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Stability of the Calabi flow near an extremal metric

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Abstract. We prove that on a Kähler manifold admitting an extremal metric ω and for any Kähler potential φ_0 close to ω, the Calabi flow starting at φ_0 exists for all time and the modified Calabi flow starting at φ_0 will always be close to ω. Furthermore, when the initial data is invariant under the maximal compact subgroup of the identity component of the reduced automorphism group, the modified Calabi flow converges to an extremal metric near ω exponentially fast.

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1. Introduction

Let M be a Kähler manifold and Ω be the Kähler class in $H^2(M, R) \cap H^{1,1}(M, C)$. By the $\partial\bar{\partial}$-lemma, any Kähler metric ω_φ in Ω can be written as

$$\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$$

for some smooth real-valued Kähler potential φ. The space of Kähler metrics is defined by

$$\mathcal{H} = \{\varphi \in C^\infty(M, R) | \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}.$$

Donaldson [17], Mabuchi [20] and Semmes [21] independently defined a Weil-Peterson type-metric on $\mathcal{H}$, under which $\mathcal{H}$ becomes a non-positively curved infinite dimensional symmetric space. Chen [5] proved that any two points in $\mathcal{H}$ can be connected by a $C^{1,1}$ geodesic and that $\mathcal{H}$ is a metric space, which verifies two of Donalson's conjectures.

In order to tackle the existence of a constant scalar curvature Kähler metric (cscK) problem, Calabi [2, 3] introduced a well-known functional

$$Ca(\varphi) = \int_M S(\varphi)^2 \, \omega_\varphi^n,$$

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where $S(\varphi)$ is the scalar curvature of ω_φ. The critical point of this Calabi functional is called an extremal Kähler metric. Calabi discovered that an extremal Kähler metric is a cscK if and only if the Calabi-Futaki invariant is equal to zero. Later, he suggested that one may study the gradient flow of the K-energy to search for the cscK. This flow is defined as

$$\frac{\partial\varphi}{\partial t} = S - \underline{S} \tag{1.1}$$

and it decreases the Calabi energy. Since (1.1) is a fourth order equation, the maximal principle fails. In [18], Donaldson proposed a programme to study the convergence of the Calabi flow. On a Riemannian surface, P. Chruściel [16] proved that the flow exists for all time and converges to a cscK metric by using the Bondi mass. Later Chen [6] and Struwe [22] gave a different proof assuming the uniformization theorem. In Chen-Zhu [15], they removed the assumption of the uniformization theorem. For higher dimensions, the Calabi flow has been studied in Chen-He [8–10] and Tosatti-Weinkove [23]. In Chen-He [8], they proved that the Calabi flow can start from a $C^{3,\alpha}$ Kähler potential and become smooth immediately as $t > 0$.

One defines the little Hölder space $c^{k,\alpha}$ to be the closure of smooth functions in the usual Hölder norm $C^{k,\alpha}$.

Theorem 1.1 ([8]). *If $\omega_{\varphi_0} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_0$ satisfies $|\varphi_0|_{c^{3,\alpha}(M,g)} \le K$, and $\lambda\omega < \omega_0 = \omega_{\varphi_0} < \Lambda\omega$, where K, λ, Λ are positive constants, then the Calabi flow initiating from φ_0 admits a unique solution*

$$\varphi(t) \in C([0,T], c^{3,\alpha}(M,g)) \cap C((0,T], c^{4,\alpha}(M,g))$$

for small $T = T(\lambda,\Lambda,K,\omega)$. More specifically, for any $t \in (0,T]$, there is a constant $C = C(\lambda,\Lambda,K,\omega)$ such that

$$t^{1/4}(|\dot{\varphi}(t)|_{c^{0,\alpha}}(M) + |\varphi(t)|_{c^{4,\alpha}(M)}) \le C, |\varphi(t)|_{c^{3,\alpha}(M)} \le C.$$

Remark 1.2. He [19] shows that the Calabi flow can start from ω_φ where $\varphi \in c^{2,\alpha}(M)$.

Theorem 1.3 ([8]). *The solution obtained above belongs to*

$$C^0([0,T], c^{3,\alpha}(M)) \cap C^0((0,T], C^\infty(M)).$$

Chen and He then use an energy argument to show that when Kähler manifolds admits a cscK ω and the initial Kähler potential is $C^{3,\alpha}$ small, the Calabi flow converges exponentially fast to a cscK nearby. In He [19], he improved this result for $C^{2,\alpha}$ small initial Kähler potentials.

