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STABILITY OF CERTAIN MINIMAL SUBMANIFOLDS OF COMPACT HERMITIAN SYMMETRIC SPACES

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Introduction. In this paper we consider a compact totally real totally geodesic submanifold M of a Hermitian symmetric space $(\bar{M}, \bar{g})$ of compact type with $\dim M = \dim_C \bar{M}$, and study their classification and stability.

We shall show that *such a submanifold M is always a symmetric R-space* (cf. §1 for definition), *and these pairs $((\bar{M}, \bar{g}), M)$ correspond in one to one fashion to symmetric R-spaces.* Furthermore we shall prove that *M is stable in $(\bar{M}, \bar{g})$ as a minimal submanifold if and only if M is simply connected.*

Lawson-Simons [6] proved that a compact stable minimal submanifold of the complex projective n-space $P_n(C)$ endowed with the Kähler metric of constant holomorphic sectional curvature is always a complex submanifold. They showed also [6] that this is not true for a general Hermitian symmetric space of compact type, by giving an example of a compact stable minimal submanifold of $P_1(C) \times P_1(C)$ which is not a complex submanifold. The simply connected ones among our submanifolds include the example of Lawson-Simons and provide many examples with the same properties. For example, the quaternion Grassmann manifold $G_{p,q}(\boldsymbol{H})$ imbedded in the complex Grassmann manifold $G_{2p,2q}(C)$ is minimal and stable, but not a complex submanifold.

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1. Totally real totally geodesic submanifolds of compact Hermitian symmetric spaces. In this section we shall classify compact totally real totally geodesic submanifolds M of a Hermitian symmetric space $(\bar{M}, \bar{g})$ of compact type with $\dim M = \dim_C \bar{M}$.

Let $(\bar{M}, \bar{g})$ be a Hermitian manifold. The inner product and the complex structure tensor on the tangent bundle $T\bar{M}$ are denoted by $\langle\ ,\ \rangle$ and J, respectively. A submanifold M of $\bar{M}$ is said to be *totally real* if $\langle JT_pM, T_pM\rangle = 0$ for each $p \in M$. A submanifold M is called a *real form* of $(\bar{M}, \bar{g})$ if there exists an involutive anti-holomorphic isometry σ of $(\bar{M}, \bar{g})$ such that

$$M = \{p \in \bar{M};\ \sigma(p) = p\}\ .$$

LEMMA 1.1. *Let $(\bar{M}, \bar{g})$ be a (complete) Hermitian manifold. Then any real form M of $(\bar{M}, \bar{g})$ is a (complete) totally real totally geodesic submanifold with* $\dim M = \dim_C \bar{M}$.

PROOF. Let σ be an involutive anti-holomorphic isometry of $(\bar{M}, \bar{g})$ which defines M. Then M coincides with the set of fixed points of the isometry σ of $(\bar{M}, \bar{g})$, and hence it is totally geodesic (cf. Kobayashi [4]).

Let $p \in M$ and σ_* denote the differential of σ at p. Then σ_* is an involutive linear isometry of $T_p\bar{M}$ with $\sigma_* J = -J\sigma_*$. Thus, denoting by $(T_p\bar{M})^{\pm}$ the (± 1)-eigenspace of σ_*, we have

$$T_p\bar{M} = (T_p\bar{M})^+ + (T_p\bar{M})^- \quad \text{(orthogonal sum)}$$

and $J(T_p\bar{M})^{\pm} = (T_p\bar{M})^{\mp}$. Since $(T_p\bar{M})^+ = T_pM$, we have that $\langle JT_pM, T_pM\rangle = 0$ and $\dim M = \dim_C \bar{M}$. q.e.d.

In the following we recall a construction of real forms, called symmetric R-spaces, of a Hermitian symmetric space of compact type (cf. Takeuchi [12]).

Let $(\mathfrak{g}, \tau)$ be a *positive definite symmetric graded Lie algebra* (cf. Satake [10]), that is,

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1\ , \quad [\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}\ ,$$

is a real semi-simple graded Lie algebra such that $\mathfrak{g}_{-1} \neq 0$ and $\mathfrak{g}_0$ acts effectively on $\mathfrak{g}_{-1}$, and τ is a Cartan involution of $\mathfrak{g}$ with $\tau\mathfrak{g}_p = \mathfrak{g}_{-p}$ $(p = -1, 0, 1)$. Then $\mathfrak{u} = \mathfrak{g}_0 + \mathfrak{g}_1$ is a subalgebra of $\mathfrak{g}$. Let G be the connected Lie group with the trivial center such that $\mathrm{Lie}\, G$, the Lie algebra of G, is $\mathfrak{g}$. Put

$$U = \{a \in G;\ \mathrm{Ad}(a)\mathfrak{u} = \mathfrak{u}\}\ .$$

Then we have $\mathrm{Lie}\, U = \mathfrak{u}$. The homogeneous space $M = G/U$ is compact and called the *symmetric R-space* associated to $(\mathfrak{g}, \tau)$. The origin U of M will be denoted by o.

Let $\bar{\mathfrak{g}}$ and $\bar{\mathfrak{u}}$ be the complexifications of $\mathfrak{g}$ and $\mathfrak{u}$, respectively and $\bar{G}$ the connected complex Lie group with the trivial center such that $\mathrm{Lie}\, \bar{G} = \bar{\mathfrak{g}}$. We regard G as a subgroup of $\bar{G}$. Put

$$\bar{U} = \{a \in \bar{G};\ \mathrm{Ad}(a)\bar{\mathfrak{u}} = \bar{\mathfrak{u}}\}\ .$$

Then $\bar{U}$ is a connected complex Lie subgroup of $\bar{G}$ with $\mathrm{Lie}\, \bar{U} = \bar{\mathfrak{u}}$ and

$\bar{U} \cap G = U$. The complex homogeneous space $\bar{M} = \bar{G}/\bar{U}$ is compact, and the identity component $\mathrm{Aut}^0(\bar{M})$ of the group of all holomorphic automorphisms of $\bar{M}$ is identified with $\bar{G}$ (cf. Takeuchi [14]). Moreover we obtain a natural G-equivariant imbedding $f: M \to \bar{M}$ by virtue of $\bar{U} \cap G = U$. It is called the *canonical imbedding* associated to $(\mathfrak{g}, \tau)$. In what follows we shall often regard M as a submanifold of $\bar{M}$ through the imbedding f.

Let σ be the complex conjugation of $\bar{\mathfrak{g}}$ with respect to $\mathfrak{g}$ and denote the extension of σ to $\bar{G}$ also by σ. Since $\bar{U}$ is connected we have $\sigma(\bar{U}) = \bar{U}$, and thus σ induces an involutive anti-holomorphic diffeomorphism σ of $\bar{M}$. Then $M \subset \bar{M}$ is given by

$$M = \{p \in \bar{M};\ \sigma(p) = p\} .$$

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition associated to τ. Then $\mathfrak{g}_u = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$ is a compact real form of $\bar{\mathfrak{g}}$. Let τ denote the complex conjugation of $\bar{\mathfrak{g}}$ with respect to $\mathfrak{g}_u$. Then $\mathfrak{g}$ is stable under τ and τ coincides with the original τ on $\mathfrak{g}$. From the semi-simplicity of $\mathfrak{g}$, there exists uniquely an element $Z \in \mathfrak{g}_0$ such that

$$\mathfrak{g}_p = \{X \in \mathfrak{g};\ [Z, X] = pX\} \quad (p = -1, 0, 1) .$$

The condition $\tau\mathfrak{g}_p = \mathfrak{g}_{-p}$ $(p = -1, 0, 1)$ implies $\tau Z = -Z$, and hence $Z \in \mathfrak{p}$. Let K and G_u be the connected subgroups of $\bar{G}$ generated by $\mathfrak{k}$ and $\mathfrak{g}_u$, respectively, and put

$$K_0 = \{a \in K;\ \mathrm{Ad}(a)Z = Z\} , \quad \mathfrak{k}_0 = \mathrm{Lie}\, K_0 ,$$
$$K_u = \{a \in G_u;\ \mathrm{Ad}(a)Z = Z\} , \quad \mathfrak{k}_u = \mathrm{Lie}\, K_u .$$

Then we have smooth identifications

$$M = K/K_0 , \quad \bar{M} = G_u/K_u .$$

We define an involutive automorphism θ of $\bar{G}$ by

$$\theta(a) = \exp(\pi\sqrt{-1}Z)a(\exp(\pi\sqrt{-1}Z))^{-1} \quad \text{for} \quad a \in \bar{G} .$$

Then $\theta(K) = K$, $\theta(G_u) = G_u$ and

$$(K_\theta)^0 \subset K_0 \subset K_\theta , \quad K_u = (G_u)_\theta ,$$

where K_θ (resp. $(G_u)_\theta$) denotes the subgroup of all fixed points of θ in K (resp. in G_u) and $(K_\theta)^0$ the identity component of K_θ. Thus both (K, K_0) and (G_u, K_u) are compact symmetric pairs. If we define

$$\mathfrak{m} = \{X \in \mathfrak{k};\ \theta X = -X\} ,$$
$$\mathfrak{m}_u = \{X \in \mathfrak{g}_u;\ \theta X = -X\} ,$$

denoting also by θ the differential of θ, we have direct sum decompositions

$$\mathfrak{k} = \mathfrak{k}_0 + \mathfrak{m} , \quad \mathfrak{g}_u = \mathfrak{k}_u + \mathfrak{m}_u$$

as vector spaces. Thus $\mathfrak{m}$ and $\mathfrak{m}_u$ are identified with T_oM and $T_o\bar{M}$, respectively. Then $H_0 = -\sqrt{-1}Z$ is the unique element of the center of $\mathfrak{k}_u$ such that $\mathrm{ad}(H_0)|\mathfrak{m}_u$ gives the complex structure tensor J_o of $\bar{M}$ at o. Denote by $(,)$ the Killing form of $\bar{\mathfrak{g}}$, and define a $\mathfrak{g}_u$-invariant inner product $\langle , \rangle$ on $\mathfrak{g}_u$ by

$$\langle X, Y\rangle = -(X, Y) \quad \text{for} \quad X, Y \in \mathfrak{g}_u .$$

The K-invariant (resp. G_u-invariant) Riemannian metric on M (resp. on $\bar{M}$) which extends $\langle , \rangle|\mathfrak{m}\times\mathfrak{m}$ (resp. $\langle , \rangle|\mathfrak{m}_u\times\mathfrak{m}_u$) is denoted by g (resp. by $\bar{g}$), and called the *canonical Riemannian metric* on M (resp. on $\bar{M}$). Then

(i) (M, g) (resp. $(\bar{M}, \bar{g})$) is a compact symmetric space (resp. a Hermitian symmetric space of compact type) such that the identity component $I^0(M, g)$ (resp. $I^0(\bar{M}, \bar{g})$) of the group of all isometries of (M, g) (resp. of $(\bar{M}, \bar{g})$) is identified with K (resp. with G_u), and the canonical imbedding $f: (M, g) \to (\bar{M}, \bar{g})$ is isometric.

