) and (S.3).

Let us take up the case $\rho = 0$ as an example. Then we have

$$\lambda_2(\rho) = 2.7935\cdots, \quad \pi/\sqrt{\lambda_2(\rho)} = 1.8796\cdots, \quad K(\lambda_2(\rho)) = 7.70\cdots .$$

Therefore the situation is as illustrated in Figure. If L is located in the shaded intervals, then $\tilde{U}(\varepsilon)$ is stable; if L is located in the unshaded intervals, then $\tilde{U}(\varepsilon)$ is unstable. It is also seen that $\tilde{U}(\varepsilon)$ are always unstable if $L > L_\rho(7)$, i.e., if $\tilde{U}(\varepsilon)$ are of mode greater than 7. (See Figure).



CONCLUDING REMARKS. All of conditions (H-1), (H-2), (S.1), $\cdots$, (US.1), $\cdots$, (E.2) and (E.3) are restated in terms of the original constants D_a, D_h, $\cdots$, ρ_0 and $\tilde{L}$ which appear in (G-M) by the change of scales (1.1).

We note that, in spite of Theorem 6.2, there may exist stable nonconstant stationary solutions for such values of ρ and L. For, our results are local in the sense that we have considered only in a neighborhood of the constant solution. It may happen that the unstable branches gain their stability if we continue the bifurcating solution curves with respect to the parameter ε. (Cf. Sections 3 and 4 of [2].)

Therefore, it would be of interest to study the global behavior of the branches of nontrivial solutions.

Appendix. Here we give the proof of Proposition 4.1.

Let $\iota: \mathscr{H} \to H$ be the natural imbedding. Then the bounded linear map $L(D_k): \mathscr{H} \to H$ has 0 as an ι-simple eigenvalue, i.e.,

(i) $\dim \operatorname{Ker}(L(D_k)) = \operatorname{codim} \operatorname{Range}(L(D_k)) = 1$;

moreover, if $\operatorname{Ker}(L(D_k)) = \operatorname{span}\{U_0\}$, then

(ii) $\iota U_0 \notin \operatorname{Range}(L(D_k))$.

In fact, (i) is obvious and (ii) is verified as follows:

$$\begin{aligned}(\iota U_0, U_0^*)_H &= (\varphi_k, \varphi_k)_{L^2} + (2\bar{u}(1+\ell_k)^{-1}\varphi_k, -[\bar{u}^2(1+\ell_k)]^{-1}\varphi_k)_{L^2} \\ &= 1 - 2/[\bar{u}(1+\ell_k)^2] \\ &= 1 - \mu/[m(1+\ell_k)] + \ell_k D_k/(1+\ell_k)\ ,\end{aligned}$$

which is positive by (H-1). Thus we see that (ii) holds by (3.26). (Note that ι-simplicity means algebraic simplicity in this case.) Therefore, we can apply the perturbation theorem due to Crandall and Rabinowitz to our case.

First, Corollary 1.13 and Theorem 1.16 of [2] lead us to the following lemma:

LEMMA A.1. *There exist a $\delta > 0$ and analytic functions $\lambda(\tau) \in \boldsymbol{R}^1$, $\mu(\varepsilon) \in \boldsymbol{R}^1$, $V(\tau) \in \mathscr{H}$, $W(\varepsilon) \in \mathscr{H}$ for $|\varepsilon| < \delta$ and $|\tau| < \delta$ such that*

(i) $\quad (L(D_k) + \tau L_1)V(\tau) = \lambda(\tau)\iota V(\tau)$ *for* $|\tau| < \delta$,

(ii) $\quad L(D(\varepsilon), \tilde{U}(\varepsilon))W(\varepsilon) = \mu(\varepsilon)\iota W(\varepsilon)$ *for* $|\varepsilon| < \delta$.

Moreover,

$\lambda(0) = \mu(0) = 0$, $V(0) = U_0 = W(0)$ *and* $V(\tau) - U_0 \in \mathscr{K}$, $W(\varepsilon) - U_0 \in \mathscr{K}$,

where $\mathscr{K}$ is a complement of span $\{U_0\}$ *in $\mathscr{H}$.*

Furthermore, $\lambda'(0) \neq 0$ *(see Lemma* A.2 *below); and near $\varepsilon = 0$, the functions $\mu(\varepsilon)$ and $-\varepsilon\tau'(\varepsilon)\lambda'(0)$ have the same zeros, and, whenever $\mu(\varepsilon) \neq 0$, the same sign.*

Note that $\mu(\varepsilon)$ is the eigenvalue of $L(D(\varepsilon), \tilde{U}(\varepsilon))$ with the largest real part. Hence the bifurcating solution $\tilde{U}(\varepsilon)$ is stable if $\mu(\varepsilon) < 0$ and unstable if $\mu(\varepsilon) > 0$.

LEMMA A.2. $\qquad \lambda'(0) < 0$.

PROOF. Observe that $\lambda = \lambda(\tau)$ satisfies the equation

$$\begin{aligned}\lambda^2 + [(D_k + \tau + 1)\ell_k + 1 - \mu/m]\lambda + (D_k + \tau)\ell_k^2 \\ + (D_k + \tau - \mu/m)\ell_k + 1/m = 0\ ,\end{aligned}$$

whence we have, by the definition of D_k,

$$\lambda^2 + [(D_k + 1)\ell_k + 1 - \mu/m + \tau\ell_k]\lambda + \ell_k(\ell_k + 1)\tau = 0\ .$$

Differentiate the both sides with respect to τ and evaluate at $\tau = 0$. Then, from $\lambda(0) = 0$,

$$[(D_k + 1)\ell_k + 1 - \mu/m]\lambda'(0) + \ell_k(1 + \ell_k) = 0 .$$

This proves Lemma A.2.

Therefore, $\mu(\varepsilon)$ and $\varepsilon\tau'(\varepsilon)$ have the same sign. Since $\varepsilon\tau'(\varepsilon)$ and $\tau(\varepsilon)$ have the same sign for sufficiently small $|\varepsilon|$, the proof of Proposition 4.1 is now completed.

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