Stability of Bounded and Unbounded Sets for Ordinary Differential Equations (*).

R. C. GRIMMER (Carbondale, Ill.) - JOHN R. HADDOCK (Memphis, Tenn.)

Summary. – *The stability properties of subsets of R^n are examined using a family of Liapunov functions and the invariance properties of the sets.*

1. – Introduction.

In this paper we consider various stability properties of certain subsets of $D \subset R^n$ for the system of differential equations.

$$x' = f(t, x) \qquad (' = d/dt) \tag{1}$$

where $f\colon [0, \infty) \times D \to R^n$ is continuous and D is an open region in R^n.

We use the following notation. For a set $B \subset R^n$, $\bar{B}$, B^c, and ∂B will denote the closure, complement and boundary of B, respectively. Also, $d(x, B) = \inf\{|x-y|: y \in B\}$ will denote the distance between B and a point x in R^n, where $|\cdot|$ is any convenient norm in R^n. Finally, for $\varepsilon > 0$, we define $S(B, \varepsilon) = \{x: d(x, B) < \varepsilon\}$ and $R(B, \varepsilon) = \{x: \varepsilon/2 \leqslant d(x, B) \leqslant \varepsilon\}$.

We shall examine stability, uniform stability, asymptotic stability and global asymptotic stability of certain subsets of D. We want our results to include sets which may have unbounded solutions of (1) in every neighborhood. In particular, we want to consider sets in which solutions in every neighborhood may have finite escape time. We give our stability definitions accordingly.

DEFINITION 1. – Let N be a closed subset of D. For a solution $x(t, t_0, x_0)$ of (1) which satisfies $x(t_0, t_0, x_0) = x_0$, let $[t_0, T)$ $(t_0 < T \leqslant +\infty)$ denote its maximal right-interval of definition.

(i) N is *stable* if, for every $t_0 \geqslant 0$ and $\varepsilon > 0$ there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that $d(x_0, N) < \delta$ implies $d(x(t, t_0, x_0), N) < \varepsilon$ for all t in $[t_0, T)$.

(ii) N is *uniformly stable* if δ in (i) is independent of t_0.

(iii) N is an *attractor* if, for each $t_0 \geqslant 0$ there exists $\delta = \delta(t_0) > 0$ with the property that $d(x_0, N) < \delta$ implies $d(x(t, t_0, x_0), N) \to 0$ as $t \to T^-$. N is a *global attractor* if $D = R^n$ and for each $t_0 \geqslant 0$, $d(x(t, t_0, x_0), N) \to 0$ as $t \to T^-$.

(*) Entrata in Redazione l'8 giugno 1972.

(iv) N is *asymptotically stable* (globally asymptotically stable) if it is a stable attractor (stable global attractor).

We note that if N is a compact set in D, then $T = \infty$ and our definitions are the usual ones.

2. – The autonomous case.

In this section we restrict our attention to the autonomous system

$$x' = f(x) \tag{2}$$

where $f: D \to R^n$ is continuous with D open in R^n. For the autonomous case we have the following definition.

DEFINITION 2. – A set $G \subset D$ is *positively invariant* for (2) if, given $x_0 \in G$ every solution $x(t, 0, x_0)$ of (2) remains in G on its right maximal interval of definition. G is a *semi-invariant* set for (2) if, given $x_0 \in G$ there is a least one solution of (2) with initial condition x_0 at $t = 0$ which remains in G on its maximal interval of definition.

Our first theorem gives the relation between asymptotic stability and attraction for a compact set.

THEOREM 1. – Let $M \subset G \subset D$ where G is open and positively invariant for (2) and M is a compact positively invariant set for (2). Then M is asymptotically stable and G is contained in its region of attraction if and only if M is an attractor with G contained in its region of attraction and if K is a compact semi-invariant set for (2) contained in G, then $K \subset M$.

PROOF. – Suppose M is a compact positively invariant attractor in G with the property that every compact semi-invariant set in G is contained in M. We will show M is stable.

Suppose M is not stable. Then there is an $\varepsilon > 0$ with $\overline{S(M, \varepsilon)} \subset G$ such that for every positive integer n, there is an x_n with $d(x_n, M) < 1/n$ and a solution $x_n(t)$ of (2) with $x_n(0) = x_n$ and $t_n > 0$ so that $d(x_n(t_n), M) = \varepsilon$. Also, without loss of generality, we may assume t_n is the first such positive t. Now $\{x_n(t_n)\}$ is a bounded set and we may assume, by passing to a subsequence if necessary, that $x_n(t_n) \to z$ as $n \to \infty$ for some z where $d(z, M) = \varepsilon$.

