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STABILITY OF CERTAIN FUNCTIONAL EQUATIONS VIA A FIXED POINT OF ĆIRIĆ

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Abstract

Let S be a non empty set. We prove the stability (in the sense of Ulam) of the functional equation: $f(t) = F(t, f(\phi(t)))$, where ϕ is a given function of S into itself and F is a function satisfying a contraction of Ćirić type ([5]). Our analysis is based on the use of a fixed point theorem of Ćirić (see [5] and [4]). In particular our result provides a generalization and a natural continuation of a paper of Baker (see [3]).

1 Introduction and preliminaries

Problems of stability for functional equations have been considered by S.M. Ulam in 1940 ([20]) and by Hyers ([9], [10]). One of the first results established in this direction is the following result, due to Hyers ([9], [10]), that answered a question of Ulam ([20]).

Theorem 1.1. *Suppose that S is an additive semigroup, E is a Banach space, $f : S \to E$, $\delta > 0$ and*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta \quad \text{for all } x, y \in S. \tag{1.1}$$

Then there is a unique function $a : S \to E$ such that

$$a(x+y) = a(x) + a(y) \quad \text{for all } x, y \in S, \tag{1.2}$$

and

$$\|f(x) - a(x)\| \leq \delta \quad \text{for all } x \in S, \tag{1.3}$$

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This theorem says that the Cauchy functional equation is stable in the sense of Heyers-Ulam.

Since the paper of Heyers, a large amount of papers and books were published offering many kind of extensions and generalizations. Now, the research in the topic is very extensive and a very rich background of results is build. (See the references).

In 1991, J. A. Baker ([3]) has studied the stability of the functional equation

$$f(t) = \alpha(t) + \beta(t)f(\phi(t)), \quad \text{for all } t \in S, \tag{1.4}$$

where S is a non empty set, α and β are given complex valued functions defined on S such that $\sup_{t\in S} |\beta(t)| < 1$ and ϕ is a given mapping of S into itself.

Based on a variant of Banach fixed point theorem, Baker has proved the following theorem.

Theorem 1.2. *(Baker [3]) Suppose that S is a nonempty set; E is a real (or complex) Banach space, $\phi : S \to S$, $\alpha : S \to E$ $\beta : S \to \mathbb{R}$ (or $\mathbb{C}$), $0 \le \lambda < 1$, and $|\beta(t)| \le \lambda$ for all $t \in S$. Also suppose that $g : S \to E$, $\delta > 0$, and*

$$\|g(t) - [\alpha(t) + \beta(t)g(\phi(t))]\| \le \delta \quad \textit{for all } t \in S. \tag{1.5}$$

Then there is a unique function $f : S \to E$ such that

$$f(t) = \alpha(t) + \beta(t)f(\phi(t)) \tag{1.6}$$

and

$$\|f(t) - g(t)\| \le \frac{\delta}{1-\lambda} \quad \textit{for all } t \in S, \tag{1.7}$$

The aim of this paper is to extend the above result by proving a stability result (in the sense of Heyers-Ulam) for the general functional equation

$$f(t) = F(t, f(\phi(t))), \quad \forall t \in S,$$

where S is a nonempty set. This equation is extensively studied in [13].

Our study will be based on a fixed point result of Ćirić (see [5] and [4]). Let us recall this result

Theorem 1.3. *(Ćirić [5]) Let (X, d) be a complete metric space, $T : X \to X$ a mapping satisfying the condition*

$$d(T(x), T(y)) \le \alpha_1(x,y)d(x,y) + \alpha_2(x,y)d(x,T(x)) + \alpha_3(x,y)d(y,T(y)) + \\ + \alpha_4(x,y)d(x,T(y)) + \alpha_5(x,y)d(y,T(x)), \tag{1.8}$$

for all $x, y \in X$, where $\alpha_i : X \times X \to [0,\infty)$, $i = 1, 2, \ldots, 5$ and $\Sigma_{i=1}^{5}\alpha_i(x,y) \le \lambda$ for each $x, y \in X$ and some $\lambda \in [0,1)$.

Then T has a unique fixed point in X.

This theorem was established in [5]. A new proof of this theorem is given by M. Balaj and S. Mureşan in [4].

2 Results

To establish our results, we need two lemmas. The first lemma is proved in [4].

Lemma 2.1. *([4], Lemma 1). Let* (X, d) *be a complete metric space,* Y *a nonempty closed bounded subset of* X *and* $T : Y \to Y$ *a mapping. Put* $Y_0 := Y$, $Y_1 := \overline{T(Y_0)}, \cdots$, $Yn := \overline{T(Y_{n-1})}, \cdots$. *If* $\lim_{n\to\infty} diam(Y_n) = 0$, *then* T *has a unique fixed point.*

Let (X, d) be a metric space. For each $a \in X$ and $\epsilon > 0$, $\overline{B}(a, \epsilon)$ means the closed ball of radius ϵ and center a.

