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Published on: 01 Jan 1990 - Kodai Mathematical Journal (Department of Mathematics, Tokyo Institute of Technology)

Topics: Simple Lie group, Representation of a Lie group, Maximal torus, Indefinite orthogonal group and Lie group

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KODAI MATH. J.
13 (1990), 181—203

STABILITY OF CLOSED LIE SUBGROUPS IN COMPACT LIE GROUPS

Dedicated to Professor Shingo Murakami on his sixtieth birthday

BY KATSUYA MASHIMO* AND HIROYUKI TASAKI**

Introduction. In [C-L-N], they dealt systematically the stability of totally geodesic submanifolds of a compact Riemannian symmetric space as minimal submanifolds. Using the method, Takeuchi [Tak] and Ohnita [O] studied the stability of some kinds of totally geodesic submanifolds. The class of closed subgroups in compact Lie groups with bi-invariant Riemannian metrics is one of the most typical totally geodesic submanifolds. On their stability, there are some results by Fomenko [F], Thi [Th] and Brothers [Br].

In [D], the index of a (complex) simple Lie subalgebra in a (complex) simple Lie algebra was defined and it played an important role. The results mentioned above and a result by the second named author (Theorem A) made us get interested in the problem to find some relationship between the index of a Lie subgroup and the stability of it as a totally geodesic submanifold.

Let U be a compact connected simple Lie group whose rank is greater than 1 and U_1 be an analytic subgroup of U associated with the highest root of U. It is known that U_1 is isomorphic to $SU(2)$ ([W]). The second named author [Tasl] proved the following: *U_1 is homologically volume minimizing (especially it is a stable minimal submanifold) with respect to a bi-invariant Riemaenian metric on U.* By the definition, U_1 is a subgroup of index 1. On the other hand, a 3-dimensional connected simple closed subgroup of index 1 in U is conjugate to U_1. Thus we can restate the above Theorem as follows:

THEOREM A. *Let U be a compact connected simple Lie group whose rank is greater than* 1. *A connected* 3*-dimensional simple closed subgroup of index* 1 *in U is a stable minimal submanifold with respect to a bi-invariant Riemannian metric on U.*

In this paper we generalize the above Theorem. Precisely speaking, we will

* Partly supported by the Grants-in-Aid for Science Research, The Ministry of Education, Science and Culture, Japan.

** Partly supported by the Grants-in-Aid for Encouragement of Young Scientists, The Ministry of Education, Science and Culture, Japan.

Received March 28, 1989; Revised July 31, 1989.

prove the following:

THEOREM B. *Let U be a compact connected simple Lie group with a bi-invariant Riemannian metric. A connected simple closed Lie subgroup G of index 1 in U is a stable minimal submanifold.*

For the case that G is isomorphic to $SU(2)$ the converse to Theorem B is true. Namely we will prove the following:

THEOREM C. *Let G be a a simple Lie subgroup which is isomorphic to $SU(2)$ in a compact connected simple Lie group U. Then, G is stable if and only if G is of index* 1.

In general, the converse to Theorem B is not true. For the case that G is isomorphic to $SO(3)$ a necessary and sufficient condition that G is stable in U will be given in section 4 (Theorem D). Moreover we will determine all stable 3-dimensional connected simple Lie subgroups in each compact connected simple Lie group (Theorem E). And we get some examples which is stable but not of index 1.

The authors wish to thank the referee for his useful advice.

1. Stability of totally geodesic submanifolds.

In this section, we give a brief review on basic results on the stability of totally geodesic submanifolds in compact symmetric spaces after [O].

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and M be a homogeneous space of G. Let o be a point in M and K be the isotropy subgroup of G at o. Let E be a G-homogeneous complex vector bundle on M. Then the fiber E_o over o is a K-module. The space of smooth sections of E on M is denoted by $\Gamma(E)$. Let $C^\infty(G; E_0)$ be the space of smooth E_o-valued functions on G and $C^\infty(G; E_o)_K=\{f\in C^\infty(G; E_o): f(uk)=k^{-1}f(u)$, for $u\in G$ and $k\in K\}$. Then G acts on $\Gamma(E)$ and $C^\infty(G; E_o)_K$ in a natural manner. Define a mapping

$$s: C^\infty(G; E_o)_K \longrightarrow \Gamma(E); f \longrightarrow [g\cdot o \longrightarrow gf(g)].$$