We claim $t_n \to \infty$ as $n \to \infty$. If not, then $\{t_n\}$ has a subsequence s_j with $s_j \to T_1 < \infty$ as $j \to \infty$ and $\{x_j(t)\}$ is a bounded uniformly equicontinuous sequence of functions on $[0, T_1]$ and, by Ascoli's Theorem, $\{x_j(t)\}$ has a uniformly convergent subsequence which converges to a solution $x(t)$ of (2) on $[0, T_1]$ where $x(0) \in M$, since $d(x_n, M) \to 0$ as $n \to \infty$, and $x(T_1) = z$ is not in M. This, however, is a contradiction of the right-invariance of M.

Consider now the solutions $y_n(t)$ of (2) where $y_n(t) = x_n(t_n + t)$ for $-t_n \leqslant t \leqslant 0$. Then $y_n(0) \to z$ as $n \to \infty$ and $d(y_n(t), M) \leqslant \varepsilon$ for $-t_n \leqslant t \leqslant 0$. As $t_n \to \infty$, we have, for sufficiently large n, $\{y_n(t)\}$ is a bounded uniformly equicontinuous sequence on any interval of the form $[-T_2, 0]$, $T_2 > 0$, and thus, $\{y_n(t)\}$ has a subsequence which is uniformly convergent on compact subintervals of $(-\infty, 0]$ to a solution $x(t)$ of (2) and $x(0) = z$. Also, since $d(y_n(t), M) \leqslant \varepsilon$ for $-t_n \leqslant t \leqslant 0$, we must have $d(x(t), M) \leqslant \varepsilon$ for $t \geqslant 0$. Since $x(0) \in G$, we may extend $x(t)$ to $(-\infty, \infty)$ and we must have $x(t) \in G$ for every t and $d(x(t), M) \to 0$ as $t \to \infty$. This, however, is a contradiction as $\bar{K}$, with $K = \{x(t): -\infty < t < \infty\}$, is a compact semi-invariant set in G. But $\bar{K}$ is not contained in M as z is not in M. Hence, M must be stable.

Suppose now that M is asymptotically stable and G is in the region of attraction. Suppose there is a compact semi-invariant set $K \subset G$ that is not contained in M. Let $z \in K \setminus M$ and $x(t)$ a solution of (2) with $x(0) = z$ and with $x(t)$ in K for all $t \leqslant 0$. As M is stable, there exists $\delta > 0$ so that if $d(x_0, M) < \delta$, then $d(x(t, 0, x_0), M) < \frac{1}{2} d(z, M)$ for all $t \geqslant 0$ for every solution $x(t, 0, x_0)$ of (2) with $x(0, 0, x_0) = x_0$. We must have, then, that $d(x(t), M) \geqslant \delta$ for $t \leqslant 0$. Since K is compact and contained in G, the α-limit set A of $x(t)$ exists, is non-empty and is a semi-invariant set for (2) contained in G. Let $y \in A$ and $y(t)$ a solution of (2) which remains in A for $t \geqslant 0$. As $d(A, M) \geqslant \delta$, we have $d(y(t), M) \geqslant \delta > 0$ and, since $y(t)$ is in G for $t \geqslant 0$, we have a contradiction. Clearly, M is an attractor and the proof is complete.

In $G = R^n$, the statement of Theorem 1 can be simplified as follows.

THEOREM 2. - Let $M \subset R^n$ be a compact positively invariant set for (2). Then M is globally asymptotically stable if and only if M is a global attractor and if K is a compact semi-invariant set for (2), then $K \subset M$.

If $f(0) = 0$, we have the following corollary.

COROLLARY 1. - Let $0 \in G \subset D$ where G is open and positively invariant for (2). Then $\{0\}$ is asymptotically stable and G is contained in its region of attraction if and only if $\{0\}$ is an attractor with G contained in its region of attraction and $\{0\}$ is the only compact semi-invariant set for (2) contained in G.

PROOF. - If $\{0\}$ is an attractor with G in its region of attraction and if $\{0\}$ is the only compact semi-invariant set for (2) in G, then the zero solution of (2) must be unique to the right and hence, $\{0\}$ is a compact positively invariant set for (2). The result now follows from Theorem 1.

In a similar fashion the next corollary follows from Theorem 2 and the fact that the closure of a bounded semi-invariant set is semi-invariant.

COROLLARY 2. - $\{0\}$ is globally asymptotically stable if and only if it is a global attractor and $\{0\}$ is the only bounded semi-invariant set for (2).