Moreover σ is an isometry of $(\bar{M}, \bar{g})$, and hence M is a real form of $(\bar{M}, \bar{g})$. Thus, by Lemma 1.1.

(ii) M is a totally real totally geodesic submanifold of $(\bar{M}, \bar{g})$ with $\dim M = \dim_C \bar{M}$.

REMARK 1. If $\mathfrak{g}$ is simple, the Riemannian metrics g and $\bar{g}$ satisfying (i) and (ii) are unique up to homothety. In this case, the symmetric R-space M or (M, g) is said to be *irreducible*.

REMARK 2. Let $\bar{M}^*$ be the symmetric bounded domain dual to $\bar{M}$ which is imbedded into $\bar{M}$ as an open submanifold of $\bar{M}$ by means of Harish-Chandra imbedding. It can be shown (Takeuchi [12]) that then $M^* = \bar{M}^* \cap M$ is a non-compact symmetric space dual to M and it is a real form of $\bar{M}^*$.

Two positive definite symmetric graded Lie algebras $(\mathfrak{g}, \tau)$ and $(\mathfrak{g}', \tau')$ are said to be *isomorphic* if there exists a Lie isomorphism $\phi: \mathfrak{g} \to \mathfrak{g}'$ such that $\phi\mathfrak{g}_p = \mathfrak{g}'_p$ $(p = -1, 0, 1)$ and $\phi\circ\tau = \tau'\circ\phi$. Let $\mathscr{S}$ denote the set of all isomorphism classes of positive definite symmetric graded Lie algebras. The set $\mathscr{S}$ was completely determined (Kobayashi-Nagano [5], Takeuchi [12]). Next we consider a pair $((\bar{M}, \bar{g}), M)$ of a connected Hermitian symmetric space $(\bar{M}, \bar{g})$ of compact type and a compact connected totally real totally geodesic submanifold M of $(\bar{M}, \bar{g})$ with $\dim M = \dim_C \bar{M}$. Such a pair is called a TRG-*pair*. For a finite number of TRG-pairs $((\bar{M}_i, \bar{g}_i), M_i)$, $1 \leqq i \leqq s$, the *direct product* $((\bar{M}, \bar{g}), M) = ((\bar{M}_1, \bar{g}_1), M_1) \times \cdots \times ((\bar{M}_s, \bar{g}_s), M_s)$, which is also a TRG-pair, is defined by $\bar{M} = \bar{M}_1 \times \cdots \times \bar{M}_s$, $\bar{g} = \bar{g}_1 \times \cdots \times \bar{g}_s$ and $M = M_1 \times \cdots \times M_s$. Two TRG-pairs $((\bar{M}, \bar{g}), M)$ and $((\bar{M}', \bar{g}'), M')$ are said to be *equivalent* if there exist direct product decompositions $((\bar{M}, \bar{g}), M) = ((\bar{M}_1, \bar{g}_1), M_1) \times \cdots \times ((\bar{M}_s, \bar{g}_s), M_s)$ and $((\bar{M}', \bar{g}'), M') =$

$((\bar{M}_1', \bar{g}_1'), M_1') \times \cdots \times ((\bar{M}_{s'}', \bar{g}_{s'}'), M_{s'}')$ with $s = s'$ and homothetic biholomorphic maps $\phi_i: (\bar{M}_i, \bar{g}_i) \to (\bar{M}_i', \bar{g}_i')$, $1 \leqq i \leqq s$, such that the product map $\phi = \phi_1 \times \cdots \times \phi_s: \bar{M} \to \bar{M}'$ satisfies $\phi(M) = M'$. Let $\mathscr{T}$ denote the set of all equivalence classes of TRG-pairs.

THEOREM 1.2. *Our correspondence* $(\mathfrak{g}, \tau) \mapsto ((\bar{M}, \bar{g}), M)$ *induces a bijection* $\Phi: \mathscr{S} \to \mathscr{T}$.

PROOF. It follows from definition that our correspondence induces a map $\Phi: \mathscr{S} \to \mathscr{T}$. Conversely, for any TRG-pair $((\bar{M}, \bar{g}), M)$ we shall associate canonically a positive definite symmetric graded Lie algebra $(\mathfrak{g}, \tau)$. Let $\bar{G} = \mathrm{Aut}^0(\bar{M})$ which is a connected complex semi-simple Lie group with the trivial center, and let $G_u = I^0(\bar{M}, \bar{g})$ which is a subgroup of $\bar{G}$ because $(\bar{M}, \bar{g})$ is a compact Kähler manifold (cf. Kobayashi [4]). Let J denote the complex structure tensor of $\bar{M}$. We identify $\bar{\mathfrak{g}} = \mathrm{Lie}\,\bar{G}$ (resp. $\mathfrak{g}_u = \mathrm{Lie}\,G_u$) with the Lie algebra of all smooth vector fields X on $\bar{M}$ such that the Lie derivative of J with respect to X vanishes (resp. of all Killing vector fields on $(\bar{M}, \bar{g})$) with Lie product $[X, Y] = YX - XY$. Then by Matsushima's theorem on compact Kähler Einstein manifolds we have

$$\bar{\mathfrak{g}} = \mathfrak{g}_u + J\mathfrak{g}_u \,, \quad \mathfrak{g}_u \cap J\mathfrak{g}_u = 0 \,. \tag{1.1}$$

Let $\mathfrak{g}(M)$ be the real subalgebra of $\bar{\mathfrak{g}}$ consisting of all $X \in \bar{\mathfrak{g}}$ such that the restriction $X|M$ is tangent to M, and $\mathfrak{k}(M)$ the Lie algebra of all Killing vector fields on M with respect to the Riemannian metric g induced from $\bar{g}$. We put

$$\mathfrak{k} = \mathfrak{g}(M) \cap \mathfrak{g}_u \,, \quad \mathfrak{p} = \mathfrak{g}(M) \cap J\mathfrak{g}_u \,,$$

and

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} \,. \tag{1.2}$$

Then $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, and hence $\mathfrak{g}$ is a real subalgebra of $\bar{\mathfrak{g}}$. We need here the following:

LEMMA 1.3. (1) *The map* $\mathfrak{k} \to \mathfrak{k}(M)$ *defined by* $X \mapsto X|M$ $(X \in \mathfrak{k})$ *is a Lie isomorphism.*

(2) *We have*

$$\mathfrak{g}_u = \mathfrak{k} + J\mathfrak{p} \,, \quad \mathfrak{k} \cap J\mathfrak{p} = 0 \,. \tag{1.3}$$

Now, it follows from (1.1), (1.2) and (1.3) that $\mathfrak{g}$ is a real form of $\bar{\mathfrak{g}}$. Let σ and τ denote the complex conjugation of $\bar{\mathfrak{g}}$ with respect to $\mathfrak{g}$ and $\mathfrak{g}_u$, respectively. Then

$$\sigma JX = -J\sigma X \quad \text{for} \quad X \in \bar{\mathfrak{g}} \,, \tag{1.4}$$

$$\sigma\mathfrak{g}_u = \mathfrak{g}_u \,. \tag{1.5}$$

We fix a point $o \in M$ and put

$$K_u = \{a \in G_u;\ a(o) = o\}\ ,$$

which is known to be connected. (See Helgason [2] for fundamental results on symmetric spaces.) Then $\bar{M} = G_u/K_u$ as smooth manifold. Let $\mathfrak{k}_u = \mathrm{Lie}\,K_u$ and $\mathfrak{g}_u = \mathfrak{k}_u + \mathfrak{m}_u$ be the associated Cartan decomposition. Let H_0 be the unique element of the center of $\mathfrak{k}_u$ such that $J_o = \mathrm{ad}(H_0)|\mathfrak{m}_u$. Putting $Z = JH_0 \in \bar{\mathfrak{g}}$, we define

$$\begin{aligned} &\bar{\mathfrak{g}}_p = \{X \in \bar{\mathfrak{g}};\ [Z, X] = pX\} \quad (p = -1, 0, 1)\ , \\ &\bar{\mathfrak{u}} = \bar{\mathfrak{g}}_0 + \bar{\mathfrak{g}}_1\ , \\ &\bar{U} = \{a \in \bar{G};\ \mathrm{Ad}(a)\bar{\mathfrak{u}} = \bar{\mathfrak{u}}\}\ . \end{aligned}$$

Then $\mathrm{Lie}\,\bar{U} = \bar{\mathfrak{u}}$, $\bar{\mathfrak{g}} = \bar{\mathfrak{g}}_{-1} + \bar{\mathfrak{g}}_0 + \bar{\mathfrak{g}}_1$ and $\bar{M} = \bar{G}/\bar{U}$ as complex manifold. Note here that $\bar{\mathfrak{g}}_0$ acts on $\bar{\mathfrak{g}}_{-1}$ effectively. We define an involutive automorphism θ of $\bar{G}$ by

$$\theta(a) = \exp(\pi JZ)a(\exp(\pi JZ))^{-1} \quad \text{for} \quad a \in \bar{G}\ .$$