Lemma 2.2. *Let* (X, d) *be a metric space,* $T : X \to X$ *be a mapping satisfying the condition of Theorem 1.3. Let* $a \in X$ *and let* ρ *be any number such that* $d(a, T(a)) \leq \rho$. *Put*

$$\theta(\rho) := \frac{(2 + \lambda)\rho}{2(1 - \lambda)}. \tag{2.1}$$

Then

$$T(\overline{B}(a, \theta(\rho)) \subset \overline{B}(a, \theta(\rho)). \tag{2.2}$$

Proof. Using the same technique of proofs as in Theorem 2 of Balaj and Mureşan [4], one can prove Lemma 2.2. □

To prove our main result, we need the following variant of Ćirić Theorem 1.3.

Theorem 2.1. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be as in Theorem 1.3. Let* $u \in X$ *be arbitrary and let* $\delta > 0$ *be any number such that*

$$d(u, T(u)) \leq \delta.$$

Then there exists a unique point $p \in X$ *such that* $p = T(p)$. *Moreover,*

$$d(u, p) \leq \frac{(2 + \lambda)\delta}{2(1 - \lambda)}. \tag{2.3}$$

Proof. Let $u \in X$ and $\delta > 0$ are such that $d(u, T(u)) \leq \delta$. Let $\theta(\delta)$ be defined by (2.1), that is,

$$\theta(\delta) := \frac{(2 + \lambda)\delta}{2(1 - \lambda)}.$$

Then by Lemma 2.2, we have

$$T(\overline{B}(u, \theta(\delta)) \subset \overline{B}(u, \theta(\delta)).$$

As in the proof of Theorem 2 in [4], we take $Y_0 = \overline{B}(u, \theta(\delta))$, $Y_n = \overline{T(Y_{n-1})}$, for $n \geq 1$. From (1.8) we get $diam(Y_n) \leq \lambda diam(Y_{n-1})$ and, since $\lambda \in [0, 1)$, $diam(Y_n) \to 0$, as $n \to \infty$. By Lemma 2.1 the restriction of T to $\overline{B}(u, \theta(\delta))$ has a fixed point $p \in \overline{B}(u, \theta(\delta))$. The uniqueness of the fixed point follows easily from (1.8). This ends the proof. □

Remark 2.1. From Theorem 2.1 it follows that the unique fixed point of T, say a, satisfies the following condition:

$$a \in \bigcap_{x\in X} \overline{B}(x, \theta(\rho(x))),$$

where $\rho(x) := d(x, Tx)$ and $\theta(t)$ is defined as in (2.1). As a consequence, for all $x, y \in X$, we have

$$d(x, y) \leq d(x, a) + d(a, y) \leq \theta(\rho(x)) + \theta(\rho(y)). \tag{2.4}$$

The main result of this paper reads is the following theorem.

Theorem 2.2. *Suppose S is a nonempty set, (X, d) is a complete metric space, $\phi : S \to S$, $F : S \times X \to X$ a function satisfying the condition*

$$\begin{aligned} d(F(t,x), F(t,y)) \leq \alpha_1(x,y)d(x,y) + \alpha_2(x,y)d(x, F(t,x)) + \alpha_3(x,y)d(y, F(t,y)) \\ +\alpha_4(x,y)d(x, F(t,y)) + \alpha_5(x,y)d(y, F(t,x)), \end{aligned} \tag{2.5}$$

for all $x, y \in X$, where $\alpha_i : X \times X \to [0, \infty)$, $i = 1, 2, \ldots, 5$ and $\Sigma_{i=1}^5 \alpha_i(x,y) \leq \lambda$.

Also suppose that for some $g : S \to X$ and some $\delta > 0$, we have

$$d(g(t), F(t, g(\phi(t)))) \leq \delta \quad \forall t \in S. \tag{2.6}$$

Then there exists a unique function $f : S \to X$ such that

$$f(t) = F(t, f(\phi(t))), \quad \forall t \in S \tag{2.7}$$

and

$$d(f(t), g(t)) \leq \frac{(2+\lambda)\delta}{2(1-\lambda)} \quad \forall\, t \in S. \tag{2.8}$$

Proof. Let $Y := \{a : S \to X \mid \sup_{t\in S} d(a(t), g(t)) < \infty\}$. Since $g \in Y$, then $Y \neq \emptyset$. For $a, b \in Y$, we set

$$d_\infty(a, b) := \sup_{t\in S} d(a(t), b(t)).$$

Then (Y, d_∞) is a metric space. Since (X, d) is complete, then (Y, d_∞) is also complete. The convergence in Y with respect to d_∞ is the uniform convergence on S.

Let $Y_\delta := \{a \in Y \mid d_\infty(a, g) \leq \theta(\delta)\}$. For $a \in Y_\delta$ define $Ta : S \to X$ by

$$(Ta)(t) := F(t, a(\phi(t))) \quad \forall t \in S.$$

Then, from computations similar to those of Lemma 2.2, and by using (2.4), one can see that T maps Y_δ into Y_δ.