In this short note, we prove a parallel theorem for extremal Kähler metrics using a different method from Chen-Ding-Zheng [7]. In that paper, they defined a flow called the pseudo-Calabi flow. They proved the short time existence from $c^{2,\alpha}$ initial Kähler potentials, the long time existence under uniform Ricci bounds and the stability near a cscK. Since the linearized operator of the pseudo-Calabi flow is not self-adjoint, they set up a unified frame to tackle the stability problem of the Kähler Ricci flow (*cf.* Zheng [24]), the Calabi flow and the pseudo-Calabi flow. This method strongly relies on the geometric structure of the space of Kähler metrics.

First, we will prove the long time existence of the Calabi flow.

Theorem 1.4. *On Kähler manifolds admitting an extremal metric ω and for any positive constant $\mathcal{K}$, λ, there is a small constant ϵ depending on ω, $\mathcal{K}$, λ, such that for any Kähler potential φ_0, if*

$$|\varphi_0|_{C^{2,\alpha}(M)} < \mathcal{K}, \quad \lambda\omega < \omega_{\varphi_0}, \quad \int_M |\varphi_0|^2 \omega^n < \epsilon,$$

then the Calabi flow exists for all time.

Next we want to study the modified Calabi flow. Let K be a maximal compact subgroup in the reduced automorphism group. Denote the corresponding Lie algebra of K by $h_0(M)$, which is the ideal of holomorphic vector fields with zeros. For any holomorphic vector field $\tilde{Y} \in h_0(M)$, denote $\tilde{Y} = Y - \sqrt{-1}JY$. Then there is a real function θ_Y such that

$$L_{\tilde{Y}}\omega = L_Y\omega = \sqrt{-1}\partial\bar{\partial}\theta_Y(t)$$

and

$$\int_M \theta_Y \omega^n = 0.$$

For an arbitrary metric

$$\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi,$$

the corresponding $\theta_{\tilde{Y}}(\varphi)$ is

$$\theta_{\tilde{Y}}(\varphi) = \theta_Y + \tilde{Y}(\varphi).$$

Following Futaki-Mabuchi [1], suppose $\tilde{X}, \tilde{Y} \in h_0(M)$, then the bilinear form

$$B(\tilde{X}, \tilde{Y}) = \int_M \theta_{\tilde{X}(\varphi)}\theta_{\tilde{Y}(\varphi)}\omega_\varphi^n$$

is independent of the choice of ω_φ in the Kähler class $\Omega = [\omega]$.

Let $\varphi(t)$ be a one parameter of Kähler potentials satisfying the Calabi flow equation and let $\sigma(t)$ be the holomorphic group generated by X, the real part of the extremal vector field $\tilde{X}$. Then $\sigma^*(t)\omega = \omega + i\partial\bar{\partial}\rho(t)$ satisfies the Calabi flow equation since

$$\frac{\partial\sigma^*(t)\omega}{\partial t} = L_X\omega(t) = i\partial\bar{\partial}(S(t) - \underline{S}).$$

Hence we can choose $\rho(t)$ to be a parameter of Kähler potentials satisfying the Calabi flow equation starting from 0.

Let $\psi(t) = \sigma(-t)^*(\varphi(t) - \rho(t))$. Notice that, by definition, $X = \sigma_*^{-1}(\frac{\partial}{\partial t}\sigma)$. So we obtain the modified Calabi flow,

$$\begin{aligned}\frac{\partial \psi}{\partial t} &= -X(\psi(t)) + \sigma(-t)^*\left(\frac{\partial \varphi(t)}{\partial t} - \frac{\partial \rho(t)}{\partial t}\right)\\ &= -X(\psi(t)) + \sigma(-t)^*(S(\varphi(t)) - \underline{S}) - (S(\omega) - \underline{S})\\ &= S_\psi - \underline{S} - \theta_X - X(\psi)\\ &= S_\psi - \underline{S} - \theta_X(\psi).\end{aligned}$$

Theorem 1.5. *On Kähler manifolds admitting an extremal metric ω, for any $\mathcal{K}$-invariant Kähler potential φ_0 close to ω (in the sense of Theorem (1.4)), the modified Calabi flow exponentially converges to a nearby extremal metric.*

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2. Long time existence

First of all, we would like to give a rough estimate of the geodesic distance between any two Kähler potentials φ_0, φ_1 when $\omega_{\varphi_0}, \omega_{\varphi_1} < \Lambda\omega$.

