Kyushu J. Math. **71** (2017), 187–196
doi:10.2206/kyushujm.71.187

STABILITY OF C^∞ CONVEX INTEGRANDS

Erica Boizan BATISTA, Huhe HAN and Takashi NISHIMURA

(Received 9 March 2016 and revised 15 June 2016)

Abstract. In this paper, it is shown that the set consisting of stable convex integrands $S^n \to \mathbb{R}_+$ is open and dense in the set consisting of C^∞ convex integrands with respect to Whitney C^∞ topology. Moreover, examples are given representing well why stable convex integrands are preferred.

1. Introduction

In the celebrated series [**14–19**], Mather gave a complete answer to the problem of density of proper stable mappings in a surprising form. For proper C^∞ mappings of special type, it is natural to ask a similar question: are generic proper mappings of special type stable? Such investigations, for instance, can be found in [**20**] for generic projections of submanifolds, in [**6**] for generic projections of stable mappings and in [**8–11**] for generic distance-squared mappings and their generalizations.

Motivated by this earlier research, in this paper, the density problem for C^∞ convex integrands is investigated. The notion of a convex integrand was first introduced in [**23**], which is defined as follows. For a positive integer n, let S^n be the unit sphere of $\mathbb{R}^{n+1}$. The set consisting of positive real numbers is denoted by $\mathbb{R}_+$. Then, a continuous function $\gamma : S^n \to \mathbb{R}_+$ is called a *convex integrand* if the boundary of the convex hull of $\mathrm{inv}(\mathrm{graph}(\gamma))$ is exactly the same set as $\mathrm{inv}(\mathrm{graph}(\gamma))$, where $\mathrm{graph}(\gamma)$ is the set $\{(\theta, \gamma(\theta)) \mid \theta \in S^n\}$ with respect to the polar plot expression for $\mathbb{R}^{n+1} - \{0\}$ and $\mathrm{inv} : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}^{n+1} - \{0\}$ is the inversion defined by $\mathrm{inv}(\theta, r) = (-\theta, 1/r)$. The notion of a convex integrand is closely related to the notion of a Wulff shape, which was first introduced in [**24**] as a geometric model of a crystal at equilibrium. Integration of a convex integrand γ over S^n represents the surface energy of the Wulff shape associated with γ. Hence, γ is called a convex integrand. For more details on convex integrands, see for instance [**22, 23**].

Set

$$C^\infty_{\mathrm{conv}}(S^n, \mathbb{R}_+) = \{\gamma \in C^\infty(S^n, \mathbb{R}_+) \mid \gamma \text{ is a convex integrand}\},$$

where $C^\infty(S^n, \mathbb{R}_+)$ is the set consisting of C^∞ functions $S^n \to \mathbb{R}_+$. The set $C^\infty(S^n, \mathbb{R}_+)$ is endowed with *Whitney C^∞ topology* (for details on Whitney C^∞ topology, see, for instance, [**2, 7**]) and the set $C^\infty_{\mathrm{conv}}(S^n, \mathbb{R}_+)$ is a topological subspace of $C^\infty(S^n, \mathbb{R}_+)$.

2010 Mathematics Subject Classification: Primary 58K05, 58K25.

Keywords: convex integrand; C^∞ convex integrand; stable convex integrand; caustic; symmetry set; wave front; index of critical point; Morse inequalities.

This work is partially supported by JSPS and CAPES under the Japan–Brazil research cooperative program and JSPS KAKENHI Grant Number 26610035.

Two C^∞ functions $\gamma_1, \gamma_2 : S^n \to \mathbb{R}_+$ are said to be $\mathcal{A}$*-equivalent* if there exist C^∞ diffeomorphisms $h : S^n \to S^n$ and $H : \mathbb{R}_+ \to \mathbb{R}_+$ such that the equality $\gamma_2 = H \circ \gamma_1 \circ h^{-1}$ holds. A C^∞ function $\gamma \in C^\infty(S^n, \mathbb{R}_+)$ is said to be *stable* if the $\mathcal{A}$-equivalence class $\mathcal{A}(\gamma)$ is an open subset of the topological space $C^\infty(S^n, \mathbb{R}_+)$. By definition, any function $\mathcal{A}$-equivalent to a stable function is stable. Set

$$S^\infty(S^n, \mathbb{R}_+) = \{\gamma \in C^\infty(S^n, \mathbb{R}_+) \mid \gamma \text{ is stable}\}.$$

By definition, $S^\infty(S^n, \mathbb{R}_+)$ is open. The following proposition is one of the corollaries of Mather's series [**14–19**].