Then s is a G-isomorphism. Each element of the Lie algebra $\mathfrak{g}$ of left invariant vector fields on G acts on $C^\infty(G; E_o)$ as a left invariant (linear) differential operator. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Then the action of $\mathfrak{g}$ on $C^\infty(G; E_o)$ is extended to that of $U(\mathfrak{g})$ in a natural manner. An element $L\otimes X$ of $\mathrm{Hom}(E_o, E_o)\otimes U(\mathfrak{g})$ acts, as a linear differential operator, on $C^\infty(G; E_o)$ by

$$(L\otimes X)(f)=L(Xf), \qquad f\in C^\infty(G; E_o).$$

Define an action of K on $\mathrm{Hom}(E_o, E_o)\otimes U(\mathfrak{g})$ by $k(L\otimes X)=(kLk^{-1})\otimes \mathrm{Ad}(k)X$ for $k\in K$. Then a K-invariant element D of $\mathrm{Hom}(E_o, E_o)\otimes U(\mathfrak{g})$ leaves the subspace

$C^\infty(G; E_o)_K$ invariant. Thus D induces a G-invariant linear differential operator of $\Gamma(E)$. Converesely every G-invariant linear differential operators of $\Gamma(E)$ can be obtained in the above manner.

Let P be a compact Riemannian symmetric space and U be the identity component of the group of isometries of P. We denote by R^P the curvature tensor of P. Let M be a compact totally geodesic submanifold of P. Take an analytic subgroup G of U which leaves M invariant and is locally isomorphic to the group of isometries of M. Then the normal bundle $N(M)$ of M in P is a G-homogeneous vector bundle. Let $\{M_t\}$ be a smooth variation of M in P and V be its variational vector field. We denote by V^N the normal component of V. Define a section S of $\mathrm{End}(N(M))$ by

$$\langle S(\xi), \eta\rangle = \sum_i \langle R^P(e_i, \xi)e_i, \eta\rangle \qquad \text{for} \quad \xi, \eta \in N_p(M),$$

taking an orthonormal basis $\{e_i\}$ of $T_p(M)$. We denote by $\Delta^{N(M)}$ the rough Laplacian of the normal connection on $N(M)$. Then the second variational formula is given by the following:

$$d^2 \mathrm{Vol}(M_t)/dt^2|_{t=0} = \int_M \langle \mathcal{J}(V^N), V^N\rangle d\, vol_M,$$

where $\mathcal{J}$ is defined by

$$\mathcal{J} = -\Delta^{N(M)} + S,$$

and is called the *Jacobi differential operator*. It is easily verified that $\mathcal{J}$ is a G-invariant linear differential operator of $\Gamma(N(M))$.

We denote by L the isotropy subgroup of U at $o \in M \subset P$ and put $K = G \cap L$. Let $\mathfrak{u}$, $\mathfrak{l}$ and $\mathfrak{k}$ be the Lie algebras of U, L and K respectively. Take an $\mathrm{Ad}(U)$-invariant inner product $\langle\,,\rangle$ on $\mathfrak{u}$ which induces the Riemannian metric on P. Take the orthogonal complement $\mathfrak{m}$ [resp. $\mathfrak{p}$] of $\mathfrak{k}$ [resp. $\mathfrak{l}$] in $\mathfrak{g}$ [resp. $\mathfrak{u}$]. Let $\mathfrak{m}^\perp$ [resp. $\mathfrak{k}^\perp$] be the orthogonal complement of $\mathfrak{m}$ [resp. $\mathfrak{k}$] in $\mathfrak{p}$ [resp. $\mathfrak{l}$]. Let $\mathfrak{g}^\perp$ be the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{u}$. Then $\mathfrak{g}^\perp = \mathfrak{k}^\perp \oplus \mathfrak{m}^\perp$. The action of $\mathfrak{g}$ on $\mathfrak{g}^\perp$ is extended to that of $U(\mathfrak{g})$ on $\mathfrak{g}^\perp$. Let C be the Casimir element of $U(\mathfrak{g})$ with respect to the inner product $\langle\,,\rangle|_{\mathfrak{g}\times\mathfrak{g}}$. Since $\mathrm{ad}_{\mathfrak{u}}(C)$ leaves $\mathfrak{m}$ invariant, it is considered as an element of $\mathrm{Hom}(\mathfrak{m}^\perp, \mathfrak{m}^\perp)$. Then the Jacobi differential operator $\mathcal{J}$ is identified with a linear differential operator on $C^\infty(G; N_o(M))_K$.