Then $\theta(G_u) = G_u$ and hence the differential of θ, denoted also by θ, satisfies $\theta\mathfrak{g}_u = \mathfrak{g}_u$. Morever we have

$$\mathfrak{k}_u = \{X \in \mathfrak{g}_u;\ \theta X = X\}\ , \tag{1.6}$$

$$\mathfrak{m}_u = \{X \in \mathfrak{g}_u;\ \theta X = -X\}\ . \tag{1.7}$$

A diffeomorphism θ of $\bar{M} = G_u/K_u$ is defined by the correspondence $a \cdot o \mapsto \theta(a) \cdot o (a \in G_u)$ because K_u is connected. It is the symmetry of $(\bar{M}, \bar{g})$ at o. Since M is totally geodesic in $(\bar{M}, \bar{g})$ we have $\theta(M) = M$, and hence $\theta\mathfrak{g}(M) = \mathfrak{g}(M)$. Therefore we have $\theta\mathfrak{k} = \mathfrak{k}$ and $\theta\mathfrak{p} = \mathfrak{p}$, and hence $\theta\mathfrak{g} = \mathfrak{g}$. Thus (1.5), (1.6) and (1.7) imply

$$\sigma\mathfrak{k}_u = \mathfrak{k}_u\ , \tag{1.8}$$

$$\sigma\mathfrak{m}_u = \mathfrak{m}_u\ . \tag{1.9}$$

Now it follows from (1.4) and (1.9) that $\sigma J_o = -J_o\sigma$ on $\mathfrak{m}_u = T_o(\bar{M})$, and thus $[\sigma H_o, \sigma X] = -J_o\sigma X$ for each $X \in \mathfrak{m}_u$, where σH_0 is an element of the center of $\mathfrak{k}_u$ by (1.8). Therefore the uniqueness of H_0 implies that $\sigma H_0 = -H_0$, and so $\sigma Z = Z$, that is, $Z \in \mathfrak{g}$. Thus, putting $\mathfrak{g}_p = \bar{\mathfrak{g}}_p \cap \mathfrak{g}$ $(p = -1, 0, 1)$ we get $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$. Moreover τ restricted to $\mathfrak{g}$ is a Cartan involution with $\tau Z = -Z$, and thus $\tau\mathfrak{g}_p = \mathfrak{g}_{-p}$ $(p = -1, 0, 1)$. The effectiveness of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ follows from that of $\bar{\mathfrak{g}}_0$ on $\bar{\mathfrak{g}}_{-1}$. Therefore $(\mathfrak{g}, \tau)$ is a positive definite symmetric graded Lie algebra.

Next we shall show that our correspondence $((\bar{M}, \bar{g}), M) \mapsto (\mathfrak{g}, \tau)$ induces a map $\Psi: \mathscr{T} \to \mathscr{S}$. Let $((\bar{M}, \bar{g}), M)$ and $((\bar{M}', \bar{g}'), M')$ be equivalent.

Various objects for $((\bar{M}', \bar{g}'), M')$ will be denoted by the same notation as $((\bar{M}, \bar{g}), M)$ but with primes. Let $\phi: \bar{M} \to \bar{M}'$ be an equivalence. Then, since both $\phi(o)$ and o' are on M', by Lemma 1.3, (1) there exists $\phi' \in I^0(\bar{M}', \bar{g}')$ such that $\phi'(M') = M'$ and $\phi'(\phi(o)) = o'$. Therefore we may assume that $\phi(o) = o'$. Then the correspondence $a \mapsto \phi \circ a \circ \phi^{-1} (a \in \bar{G})$ defines an isomorphism $\phi: \bar{G} \to \bar{G}'$ such that the differential $\phi: \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}'$ is a Lie isomorphism with $\phi \circ J = J' \circ \phi$, $\phi \mathfrak{g}_u = \mathfrak{g}'_u$, $\phi \mathfrak{g}(M) = \mathfrak{g}(M')$ and $\phi Z = Z'$. Thus we get $\phi \mathfrak{g} = \mathfrak{g}'$ and $\phi \mathfrak{k} = \mathfrak{k}'$. Therefore ϕ gives an isomorphism $(\mathfrak{g}, \tau) \to (\mathfrak{g}', \tau')$, and so $(\mathfrak{g}, \tau)$ is isomorphic to $(\mathfrak{g}', \tau')$.

Now we have $\Psi \circ \Phi = I_{\mathscr{S}}$ by definitions, and $\Phi \circ \Psi = I_{\mathscr{T}}$ by Remark 1, where I indicates the identity map. Thus our map Φ is a bijection.

q.e.d.

PROOF OF LEMMA 1.3. (1) Since (M, g) is a compact connected symmetric space, $I^0(M, g)$ is generated by symmetries. Thus the map $\mathfrak{k} \to \mathfrak{k}(M)$ is surjective, because M is totally geodesic in $(\bar{M}, \bar{g})$. So it suffices to show

$$X \in \mathfrak{k},\ X|M = 0 \Rightarrow X = 0 . \tag{1.10}$$

We fix a point $p \in M$ and define an endomorphism $\tilde{X}_p$ of $T_p\bar{M}$ by

$$\tilde{X}_p(y) = \bar{\nabla}_y X \quad \text{for} \quad y \in T_p\bar{M} ,$$

where $\bar{\nabla}$ is the Riemannian connection of $(\bar{M}, \bar{g})$. It suffices to show $\tilde{X}_p = 0$ since X is a Killing vector field on $(\bar{M}, \bar{g})$. For any $y \in T_pM$ we have

$$\tilde{X}_p(y) = \bar{\nabla}_y X = \nabla_y X = 0 ,$$
$$\tilde{X}_p(Jy) = \bar{\nabla}_{Jy} X = J\bar{\nabla}_y X = 0 ,$$

where ∇ is the Riemannian connection of (M, g). Here we have used the facts that M is totally geodesic, $X|M = 0$ and X is a holomorphic vector field on the Kähler manifold $(\bar{M}, \bar{g})$. Now $T_p\bar{M} = T_pM \oplus JT_pM$ implies $\tilde{X}_p = 0$.

(2) Let $X \in \mathfrak{g}_u$ and decompose $X|M$ as $X|M = X^T + X^N$, where X^T is tangent to M and X^N is normal to M. Then

$$0 = \langle \bar{\nabla}_y X, z \rangle + \langle \bar{\nabla}_z X, y \rangle = \langle \nabla_y X^T, z \rangle + \langle \nabla_z X^T, y \rangle$$

for any $y, z \in T_pM$, $p \in M$, and thus $X^T \in \mathfrak{k}(M)$. Now by (1) there is $X' \in \mathfrak{k}$ such that $X'|M = X^T$. Put $X'' = X - X' \in \mathfrak{g}_u$. Then $X''|M = X^N$ and $(JX'')|M = JX^N$ which is tangent to M. Therefore $JX'' \in \mathfrak{g}(M) \cap J\mathfrak{g}_u = \mathfrak{p}$, and hence $X = X' + X'' \in \mathfrak{k} + J\mathfrak{p}$. Thus we have shown that $\mathfrak{g}_u \subset \mathfrak{k} + J\mathfrak{p}$ and so $\mathfrak{g}_u = \mathfrak{k} + J\mathfrak{p}$. On the other hand, any $X \in \mathfrak{k} \cap J\mathfrak{p}$ satisfies $X|M = 0$, and hence $X = 0$ by (1.10). This shows $\mathfrak{k} \cap J\mathfrak{p} = 0$. q.e.d.

REMARK 3. Actually the subalgebra $\mathfrak{g}(M)$ of $\bar{\mathfrak{g}}$ in Theorem 1.2 coincides with $\mathfrak{g}$. In fact, for each point p of a symmetric R-space $M \subset \bar{M}$ there exists a holomorphic coordinate (z^α) of $\bar{M}$ around p such that M is given by $\operatorname{Im} z^\alpha = 0$ around p. Therefore we get

$$X \in \bar{\mathfrak{g}},\ X|M = 0 \Rightarrow X = 0\ ,$$

which implies $\mathfrak{g}(M) \cap J\mathfrak{g}(M) = 0$ and so $\mathfrak{g}(M) = \mathfrak{g}$.

REMARK 4. For any connected Hermitian symmetric space $(\bar{M}, \bar{g})$ of compact type, there exists at least one involutive anti-holomorphic isometry of $(\bar{M}, \bar{g})$ (Satake [10]).

2. First eigenvalues of symmetric R-spaces. In this section we shall compute the first eigenvalue of the Laplacian on smooth functions of an irreducible symmetric R-space.

Let $(\mathfrak{g}, \tau)$ be a positive definite symmetric graded Lie algebra and $M = G/U = K/K_0$ be the symmetric R-space associated to $(\mathfrak{g}, \tau)$. We use the same notation as in §1.