PROPOSITION 1.

(1) *A C^∞ function $\gamma \in C^\infty(S^n, \mathbb{R}_+)$ is stable if and only if all critical points of γ are non-degenerate and $\gamma(\theta_1) \neq \gamma(\theta_2)$ holds for any two distinct critical points $\theta_1, \theta_2 \in S^n$.*

(2) *The open subset $S^\infty(S^n, \mathbb{R}_+)$ is dense in $C^\infty(S^n, \mathbb{R}_+)$.*

Here, a critical point $\theta \in S^n$ is said to be *non-degenerate* if there exists a local neighborhood (U, φ) such that $\theta \in U$ and the following expression holds for an integer i $(0 \leq i \leq n)$:

$$\gamma \circ \varphi^{-1}(x_1, \ldots, x_n) = -x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots x_n^2.$$

The integer i given above is called the *index* of γ at the non-degenerate critical point θ.

Assertion (1) of Proposition 1 well explains reasons why we prefer stable functions rather than general C^∞ functions. First of all, by using the normal form of a stable function germ $\gamma : (S^n, \theta) \to \mathbb{R}_+$ at a critical point $\theta \in S^n$ given above, it is easy to investigate the local differentiable type of a level set $\gamma^{-1}(\gamma(\theta))$ at $\theta \in S^n$. This is an advantage of stable functions because, in general, it is almost impossible to study it for a general C^∞ function. Moreover, by using the Morse inequalities [**21**], it is possible to study even restrictions on the global differentiable types of stable functions. Thus, we want to perturb a given C^∞ function γ to a stable function $\widetilde{\gamma}$.

Assertion (2) of Proposition 1 asserts that any C^∞ function $\gamma : S^n \to \mathbb{R}_+$ can be perturbed to a stable function $\widetilde{\gamma}$ by a sufficiently small perturbation, and for any sufficiently small $\varepsilon > 0$, any continuous mapping $\Phi : (-\varepsilon, \varepsilon) \to C^\infty(S^n, \mathbb{R}_+)$ such that $\Phi(0) = \widetilde{\gamma}$ and any two $t_1, t_2 \in (-\varepsilon, \varepsilon)$, there exist C^∞ diffeomorphisms $h : S^n \to S^n$ and $H : \mathbb{R}_+ \to \mathbb{R}_+$ such that the equality $\Phi(t_2) = H \circ \Phi(t_1) \circ h^{-1}$ holds.

The main purpose of this paper is to show the following.

THEOREM 1. *The open subset $S^\infty(S^n, \mathbb{R}_+) \cap C^\infty_{\text{conv}}(S^n, \mathbb{R}_+)$ is dense in $C^\infty_{\text{conv}}(S^n, \mathbb{R}_+)$.*

Similar to assertion (2) of Proposition 1, Theorem 1 asserts that any C^∞ convex integrand $\gamma : S^n \to \mathbb{R}_+$ can be perturbed to a stable convex integrand $\widetilde{\gamma}$ by a sufficiently small perturbation, and for any sufficiently small $\varepsilon > 0$, any continuous mapping $\Phi : (-\varepsilon, \varepsilon) \to C^\infty_{\text{conv}}(S^n, \mathbb{R}_+)$ such that $\Phi(0) = \widetilde{\gamma}$ and any two $t_1, t_2 \in (-\varepsilon, \varepsilon)$, there exist C^∞ diffeomorphisms $h : S^n \to S^n$ and $H : \mathbb{R}_+ \to \mathbb{R}_+$ such that the equality $\Phi(t_2) = H \circ \Phi(t_1) \circ h^{-1}$ holds. Then, for $\Phi(t)$ $(\forall t \in (-\varepsilon, \varepsilon))$, detailed investigation from the differentiable viewpoint is possible by applying assertion (1) of Proposition 1.

In Section 2, preliminaries for the proof of Theorem 1 are given. Theorem 1 is proved in Section 3. Finally, in Section 4, several examples of stable convex integrands are given.