THEOREM 1.1.

$$\mathcal{J} = \mathrm{ad}_{\mathfrak{u}}(C) \otimes I - I \otimes C.$$

Proof. We refer to [0].

Since the Jacobi differential operator $\mathcal{J}$ is a strongly elliptic linear differential operator, it has discrete eigenvalues

$$\lambda_1 < \lambda_2 < \cdots \longrightarrow \infty$$

and all eigenspaces are of finite dimension. We put $E_\lambda = \{V \in \Gamma(N(M)) : \mathcal{J}(V) =$

$\lambda V\}$. We call the number $i(M)=\sum_{\lambda<0}\dim E_\lambda$ the *index* of M in P, and $n(M)=\dim E_0$ the *nullity* of M in P. When the index $i(M)=0$, the submanifold M is said to be *stable* in P. We call the dimension of the subspace $\{X^N : X$ is a Killing vector field on $P\}$ of E_0 the *Killing nullity* of M in P and denote it by $n_K(M)$.

We denote by $\mathcal{D}(G)$ the set of all equivalence classes of the complex irreducible representations of G. Let $V(\lambda)$ be a representation space of an element λ of $\mathcal{D}(G)$. Then $\lambda(C)$ is a scalar operator $a_\lambda I$ on $V(\lambda)$. Let θ be the involutive automorphism of $\mathfrak{u}$ defining the symmetric structure of $P=U/L$. We can take a direct sum decomposition $\mathfrak{g}^\perp=\mathfrak{g}_1{}^\perp\oplus\cdots\oplus\mathfrak{g}_k{}^\perp$, where each $\mathfrak{g}_i{}^\perp$ is θ-stable G-invariant and has no nontrivial θ-stable, G-invariant subspace. By Schur's lemma and $\theta(C)=C$, we have $\mathrm{ad}_\mathfrak{u}(C)=a_i I$ on each $\mathfrak{g}_i{}^\perp$ for some scalar a_i. Put $\mathfrak{k}_i{}^\perp=\mathfrak{k}^\perp\cap\mathfrak{g}_i{}^\perp$ and $\mathfrak{m}_i{}^\perp=\mathfrak{m}^\perp\cap\mathfrak{g}_i{}^\perp$. Then we have $\mathfrak{g}_i{}^\perp=\mathfrak{k}_i{}^\perp\oplus\mathfrak{m}_i{}^\perp$ and each $\mathfrak{m}_i{}^\perp$ is K-invariant.

THEOREM 1.2. *The index, nullity and Killing nullity are given as follows*:

(i) $i(M)=\sum_{i=1}^{k}\sum_{\substack{a_\lambda>a_i\\ \lambda\in\mathcal{D}(G)}}\dim \mathrm{Hom}_K(V(\lambda),(\mathfrak{m}_i{}^\perp)^C)\dim V(\lambda)$

(ii) $n(M)=\sum_{i=1}^{k}\sum_{\substack{a_\lambda=a_i\\ \lambda\in\mathcal{D}(G)}}\dim \mathrm{Hom}_K(V(\lambda),(\mathfrak{m}_i{}^\perp)^C)\dim V(\lambda)$

(iii) $n_K(M)=\sum_{i=1,\,\mathfrak{m}_i{}^\perp\neq 0}^{k}\dim\mathfrak{g}_i{}^\perp$.

Proof. We refer to [0].

2. Lie subgroups.

In this section we consider the case that P is a compact connected semisimple Lie group U with a bi-invariant Riemannian metric $\langle\,,\rangle$ and M is a connected closed semisimple subgroup G. We denote by $\mathfrak{u}$ and $\mathfrak{g}$ the Lie algebras of U and G respectively. The bi-invariant Riemannian metric $\langle\,,\rangle$ on U induces an $\mathrm{Ad}(U)$-invariant inner product on $\mathfrak{u}$, which we also denote by $\langle\,,\rangle$. Let $\mathfrak{g}^\perp$ be the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{u}$. Take the identity element as the point o. We use the following notation:

$$U^*=U\times U\,,$$
$$G^*=G\times G\,,$$
$$L=\{(u,\ u)\in U^* : u\in U\}\,,$$
$$K=\{(g,\ g)\in G^* : g\in G\}\,,$$
$$\mathfrak{p}=\{(X,\ -X) : X\in\mathfrak{u}\}\,,$$
$$\mathfrak{m}=\{(X,\ -X) : X\in\mathfrak{g}\}\,,$$
$$(\mathfrak{g}^*)^\perp=\{(X,\ Y) : X,\ Y\in\mathfrak{g}^\perp\}\,,$$

$$\mathfrak{k}^{\perp}=\{(X, X): X\in\mathfrak{g}^{\perp}\},$$
$$\mathfrak{m}^{\perp}=\{(X, -X): X\in\mathfrak{g}^{\perp}\}.$$