LEMMA 2.1. *Let $C_\mathfrak{p}$ be the Casimir operator on the $\mathfrak{k}$-module $\mathfrak{p}$ relative to the $\mathfrak{k}$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{k}$. Then $C_\mathfrak{p} = (1/2)I_\mathfrak{p}$.*

PROOF. Let $\{E_\alpha\}$ be an orthonormal basis for $\mathfrak{k}$ with respect to $\langle\ ,\ \rangle$. Then, by definition

$$C_\mathfrak{p} = -\sum_\alpha (\operatorname{ad}(E_\alpha)|\mathfrak{p})^2\ .$$

For each $X \in \mathfrak{p}$ we have

$$\begin{aligned} -([E_\alpha, [E_\alpha, X]], X) &= ([E_\alpha, X], [E_\alpha, X]) \\ &= (E_\alpha, [X, [E_\alpha, X]]) = -(E_\alpha, [X, [X, E_\alpha]]) \\ &= -(\operatorname{ad}(X)^2 E_\alpha, E_\alpha) = \langle \operatorname{ad}(X)^2 E_\alpha, E_\alpha \rangle\ . \end{aligned}$$

Therefore $(C_\mathfrak{p}X, X) = \operatorname{Tr}(\operatorname{ad}(X)^2|\mathfrak{k})$. On the other hand, from $\operatorname{ad}(X)\mathfrak{k} \subset \mathfrak{p}$, $\operatorname{ad}(X)\mathfrak{p} \subset \mathfrak{k}$ we get $(X, X) = \operatorname{Tr}(\operatorname{ad}(X)^2) = 2\operatorname{Tr}(\operatorname{ad}(X)^2|\mathfrak{k})$. Thus we obtain $(C_\mathfrak{p}X, X) = (X, X)/2$ for each $X \in \mathfrak{p}$, and hence

$$(C_\mathfrak{p}X, Y) = (X, Y)/2 \quad \text{for any } X, Y \in \mathfrak{p}\ .$$

This implies the assertion. q.e.d.

Let $\mathfrak{h}^- \subset \mathfrak{p}$ be a maximal abelian subalgebra in $\mathfrak{p}$ with $Z \in \mathfrak{h}^-$ and take an abelian subalgebra $\mathfrak{h}^+$ of $\mathfrak{k}$ such that $\mathfrak{h} = \mathfrak{h}^+ + \mathfrak{h}^-$ is a Cartan subalgebra of $\mathfrak{g}$. Then the complexification $\bar{\mathfrak{h}}$ of $\mathfrak{h}$ is a Cartan subalgebra of $\bar{\mathfrak{g}}$, whose real part $\mathfrak{h}_R$ is given by $\mathfrak{h}_R = \sqrt{-1}\mathfrak{h}^+ + \mathfrak{h}^-$. Let $\Sigma \subset \mathfrak{h}_R$ be the root system of $\bar{\mathfrak{g}}$ relative to $\bar{\mathfrak{h}}$ and put

$$\bar{\Sigma}_0 = \{\alpha \in \bar{\Sigma};\ (\alpha, Z) = 0\} .$$

Choose a σ-order on $\mathfrak{h}_R$ in the sense of Satake [9] such that $(\alpha, Z) \geqq 0$ for each α in $\bar{\Sigma}^+$, the set of positive roots. Then we have

$$\bar{\Sigma}^+ - \bar{\Sigma}_0 = \{\alpha \in \bar{\Sigma};\ (\alpha, Z) = 1\} .$$

In what follows in this section we assume that $\mathfrak{g}$ is simple. Then the followings are known (Takeuchi [12]):

There exists a maximal system $\{\gamma_1, \cdots, \gamma_s\}$, $s = \mathrm{rank}(\bar{M}, \bar{g})$, of strongly orthogonal roots in $\bar{\Sigma}^+ - \bar{\Sigma}_0$ with the same length such that $\sigma\{\gamma_1, \cdots, \gamma_s\} = \{\gamma_1, \cdots, \gamma_s\}$. Moreover, if $r = \mathrm{rank}(M, g)$, we have

(a) $r = s$, $\sigma\gamma_i = \gamma_i$ $(1 \leqq i \leqq r)$; or

(b) $2r = s$, $\sigma\gamma_i = \gamma_{r+i}$ $(1 \leqq i \leqq r)$, changing indices of γ_j's if necessary.

We define $\beta_i \in \mathfrak{h}^-(1 \leqq i \leqq r)$ by

$$\beta_i = \begin{cases} \gamma_i & \text{if} \quad r = s , \\ (1/2)(\gamma_i + \sigma\gamma_i) & \text{if} \quad 2r = s. \end{cases}$$

Then

$$(\beta_i, \beta_i) = \begin{cases} (\gamma_i, \gamma_i) & \text{if} \quad r = s , \\ (\gamma_i, \gamma_i)/2 & \text{if} \quad 2r = s . \end{cases} \tag{2.1}$$

Let $\mathfrak{a}^- = \{\beta_1, \cdots, \beta_r\}_R$ be the $\boldsymbol{R}$-span of $\{\beta_1, \cdots, \beta_r\}$, and $\pi_{\mathfrak{a}^-}\colon \mathfrak{h}_R \to \mathfrak{a}^-$ denote the orthogonal projection with respect to $(\ ,\ )$. By Satake [10] (cf. also Moore [7]) we have then

$$\begin{aligned} \pi_{\mathfrak{a}^-}(\bar{\Sigma}) - \{0\} &= \{\pm(1/2)(\beta_i \pm \beta_j)\ (1 \leqq i < j \leqq r),\ \pm\beta_i\ (1 \leqq i \leqq r)\} , \\ \text{or}\ \ &\{\pm(1/2)(\beta_i \pm \beta_j)\ (1 \leqq i < j \leqq r),\ \pm\beta_i,\ \pm(1/2)\beta_i\ (1 \leqq i \leqq r)\} . \end{aligned} \tag{2.2}$$

We may choose (cf. Takeuchi [12]) root vectors $X_\alpha \in \bar{\mathfrak{g}}$ $(\alpha \in \bar{\Sigma})$ in such a way that

$$[X_\alpha, X_{-\alpha}] = -\frac{2}{(\alpha, \alpha)}\alpha , \quad \tau X_\alpha = X_{-\alpha} , \quad \sigma X_\alpha = X_{\sigma\alpha} .$$

We put $U_{\gamma_j} = X_{\gamma_j} + X_{-\gamma_j} \in \mathfrak{m}_u$ $(1 \leqq j \leqq s)$ and define $S_i \in \mathfrak{m}$ $(1 \leqq i \leqq r)$ by

$$S_i = \begin{cases} U_{\gamma_i} & \text{if} \quad r = s , \\ U_{\gamma_i} + U_{\sigma\gamma_i} & \text{if} \quad 2r = s , \end{cases}$$

whose length with respect to $\langle\ ,\ \rangle$ are the same. Then $\mathfrak{t}^- = \{S_1, \cdots, S_r\}_R$ is a maximal abelian subalgebra in $\mathfrak{m}$. We define elements V_i, V_i' $(1 \leqq i \leqq r)$ of $\bar{\mathfrak{g}}$ by

$$V_i = \begin{cases} X_{\gamma_i} & \text{if} \quad r = s , \\ X_{\gamma_i} + X_{\sigma\gamma_i} & \text{if} \quad 2r = s , \end{cases}$$

$$V_i' = \begin{cases} \frac{1}{2}\Big(X_{\gamma_i} - X_{-\gamma_i} + \frac{2\sqrt{-1}}{(\beta_i, \beta_i)}\beta_i\Big) & \text{if} \quad r = s\,, \\ \frac{1}{2}\Big(X_{\gamma_i} + X_{\sigma\gamma_i} - X_{-\gamma_i} - X_{-\sigma\gamma_i} + \frac{2\sqrt{-1}}{(\beta_i, \beta_i)}\beta_i\Big) & \text{if} \quad 2r = s\,. \end{cases}$$

Note that the V_i''s are non-zero elements of the complexification $\bar{\mathfrak{p}}$ of $\mathfrak{p}$. Moreover we define $c' \in G_u$ by

$$c' = \prod_{j=1}^{s} \exp \frac{\pi}{4\sqrt{-1}}\Big(X_{\gamma_j} - X_{-\gamma_j}\Big)\,.$$

LEMMA 2.2. (1) *For each i $(1 \leqq i \leqq r)$ we have*

$$\mathrm{Ad}(c')\Big(\frac{2}{(\beta_i, \beta_i)}\beta_i\Big) = \sqrt{-1}S_i\,, \tag{2.3}$$

$$\mathrm{Ad}(c')V_i = V_i'\,. \tag{2.4}$$

(2) *We have*

$$[H, V_i] = (\beta_i, H)V_i \quad \textit{for each } H \in \mathfrak{a}^-, 1 \leqq i \leqq r\,.$$

PROOF. (1) If we put

$$X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

then $\{X_+, X_-, H\}$ is a basis for $\mathfrak{sl}(2, C)$ with relations $[X_+, X_-] = -H$, $[H, X_\pm] = \pm 2X_\pm$. On the other hand we have relations $[X_{\gamma_j}, X_{-\gamma_j}] = -(2/(\gamma_j, \gamma_j))\gamma_j$, $[(2/(\gamma_j, \gamma_j))\gamma_j, X_{\pm\gamma_j}] = \pm 2X_{\pm\gamma_j}$ $(1 \leqq j \leqq s)$. Thus the correspondence $X_\pm \mapsto X_{\pm\gamma_j}$, $H \mapsto (2/(\gamma_j, \gamma_j))\gamma_j$ defines an injective Lie homomorphism $\mathfrak{sl}(2, C) \to \bar{\mathfrak{g}}$ such that $U \mapsto U_{\gamma_j}$, where $U = X_+ + X_-$. Since the element c_0' of $SU(2)$ defined by

$$c_0' = \exp \frac{\pi}{4\sqrt{-1}}(X_+ - X_-) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\sqrt{-1} \\ -\sqrt{-1} & 1 \end{pmatrix}$$

satisfies $\mathrm{Ad}(c_0')H = \sqrt{-1}U$, $\mathrm{Ad}(c_0')X_+ = (1/2)(X_+ - X_- + \sqrt{-1}H)$, we get for each j $(1 \leqq j \leqq s)$

$$\mathrm{Ad}(c')\Big(\frac{2}{(\gamma_j, \gamma_j)}\gamma_j\Big) = \sqrt{-1}U_{\gamma_j}\,, \tag{2.3$'$}$$

$$\mathrm{Ad}(c')X_{\gamma_j} = \frac{1}{2}\Big(X_{\gamma_j} - X_{-\gamma_j} + \frac{2\sqrt{-1}}{(\gamma_j, \gamma_j)}\gamma_j\Big)\,. \tag{2.4$'$}$$

Thus we obtain (2.2), (2.3) in case $r = s$. In case $2r = s$, we have for each i $(1 \leqq i \leqq r)$

$$\mathrm{Ad}(c')\Big(\frac{2}{(\gamma_i,\gamma_i)}\sigma\gamma_i\Big) = \sqrt{-1}\,U_{\sigma\gamma_i}\,, \tag{2.3}''$$

$$\mathrm{Ad}(c')X_{\sigma\gamma_i} = \frac{1}{2}\Big(X_{\sigma\gamma_i} - X_{-\sigma\gamma_i} + \frac{2\sqrt{-1}}{(\gamma_i,\gamma_i)}\sigma\gamma_i\Big)\,. \tag{2.4}''$$

Adding (2.3)′ and (2.3)″ (resp. (2.4)′ and (2.4)″) we get (2.3) (resp. (2.4)), by virtue of the equality

$$\frac{2}{(\gamma_i,\gamma_i)}\gamma_i + \frac{2}{(\gamma_i,\gamma_i)}\sigma\gamma_i = \frac{2}{(\beta_i,\beta_i)}\beta_i\,,$$

which follows from (2.1).