Lemma 2.1. $d(\varphi_0, \varphi_1) < C(\Lambda)\left(\int_M |\varphi_0 - \varphi_1|^2 \omega^n\right)^{\frac{1}{2}}$.

Proof. Let $\varphi_t = (1-t)\varphi_0 + t\varphi_1$ for $0 \le t \le 1$. Then

$$\begin{aligned}d(\varphi_0, \varphi_1) \le L(\gamma_t) &= \int_0^1 \left(\int_M \left(\frac{\partial \gamma_t}{\partial t}\right)^2 \omega_{\gamma_t}^n\right)^{\frac{1}{2}} dt\\ &\le \int_0^1 \left(\int_M (\varphi_0 - \varphi_1)^2 \omega_{\gamma_t}^n\right)^{\frac{1}{2}} dt\\ &\le C(\Lambda)\left(\int_M |\varphi_0 - \varphi_1|^2 \omega^n\right)^{\frac{1}{2}}.\end{aligned}$$

□

We are ready to give a proof of Theorem 1.4.

Proof. Suppose that the conclusion fails, then there exist positive constants $\mathcal{K}$, λ, Λ and a sequence of φ_s^0 such that

$$|\varphi_s^0|_{C^{2,\alpha}} < \mathcal{K},\ \lambda\omega < \omega_{\varphi_0^s} < \Lambda\omega,\ \int_M |\varphi_s^0|^2\omega^n < \frac{1}{s} \quad s = 1, 2, 3 \cdots .$$

By virtue of the short time existence theorem, we get a sequence of solutions $\varphi_s(t)$ satisfying the flow equation (1.1) with $\varphi_s(0) = \varphi_s^0$. Let T_s be the first time such that

$$|\varphi_s(T_s) - \rho(T_s)|_{c^{2,\alpha}(\omega(T_s))} = 2C \quad \text{holds}$$

or

$$\lambda\omega(T_s) < \omega_{\varphi_s(T_s)} < \Lambda\omega(T_s) \quad \text{fails},$$

where the constant C is from Theorem 1.1. Then T_s is bounded from below for sufficiently large s. Otherwise there is a subsequence of $\varphi_s(T_s)$ converging to φ_∞ in the $C^{2,\alpha'}(\omega)$ sense, where $\alpha' < \alpha$. Notice that $\lambda\omega \le \omega_{\varphi_\infty} \le \Lambda\omega$, but that $\lambda\omega < \omega_{\varphi_\infty} < \Lambda\omega$ fails.

On the other hand, Lemma 2.1 shows that $d(0, \varphi_s^0) \to 0$ as $s \to \infty$. Since the distance function decreases under the Calabi flow, we have

$$d(\rho(T_s), \varphi_s(T_s)) \to 0$$

as $s \to \infty$. Let $\varphi_\infty(t)$ be one parameter potentials satisfying the Calabi flow equation initiating from φ_∞. Then for $t_0 \ge t$,

$$\begin{aligned} d(\rho(t_0), \varphi_\infty(t_0)) &\le d(\rho(t), \varphi_\infty(t)) \\ &\le d(\rho(t), \rho(T_s)) + d(\rho(T_s), \varphi_s(T_s)) + d(\varphi_s(T_s), \varphi_\infty(t)). \end{aligned}$$

By Lemma 2.1, $d(\varphi_s(T_s), \varphi_\infty(t)) \to 0$ as $s \to \infty$ and $t \to 0$. Hence $\rho(t_0) = \varphi_\infty(t_0)$, which implies $0 = \varphi_\infty$, a contradiction.