We take the direct sum of $\langle\,,\rangle$ as an inner product on the Lie algebra $\mathfrak{u}\oplus\mathfrak{u}$ of U^*. Take a G-irreducible decomposition of $\mathfrak{g}^{\perp}$:

$$\mathfrak{g}^{\perp}=\mathfrak{g}_1{}^{\perp}\oplus\cdots\oplus\mathfrak{g}_k{}^{\perp}.$$

Then it induces a G^*-invariant decomposition:

$$(\mathfrak{g}^*)^{\perp}=\bigoplus_{i=1}^{k}(\mathfrak{g}_i^*)^{\perp}, \qquad (\mathfrak{g}_i^*)^{\perp}=\{(X, Y): X, Y\in\mathfrak{g}_i{}^{\perp}\}$$

Each $(\mathfrak{g}_i^*)^{\perp}$ is θ-stable and G^*-invariant, where θ is the involutive automorphism $\theta:\mathfrak{u}\oplus\mathfrak{u}\to\mathfrak{u}\oplus\mathfrak{u}\,;(X, Y)\to(Y, X)$. We have a decomposition $(\mathfrak{g}_i^*)^{\perp}=\mathfrak{k}_i{}^{\perp}\oplus\mathfrak{m}_i{}^{\perp}$, where

$$\mathfrak{k}_i{}^{\perp}=\{(X, X): X\in\mathfrak{g}_i{}^{\perp}\}.$$
$$\mathfrak{m}_i{}^{\perp}=\{(X, -X): X\in\mathfrak{g}_i{}^{\perp}\}.$$

If $\dim\mathfrak{g}_i{}^{\perp}=1$, then both of $\mathfrak{k}_i{}^{\perp}$ and $\mathfrak{m}_i{}^{\perp}$ are θ-stable and G^*-irreducible. It is well-known that $\mathcal{D}(G^*)=\{(V(\lambda), \lambda)\boxtimes(V(\mu), \mu): \lambda, \mu\in\mathcal{D}(G)\}$, where $\boxtimes$ means the outer tensor product. Let C^* be the Casimir element of $U(\mathfrak{g}^*)$ and let C be the Casimir element of $U(\mathfrak{g})$. Let a_λ be the eigenvalue of $\lambda(C)$ on $V(\lambda)$ for each $\lambda\in\mathcal{D}(G)$. Since $(\lambda\boxtimes\mu)(C^*)=\lambda(C)\otimes I+I\otimes\mu(C)$, we have $(\lambda\boxtimes\mu)(C^*)=(a_\lambda+a_\mu)I$. We simply denote by a_i the eigenvalue of $\mathrm{ad}_{\mathfrak{u}}(C)$ on $\mathfrak{g}_i{}^{\perp}$, then $\mathrm{ad}_{\mathfrak{u}^*}(C^*)=a_iI$ on each $(\mathfrak{g}_i^*)^{\perp}$. Since $\mathfrak{m}_i{}^{\perp}$ is a K-irreducible module, we must decompose each G^*-irreducible module into a direct sum of K-irreducible modules. Since K is the diagonal subgroup of G^*, the problem is to decompose the (inner) tensor product $V(\lambda)\otimes V(\mu)$ into a direct sum of G-irreducible modules. We can reagard each K-module $\mathfrak{m}_i{}^{\perp}$ as a G-module $\mathfrak{g}_i{}^{\perp}$. Applying the Theorem 1.2 to our case, we have the following:

THEOREM 2.1. *The index, nullity and Killing nullity are given as follows*:

(i) $$i(G)=\sum_{i=1}^{k}\sum_{\substack{a_\lambda+a_\mu>a_i\\ \lambda,\mu\in\mathcal{D}(G)}}\dim\mathrm{Hom}_G(V(\lambda)\otimes V(\mu), (\mathfrak{g}_i{}^{\perp})^C)\dim(V(\lambda)\otimes V(\mu))$$