(2) This follows from a direct calculation. q.e.d.

The eigenvalues of the Laplacian Δ with respect to the canonical Riemannian metric g acting on the space $C^\infty(M)$ of smooth functions on $M = K/K_0$ are obtained in the following way (cf. Takeuchi [13]).

Take an abelian subalgebra $\mathfrak{t}^+$ of $\mathfrak{k}_0$ such that $\mathfrak{t} = \mathfrak{t}^+ + \mathfrak{t}^-$ is a maximal abelian subalgebra of $\mathfrak{k}$. The complexification $\bar{\mathfrak{t}}$ of $\mathfrak{t}$ is a Cartan subalgebra of the complexification $\bar{\mathfrak{k}}$ of $\mathfrak{k}$ and the real part $\mathfrak{t}_R$ of $\bar{\mathfrak{t}}$ is given by $\mathfrak{t}_R = \sqrt{-1}\mathfrak{t}^+ + \sqrt{-1}\mathfrak{t}^-$. Taking a basis $\{H_{r+1}, \cdots, H_l\}$ for $\sqrt{-1}\mathfrak{t}^+$, we define a lexicographic order $>$ on $\mathfrak{t}_R$ by the basis $\{\sqrt{-1}S_1, \cdots, \sqrt{-1}S_r, H_{r+1}, \cdots, H_l\}$ for $\mathfrak{t}_R$. Let $\Sigma \subset \sqrt{-1}\mathfrak{t}^-$ be the root system of the symmetric pair $(\mathfrak{k}, \mathfrak{k}_0)$ and Σ^+ the set of positive roots in Σ (with respect to $>$). We set

$$\begin{aligned} \Gamma &= \{H \in \mathfrak{t}^-;\ \exp H \in K_0\}\,, \\ \Gamma^\perp &= \{\lambda \in \sqrt{-1}\mathfrak{t}^-;\ (\lambda, \Gamma) \subset 2\pi\sqrt{-1}\boldsymbol{Z}\}\,, \\ D &= \{\lambda \in \Gamma^\perp;\ (\lambda, \alpha) \geqq 0 \quad \text{for each } \alpha \in \Sigma^+\}\,. \end{aligned}$$

Let $\delta \in \sqrt{-1}\mathfrak{t}^-$ be the half-sum of all roots in Σ^+ with multiplicity counted. Then the set $\mathrm{Spec}(M, g)$ of eigenvalues of Δ is given by

$$\mathrm{Spec}(M, g) = \{(2\delta + \lambda, \lambda);\ \lambda \in D\}\,. \tag{2.5}$$

Here the multiplicity of $(2\delta + \lambda, \lambda)$ is equal to the dimension of the irreducible $\bar{\mathfrak{k}}$-module V_λ with the highest weight λ, and $(2\delta + \lambda, \lambda)$ is nothing but the eigenvalue of the Casimir operator on V_λ relative to the inner product $\langle\ ,\ \rangle$. In our case we have (Takeuchi [12])

$$\Gamma = \pi\{S_1, \cdots, S_r\}_Z\,,$$

where $\{S_1, \cdots, S_r\}_Z$ denotes the subgroup of $\mathfrak{t}^-$ generated by $\{S_1, \cdots, S_r\}$. Thus, if we define $h_i \in \sqrt{-1}\mathfrak{t}^-(1 \leqq i \leqq r)$ by $(h_i, \sqrt{-1}S_j) = \delta_{ij}$, then they have the same length with respect to $(\ ,\ )$ and

$$\Gamma^{\perp} = 2\{h_1, \cdots, h_r\}_Z , \quad h_1 > \cdots > h_r > 0 . \tag{2.6}$$

LEMMA 2.3. *The highest weight Λ relative to $\bar{\mathfrak{t}}$ of the $\bar{\mathfrak{k}}$-module $\bar{\mathfrak{p}}$ is given by $\Lambda = 2h_1$.*

PROOF. Take an abelian subalgebra $\mathfrak{s}$ in $\mathfrak{p}$ such that $\mathfrak{h}' = \mathfrak{t} + \mathfrak{s}$ is a Cartan subalgebra of $\mathfrak{g}$. Then the real part $\mathfrak{h}'_R$ of the complexification $\bar{\mathfrak{h}}'$ of $\mathfrak{h}'$ is given by $\mathfrak{h}'_R = \sqrt{-1}\mathfrak{t} + \mathfrak{s}$. Let $\bar{\Sigma}' \subset \mathfrak{h}'_R$ be the root system of $\bar{\mathfrak{g}}$ relative to $\bar{\mathfrak{h}}'$. Let $\pi_{\mathfrak{t}}: \mathfrak{h}'_R \to \sqrt{-1}\mathfrak{t}$ and $\pi_{\mathfrak{t}^-}: \mathfrak{h}'_R \to \sqrt{-1}\mathfrak{t}^-$ be orthogonal projections with respect to (,).

Since $\mathrm{Ad}(c')\mathfrak{a}^- = \sqrt{-1}\mathfrak{t}^-$ by (2.3), both $\mathrm{Ad}(c')\bar{\mathfrak{h}}$ and $\bar{\mathfrak{h}}'$ are Cartan subalgebras of the centralizer in $\bar{\mathfrak{g}}$ of $\mathfrak{t}^-$. Thus there exists an element c'' of the centralizer in $\bar{G}$ of $\mathfrak{t}^-$ such that $\mathrm{Ad}(c'')\mathrm{Ad}(c')\bar{\mathfrak{h}} = \bar{\mathfrak{h}}'$. Put $c = c''c' \in \bar{G}$. Then $\mathrm{Ad}(c)\bar{\mathfrak{h}} = \bar{\mathfrak{h}}'$, and hence

$$\mathrm{Ad}(c)\mathfrak{h}_R = \mathfrak{h}'_R , \quad \mathrm{Ad}(c)\bar{\Sigma} = \bar{\Sigma}' , \tag{2.7}$$

$$\pi_{\mathfrak{t}^-}\circ\mathrm{Ad}(c) = \mathrm{Ad}(c)\circ\pi_{\mathfrak{a}^-} \quad \text{on} \quad \mathfrak{h}_R . \tag{2.8}$$

Moreover, by (2.3) we have

$$\mathrm{Ad}(c)((1/2)\beta_i) = h_i \quad (1 \leqq i \leqq r) . \tag{2.9}$$

Next we show

$$\Lambda = \mathrm{Max}\{\pi_{\mathfrak{t}^-}(\alpha);\ \alpha \in \bar{\Sigma}', \exists V \in \bar{\mathfrak{p}} - \{0\} \ \text{with} \ [H, V] = (\alpha, H)V \ \text{for each} \ H \in \sqrt{-1}\mathfrak{t}^-\} . \tag{2.10}$$

In fact, the set of weights relative to $\bar{\mathfrak{t}}$ of the $\bar{\mathfrak{k}}$-module $\bar{\mathfrak{p}}$ coincides with the set of $\pi_{\mathfrak{t}}(\alpha)$ such that $\alpha \in \bar{\Sigma}' \cup \{0\}$ and that there exists $V \in \bar{\mathfrak{p}} - \{0\}$ with $[H, V] = (\alpha, H)V$ for each $H \in \mathfrak{t}_R$. Since $\mathfrak{p}$ is K-isomorphic with a K-submodule of $C^\infty(M)$, we have $\Lambda \in \sqrt{-1}\mathfrak{t}^-$ (cf. Takeuchi [13]). On the other hand, from the definition of the order $>$ on $\mathfrak{t}_R$ we have

$$\mu, \mu' \in \mathfrak{t}_R , \quad \pi_{\mathfrak{t}^-}(\mu) > \pi_{\mathfrak{t}^-}(\mu') \Rightarrow \mu > \mu' .$$

These imply the assertion (2.10). Finally we show that

$$[H', V'_i] = (2h_i, H')V'_i \quad \text{for each } H' \in \sqrt{-1}\mathfrak{t}^- , \quad 1 \leqq i \leqq r . \tag{2.11}$$

Put $H = \mathrm{Ad}(c)^{-1}H' \in \mathfrak{a}^-$, so $\mathrm{Ad}(c')H = \mathrm{Ad}(c)H$. Applying $\mathrm{Ad}(c')$ to the equality in Lemma 2.2, (2) we get

$$[\mathrm{Ad}(c)H, \mathrm{Ad}(c')V_i] = (\beta_i, H)\mathrm{Ad}(c')V_i ,$$

and hence by (2.4), (2.9)

$$[H', V'_i] = (\beta_i, \mathrm{Ad}(c)^{-1}H')V'_i = (2h_i, H')V'_i .$$

Now, by (2.7), (2.8), (2.9) and (2.2) we have

$$\pi_{t^-}(\bar{\Sigma}') - \{0\} = \{\pm(h_i \pm h_j)\ (1 \leqq i < j \leqq r),\ \pm 2h_i\ (1 \leqq i \leqq r)\}\ , \quad \text{or}$$
$$\{\pm(h_i \pm h_j)\ (1 \leqq i < j \leqq r),\ \pm 2h_i,\ \pm h_i\ (1 \leqq i \leqq r)\}\ ,$$

and thus $\Lambda = 2h_1$ by (2.10) and (2.11). q.e.d.