If $u, v \in Y_\delta$ then for all $t \in S$, we have

$$d(Tu(t), Tv(t)) = d(F(t, u(\phi(t))), F(t, v(\phi(t))))$$
$$\leq \alpha_1 d(u(\phi(t)), v(\phi(t))) + \alpha_2 d(u(\phi(t)), F(t, u(\phi(t)))) + \alpha_3 d(v(\phi(t)), F(t, v(\phi(t))))$$
$$+\alpha_4 d(u(\phi(t)), F(t, v(\phi(t)))) + \alpha_5 d(v(\phi(t)), F(t, u(\phi(t)))). \tag{2.9}$$

¿From (2.9), we get

$$d_\infty(Tu, Tv) \leq \gamma_1(u, v) d_\infty(u, v) + \gamma_2(u, v) d_\infty(u, Tu) + \gamma_3(u, v) d_\infty(v, Tv)$$
$$+\gamma_4(u, v) d_\infty(u, Tv) + \gamma_5(u, v) d_\infty(v, Tu), \tag{2.10}$$

where

$$\gamma_i(u, v) := \sup_{t \in S} \alpha_i(u(\phi(t)), v(\phi(t)))$$

for $i = 1, 2 \ldots, 5$. We have still the condition $\Sigma_{i=1}^5 \gamma_i(u, v) \leq \lambda < 1$. So all the conditions of Theorem 2.1 are satisfied. Moreover, the condition (2.6) means that $d_\infty(g, Tg) \leq \delta$. Hence, according to Theorem 2.1, there exists a unique element $f \in Y_\delta$ such that $f = Tf$ and $d_\infty(g, Tg) \leq \theta(\delta)$. Therefore, the conditions (2.7) and (2.8) hold. This ends the proof. □

Remark 2.2. One can observe that the mapping θ involved in Theorem 2.2, has the expression :

(i) $\theta(\delta) := \frac{\delta}{1-\lambda}$ for the Banach's contraction principle.

(ii) $\theta(\delta) := \frac{(1+\alpha)\delta}{1-2\alpha}$ for the Kannan's fixed point theorem ([12]) ($\alpha_1 = \alpha_4 = \alpha_5 = 0$ and $\alpha_2 = \alpha_3 = \alpha \in [0, \frac{1}{2})$).

(iii) $\theta(\delta) := \frac{(1+\beta)\delta}{1-\alpha-2\beta}$ for the Ćirić-Reich-Rus fixed point theorem ([18]) ($\alpha_1 = \alpha$, $\alpha_2 = \alpha_3 = \beta$ $\alpha_4 = \alpha_5 = 0$, and $\alpha + 2\beta \in [0, 1)$).

(iv) $\theta(\delta) := \frac{(2+\alpha)\delta}{2-\alpha}$ for the Hardy and Rogers fixed point theorem ([8]) ($\alpha_1, \alpha_2, \alpha_3$, α_4, α_5 are nonnegative constants such that $\alpha := \sum_{i=1}^5 \alpha_i \in [0, 1)$).

We have the following consequence.

Theorem 2.3. *Suppose that S is a nonempty set; E is a real (or complex) Banach space, $\phi : S \to S$, $\alpha : S \to E$, $B : S \to \mathcal{L}(E)$ ($\mathcal{L}(E)$ is the Banach algebra of all bounded linear operators on E) $0 \leq \lambda < 1$, and $\|B(t)\| \leq \lambda$ for all $t \in S$. Also suppose that $g : S \to E$, $\delta > 0$, and*

$$\|g(t) - [\alpha(t) + B(t)(g(\phi(t)))]\| \leq \delta \quad \text{for all } x, y \in S. \tag{2.11}$$

Then there is a unique function $f : S \to E$ such that

$$f(t) = \alpha(t) + B(t)(f(\phi(t))) \tag{2.12}$$

and

$$\|f(t) - g(t)\| \leq \frac{\delta}{1-\lambda} \quad \text{for all } t \in S, \tag{2.13}$$

Proof. For all $(t, x) \in S \times E$, we set

$$F(t, x) = \alpha(t) + B(t)(x).$$

Then we have

$$\|F(t, x) - F(t, y)\| = \|B(t)(x - y)\| \leq \|B(t)\| \, \|x - y\| \leq \lambda \, \|x - y\| \, .$$

Thus, F satisfies the condition (2.5) of Theorem 2.2 with

$$\alpha_1(x, y) = \lambda, \quad \text{and} \quad \alpha_j(x, y) = 0, \quad \text{for} \quad j = 2, 3, 4, 5.$$

By Remark 2.2, we know that , in this case, the map θ is given by $\theta(\delta) = \frac{\delta}{1-\lambda}$. By application of Theorem 2.2, we obtain the required conclusions expressed in (2.12) and (2.13). This ends the proof. □

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