2. Preliminaries

Let $\phi : S^n \to \mathbb{R}^{n+1}$ be a C^∞ embedding. Consider the family of functions $F : \mathbb{R}^{n+1} \times S^n \to \mathbb{R}$ defined by

$$F(v, z) = \frac{1}{2}\|\phi(z) - v\|^2.$$

Notice that F may be regarded as a mapping from $\mathbb{R}^{n+1}$ to $C^\infty(S^n, \mathbb{R})$ which maps each $v \in \mathbb{R}^{n+1}$ to the function $f_v(z) = F(v, z) \in C^\infty(S^n, \mathbb{R}) = \{g : S^n \to \mathbb{R}\ C^\infty\}$. The set of values v for which $f_v(z)$ has a degenerate critical point, denoted by $\mathrm{Caust}(\phi)$, is called the *Caustic* of ϕ (for details on caustics, see for instance, [**1, 2, 12, 13**]). The set of values v for which $f_v(z)$ has a multiple critical value forms the *Symmetry set* of ϕ, denoted by $\mathrm{Sym}(\phi)$ (for details on symmetry sets, see, for instance, [**3–5**]). By assertion (1) of Proposition 1, these two sets $\mathrm{Caust}(\phi)$ and $\mathrm{Sym}(\phi)$ constitute the set of points v for which the function $f_v \in C^\infty(S^n, \mathbb{R})$ is not stable.

PROPOSITION 2. *Let $\phi : S^n \to \mathbb{R}^{n+1}$ be a C^∞ embedding. Then,* $\mathrm{Caust}(\phi)$ *has Lebesgue measure zero in* $\mathbb{R}^{n+1}$.

For the proof of Proposition 2, see [**21**, §6 'Manifolds in Euclidean space'].

PROPOSITION 3. *Let $\phi : S^n \to \mathbb{R}^{n+1}$ be a C^∞ embedding. Then,* $\mathrm{Sym}(\phi)$ *has Lebesgue measure zero in* $\mathbb{R}^{n+1}$.

Proof. Since ϕ is an embedding, the complement of $\phi(S^n)$ constitutes two connected components. Denote the bounded connected component by V_ϕ. Set $M = \phi(S^n)$. For each $\theta \in S^n$, consider the normal vector space $N_{\phi(\theta)}(M)$ to M at θ. Notice that $N_{\phi(\theta)}(M)$ is a one-dimensional vector space. Thus, we can uniquely specify the unit vector $\mathbf{n}(\theta)$ of $N_{\phi(\theta)}(M)$ so that $\phi(\theta) + \varepsilon\mathbf{n}(\theta)$ belongs to V_ϕ for any sufficiently small $\varepsilon > 0$. For any $t \in \mathbb{R}$, let $\phi_t : S^n \to \mathbb{R}^{n+1}$ be the C^∞ mapping defined by $\phi_t(\theta) = \phi(\theta) + t\mathbf{n}(\theta)$. The mapping ϕ_t is called a *wave front* of ϕ (for details on wave fronts, see, for instance, [**1, 2, 12, 13**]). It is clear that, by using wave fronts $\{\phi_t\}_{t\in\mathbb{R}}$, the set $\mathrm{Sym}(\phi)$ can be characterized as follows:

$$\mathrm{Sym}(\phi) = \bigcup_{t\in\mathbb{R}} \{\phi_t(\theta_1) = \phi_t(\theta_2) \mid \theta_1 \neq \theta_2\}.$$

By Proposition 2, the intersection $\mathrm{Sym}(\phi) \cap \mathrm{Caust}(\phi)$ is of Lebesgue measure zero. Thus, in order to show Proposition 3, it is sufficient to show that $\mathrm{Sym}(\phi) \cap (\mathbb{R}^{n+1} - \mathrm{Caust}(\phi))$ is of Lebesgue measure zero. Take one point $\phi_{t_0}(\theta_1) = \phi_{t_0}(\theta_2)$ of $\mathrm{Sym}(\phi) \cap (\mathbb{R}^{n+1} - \mathrm{Caust}(\phi))$, where θ_1, θ_2 are two distinct points of S^n. Set $x_0 = \phi_{t_0}(\theta_1) = \phi_{t_0}(\theta_2)$, and let U_0 be a sufficiently small open neighborhood of x_0. Notice that, since $\mathrm{Caust}(\phi)$ is compact, U_0 may be chosen so that $U_0 \cap \mathrm{Caust}(\phi) = \emptyset$.

Let i be 1 or 2. For i, define the mapping $(t_i, \widetilde{\theta}_i) : U_0 \to \mathbb{R} \times S^n$ as follows:

$$x = \phi_{t_i(x)}(\widetilde{\theta}_i(x)) \quad (t_i(x_0) = t_0, \widetilde{\theta}_i(x_0) = \theta_i).$$

Notice that, since $U_0 \cap \mathrm{Caust}(\phi) = \emptyset$, both of the following are well-defined C^∞ diffeomorphisms:

$$(t_1, \widetilde{\theta}_1) : U_0 \to (t_1, \widetilde{\theta}_1)(U_0),$$

$$(t_2, \widetilde{\theta}_2) : U_0 \to (t_2, \widetilde{\theta}_2)(U_0).$$

Set $T = t_1 - t_2$. Then, it is clear that $\mathrm{Sym}(\phi) \cap U_0 = T^{-1}(0)$.