It is known (Takeuchi [12], [15]) that irreducible symmetric R-spaces are devided into the following five classes.

(I) Hermitian type

$2r = s$, $\bar{\Sigma}$ is reducible, $\pi_1(M) = 0$.

$\Sigma = \{\pm(h_i \pm h_j)(1 \leqq i < j \leqq r),\ \pm 2h_i(1 \leqq i \leqq r)\}$, or

$\{\pm(h_i \pm h_j)\ (1 \leqq i < j \leqq r),\ \pm 2h_i,\ \pm h_i\ (1 \leqq i \leqq r)\}$.

(II) type $Sp(r)$

$2r = s$, $\bar{\Sigma}$ is irreducible, $\pi_1(M) = 0$.

Σ is the same as (I).

(III) type $SO(2r+1)$

$r = s$, $\bar{\Sigma}$ is irreducible, $\pi_1(M) = \boldsymbol{Z}_2$.

$\Sigma = \{\pm(h_i \pm h_j)(1 \leqq i < j \leqq r),\ \pm h_i(1 \leqq i \leqq r)\}$.

(IV) type $SO(2r)$

$r = s \geqq 2$, $\bar{\Sigma}$ is irreducible, $\pi_1(M) = \boldsymbol{Z}_2$.

$\Sigma = \{\pm(h_i \pm h_j)\ (1 \leqq i < j \leqq r)\}$.

(V) type $U(r)$

$r = s$, $\bar{\Sigma}$ is irreducible, $\pi_1(M) = \boldsymbol{Z}$.

$\Sigma = \{\pm(h_i - h_j)\ (1 \leqq i < j \leqq r)\}$.

REMARK 1. If M is of Hermitian type, then (M, g) is an irreducible Hermitian symmetric space of compact type and the canonical imbedding f is given as follows. Let M^* be the complex manifold which is the same as M as smooth manifold, but with the complex structure such that the identity map $M \to M^*$, denoted by $p \mapsto p^*$, is anti-holomorphic. We put $\bar{M} = M \times M^*$ and $\bar{g} = (1/2)(g \times g)$. Then the map $f: (M, g) \to (\bar{M}, \bar{g})$ defined by $f(p) = p \times p^*(p \in M)$ is the canonical imbedding.

THEOREM 2.4. *Let (M, g) be an irreducible symmetric R-space with the canonical Riemannian metric g. Let λ_1 be the least positive eigenvalue of the Laplacian Δ on $C^\infty(M)$. Suppose that the fundamental group $\pi_1(M)$ of M is finite and g is an Einstein metric. Then $\lambda_1 = 1/2$ with the multiplicity equal to* dim $\mathfrak{p}$.

PROOF. From the classification of irreducible symmetric R-spaces (cf. §3) we know that the only non-Einstein irreducible symmetric R-spaces M with finite $\pi_1(M)$ are

$$M = Q_{p,q}(\boldsymbol{R}) = \{[x] \in P_{p+q-1}(\boldsymbol{R});\ x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 = 0\}\ , \quad 3 \leqq p < q\ ,$$

where $[x]$ denotes the line of $\boldsymbol{R}^{p+q}$ through $x = (x_i) \in \boldsymbol{R}^{p+q} - \{0\}$. They are characterized by the property that M is of type $SO(4)$ and the multiplicities of roots $h_1 + h_2$ and $h_1 - h_2$ are different.

We introduce a new inner product $((\ ,\ ))$ on $\mathfrak{t}_R$ with $((h_i, h_j)) = \delta_{ij}$ by

$$((H, H')) = \frac{1}{(h_1, h_1)}(H, H') \quad \text{for} \quad H, H' \in \mathfrak{t}_R\ .$$

We shall show that $\Lambda = 2h_1$ is the unique element of $D - \{0\}$ such that

$$((2\delta + \Lambda, \Lambda)) = \mathrm{Min}\{((2\delta + \lambda, \lambda));\ \lambda \in D - \{0\}\}\ .$$

If M is of Hermitian type, we have

$$\Sigma^+ = \{h_i \pm h_j\ (1 \leqq i < j \leqq r),\ 2h_i\ (1 \leqq i \leqq r)\}\ , \quad \text{or}$$
$$\{h_i \pm h_j\ (1 \leqq i < j \leqq r),\ 2h_i,\ h_i\ (1 \leqq i \leqq r)\}\ ,$$

and hence by (2.6)

$$D = \{\lambda = 2(m_1 h_1 + \cdots + m_r h_r);\ m_i \in \boldsymbol{Z},\ m_1 \geqq \cdots \geqq m_r \geqq 0\}\ .$$

Since the Weyl group W of Σ consists of transformations $h_i \mapsto \varepsilon_i h_{s(i)}$, $\varepsilon_i = \pm 1$, $s \in \mathfrak{S}_r$, and leaves the multiplicities of roots invariant, 2δ is of the form

$$2\delta = n_1 h_1 + \cdots + n_r h_r\ , \quad n_i \in \boldsymbol{Z},\ n_1 > \cdots > n_r > 0\ .$$

Thus, for $\lambda \in D - \{0\}$ as above, we have

$$\begin{aligned} ((2\delta + \lambda, \lambda)) &= ((2\delta, \lambda)) + ((\lambda, \lambda)) \\ &= 2\Sigma n_i m_i + 4\Sigma m_i^2 \\ &\geqq 2n_1 + 4 = ((2\delta + 2h_1, 2h_1))\ . \end{aligned}$$

If $\lambda \neq 2h_1$, then $((2\delta, \lambda)) \geqq 2n_1$, $((\lambda, \lambda)) > 4$ and so $((2\delta + \lambda, \lambda)) > 2n_1 + 4$. Thus $\Lambda = 2h_1$ has the required property. In the same way we can show the assertion for a space M of type $Sp(r)$ or of type $SO(2r + 1)$. If M is of type $SO(2r)$, we have

$$\Sigma^+ = \{h_i \pm h_j\ (1 \leqq i < j \leqq r)\}\ ,$$

and hence

$$D = \{\lambda = 2(m_1 h_1 + \cdots + m_r h_r);\ m_i \in \boldsymbol{Z},\ m_1 \geqq \cdots \geqq m_{r-1} \geqq |m_r|\}\ .$$

The Weyl group W consists of transformations $h_i \mapsto \varepsilon_i h_{s(i)}$, $\varepsilon_i = \pm 1$, $\prod \varepsilon_i =$

1, $s\in\mathfrak{S}_r$. Moreover the multiplicities of h_1+h_2 and h_1-h_2 are the same if $r=2$. Therefore 2δ is of the form

$$2\delta = n_1h_1+\cdots+n_rh_r\,,\quad n_i\in \boldsymbol{Z},\ n_1>\cdots>n_{r-1}>n_r=0\,.$$

For $\lambda\in D-\{0\}$ as above, we have

$$((2\delta+\lambda,\lambda)) = 2\Sigma n_im_i+4\Sigma m_i^2\,.$$

Theorefore the assertion for M of type $SO(2r)$ follows in the same way as above. Thus the assertion is proved for each (M, g) in consideration.

Now, since $\pi_1(M)$ is finite, K is semi-simple, and hence the $\bar{\mathfrak{k}}$-module $\bar{\mathfrak{p}}$ is irreducible. Thus Lemmas 2.1 and 2.3 imply that $(2\delta+\Lambda,\Lambda)=1/2$. The theorem follows from this and (2.5). q.e.d.

REMARK 2. The first eigenvalues λ_1 for the other irreducible symmetric R-spaces are calculated in the same way as follows.

(i) $M=Q_{p,q}(\boldsymbol{R})$ $(3\leqq p<q)$, $\pi_1(M)=\boldsymbol{Z}_2$.

$$\lambda_1=\begin{cases}1/2 \ \text{ with multiplicity} = p(p+1)=\dim\mathfrak{p} & \text{if}\quad q=q+1\,,\\ 1/2 \ \text{ with multiplicity} = (p+2)(3p-1)/2 & \text{if}\quad q=p+2\,,\\ p/(p+q-2)(<1/2)\ \text{ with multiplicity} = (p+2)(p-1)/2 & \\ & \text{if}\quad q\geqq p+3\,.\end{cases}$$

(ii) M is of type $U(r)$, $\pi_1(M)=\boldsymbol{Z}$.

Let $\nu\geqq 0$ be the multiplicity of the root h_1-h_2. Then

$$\lambda_1=\begin{cases}1/2 \ \text{ with multiplicity} = \dim\mathfrak{p} & \text{if}\quad \nu\leqq 1\,,\\ 1/2 \ \text{ with multiplicity} = \dim\mathfrak{p}+2 & \text{if}\quad \nu=2\,,\\ r/(\nu(r-1)+2)(<1/2)\ \text{ with multiplicity} = 2 & \text{if}\quad \nu\geqq 3\,.\end{cases}$$

3. Ricci curvatures of symmetric R-spaces. In this section we shall study the Ricci curvature tensor of an irreducible symmetric R-space.

In general, for a symmetric space (M, g) expressed as $M=K/K_0$ by a symmetric pair (K, K_0) with a K-invariant Riemannian metric g, the Ricci curvature tensor S is given at the origin $o=K_0\in M$ by

$$S(X, Y) = -(X, Y)_{\mathfrak{k}}/2 \quad \text{for}\quad X, Y\in\mathfrak{m}=T_oM\,, \tag{3.1}$$

where $(\ ,\ )_{\mathfrak{k}}$ is the Killing form of $\mathfrak{k}=\operatorname{Lie} K$ and $\mathfrak{k}=\mathfrak{k}_0+\mathfrak{m}$ is the Cartan decomposition (cf. Takeuchi-Kobayashi [16]).