Moreover, from Theorem 1.3, we obtain the higher order uniform bounds of the sequence of the solutions:

$$|\varphi_s(T_s) - \rho(T_s)|_{C^{k,\alpha}(\omega(T_s))} \le C(k),\ \forall k \ge 0.$$

Therefore we can choose a subsequence of $\phi_s = \sigma(-T_s)^*(\varphi_s(T_s) - \rho(T_s))$ so that

$$\phi_s \to \phi_\infty \text{ in } C^{k,\alpha}(\omega), \forall k \ge 0,$$

and

$$|\phi_\infty|_{C^{2,\alpha}(\omega)} = 2C \quad (\text{or} \quad \lambda\omega < \omega_{\phi_\infty} < \Lambda\omega \text{ fails}).$$

However, this contradicts the fact that $d(0, \phi_\infty) = 0$. □

Corollary 2.2. *Given a Kähler potential φ_0 close to* 0 *in the sense of Theorem* 1.4*, then the modified Calabi flow stays in a neighborhood of* 0*. If φ_0 is $\mathcal{K}$-invariant, then the modified Calabi flow converges to an extremal metric nearby.*

Proof. That the modified Calabi flow stays in a neighborhood of 0 can be easily seen from the regularity Theorem 1.3. Notice that the Calabi flow decreases the Calabi energy, *i.e.*

$$\frac{\partial}{\partial t} Ca(\omega_\varphi) = -2 \int_M \mathcal{L}_\varphi(S_\varphi) S_\varphi \, \omega_\varphi^n,$$

where $\mathcal{L}_\varphi$ is the Lichnerowicz operator with respect to ω_φ. It follows that we can take a sequence of $t_j \to \infty$ such that

$$\lim_{j\to\infty} \int_M \mathcal{L}_{\varphi(t_j)}(S_{\varphi(t_j)}) S_{\varphi(t_j)} \, \omega^n_{\varphi(t_j)} = 0.$$

Then there is a subsequence of t_j such that $\psi(t_j)$ converges to a potential ψ_∞ in C^∞ and

$$\int_M \mathcal{L}_{\psi_\infty}(S_{\psi_\infty}) S_{\psi_\infty} \, \omega^n_{\psi_\infty} = 0.$$

Hence ω_{ψ_∞} is an extremal metric. If φ_0 is $\mathcal{K}$-invariant, then ψ_∞ is a fixed point under the modified Calabi flow and the modified Calabi flow decreases the geodesic distance between $\psi(t)$ and ψ_∞. Hence the flow converges to ψ_∞. □

3. Exponential decay

We define the modified Calabi energy as

$$\widetilde{Ca}(\psi) = \int_M (S(\psi) - \underline{S} - \theta_X(\psi))^2 \omega_\psi^n.$$

The evolution of the modified Calabi energy along the modified Calabi flow is

$$\begin{aligned}
\partial_t \int_M \dot\psi^2 \omega_\psi^n &= \int_M (2\dot\psi\ddot\psi + \dot\psi^2 \triangle_\psi \dot\psi)\omega_\psi^n \\
&= 2\int_M \dot\psi(\dot S_\psi - \dot\theta_X(t) - \dot\psi_i \dot\psi^i)\omega_\psi^n \\
&= 2\int_M \dot\psi(-L\dot\psi + \dot\psi^i S_i - \dot\theta_X(t) - \dot\psi_i \dot\psi^i)\omega_\psi^n \\
&= 2\int_M \dot\psi(-L\dot\psi + \dot\psi^i(\dot\psi_i + \theta_X(t)_i) - \dot\theta_X(t) - \dot\psi_i \dot\psi^i)\omega_\psi^n \\
&= 2\int_M \dot\psi(-L\dot\psi + \dot\psi^i \dot\psi_i + X(\dot\psi) - X(\dot\psi) - \dot\psi_i \dot\psi^i)\omega_\psi^n \\
&= -2\int_M \dot\psi L \dot\psi \omega_\psi^n.
\end{aligned}$$

In this computation we use the identities

$$\dot\psi^i(\theta_{Xi} + (X(\psi))_i) = \dot\psi^i(\theta_X(t))_i = X(\dot\psi).$$

The modified Calabi-Futaki invariant

$$\tilde{F}(\tilde{Y}) = F(\tilde{Y}) - B(\tilde{X}, \tilde{Y}) = \int_M \theta_{\tilde{Y}}(\psi)(S - \underline{S} - \theta_{\tilde{X}}(\psi))\omega_\psi^n \tag{3.1}$$

is equal to zero when M admits an extremal metric ω. This shows that the direction of the modified Calabi flow is always perpendicular to the kernel of the Lichnerowicz operator. To obtain exponential convergence, one needs to give a uniform lower bound of the the first eigenvalue of L_t along the modified Calabi flow. More precisely, we have the following lemma which is similar to Chen-Li-Wang [11].