For any $i = 1, 2$, let $\nabla t_i(x_0)$ be the gradient vector of t_i at x_0. Since both t_1, t_2 are non-singular functions, it follows that neither $\nabla t_1(x_0)$ nor $\nabla t_2(x_0)$ is the zero vector. Moreover, from the construction, it is easily seen that even when $\nabla t_1(x_0)$ and $\nabla t_2(x_0)$ are linearly dependent, $\nabla T(x_0) = \nabla t_1(x_0) - \nabla t_2(x_0)$ is a non-zero vector. Therefore, taking a smaller open neighborhood $\widetilde{U}_0$ of x_0 if necessary, it follows that $T^{-1}(0) \cap \widetilde{U}_0$ is an n-dimensional submanifold of $\mathbb{R}^{n+1}$. Therefore, Proposition 3 follows. □

Propositions 2 and 3 clearly yield the following.

COROLLARY 1. *Let $\phi : S^n \to \mathbb{R}^{n+1}$ be a C^∞ embedding. Then, the union* Caust$(\phi) \cup$ Sym(ϕ) *is a subset of Lebesgue measure zero in $\mathbb{R}^{n+1}$.*

3. Proof of Theorem 1

Let $\gamma : S^n \to \mathbb{R}_+$ be a C^∞ convex integrand, and let V be a neighborhood of γ in $C^\infty_{\text{conv}}(S^n, \mathbb{R}_+)$. It is sufficient to show that $V \cap S^\infty(S^n, \mathbb{R}_+) \neq \emptyset$. In order to construct an element of $V \cap S^\infty(S^n, \mathbb{R}_+)$, we consider the C^∞ embedding $\phi : S^n \to \mathbb{R}^{n+1} - \{0\}$ defined as follows:

$$\phi(\theta) = \left(\theta, \frac{1}{\gamma(-\theta)}\right).$$

Let W be the convex hull of $\phi(S^n)$. Then, since γ is a convex integrand, it follows that

$$\phi(S^n) = \partial W, \tag{$*$}$$

where ∂W stands for the boundary of W.

Next, for any $v \in \text{int}(W)$, consider the parallel translation $T_v : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ defined by $T_v(x) = x - v$, where int(W) means the set consisting of interior points of W. Moreover, for any $\theta \in S^n$, set $L_\theta = \{(\theta, r) \in \mathbb{R}^{n+1} - \{0\} \mid r \in \mathbb{R}_+\}$ and for any $v \in \text{int}(W)$, define $\tilde{\gamma}_v : S^n \to \mathbb{R}_+$ as follows:

$$(\theta, \tilde{\gamma}_v(\theta)) = T_v(\partial W) \cap L_\theta.$$

Notice that, by $(*)$ and $v \in \text{int}(W)$, $\widetilde{\gamma}_v$ is a well-defined function. Notice also that graph$(\tilde{\gamma}_v) = T_v(\partial W)$. By $(*)$ and $v \in \text{int}(W)$ again, it follows that $\|\phi(\theta) - v\| > 0$ for any $\theta \in S^n$. Thus, it follows that the mapping $h_v : S^n \to S^n$ defined by

$$h_v(\theta) = \frac{\phi(\theta) - v}{\|\phi(\theta) - v\|}$$

is a C^∞ diffeomorphism and the following holds:

$$(\tilde{\gamma}_v \circ h_v)(\theta) = \|\phi(\theta) - v\|.$$

Let $H : \mathbb{R}_+ \to \mathbb{R}_+$ be the C^∞ diffeomorphism defined by $H(X) = \frac{1}{2}X^2$. Then, we have the following:

$$F(v, \theta) = \tfrac{1}{2}\|\phi(\theta) - v\|^2 = (H \circ \tilde{\gamma}_v \circ h_v)(\theta).$$