Now let $(\mathfrak{g},\tau)$ be a simple positive definite symmetric graded Lie algebra and (M, g) the irreducible symmetric R-space associated to $(\mathfrak{g},\tau)$ with the canonical Riemannian metric g. We retain the notation in §1.

If (M, g) is an Einstein manifold: $S=cg$, $c\geqq 0$, we can compute the constant c by (3.1).

For example, let M be of Hermitian type. Then there exists a a complex simple Lie algebra $\mathscr{G}$ such that $\mathfrak{g}$ is the scalar restriction to $\boldsymbol{R}$ of $\mathscr{G}$, and $\mathfrak{k}$ is a compact real form of $\mathscr{G}$ and $\mathfrak{p} = J\mathfrak{k}$, where J is the complex structure of $\mathfrak{g}$. Thus we have

$$(X, Y) = 2(X, Y)_{\mathfrak{k}} \quad \text{for} \quad X, Y \in \mathfrak{k} ,$$

and hence by (3.1)

$$S(X, Y) = -(X, Y)_{\mathfrak{k}}/2 = -(X, Y)/4 = \langle X, Y \rangle/4$$

for $X, Y \in \mathfrak{m}$. Therefore (M, g) is an Einstein manifold: $S = cg$ with

$$c = 1/4 . \tag{3.2}$$

If $M = Q_{p,q}(\boldsymbol{R})$ $(3 \leqq p < q)$, we have decompositions

(3.3) $(M, g) \sim (M_1, g_1) \times (M_2, g_2)$ (locally isometric); and
$K \sim K_1 \times K_2$ (locally isomorphic),

where (M_i, g_i) is a compact connected Einstein symmetric space: $S_i = c_i g_i$ $(i = 1, 2)$ with $0 \leqq c_1 < c_2$ and $K_i = I^0(M_i, g_i)$ $(i = 1, 2)$. That is, $M_1 = S^{p-1}$, $M_2 = S^{q-1}$, $K_1 = SO(p)$ and $K_2 = SO(q)$. The remaining irreducible symmetric R-spaces are those of type $U(r)$ $(r \geqq 2)$. In this case we have also the decompositions (3.3) with $M_1 = S^1$, $K_1 = SO(2)$ and $c_1 = 0$. These constants c_1, c_2 are also computed by (3.1).

We give here the constants c or c_1, c_2 for each non-Hermitian irreducible symmetric R-space.

(1) $\bar{M} = G_{p,q}(\boldsymbol{C})$ $(1 \leqq p \leqq q)$, $M = G_{p,q}(\boldsymbol{R})$.

(a) $p = q = 1$. $r = 1$, type $U(1)$, $\nu = 0$, $\pi_1(M) = \boldsymbol{Z}$, Einstein, $c = 0$.

(b) $p = q \geqq 2$. $r = p$, type $SO(2p)$, $\pi_1(M) = \boldsymbol{Z}_2$, Einstein, $c = (p-1)/4p$.

(c) Otherwise. $r = p$, type $SO(2p+1)$, $\pi_1(M) = \boldsymbol{Z}_2$, Einstein, $c = (p+q-2)/4(p+q)$.

(2) $\bar{M} = G_{2p,2q}(\boldsymbol{C})$ $(1 \leqq p \leqq q)$, $M = G_{p,q}(\boldsymbol{H})$. $r = p$, type Sp(p), $\pi_1(M) = 0$, Einstein, $c = (p+q+1)/4(p+q)$.

(3) $\bar{M} = G_{n,n}(\boldsymbol{C})$ $(n \geqq 2)$, $M = U(n)$. $r = n$, type $U(n)$, $\nu = 2$, $\pi_1(M) = \boldsymbol{Z}$, $c_1 = 0$, $c_2 = 1/4$.

(4) $\bar{M} = SO(2n)/U(n)$ $(n \geqq 5)$, $M = SO(n)$. $r = [n/2]$, type $SO(n)$, $\pi_1(M) = \boldsymbol{Z}_2$, Einstein, $c = (n-2)/4(n-1)$.

(5) $\bar{M} = SO(4n)/U(2n)$ $(n \geqq 3)$, $M = U(2n)/Sp(n)$. $r = n$, type $U(n)$, $\nu = 4$, $\pi_1(M) = \boldsymbol{Z}$, $c_1 = 0$, $c_2 = n/2(2n-1)$.

(6) $\bar{M} = Sp(2n)/U(2n)$ $(n \geqq 2)$, $M = Sp(n)$. $r = n$, type $Sp(n)$, $\pi_1(M) = 0$, Einstein, $c = (n+1)/2(2n+1)$.

(7) $\bar{M} = Sp(n)/U(n)$ $(n \geqq 3)$, $M = U(n)/O(n)$. $r = n$, type $U(n)$, $\nu = 1$, $\pi_1(M) = \boldsymbol{Z}$, $c_1 = 0$, $c_2 = n/4(n+1)$.

(8) $\bar{M} = Q_{p+q-2}(\boldsymbol{C})$ $(p + q \geqq 3, 1 \leqq p \leqq q)$, $M = Q_{p,q}(\boldsymbol{R})$.

(a) $p = 1$, $q \geqq 4$ $(q \neq 5)$. $r = 1$, type $Sp(1)$, $\pi_1(M) = 0$, Einstein, $c = (q-2)/2(q-1)$.

(b) $p = 2$, $q \geqq 3$ $(q \neq 4)$. $r = 2$, type $U(2)$, $\nu = q - 2$, $\pi_1(M) = \boldsymbol{Z}$, $c_1 = 0$, $c_2 = (q-2)/2q$.

(c) $p = q \geqq 4$. $r = 2$, type $SO(4)$, $\pi_1(M) = \boldsymbol{Z}_2$, Einstein, $c = (p-2)/4(p-1)$.

(d) $3 \leqq p < q$. $r = 2$, type $SO(4)$, $\pi_1(M) = \boldsymbol{Z}_2$, $c_1 = (p-2)/2(p+q-2)$, $c_2 = (q-2)/2(p+q-2)$.

(9) $\bar{M} = E_6/T\cdot Spin(10)$, $M = G_{2,2}(\boldsymbol{H})/\boldsymbol{Z}_2$. $r = 2$, type $SO(5)$, $\pi_1(M) = \boldsymbol{Z}_2$, Einstein, $c = 5/24$.

(10) $\bar{M} = E_6/T\cdot Spin(10)$, $M = P_2(\boldsymbol{K})$. $r = 1$, type $Sp(1)$, $\pi_1(M) = 0$, Einstein, $c = 3/8$.

(11) $\bar{M} = E_7/T\cdot E_6$, $M = SU(8)/Sp(4)\cdot \boldsymbol{Z}_2$. $r = 4$, type $SO(8)$, $\pi_1(M) = \boldsymbol{Z}_2$, Einstein, $c = 2/9$.

(12) $\bar{M} = E_7/T\cdot E_6$, $M = T\cdot E_6/F_4$. $r = 3$, type $U(3)$, $\nu = 8$, $\pi_1(M) = \boldsymbol{Z}$, $c_1 = 0$, $c_2 = 1/3$.

In the above list,

$G_{p,q}(\boldsymbol{F})$: Grassmann manifold of all p-subspaces in $\boldsymbol{F}^{p+q}$, for $\boldsymbol{F} = \boldsymbol{R}, \boldsymbol{C}$ or real quaternion algebra $\boldsymbol{H}$,

$P_2(\boldsymbol{K})$: Cayley projective plane,

$Q_n(\boldsymbol{C})$: Complex quadric of dimension n,

Einstein: (M, g) is an Einstein manifold.

4. Stability of TRG-pairs. In this section we shall study the stability as a minimal submanifold of M in $(\bar{M}, \bar{g})$ for a TRG-pair $((\bar{M}, \bar{g}), M)$.

In general, let $f: (M, g) \to (\bar{M}, \bar{g})$ be a minimal isometric immersion of a compact Riemannian manifold (M, g) into a Riemannian manifold $(\bar{M}, \bar{g})$. Let f_t be a smooth variation of f with $f_0 = f$ and $\mathscr{V}(t)$ the volume of $(M, f_t^*\bar{g})$. Then the second derivative of $\mathscr{V}(t)$ is described as follows (cf. Simons [11]). We define a vector field V along f by

$$V_p = \left[\frac{d}{dt} f_t(p)\right]_{t=0} \quad \text{for} \quad p \in M .$$

We define furthermore an elliptic self-adjoint differential operator L of order 2 on the space $C^\infty(NM)$ of all smooth sections of the normal bundle NM for f, called the *Jacobi operator* for f, by

$$L = \Delta^{\perp} + S^{\perp} - \tilde{\alpha} .$$

Here $\Delta^{\perp} = -\mathrm{Tr}_g(\nabla^{\perp})^2$ is the Laplacian on NM; $\tilde{\alpha} \in C^\infty(\mathrm{End}\, NM)$ is defined by $\tilde{\alpha} = \alpha \circ {}^t\alpha$ regarding the second fundamental form α of f as $\alpha \in C^\infty(\mathrm{Hom}(TM \otimes TM, NM))$; $S^{\perp} \in C^\infty(\mathrm{End}\, NM)$ is defined by

$$\langle S^{\perp}(u), v\rangle = \sum_i \langle \bar{R}(e_i, u)e_i, v\rangle \quad \text{for} \quad u, v \in N_pM , \quad p \in M ,$$

where $\bar{R}$ is the curvature tensor of $(\bar{M}, \bar{g})$ and $\{e_i\}$ is an orthonormal basis for T_pM. We have then

$$\frac{d^2\mathscr{V}}{dt^2}(0) = \int_M \langle L V^N, V^N\rangle dv ,$$

where V^N denotes the normal component of V and dv the Riemannian measure of (M, g).