(ii) $$n(G)=\sum_{i=1}^{k}\sum_{\substack{a_\lambda+a_\mu=a_i\\ \lambda,\mu\in\mathcal{D}(G)}}\dim\mathrm{Hom}_G(V(\lambda)\otimes V(\mu), (\mathfrak{g}_i{}^{\perp})^C)\dim(V(\lambda)\otimes V(\mu))$$

(iii) $$n_K(G)=\#\{i:\dim\mathfrak{g}_i{}^{\perp}=1\}+2\sum_{\dim(\mathfrak{g}_i{}^{\perp})\neq1}\dim(\mathfrak{g}_i{}^{\perp})$$

In order to count $\dim\mathrm{Hom}_G(V(\lambda)\otimes V(\mu), (\mathfrak{g}_i{}^{\perp})^C)$ we must remember that there are two possibilities for $(\mathfrak{g}_i{}^{\perp})^C$:

(i) $(\mathfrak{g}_i{}^{\perp})^C$ is G-irreducible,

(ii) $(\mathfrak{g}_i{}^{\perp})^C$ is decomposed into a direct sum of G-irreducible modules V and $\overline{V}$, the conjugate module of V.

Let T be a maximal torus of G and $\mathfrak{t}$ be its Lie algebra. Let (V, ρ) be a complex representation of G. For each element λ in $\mathfrak{t}$, put

$$V_{\lambda}=\{X\in V : \rho(H)(X)=\sqrt{-1}\langle\lambda, H\rangle X, \text{ for any } H\in\mathfrak{t}\}.$$

If $V_{\lambda}\neq\{0\}$, then λ is called a weight and V_{λ} is called a weight space. Especially, if $(V, \rho)=(\mathfrak{g}^{C}, \mathrm{ad})$, then a weight is called a root of G and a weight space is called a root space. We denote by $\Sigma(G)$ the set of all non-zero roots of G. Fix a lexicographic ordering on $\mathfrak{t}$.

THEOREM 2.2 (Freudenthal). *Let (V, ρ) be a complex irreducible representation of G with highest weight λ. Then the eigenvalue a_{λ} of the Casimir operator $\rho(C)$ with respect to $\langle , \rangle|_{\mathfrak{g}\times\mathfrak{g}}$ is given by the following*

$$a_{\lambda}=-\langle\lambda+2\delta, \lambda\rangle \tag{2.1}$$

where δ is half the sum of positive roots of G.

If we assume that U is a compact simple Lie group, then an $Ad(U)$-invariant inner product on $\mathfrak{u}$ is unique up to a constant. As a normalizing condition, we assume that the square of the length of the longest root is equal to 2. We call such an inner product the *canonical inner product*. We assume that both of U and G are simple. Let $\langle , \rangle_{\mathfrak{u}}$ and $\langle , \rangle_{\mathfrak{g}}$ be the canonical inner products on $\mathfrak{u}$ and $\mathfrak{g}$ respectively. Since $\langle , \rangle_{\mathfrak{u}}$ is also an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$, there exists a real number j such that

$$\langle X, Y\rangle_{\mathfrak{u}}=j\langle X, Y\rangle_{\mathfrak{g}} \quad \text{for any } X, Y\in\mathfrak{g},$$

which we call the *index* of $\mathfrak{g}$ in $\mathfrak{u}$ or the *index* of G in U. In fact it is known that the index is a positive integer ([A-H-S], [D], [Y]).

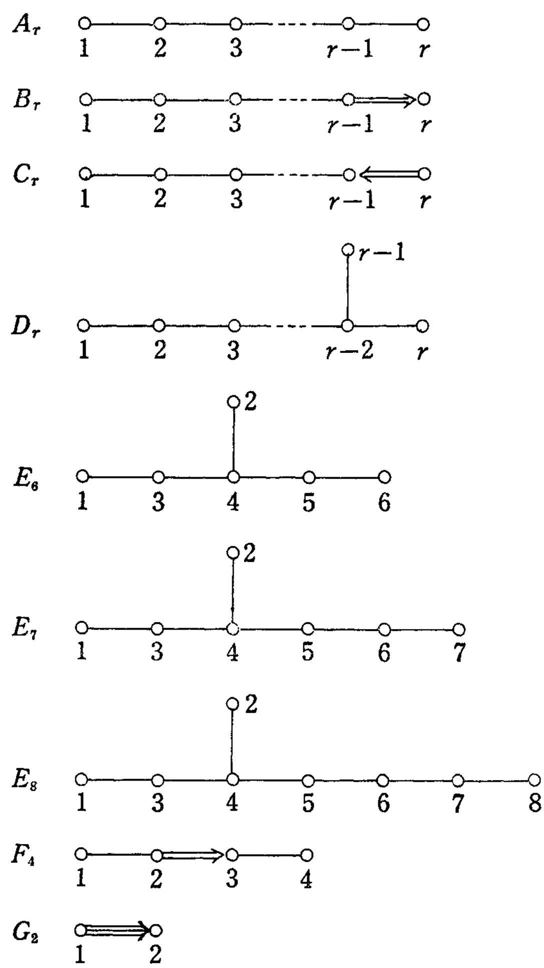
THEOREM 2.3 (Dynkin, [D, p. 133]). *Let U be a compact connected simple Lie group and G be a closed connected simple Lie subgroup. If the index of G in U is equal to* 1, *then roots of maximal length, and the corresponding root vectors in $\mathfrak{g}^{C}$ are roots and root vectors in $\mathfrak{u}^{C}$ respectively with respect to a maximal torus of U containing T.*