Lemma 3.1. *Along the modified Calabi flow, there is a positive constant $\lambda > 0$ such that for sufficiently large t and for any*

$$f \in A_t = \{f \in C_R^\infty(M) | \int_M f\omega_{\psi(t)}^n = 0 \text{ and } \int_M \theta_Y(t) f\omega_{\psi(t)}^n = 0, \forall \tilde{Y} \in h_0(M)\},$$

we have

$$\int_M L_t(f) f\omega_{\psi(t)}^n \geq \lambda \int_M f^2 \omega_{\psi(t)}^n.$$

Proof. If not, there must be a sequence $\psi_s = \psi(s)$ and f_s such that

$$\int_M |(f_s)_{ij}|^2 \omega_{\psi_s}^n < \frac{1}{s}; \int_M f_s^2 \omega_{\psi_s}^n = 1; \int_M f_s \omega_{\psi_s}^n = 0. \tag{3.2}$$

Since the C^l norm of ψ_s is uniformly bounded for any $l \geq 0$. Using the Ricci identity

$$\int_M |(f_s)_{i\bar{j}}|^2 \omega_{\psi_s}^n = \int_M |(f_s)_{ij}|^2 \omega_{\psi_s}^n + \int_M R^{i\bar{j}}(f_s)_i (f_s)_{\bar{j}} \omega_{\psi_s}^n$$

and the interpolation inequality, we conclude that f_s are uniformly $W^{2,2}$ bounded. So we can pass to the limit and get

$$\int_M |(f_\infty)_{ij}|^2 \omega_{\psi_\infty}^n = 0; \int_M f_\infty^2 \omega_{\psi_\infty}^n = 1; \int_M f_\infty \omega_{\psi_\infty}^n = 0. \tag{3.3}$$

Since in local coordinates, $\uparrow \bar{\partial} f_\infty$ is holomorphic in the weak sense, f_∞ is smooth indeed. From the assumption of A_t we have

$$\int_M \theta_Y(\psi_\infty) f_\infty \omega_{\psi_\infty}^n = 0, \forall \tilde{Y} \in h_0(M).$$

In particular, we may choose $\tilde{Y} = \uparrow \bar{\partial} f_\infty \in h_0(M)$. Hence,

$$\int_M f_\infty^2 \omega_\infty^n = 0.$$

This contradicts (3.3). □

It is easy to see that $\widetilde{Ca}(\psi(t)) \leq Ce^{-\lambda t}$. To get exponential convergence of $\psi(t)$, we calculate the evolution formula for $\int_M |\nabla^k(\psi(t) - \psi_\infty)|^2 \, \omega^n$:

$$
\begin{aligned}
&\frac{\partial}{\partial t} \int_M |\nabla^k(\psi(t) - \psi_\infty)|^2 \, \omega^n \\
&= \int_M \nabla^k(S - \underline{S} - \theta_X(\psi)) * \nabla^k(\psi(t) - \psi_\infty) \, \omega^n \\
&= \int_M (S - \underline{S} - \theta_X(\psi)) * \nabla^{2k}(\psi(t) - \psi_\infty) \, \omega^n \\
&\leq \left(\int_M (S - \underline{S} - \theta(X))^2 \, \omega^n \right)^{1/2} \left(\int_M |\nabla^{2k}(\psi(t) - \psi_\infty)|^2 \, \omega^n \right)^{1/2} \\
&\leq C \parallel S - \underline{S} - \theta(X) \parallel_{L^2(\omega)} \\
&\leq C \parallel S - \underline{S} - \theta(X) \parallel_{L^2(\omega_t)} \\
&\leq Ce^{-\lambda_2 t}.
\end{aligned}
$$

By the Sobolev embedding, we conclude that

$$
\parallel \psi_t - \psi_\infty \parallel_{C^l(\omega)} \leq \parallel \psi_t - \psi_\infty \parallel_{W^{k,2}(\omega)} \leq \, Ce^{-\lambda_2 t}.
$$

Hence we obtain the result stated in Theorem 1.5.

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