Hence, we have

$$\begin{aligned}\text{Caust}(\phi) &= \{v : \exists\theta : \nabla(H \circ \tilde{\gamma}_v \circ h_v)(\theta) = 0 \text{ and } \det(\text{Hess}(H \circ \tilde{\gamma}_v \circ h_v)(\theta)) = 0\} \\ &= \{v : \exists\theta : \nabla(\tilde{\gamma}_v \circ h_v)(\theta) = 0 \text{ and } \det(\text{Hess}(\tilde{\gamma}_v \circ h_v)(\theta)) = 0\}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Sym}(\phi) &= \{v : \exists\theta_1 \neq \theta_2 : \nabla(H \circ \tilde{\gamma}_v \circ h_v)(\theta_1) = \nabla(H \circ \tilde{\gamma}_v \circ h_v)(\theta_2) = 0 \\ &\qquad \text{and } (H \circ \tilde{\gamma}_v \circ h_v)(\theta_1) = (H \circ \tilde{\gamma}_v \circ h_v)(\theta_2)\} \\ &= \{v : \exists\theta_1 \neq \theta_2 \colon \nabla(\tilde{\gamma}_v \circ h_v)(\theta_1) = \nabla(\tilde{\gamma}_v \circ h_v)(\theta_2) = 0 \\ &\qquad \text{and } (\tilde{\gamma}_v \circ h_v)(\theta_1) = (\tilde{\gamma}_v \circ h_v)(\theta_2)\}.\end{aligned}$$

For any $r \in \mathbb{R}_+$, let $B(0, r)$ be the open disk with radius r centered at 0. Then, by Corollary 1, for any sufficiently small $\varepsilon > 0$ there exists a point $v \in B(0, \varepsilon)$ such that $\widetilde{\gamma}_v \circ h_v$ is stable. This implies that there exists a sequence $\{v_n \in \mathrm{int}(W)\}_{n=1,2,\ldots}$ converging to the origin such that $\widetilde{\gamma}_{v_n}$ is stable for any $n \in \mathbb{N}$.

For any $v \in \mathrm{int}(W)$, define the convex integrand $\gamma_v : S^n \to \mathbb{R}_+$ as follows:

$$\gamma_v(\theta) = \frac{1}{\widetilde{\gamma}_v(-\theta)}.$$

Since S^n is compact, the mapping $\Phi : B(0, \varepsilon) \to C^\infty_{\mathrm{conv}}(S^n, \mathbb{R}_+)$ defined by $\Phi(v) = \gamma_v$ is continuous. Since $\gamma_0 = \gamma$, it follows that if n is sufficiently large, then the convex integrand γ_{v_n} must be inside the given neighborhood V of γ.

4. Examples

4.1. Stable convex integrands with few critical points

Recall that for a given C^∞ convex integrand $\gamma : S^n \to \mathbb{R}_+$, $\phi : S^n \to \mathbb{R}^{n+1}$ is the C^∞ embedding defined by

$$\phi(\theta) = \mathrm{inv}\left(\theta, \frac{1}{\gamma(-\theta)}\right)$$

with respect to the polar coordinate expression.

Suppose that ϕ is parameterized as follows with respect to the Euclidean coordinate expression:

$$\phi(\cos t, \sin t) = (a \cos t + c, b \sin t + d),$$

where $a, b \in \mathbb{R}_+$ and $c, d \in \mathbb{R}$. In the case where $a = b$, the image $\phi(S^1)$ is the circle centered at (c, d) with radius $a = b$. In this case, the caustic of ϕ and the symmetry set of ϕ are exactly the same set, that is, the center of the circle $\{(c, d)\}$. Thus, it is clear that the given C^∞ convex integrand γ is stable if and only if $(c, d) \neq (0, 0)$. Notice that in the case $(c, d) \neq (0, 0)$, the number of normals to ϕ passing through $(0, 0)$ is two. Thus, in this case, the stable convex integrand γ has exactly two critical points (one gives the minimum of γ and another gives the maximum of γ).

Next, suppose that $a > b$. In this case, the image $\phi(S^1)$ is an ellipse. It is not difficult to see that $\mathrm{Caust}(\phi)$ and $\mathrm{Sym}(\phi)$ are expressed as follows (for instance, refer to [**4**, p. 13]):

$$\mathrm{Caust}(\phi) = \left\{\left(\frac{a^2 - b^2}{a}\cos^3 t + c, \frac{b^2 - a^2}{b}\sin^3 t + d\right) \,\middle|\, 0 \le t < 2\pi\right\},$$

$$\mathrm{Sym}(\phi) = \left\{(s, d) \,\middle|\, \frac{b^2 - a^2}{a} < s < \frac{a^2 - b^2}{a}\right\} \bigcup \left\{(c, s) \,\middle|\, \frac{b^2 - a^2}{b} < s < \frac{a^2 - b^2}{b}\right\}.$$