An interesting application of Corollary 2 can be made to the problem of perturbing equations that have $\{0\}$ as a global attractor (see for example [7]). Suppose the equation

$$y' = f(y) + g(t, y) \tag{4}$$

is « almost autonomous »; that is, $|g(t, y)|$ gets « small » as $t \to \infty$. Then the ω-limit set a bounded solution of (3) is a semi-invariant set for (2) under very general definitions of « small » (cf. [6] and [7]). However, if $\{0\}$ is a global attractor for (2) but not globally asymptotically stable, then there must exist bounded semi-invariant sets of (2) other than $\{0\}$ and hence, bounded solutions of (3) may not tend to zero as $t \to \infty$ in which case $\{0\}$ will not be a global attractor for (3) (see [7, Corollary 3.3]).

3. – Liapunov functions and stability of sets.

We now return our attention to the nonautonomous system

$$x' = f(t, x)\,. \tag{1}$$

We shall say that $f(t, x)$ satisfies *hypothesis* (A) *in a set* $N \subset D$ if, for every x_0 in N there exists $\eta > 0$ and a continuous function $g\colon [0, \infty) \to [0, \infty)$ with $\int_0^\infty g(t)\,dt = \infty$ such that x in N and $|x - x_0| < \eta$ implies $|f(t, x)| \geq g(t)$.

The above definition is due to BURTON [2]. In this section we combine the ideas used in recent papers by BURTON [1] and HADDOCK [4] and examine stability properties of certain subsets of D. In [1] BURTON established conditions on Liapunov functions which guarantee that a closed set H is a global attractor. Unlike other results (cf., for example, [8, Theorem 1]), his results do not depend on boundedness of solutions of (1). Indeed, his main results allow solutions of (1) to have finite escape time. In the same spirit, we shall give sufficient conditions for a set H to be stable, uniformly stable, asymptotically stable and globally asymptotically stable and we shall allow solutions in an arbitrarily small neighborhood of H to have finite escape time.

DEFINITION 3. – A scalar function $V\colon [0, \infty) \times D \to [0, \infty)$ is a *Liapunov function* for (1) if:

(i) V is C' and

(ii) $V'(t, x) = \partial V/\partial t + \sum_{i=1}^{n} \partial V/\partial x_i \cdot f_i(t, x) \leq 0$ for all $(t, x) \in [0, \infty) \times D$ (where $x = \mathrm{col}(x_1, \ldots, x_n)$ and $f(t, x) = \mathrm{col}(f_1(t, x), \ldots, f_n(t, x))$.

For the sake of simplicity we have asked that our Liapunov functions be C'. However, for the most part, our results can be carried out using Liapunov functions which are locally Lipschitzian (cf. [5, pp. 57-59]). Henceforth, when we refer to a Liapunov function, we shall mean a Liapunov function for (1).

In [1] and [2] BURTON proved that if V' satisfies certain « strong » conditions relative to some closed set H (either bounded or unbounded) and if $f(t, x)$ satisfies hypothesis (A) in H^c, then H is a global attractor. Unlike BURTON, we shall restrict our attention, for the most part, to a neighborhood of H. In particular, we shall ask that V' satisfy conditions which are similar to Burton's in a neighborhood of H and we shall seek conditions on H and V which will yield the stability of H. Although we shall ask that V satisfy certain properties with respect to H, it is significant that we do not require any « definiteness » properties of V.

We shall say that $V(t, x)$ *satisfies hypotheses* (B) *with respect to a closed set* N if, for each $t_0 \geqslant 0$ and each $\varepsilon > 0$ there exists $\delta = \delta(t_0, \varepsilon) > 0$ with the property that $V(t_0, x) < \varepsilon$ whenever $0 < d(x, N) < \delta$.

For a Liapunov function V define $H = \{x\colon V(t, x) = 0 \text{ for all } t \geqslant 0\}$. It is easily seen that if $H \neq \varphi$, then V satisfies hypothesis (B) with respect to any compact subset of H.

The following definition was given in [3].

DEFINITION 4. – $V'(t, x)$ *is strongly negative definite in a set* $N \subset R^n$ if there exists $\delta > 0$ so that $V'(t, x) \leqslant -\delta |f(t, x)|$ for all x in N and all $t \geqslant 0$.

We shall list the main theorems of this section cumulatively before giving the proofs.

THEOREM 3. – Suppose there exists $\alpha > 0$ and a closed set H with $S(H, \alpha) \subset D$ such that, for every ε, $0 < \varepsilon < \alpha$, there exists a Liapunov function V_ε with the property that V_ε satisfies hypothesis (B) with respect to H and V_ε' is strongly negative definite in $R(H, \varepsilon)$. Then H is stable.