A complex subalgebra $\bar{\mathfrak{g}}$ of a complex semisimple Lie algebra $\bar{\mathfrak{u}}$ is said to be a *regular subalgebra,* if there exists a basis of $\bar{\mathfrak{g}}$ consisting of elements of some Cartan subalgebra $\bar{\mathfrak{h}}$ of $\bar{\mathfrak{u}}$ and root vectors of the Lie algebra $\bar{\mathfrak{u}}$ with respect to $\bar{\mathfrak{h}}$. In our compact case, if $\mathfrak{g}^{C}$ is a regular subalgebra of $\mathfrak{u}^{C}$, $\mathfrak{g}$ is said to be a *regular subalgebra* of $\mathfrak{u}$. Theorem 2.3 asserts that if every roots of G is of the same length and G is of index 1, then $\mathfrak{g}$ is a regular subalgebra of $\mathfrak{u}$.

We denote by $\{\alpha_{1}, \cdots, \alpha_{r}\}$ a fundamental root system of $\Sigma(G)$ and by α_{0} the highest root of $\Sigma(G)$. Let $\alpha_{0}=\sum_{j=1}^{r} m_{j}\alpha_{j}$. Throughout this paper the fundamental roots are numbered as in the Table 1 at the end of this section. The fundamental weights $\bar{\omega}_j$ are numbered correspondingly.

Let T' be a maximal torus of U which contains T and $\Sigma(U)$ be the set of non-zero roots of U with respect to T'. We denote by $\{\beta_1, \cdots, \beta_{r'}\}$ a fundamental root system of $\Sigma(U)$ and by β_0 the highest root of $\Sigma(U)$.

Table 1. Numbering of the simple roots



3. Stability of Lie subgroups of index 1.

Let U be a compact connected simple Lie group with a bi-invariant Riemannian metric and G be a simple connected closed Lie subgroup. In this section we assume that G is a subgroup of index 1 and study the stability of G in U

as a totally geodesic submanifold. The purpose of this section is to prove the following:

THEOREM B. *Let U be a compact connected simple Lie group with a bi-invariant Riemannian metric. A simple connected closed Lie subgroup G of index 1 in U is a stable minimal submanifold.*

We will employ the same notation as in section 2. By our assumption, there are no need to distinguish the canonical inner product on $\mathfrak{g}$ and $\mathfrak{u}$. Thus we denote them by $\langle\, ,\rangle$, which will be used to define a bi-invariant Riemannian metric.

3.1 First we determine the structure of the normal space of $\mathfrak{g}$ in $\mathfrak{u}$. Since the index of G in U is 1, we may assume that $\alpha_0=\beta_0$, by Theorem 2.3. Let λ be the highest weight of an irreducible component V of the G-module $(\mathfrak{u}^C, \mathrm{ad})$. Let $\pi: \mathfrak{t}'\to\mathfrak{t}$ be the orthogonal projection. Take $\beta\in\Sigma(U)$ such that $\pi(\beta)=\lambda$. Then by Schwarz' inequality, we have $\langle\pi(\beta), \beta_0\rangle=\langle\beta, \beta_0\rangle\leqq 2$, where the equality holds if and only if $\beta=\beta_0$. If $\beta=\beta_0$, then the component V must coincide with $\mathfrak{g}^C$. Thus if V is an irreducible componenet of the G-module $((\mathfrak{g}^\perp)^C, \mathrm{ad})$, then we have $\langle\pi(\beta), \beta_0\rangle<2$. Since λ is a dominant integral weight, we have $\langle\lambda, \alpha_0\rangle= 2\langle\lambda, \alpha_0\rangle/\langle\alpha_0, \alpha_0\rangle=0, 1$. We put $\lambda=\sum_{j=1}^{r} n_j\bar{\omega}_j$. Then we have $\langle\lambda, \alpha_0\rangle=\sum_{j=1}^{r} n_j m_j\langle\alpha_j, \alpha_j\rangle/2 =0, 1$. Since $m_j\langle\alpha_j, \alpha_j\rangle/2=m_j\langle\alpha_j, \bar{\omega}_j\rangle=2\langle\alpha_0, \bar{\omega}_j\rangle/\langle\alpha_0, \alpha_0\rangle$ is a positive integer, $\langle\lambda, \alpha_0\rangle$ is equal to