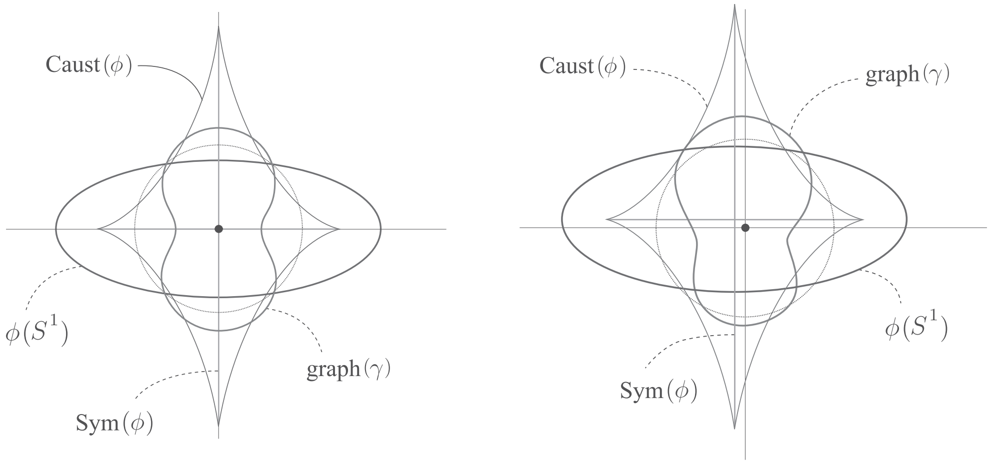

FIGURE 1. Left: the convex integrand γ is not stable though all critical points of γ are non-degenerate. Right: the convex integrand γ is stable.

In Figure 1, (c, d) is $(0, 0)$ for the left ellipse $\phi(S^1)$ while for the right ellipse, (c, d) is a point such that $cd \neq 0$ and $c^2 + d^2$ is sufficiently small. For each ellipse $\phi(S^1)$ in Figure 1, Caust(ϕ), Sym(ϕ) and graph(γ) are depicted as well. It is easily checked that, in the case $(c, d) = (0, 0)$, the number of normals to ϕ passing through $(0, 0)$ is four: namely the normals at $t = 0, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi$. And, in the same case, it is not difficult to check that all $(1, 0), (0, 1), (-1, 0), (0, -1) \in S^1$ are non-degenerate critical points of γ. Therefore, by using the following proposition called the *Morse lemma with parameters*, it is concluded that even in the case where (c, d) is a point such that $cd \neq 0$ and $c^2 + d^2$ is sufficiently small, the number of critical points of γ is exactly four.

PROPOSITION 4. (Morse lemma with parameters [**4**, p. 97]) *Let* $F\colon (\mathbb{R}^n \times \mathbb{R}^r, 0) \to \mathbb{R}$ *be a* C^∞ *function-germ, where we use coordinates* x_i *in* $\mathbb{R}^n$ *and* u_j *in* $\mathbb{R}^r$, *and* 0 *here stands for all* x_i *and* u_j *equal to* 0. *Suppose that* $\partial F/\partial x_i(0) = 0$ *for all* i *and that the matrix* $(\partial^2 F/\partial x_i \partial x_j(0))$ *is non-singular. (Thus* F_0, *defined by* $F_0(x) = F(x, 0)$, *is a Morse function.) Then there is a map-germ* $\psi : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}^n, 0)$ *with:*

(1) *the matrix* $(\partial \psi_i/\partial x_j)$ *non-singular, so that* $x \mapsto \psi(x, u)$ *is a germ of* C^∞ *diffeomorphism for* $u = 0$ *and therefore for all* u *close to* 0*;*

(2) $F(\psi(x, u), u) = F(0, 0) + \sum_{i=1}^{n} \varepsilon_i x_i^2 + h(u)$, *where each* ε_i *is* ± 1 *and* $h : (\mathbb{R}^r, 0) \to (\mathbb{R}, 0)$ *is a* C^∞ *function-germ.*

4.2. *Stable convex integrands with many critical points*

In this subsection, an example is given of a stable convex integrand with ten critical points. By this example and Subsection 4.1, it is easily seen that for any positive integer k there exists a stable convex integrand $\gamma : S^1 \to \mathbb{R}_+$ with $2k$ critical points.