THEOREM 4. – In addition to the conditions of Theorem 3, suppose $f(t, x)$ satisfies hypothesis (A) in $H^c \cap S(H, \alpha)$. Then H is asymptotically stable.

THEOREM 5. – Suppose $D = R^n$ and suppose there exists a closed set $H \subset R^n$ such that, for every $\varepsilon > 0$ there is a Liapunov function V_ε with the property that V_ε satisfies hypothesis (B) with respect to H and V_ε' is strongly negative definite in $S^c(H, \varepsilon)$. If $f(t, x)$ satisfies hypothesis (A) in H^c, then H is globally asymptotically stable.

REMARK 1. – If there exists a Liapunov function V with V' strongly negative definite in $S^c(H, \varepsilon)$ for each $\varepsilon > 0$, then V' is strongly negative definite relative to H in the sense of BURTON [1]. BURTON [1, p. 547] proved that the existence of a Liapunov function V with V' strongly negative definite in $S^c(H, \varepsilon)$ for arbitrary ε is sufficient to guarantee that H is a global attractor. In view of this, it appears at a glance that Theorem 5 is an immediate consequence of our Theorem 3 and Theorem 1 of [1]. However, BURTON asks for a Liapunov function with strongly negative definite derivative in each $S^c(H, \varepsilon)$; we ask that for each $\varepsilon > 0$ there exists a Liapunov function (depending on ε) with strongly negative definite derivative in $S^c(H, \varepsilon)$. It is currently unknown if the two concepts are equivalent.

REMARK 2. – In [2] BURTON gave a revised definition of V' strongly negative definite. Using our present terminology, we can state his new definition as follows: V' is strongly negative definite relative to a closed set H if, for every $\varepsilon > 0$, either

(i) V' is strongly negative definite in $S^c(H, \varepsilon)$ (in the sense of Definition 4) if H is unbounded or

(ii) there exists $\delta > 0$ so that x in $S^c(H, \varepsilon)$ implies $V'(t, x) \leqslant -\delta |f(t, x)|(1 + |x|)$ if H is bounded. (Here $|\cdot|$ denotes euclidean length.)

Since we are working primarily in neighborhoods of closed sets, we are not greatly benefitted by considering bounded sets as a special case in Definition 4. For instance, suppose H is bounded and $V'(t, x) \leqslant -\delta |f(t, x)|$ for all x in $R(H, \varepsilon)$ and some $\delta > 0$. Then with $\delta_1 = \delta(1 + \sup\{|x| : x \in R(H, \varepsilon)\})$, we have $V'(t, x) \leqslant -\delta_1 f(t, x)/(1 + |x|)$. The using of Burton's definition of V' strongly negative definite does, however, strengthen Theorem 5 to some extent since, in this case, we want each V'_ε to be strongly negative definite in each $S^c(H, \varepsilon)$.

Before stating a theorem which provides sufficient conditions for the set H of Theorem 3 to be uniformly stable, we need the following definition.

DEFINITION 5. – *$V(t, x)$ is decrescent with respect to a closed set $N \subset R^n$* if there exists $\alpha > 0$ and a continuous scalar function $Q(x)$ such that:

(i) $Q(x) \geqslant 0$ for all $x \in S(N, \alpha)$,

(ii) for every $\varepsilon > 0$ there exists $\eta > 0$ so that $Q(x) < \varepsilon$ whenever $0 < d(x, N) < \eta$. and

(iii) $V(t, x) \leqslant Q(x)$ for all $(t, x) \in [0, \infty) \times S(N, \alpha)$.

THEOREM 6. – In addition to the conditions of Theorem 3, suppose, for each $\varepsilon > 0$, V_ε is decrescent with respect to H. Then H is uniformly stable.

We note that V decrescent with respect to a closed set H implies that V satisfies hypothesis (B) with respect to H.

The following simple example shows that Theorem 4 cannot be extended in the same manner as Theorem 3. That is, decrescence of each V_ε in Theorem 4 does not guarantee uniform asymptotic stability of H. For the scalar linear equation

$$x' = -x/(t+1) \tag{4}$$

$\{0\}$ is not uniformly asymptotically stable. However, for $V(t, x) = x^2/2$ we have V decrescent with respect to $\{0\}$ and $V'(t, x) = -x^2/(t+1)$ is strongly negative definite in each $R(\{0\}, \varepsilon)$. Also, $f(t, x) = -x/(t+1)$ satisfies hypothesis (A).