(1) 0, if and only if all of the n_j's are 0,
(2) 1, if and only if there exists k such that

$$n_k=m_k\langle\alpha_k, \alpha_k\rangle/2=1,$$
$$n_j=0 \quad \text{if} \quad j\neq k.$$

For each simple Lie algebra, we can calculate the number $m_k\langle\alpha_k, \alpha_k\rangle/2$ (cf. [B]), and we can pick up all possible k's with the above property.

PROPOSITION 3.1. *Let λ be the highest weight of an irreducible component of the G-module $((\mathfrak{g}^\perp)^C, \mathrm{ad})$. Then λ is one in the following table 2.*

Now we inspect the possibility of λ more carefully.

Case 1. $\mathfrak{g}=\mathfrak{su}(r+1)$, $r\geqq 1$. Let V be an irreducible component of the G-module $((\mathfrak{g}^\perp)^C, ad)$ and λ be the highest weight of V. The highest weight vector Y is expressed as follows

$$Y=\sum_{\substack{\beta\in\Sigma(U)\\ \pi(\beta)=\lambda}} c_\beta X_\beta,$$

where X_β is a root vector of $\mathfrak{u}^C$ corresponding to β. Since λ is the highest

Table 2.

Type of $\mathfrak{g}^C$	λ
A_r $(r\geqq 1)$	$0, \bar{\omega}_1, \cdots, \bar{\omega}_r$
B_r $(r\geqq 2)$	$0, \bar{\omega}_1, \bar{\omega}_r$
C_r $(r\geqq 3)$	$0, \bar{\omega}_1, \cdots, \bar{\omega}_r$
D_r $(r\geqq 4)$	$0, \bar{\omega}_1, \bar{\omega}_{r-1}, \bar{\omega}_r$
E_6	$0, \bar{\omega}_1, \bar{\omega}_6$
E_7	$0, \bar{\omega}_7$
E_8	0
F_4	$0, \bar{\omega}_4$
G_2	$0, \bar{\omega}_1$

weight, we have

$$0=[X_{\alpha_i}, Y]=\sum_{\substack{\beta\in\Sigma(U)\\ \pi(\beta)=\lambda}} c_\beta[X_{\alpha_i}, X_\beta], \tag{3.1}$$

for each i. Take and fix β with $c_\beta\neq 0$. By Theorem 2.3, $\alpha_i\in\Sigma(U)$ and $[X_{\alpha_i}, X_\beta]\in\mathfrak{u}^C_{\alpha_i+\beta}$. Thus by (3.1), $[X_{\alpha_i}, X_\beta]=0$, $\alpha_i+\beta\notin\Sigma(U)$. Put

$$\Gamma=\{\alpha_1, \cdots, \alpha_r, -\beta\}.$$

Then Γ satisfies the following property:

$$\gamma-\delta\notin\Sigma(U) \text{ holds for any } \gamma, \delta\in\Gamma. \tag{C_1}$$

If a subset of $\Sigma(U)$ with the property (C_1) is linearly independent, then it corresponds uniquely to a Dynkin diagram [He, p. 470]. However even if a subset of $\Sigma(U)$ is linearly dependent, we associate with it a diagram in an analogous fashion to the construction of the Dynkin diagram. The subsets of $\Sigma(U)$ with the property (C_1) are classified in [He, p. 503]. In our case, the set Γ has two restrictive conditions:

(i) $\alpha_1, \cdots, \alpha_r$ forms a fundamental root system of $\mathfrak{su}(r+1)$,

(ii) $-\beta$ is joined to only one vertex in $\{\alpha_1, \cdots, \alpha_r\}$, if $\lambda\neq 0$ (by Proposition 3.1).