In Figure 2, each outer curve is given by a C^∞ embedding ϕ and each inner curve is the graph of the C^∞ convex integrand γ. Since the image $\phi(S^1)$ of the left-hand side is

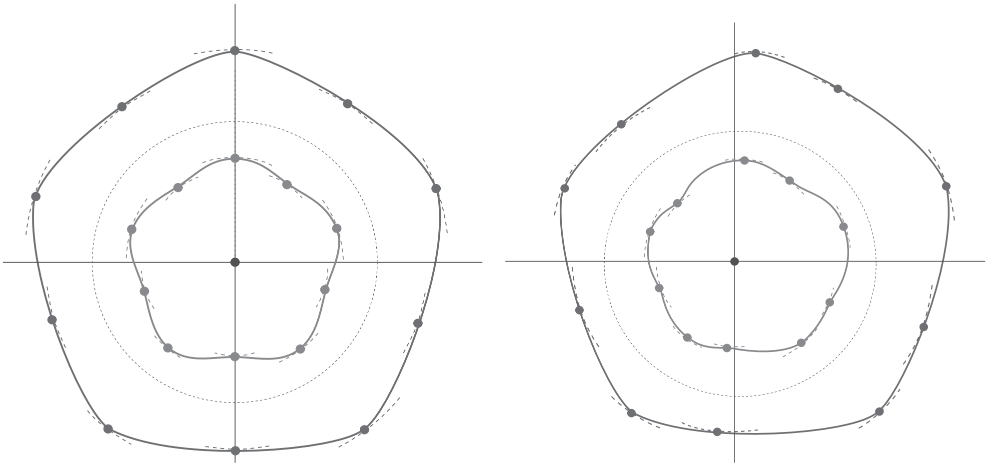

FIGURE 2. Left: the convex integrand γ is not stable though all ten critical points of γ are non-degenerate. Right: the convex integrand γ is stable and has exactly ten critical points.

symmetric about the origin (0, 0) with respect to the rotation by angle $\frac{2}{5}\pi$, it follows that (0, 0) is inside Sym(ϕ). Thus, the left-hand side γ is not stable. It is not difficult to construct a C^∞ embedding ϕ so that in the left-hand side case, the number of normals to ϕ passing through (0, 0) is exactly ten and the origin (0, 0) is not contained in Caust(ϕ). Thus, the ten normals passing through (0, 0) correspond to non-degenerate critical points of γ. On the other hand, in the right-hand side, the image $\phi(S^1)$ of the left-hand side is translated slightly so that the origin (0, 0) lies outside Caust(ϕ) $\cup$ Sym(ϕ). Thus, the right-hand side γ is stable. And, by using the Morse lemma with parameters (Proposition 4), we can conclude that also in the right-hand side case, the number of non-degenerate critical points of γ is exactly ten.

4.3. Restrictions on global differentiable types of stable convex integrands

As the title of this subsection shows, in Subsection 4.3, restrictions on global differentiable types of stable convex integrands are investigated.

Suppose that $n = 1$. Then, as in Subsection 4.2, for any positive integer k there exists a stable convex integrand γ such that the number of non-degenerate critical points is $2k$. In this case, there must exist exactly k critical points with index 0 and also exactly k critical points with index 1.

Next, suppose that $2 \leq n$. The restrictions in the case $n = 1$ obtained above suggest the following question.

Question 1. Let k be a positive integer. Then, is there any stable convex integrand γ which has k non-degenerate critical points with index 0 and k non-degenerate critical points with index n?

LEMMA 1. *Let n be an integer satisfying $n \geq 2$. Let $\gamma : S^n \to \mathbb{R}_+$ be a convex integrand with only non-degenerate critical points such that the index of γ at any critical point is zero or n. Then, γ has only two critical points: the index is zero at one point and it is n at another point.*

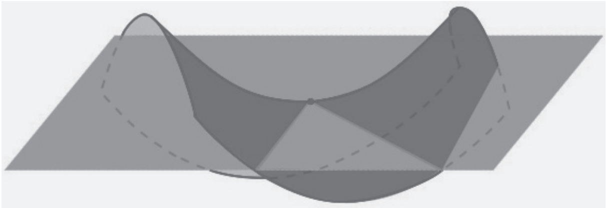

FIGURE 3. A horse saddle as the graph of a height function h. The height function h has a non-degenerate critical point of index 1 at the saddle point.

Lemma 1 may be proved by using the Morse inequalities as follows.