We now give proofs of Theorem 3-6 and then we shall discuss how these theorems relate to results given in [5].

PROOF OF THEOREM 3. – Suppose H is not stable. Then there is $t_0 \geqslant 0$ and $\varepsilon > 0$ so that for every $\eta > 0$ $(\eta < \varepsilon)$ there exists x_0 and $t^* > t_0$ with $d(x_0, H) < \eta$ and $d(x(t^*, t_0, x_0), H) = \varepsilon$ for some solution $x(t, t_0, x_0)$ of (1) where $x(t_0, t_0, x_0) = x_0$. By hypothesis, there is a Liapunov function $V = V_\varepsilon$ and $\delta > 0$ so that V satisfies hypothesis (B) with respect to H and $V'(t, x) \leqslant -\delta|f(t, x)|$ for all $t \geqslant 0$ and x in $R(H, \varepsilon)$. Since V satisfies hypothesis (B) with respect to H, there exists $\delta_0 > 0$, $\delta_0 < \varepsilon/2$, such that $V(t_0, x) < \delta\varepsilon/2$ whenever $d(x, H) < \delta_0$. Since H is not stable, there exist x_0, t_1 and t_2 with the property that $d(x_0, H) < \delta_0$, $d(x(t_1, t_0, x_0), H) = \varepsilon/2$, $d(x(t_2, t_0, x_0), H) = \varepsilon$ and $\varepsilon/2 \leqslant d(x(t, t_0, x_0), H) \leqslant \varepsilon$ whenever $t_0 < t_1 \leqslant t \leqslant t_2$. Integrating V' along $x(t) = x(t, t_0, x_0)$ we obtain

$$
\begin{aligned}
V(t_2, x(t_2)) &= V(t_1, x(t_1)) + \int_{t_1}^{t_2} V'(s, x(s))\, ds \\
&\leqslant V(t_0, x_0) - \delta \int_{t_1}^{t_2} |f(s, x(s))|\, ds \\
&\leqslant V(t_0, x_0) - \delta \left| \int_{t_1}^{t_2} x'(s)\, ds \right| \\
&= V(t_0, x_0) - \delta |x(t_2) - x(t_1)| \\
&< \delta\varepsilon/2 - \delta\varepsilon/2 = 0\,.
\end{aligned}
$$

This contradicts $V \geqslant 0$ and hence, H is stable.

PROOF OF THEOREM 4. – Let $t_0 \geqslant 0$ and $\beta > 0$ $(3\beta/2 < \alpha)$ be given. By Theorem 3, H is stable and thus, there exists $\delta_0 = \delta_0(t_0, \beta) > 0$ such that $d(x_0, H) < \delta_0$ implies $d(x(t, t_0, x_0), H) < \beta$ for all t in the maximal right interval of definition $[t_0, T)$ of $x(t, t_0, x_0)$. Suppose there exists x_0 such that $d(x_0, H) < \delta_0$ and $x(t) = x(t, t_0, x_0) \nrightarrow H$ as $t \to T^-$. We will obtain in a contradiction by considering cases in which $x(t)$ is unbounded and $x(t)$ is bounded.

Suppose $x(t)$ is unbounded and $d(x(t), H) \to c > 0$ as $t \to T^-$ for some $c \leqslant \beta$. Let $\varepsilon = 3c/2$. Then $\varepsilon/2 < c < \varepsilon < \alpha$ and there exists $t^* \geqslant t_0$ so that $x(t)$ is in $R(H, \varepsilon)$ for all t in $[t^*, T)$. By hypothesis, there exist a Liapunov function $V = V_\varepsilon$ and $\delta > 0$ with $V'(t, x(t)) \leqslant -\delta|f(t, x(t))|$ for t in $[t^*, T)$. Since $x(t)$ is unbounded, there exists an increasing sequence $\{t_n\}$ with $t_n \to T$ such that $|x(t_n)| \to \infty$ as $n \to \infty$ and $|x(t_n)| \geqslant |x(t^*)|$ for each n. Integrating V' along $x(t)$ we obtain

$$
V(t, x(t)) \leqslant V(t^*, x(t^*)) - \delta \int_{t^*}^{t_k} |f(s, x(s))|\, ds
$$