The multiplicity $n(f)$ of the eigenvalue 0 of L is called the *nullity* of f. The sum $i(f)$ of multiplicities of negative eigenvalues of L is called the *index* of f. The minimal immersion f is said to be *stable* if $i(f) = 0$. We define moreover a subspace P of $C^\infty(NM)$ by

$$P = \{(X|M)^N;\ X \text{ is a Killing vector field on } (\bar{M}, \bar{g})\} ,$$

and call the dimension $n_k(f)$ of P the *Killing nullity* of f. It is known (cf. Simons [11]) that $L|P = 0$, and hence $n_k(f) \leqq n(f)$.

LEMMA 4.1. (Chen-Leung-Nagano [1]) *Let (M, g) be a compact connected symmetric space expressed as $M = K/K_0$ by an almost effective compact symmetric pair (K, K_0). Suppose that g is defined by a K-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{k} = \mathrm{Lie}\, K$ and let C denote the Casimir operator of $\mathfrak{k}$ relative to $\langle\ ,\ \rangle$. Let $f\colon (M, g) \to (\bar{M}, \bar{g})$ be a totally geodesic isometric immersion of (M, g) into a symmetric space $(\bar{M}, \bar{g})$. Then $\mathfrak{k}$ acts on the normal bundle NM and there exists a $\mathfrak{k}$-invariant symmetric endomorphism Q of NM such that the Jacobi operator L for f is given by*

$$L = C + Q . \tag{4.1}$$

We retain the notation in §1 for symmetric R-spaces. By a method in [1] we prove the following:

THEOREM 4.2. *Let (M, g) be a symmetric R-space with the canonical Riemannian metric g associated to a positive definite symmetric graded Lie algebra $(\mathfrak{g}, \tau)$, and $f\colon (M, g) \to (\bar{M}, \bar{g})$ the canonical isometric imbedding. Then*

(1) $n_k(f) = \dim \mathfrak{p}$,
(2) $Q = -(1/2)I_{NM}$.

PROOF. (1) Identifying $\mathfrak{p}$ with a space of vector fields on M, we define a linear map $\mathfrak{p} \rightarrow P$ by the correspondence $X \mapsto (JX)|M$ $(X \in \mathfrak{p})$. Then it is a K-isomorphism since $\mathfrak{p} = \mathfrak{g} \cap J\mathfrak{g}_u$, and thus the assertion follows.

(2) Let C be the Casimir operator of $\mathfrak{k}$ relative to $\langle X, Y \rangle = -(X, Y)$. By the proof of (1) and Lemma 2.1 we have $C|P = (1/2)I_P$. Thus, by $L|P = 0$ and (4.1) we get $Q|P = -(1/2)I_P$. On the other hand, since G_u is transitive on $\bar{M}$ we have

$$T_p\bar{M} = \{X_p;\ X \in \mathfrak{g}_u\} \quad \text{for any } p \in M .$$

Therefore, by $\mathfrak{g}_u = \mathfrak{k} + J\mathfrak{p}$ we have

$$N_pM = \{X_p;\ X \in P\} \quad \text{for any } p \in M .$$

This and $Q|P = -(1/2)I_P$ imply the assertion. q.e.d.

REMARK 1. Let $f: (M, g) \rightarrow (\bar{M}, \bar{g})$ be as in Theorem 4.2. We define an endomorphism $\bar{S}^{\perp}$ of NM by

$$\langle \bar{S}^{\perp}(u), v \rangle = \bar{S}(u, v) \quad \text{for} \quad u, v \in N_pM , \quad p \in M ,$$

where $\bar{S}$ denotes the Ricci curvature tensor of $(\bar{M}, \bar{g})$. It can be proved by a direct calculation that then $Q = -\bar{S}^{\perp}$, and hence the assertion (2) follows also from the formula (3.1) for our $(\bar{M}, \bar{g})$.

Recalling (Ikeda-Taniguchi [3]) that the Laplacian acting on forms on a compact symmetric space M coincides with the Casimir operator, we get the following:

COROLLARY. *Let $\hat{L}$ be the differential operator on $C^\infty(T^*M)$ corresponding to L on $C^\infty(NM)$ under the K-isomorphism:*

$$NM \underset{J\cdot}{\overset{\cong}{\rightarrow}} TM \underset{\hat{g}}{\overset{\cong}{\rightarrow}} T^*M ,$$

where T^*M is the cotangent bundle of M, $J\cdot$ is the multiplication by J and $\hat{g}$ is the duality by means of g. Then

$$\hat{L} = \Delta - (1/2)I_{T^*M} ,$$

where Δ denotes the Laplacian of (M, g) acting on the space $C^\infty(T^*M)$ of 1-forms on M.

Here we recall some results on the Laplacian Δ on 1-forms on a general compact connected Riemannian manifold (M, g). For $\lambda \geqq 0$ we put

$$F_\lambda = \{f \in C^\infty(M);\ \Delta f = \lambda f\} ,$$
$$E_\lambda = \{\xi \in C^\infty(T^*M);\ \Delta\xi = \lambda\xi\} ,$$

$$B_\lambda = \{\xi \in E_\lambda;\ d\xi = 0\}\ ,$$
$$C_\lambda = \{\xi \in E_\lambda;\ d^*\xi = 0\}\ ,$$

where d^* denotes the formal adjoint operator of d with respect to the Riemannian measure for g. If $\lambda > 0$, we have

$$E_\lambda = B_\lambda + C_\lambda \quad \text{(direct sum)}\ , \tag{4.2}$$

and d induces an isomorphism

$$d\colon F_\lambda \xrightarrow{\cong} B_\lambda\ . \tag{4.3}$$

THEOREM OF YANO. (cf. Kobayashi [4]) *If (M, g) is an Einstein manifold: $S = cg$, then C_{2c} coincides with the space of all Killing 1-forms on (M, g).*

THEOREM OF NAGANO [8]. *If (M, g) is an Einstein manifold: $S = cg$ with $c > 0$, then $C_\lambda = 0$ for each λ with $0 < \lambda < 2c$.*

THEOREM 4.3. *Let $f\colon (M, g) \to (\bar{M}, \bar{g})$ be the canonical isometric imbedding of an irreducible symmetric R-space (M, g). Then, f is stable if and only if M is simply connected.*

PROOF. By Corollary of Theorem 4.2, f is stable if and only if $E_\lambda = 0$ for each λ with $0 \leqq \lambda < 1/2$. We prove the assertion in the following four cases separately.

(i) M is of Hermitian type.

(ii) M is not of Hermitian type, $\pi_1(M)$ is finite and g is an Einstein metric: $S = cg$.

(iii) M is not of Hermitian type, $\pi_1(M)$ is finite and g is not an Einstein metric.

(iv) M is of type $U(r)$.

In case (i), $\pi_1(M) = 0$ and (M, g) is an Einstein manifold: $S = cg$ with $c = 1/4$ by (3.2). Thus $E_0 = 0$ and $\lambda_1 = 1/2$ by Theorem 2.4. Therefore $B_\lambda = 0$ for $0 < \lambda < 1/2$ by (4.3). Moreover, by Theorem of Nagano $C_\lambda = 0$ for $0 < \lambda < 1/2$. Thus by (4.2) $E_\lambda = 0$ for $0 < \lambda < 1/2$, and hence f is stable.

In case (ii), in the same way as (i) we get $E_0 = 0$ and $B_\lambda = 0$ for $0 < \lambda < 1/2$. From §3 we see that

$$\pi_1(M) = 0 \Leftrightarrow c > 1/4\ ,$$
$$\pi_1(M) \neq 0 \Leftrightarrow 0 < c < 1/4\ .$$

Thus, if $\pi_1(M) = 0$ f is stable by the same reasoning as in case (i). If $\pi_1(M) \neq 0$, we have $0 < 2c < 1/2$ and $\dim E_{2c} = \dim C_{2c} = \dim \mathfrak{k} > 0$ by

Theorem of Yano. Thus f is not stable.

In case (iii), $M = Q_{p,q}(\boldsymbol{R})(3 \leqq p < q)$, $\pi_1(M) = \boldsymbol{Z}_2$ and $0 < c_1 = (p-2)/2(p+q-2) < 1/4$. Thus $0 < 2c_1 < 1/2$ and $\dim E_{2c_1} \geqq \dim C_{2c_1} \geqq \dim SO(p) > 0$ by Theorem of Yano. Thus f is not stable.

In case (iv), $\pi_1(M) = \boldsymbol{Z}$ and so $\dim E_0 = 1$. Hence f is not stable.

q.e.d.

REMARK 2. From the proof we see:

In case (i), $n(f) = \dim_{\boldsymbol{R}} \mathrm{Aut}^0(M)$;

In case (ii), $n(f) = \dim \mathfrak{p}$ if $\pi_1(M) = 0$, and $i(f) \geqq \dim I^0(M, g)$ if $\pi_1(M) \neq 0$.

THEOREM 4.4. *Let $(\bar{M}, \bar{g})$ be a connected Hermitian symmetric space of compact type and M a compact connected totally real totally geodesic submanifold of $(\bar{M}, \bar{g})$ with $\dim M = \dim_C \bar{M}$. Then, M is a stable minimal submanifold if and only if M is simply connected.*

PROOF. It is easily seen that the stability of M in $(\bar{M}, \bar{g})$ for a TRG-pair $((\bar{M}, \bar{g}), M)$ is invariant under the equivalence of TRG-pairs and that for the direct product $((\bar{M}, \bar{g}), M) = ((\bar{M}_1, \bar{g}_1), M_1) \times ((\bar{M}_2, \bar{g}_2), M_2)$, M is stable in $(\bar{M}, \bar{g})$ if and only if each M_i is stable in $(\bar{M}_i, \bar{g}_i)$ $(i = 1, 2)$. Thus the assertion follows from Theorems 1.2 and 4.3. q.e.d.

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