Proof. For any non-negative integer λ, denote by C_λ the number of non-degenerate critical points with index λ and by R_λ the λth Betti number of S^n. Set

$$S_\lambda = R_\lambda - R_{\lambda-1} + \cdots \pm R_0.$$

Then, the following inequalities, called the *Morse inequalities*, hold [**21**]:

$$S_\lambda \leq C_\lambda - C_{\lambda-1} + \cdots \pm C_0. \qquad (i_\lambda)$$

The inequality (i_0) implies $1 \leq C_0$. Since $n \geq 2$, the inequality (i_1) implies $0 - 1 \leq 0 - C_0$, which is equivalent to $C_0 \leq 1$. Thus, we have $C_0 = 1$. In the case $\lambda > n$, the inequalities (i_λ), $(i_{\lambda+1})$ imply the following:

$$\sum_{\lambda=0}^{n}(-1)^\lambda R_\lambda = \sum_{\lambda=0}^{n}(-1)^\lambda C_\lambda.$$

By using this equality, it is easily seen that $C_n = 1$, and thus the proof of Lemma 1 is complete. □

Lemma 1 gives a restriction on the global differentiable types of stable convex integrands. In fact, Lemma 1 answers Question 1 negatively except for $k = 1$. In the case that $k = 1$, it is easily seen that a small translation of the unit sphere S^n gives a concrete example of a stable convex integrand γ such that $C_0 = C_n = 1$ and $C_1 = \cdots = C_{n-1} = 0$.

Lemma 1 asserts that $C_1 + \cdots + C_{n-1} = 0$ must imply $C_0 = C_n = 1$. It seems that stable convex integrands satisfying $C_0 = C_n = 1$ are very special. Hence, for most stable convex integrands, it seems that $C_1 + \cdots + C_{n-1}$ must be positive. However, since inv(graph(γ)) is the boundary of a convex body for any convex integrand γ, it follows that a horse saddle shape (see Figure 3) never appears as a local shape of inv(graph(γ)). Thus, the following question naturally arises.

Question 2. Are there stable convex integrands γ satisfying $C_1 + \cdots + C_{n-1} > 0$?

Figure 4 gives an affirmative answer to Question 2. On the boundary of 'GOISHI', there is the circle consisting of points at which the absolute value of the difference of two principal curvatures attains the maximum. Suppose that the origin of $\mathbb{R}^3$ is inside the intersection of 'GOISHI' and the plane containing this circle, and is outside the caustic of the boundary of

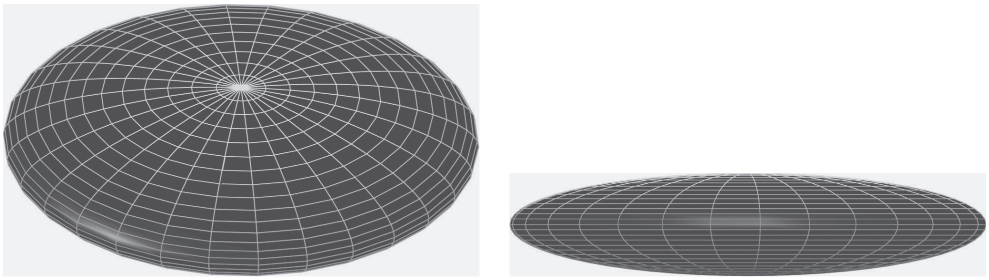

FIGURE 4. The boundary of 'GOISHI' which is a stone used in the game of 'GO'.

'GOISHI'. Suppose, moreover, that the origin of $\mathbb{R}^3$ is different from the center of the circle. Consider the sphere centered at the origin with radius r where r is the distance between the origin and the unique nearest point of the circle from the origin. Then, the set-germ of the intersection of the sphere and the boundary of 'GOISHI' at the nearest point is C^∞ diffeomorphic to the set-germ of the intersection of the horse saddle and the plane at the saddle point given in Figure 3.

By Figure 4, it is natural to ask the following.

Question 3. Suppose that $n \geq 2$. Let k be a non-negative integer and let $\gamma : S^n \to \mathbb{R}_+$ be a stable convex integrand such that

$$C_1 + \cdots + C_{n-1} = k.$$

Then, are there restrictions on global differentiable types of γ?

Question 3 seems to be open except for $k = 0$. In the authors' opinion, Question 3 seems to be interesting.

Acknowledgement. The authors would like to thank the reviewer for appropriate comments.

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Erica Boizan Batista
Departamento de Matemática
Universidade Federal de São Carlos
Caixa Postal 676, 13560-905
São Carlos-SP, Brazil
(E-mail: ericabbatista@gmail.com)

Huhe Han
Graduate School of Environment and Information Sciences
Yokohama National University
Yokohama 240-8501
Japan
(E-mail: han-huhe-bx@ynu.jp)

Takashi Nishimura
Research Institute of Environment and Information Sciences
Yokohama National University
Yokohama 240-8501
Japan
(E-mail: nishimura-takashi-yx@ynu.jp)