where $t_k \leqslant t < t_{k+1}$. Hence,

$$V(t, x(t)) \leqslant V(t^*, x(t^*)) - \delta |x(t_k) - x(t^*)|$$

and $V(t, x(t)) \to -\infty$ as $t \to T^-$. This contradicts V bounded from below. Thus, there exists $a, b > 0$, $b/2 \leqslant a < b$, and increasing sequences $\{t_n'\}$, $\{t_n''\}$, $t_n' < t_n'' < t_{n+1}'$, with $t_n' \to T^-$ such that $d(x(t_n'), H) = a$, $d(x(t_n''), H) = b$ for each n and $a \leqslant d(x(t), H) \leqslant b$ for $t_n' \leqslant t \leqslant t_n''$. Thus, there exists $\delta_1 > 0$ and a Liapunov function $V = V_b$ such that $V'(t, x(t)) \leqslant -\delta_1 |f(t, x(t))|$ if $t_n' \leqslant t \leqslant t_n''$. Integrating, we obtain

$$V(t, x(t)) = V(t_0, x_0) + \int_{t_0}^{t} V'(s, x(s))\, ds$$

$$\leqslant V(t_0, x_0) - \delta_1 \sum_{n=1}^{k} \int_{t_n'}^{t_n''} |f(s, x(s))|\, ds$$

where $t_k'' \leqslant t \leqslant t_{k+1}'$. Hence,

$$V(t, x(t)) \leqslant V(t_0, x_0) - \delta_1 \sum_{n=1}^{k} |x(t_n'') - x(t_n')|$$

$$\leqslant V(t_0, x_0) - \delta_1 k(b - a)\,.$$

Again, we have $V(t, x(t)) \to -\infty$ as $t \to T^-$ which is a contradiction and thus we may assume $x(t)$ is bounded for $t \geqslant t_0$. There must be, then, an increasing sequence $\{t_m\}$ with $t_m \to \infty$ and $x(t_m) \to y$ as $m \to \infty$ for some y. If $x(t)$ does not converge to y as $t \to \infty$, there exists $\varepsilon > 0$, $\varepsilon < \alpha$, so that y is in the interior of $R(H, \varepsilon)$ and $\varepsilon_1 > 0$ so that $S(y, \varepsilon_1) \subset R(H, \varepsilon)$. Also, as $x(t)$ does not converge to y, we may assume ε_1 is sufficiently small so that there are increasing sequences $\{t_m'\}$ and $\{t_m''\}$ with $t_m' < t_m'' < t_{m+1}'$ and $t_m' \to \infty$ as $m \to \infty$ such that $|x(t_m') - y| = \varepsilon_1/2$, $|x(t_m') - y| = \varepsilon_1$ and $\varepsilon_1/2 \leqslant |x(t) - y| \leqslant \varepsilon_1$ for $t_m' \leqslant t \leqslant t_m''$. Arguing, now, as above we obtain the contradiction $V(t, x(t)) \to -\infty$ as $t \to \infty$ where $V = V_\varepsilon$.

Finally, suppose $x(t) \to y$ as $t \to \infty$. As $f(t, x)$ satisfies hypothesis (A) in $H^c \cap S(H, \alpha)$, there exists $\eta > 0$ and $g(t)$ with the property that $|f(t, x)| \geqslant g(t)$ for $|x - y| < \eta$, x in $H^c \cap S(H, \alpha)$ where $\int_0^\infty g(t)\, dt = \infty$. Without loss of generality we may assume η is chosen sufficiently small so that there is $\varepsilon > 0$ with $S(y, \eta) \subset$ $\subset R(H, \varepsilon) \subset S(H, \alpha)$. Let $t^* \geqslant t_0$ be chosen so that $t \geqslant t^*$ implies $x(t) \in S(y, \eta)$. Then there exist $\delta > 0$ and a Liapunov function $V = V_\varepsilon$ such that $V'(t, x(t)) \leqslant -\delta |f(t, x(t))|$ for all $t \geqslant t^*$. Again, we integrate V' along $x(t)$ to obtain for $t > t^*$.

$$V(t, x(t)) \leqslant V(t^*, x(t^*)) - \delta \int_{t^*}^{t} |f(s, x(s))|\, ds$$

$$\leqslant V(t^*, x(t^*)) - \delta \int_{t^*}^{t} g(s)\, ds$$

and hence, $V(t, (x)) \to -\infty$ as $t \to \infty$ which contradicts V bounded below. All possibilities have now been exhausted and the proof is complete.

PROOF OF THEOREM 5. – By Theorem 3, H is stable. It remains to be shown that H is a global attractor. The proof of this will be omitted since it is very similar to the proof of Theorem 4 and Theorem 1 of [1].

PROOF OF THEOREM 6. – Let $\varepsilon > 0$ be given with $\varepsilon < \alpha$. Then there exist $\delta > 0$ and a Liapunov function $V = V_\varepsilon$ which is decrescent with respect to H and $V'(t, x) \leqslant -\delta|f(t, x)|$ for all x in $R(H, \varepsilon)$. Since V is decrescent with respect to H, there exists $\delta_0 > 0$ so that

$$V(t_0, x_0) \leqslant \delta\varepsilon/2 \tag{5}$$

for any t_0 and x_0 satisfying $d(x_0, H) < \delta_0$. Since (5) holds for arbitrary t_0, it follows, using arguments similar to these used in the proof of Theorem 3, that H is uniformly stable.

As was previously promised, we shall now compare our results with results given by LASALLE in [5]. For simplicity, we assume $D = R^n$.

Let $V(t, x)$ be a Liapunov function such that $V'(t, x) \leqslant -W(x) \leqslant 0$ for all (t, x) in $[0, \infty) \times R^n$ where W is continuous on R^n. Define $E = \{x : W(x) = 0\}$. LASALLE proved (among the results of Theorem 1 of [5]) that if $|f(t, x)|$ is bounded on $[0, \infty) \times A$ for each compact set $A \subset R^n$, then all bounded solutions approach E as $t \to \infty$. We shall modify LaSalle's conditions to some extent in order to determine conditions which guarantee that the set E is asymptotically stable.

COROLLARY 3. – Suppose $W(x)$ is positive definite with respect to E, that is, for $\varepsilon > 0$ sufficiently small, there is $k = k(\varepsilon) > 0$ so that $W(x) \geqslant k$ for x in $R(E, \varepsilon)$, and suppose V satisfies hypothesis (B) with respect to E. If there exists $\alpha > 0$ so that $|f(t, x)|$ is bounded on $[0, \infty) \times S(E, \alpha)$, then E is asymptotically stable.

PROOF. – To prove that E is stable it suffices to show V' is strongly negative definite in $R(E, \varepsilon)$ for $\varepsilon > 0$ sufficiently small. Since $W(x)$ is positive definite with respect to E, for each $\varepsilon > 0$ sufficiently small there exists $k_1 > 0$ so that $W(x) \geqslant k_1$ whenever $\varepsilon/2 \leqslant d(x, E) \leqslant \varepsilon$. Also, $\varepsilon/2 \leqslant d(x, E) \leqslant \varepsilon$ implies $|f(t, x)| \leqslant k_2$ for all $t \geqslant 0$ and some $k_2 \geqslant 0$. Let $\delta = k_1/k_2$. Then, for $\varepsilon/2 \leqslant d(x, E) \leqslant \varepsilon$, we have

$$\begin{aligned} V'(t, x) &\leqslant -W(x) \leqslant -k_1 = -(k_1/k_2)k_2 \\ &\leqslant -(k_1/k_2)|f(t, x)| \\ &= -\delta|f(t, x)|. \end{aligned}$$

Hence, V' is strongly negative definite in $R(E, \varepsilon)$. Since ε was arbitrary, the conditions of Theorem 3 are satisfied and E is stable.

We cannot immediately conclude from Theorem 4 that E is asymptotically stable since we have not assumed that $f(t, x)$ satisfies hypothesis (A). However, in the proof of Theorem 4, hypothesis (A) was used to obtain a contradiction only for the case where $x(t) \to y$ with $x(t)$ bounded. This case can be discarded in the present proof by using the fact that $W(x)$ is positive definite with respect to E. Hence, E is asymptotically stable and the corollary is proven.

An interesting consequence of the above corollary is a stability theorem which was originally proved by MARACHKOFF in 1940 (see [4, Corollary 2.2] for details).

We once again examine the autonomous system

$$x' = f(x) \tag{2}$$

in which case our Liapunov function has a simpler definition. In particular, if $V(x)$ is a Liapunov function for (2), then $V'(x) = \sum_{i=1}^{n} (\partial V / \partial x_i) f_i(x)$. Let $V(x)$ be a Liapunov function for (2) and define $E_V = \{x\colon V'(x) = 0\}$. Let M_V denote the largest semi-invariant subset of E_V. LASALLE recently proved the following theorem.

THEOREM 7 ([5, Theorem 3]). - Let G be bounded, open, positively invariant set. If solutions of (2) are unique and if $V(x)$ is a Liapunov function for (2) such that (i) $M_V \subset G$ and (ii) V is constant on the boundary of M_V, then M_V is asymptotically stable.

Since solutions are unique in Theorem 7, it follows that M_V is the largest invariant subset of E_V. We now show, by using Theorem 1, that this theorem can be proven without assuming uniqueness of solutions.

THEOREM 8. - Let G be a bounded, open, positively invariant set. If there exists a Liapunov function $V(x)$ for (2) such that (i) $M_V \subset G$ and (ii) V is constant on the boundary of M_V, then M_V is asymptotically stable and G is in the region of attraction.

PROOF. - By Theorem 2 of [5], each solution starting in G approaches M_V and so M_V is an attractor and G is in its region of attraction. Also, as $M_V \subset G$ and G bounded, it is easy to see that $\bar{M}_V$ is semi-invariant using standard arguments except that Ascolis' Theorem is used in place of continuity with respect to initial conditions. Then as $\bar{M}_V \subset E_V$, we must have M_V closed and, thus, compact.

Now M_V must be positively invariant. If not, there is a solution $x(t)$ of (2) with $x(0)$ in M_V and $x(t_1) \notin M_V$ for some $t_1 > 0$ and, as M_V is an attractor, we have $d(x(t), M_V) \to 0$ as $t \to \infty$. Also, we may extend $x(t)$ negatively on $(-\infty, 0]$ with $x(t)$ in M_V for $-\infty < t \leqslant 0$ as M_V is semi-invariant. Now $\{x(t)\colon -\infty < t < \infty\}$ is a semi-invariant set in G but not in E_V as $x(t_1)$ is not in M_V. As $x(0)$ is in M_V and $x(t_1)$ is not, there exists a value of t, say t_2, where $x(t_2) \in \partial M_V$, $0 \leqslant t_2 < t_1$. As $\{x(t)\}$ is not contained in E_V, $V' < 0$ along some nonempty t interval (t_3, t_4), $t_3 \geqslant t_2$, and so

$V(x(t_4)) < V(x(t_2))$. This is a contradiction, however, as $d(x(t), M_V) \to 0$ as $t \to \infty$ implies $V(x(t)) \to V(x(t_2))$ as $t \to \infty$ and $V(x(t)) \leqslant V(x(t_4))$ for $t \geqslant t_4$.

Now, suppose M_V is not stable. Then by Theorem 1, there exists a compact semi-invariant set $K \subset G$ but $K \not\subset M_V$. Let $x(t)$ be a solution of (2) in K such that $x(0)$ is not in M_V. As above, $\{x(t): -\infty < t < \infty\}$ cannot be contained in E_V and by Theorem 2 of [5], we have $d(x(t), M) \to 0$ as $t \to \infty$ and as $t \to -\infty$. Thus, $V(x(t)) \to C$ as $t \to \infty$ and $t \to -\infty$ where $V(x) = C$ for all $x \in \partial M_V$. As above, however, we must have $V' < 0$ along $x(t)$ on some t interval and we again have a contradiction and the proof is complete.

It is interesting to note that in general we cannot assume M_V is an invariant set as it is possible for E_V to have no invariant subsets. For example, consider the scalar equation

$$x' = -|x|^\gamma \operatorname{sgn} x \tag{6}$$

where $0 < \gamma < 1$ and let $V = x^2/2$. Then $V' = -|x|^\gamma x \operatorname{sgn} x$ and $E_V = \{0\}$. However, the zero solution of (6) is not unique to the left and, thus, there are no invariant sets in E_V.

REFERENCES

[1] T. A. BURTON, *An extension of Liapunov's method*, J. Math. Anal. Appl., **28** (1969), pp. 545-552.

[2] T. A. BURTON, *Correction to an extension of Liapunov's direct method*, J. Math. Anal. Appl., **32** (1970), pp. 689-691.

[3] J. R. HADDOCK, *Liapunov functions and boundedness and global existence of solutions*, Appl. Anal., **1** (1972), pp. 321-330.

[4] J. R. HADDOCK, *Some refinements of asymptotic stability theory*, Ann. Mat. Pura Appl., (4), **89** (1971), pp. 393-402.

[5] J. P. LASALLE, *Stability theory for ordinary differential equations*, J. Diff. Eqs., **4** (1968), pp. 57-65.

[6] Z. OPIAL, *Sur la dependance des solutions d'un systeme d'equations differentielles de leur second membres. Application aux systems presque autonomes*, Ann. Polon. Math., **8** (1960), pp. 75-89.

[7] A. STRAUSS - J. A. YORKE, *On asymptotically autonomous differential equations*, Math. Syst. Th., **1** (1967), pp. 175-182.

[8] T. YOSHIZAWA, *Asymptotic behavior of solutions of a system of differential equations*, Contrib. Differential Eqs., **1** (1963), pp. 371-387.