From the classification given in [He], we pick up diagrams which is possible for our Γ. And we get the following:

PROPOSITION 3.2. *If $\mathfrak{g}=\mathfrak{su}(r+1)$, $r\geqq 1$, then the highest weight λ of an irreducible component of the G-module $((\mathfrak{g}^\perp)^C, ad)$ is one of the following*:

(1) $0, \bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_{r-1}, \bar{\omega}_r$, *if* $r\geqq 9$,

(2) $0, \bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \bar{\omega}_6, \bar{\omega}_7, \bar{\omega}_8$, *if* $r=8$,

(3) $0, \bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_r$, *if* $1\leqq r\leqq 7$.

Case 2. $\mathfrak{g}=\mathfrak{so}(2r)$, $r\geqq 4$. Since each root is long, we can discuss similarly to the case 1. We get the following:

PROPOSITION 3.3. *If* $\mathfrak{g}=\mathfrak{so}(2r)$, $r\geqq 4$, *then the highest weight* λ *of an irreducible component of the G-module* $((\mathfrak{g}^{\perp})^{C}, ad)$ *is one of the following*:

(1) $0, \bar{\omega}_1$, *if* $r\geqq 9$,
(2) $0, \bar{\omega}_1, \bar{\omega}_{r-1}, \bar{\omega}_r$, *if* $4\leqq r\leqq 8$.

Case 3. $\mathfrak{g}=\mathfrak{so}(2r+1)$, $r\geqq 2$. A Lie algebra $\mathfrak{l}$ which is isomorphic to $\mathfrak{so}(2r)$ is canonically embedded in $\mathfrak{g}$. If $r\geqq 3$, $\mathfrak{l}$ is also a simple subalgebra of $\mathfrak{u}$ of index 1. We denote by $V(\bar{\omega})$ the complex irreducible $\mathfrak{g}$-module with highest weight $\bar{\omega}$ and by $W(\rho)$ the complex irreducible $\mathfrak{l}$-module with highest weight ρ. Let $\rho_1, \cdots, \rho_r$ penote the fundamental weights of $\mathfrak{l}$. It is easily verified that

$$V(\bar{\omega}_r)=W(\rho_{r-1})\oplus W(\rho_r).$$

If $((\mathfrak{g}^{\perp})^{C}, \mathrm{ad})$ contains a $\mathfrak{g}$-irreducible component $V(\bar{\omega}_r)$, then $((\mathfrak{l}^{\perp})^{C}, \mathrm{ad})$ contains an $\mathfrak{l}$-irreducible component $W(\rho_r)$. Thus by Propositions 3.2 and 3.3, r must be smaller than or equal to 8 and we get the following:

PROPOSITION 3.4. *If* $\mathfrak{g}=\mathfrak{so}(2r+1)$, $r\geqq 2$, *then the highest weight* λ *of an irreducible component of the G-module* $((\mathfrak{g}^{\perp})^{C}, ad)$ *is one of the following*:

(1) $0, \bar{\omega}_1$, *if* $r\geqq 9$,
(2) $0, \bar{\omega}_1, \bar{\omega}_r$, *if* $2\leqq r\leqq 8$.

Case 4. $\mathfrak{g}=\mathfrak{sp}(r)$, $r\geqq 3$. In this case we have the following:

PROPOSITION 3.5. *If* $\mathfrak{g}=\mathfrak{sp}(r), r\geqq 3$, *then the highest weight* λ *of an irreducible component of the G-module* $((\mathfrak{g}^{\perp})^{C}, ad)$ *is one of the following*:

(1) $0, \bar{\omega}_1, \bar{\omega}_2$, *if* $r\geqq 5$,
(2) $0, \bar{\omega}_1, \cdots, \bar{\omega}_r$, *if* $r=3, 4$.

Proof. If $\mathfrak{u}$ is of exceptional type, then $\mathfrak{sp}(r)$, $r\geqq 5$, cannot be realized as a subalgebra of index 1 ([D]). So we assume that $\mathfrak{u}$ is of classical type. If $\mathfrak{g}$ is a regular subalgebra of $\mathfrak{u}$, then we can argue similarly to the case 1. And we have one possibility that $\lambda=\bar{\omega}_1$. Tasaki classified complex simple Lie subalgebra of index 1 in classical complex simple Lie algebras (see Remark 3.9(1)). By his classification, if $\mathfrak{g}$ is not a regular subalgebra, then there exists a Lie subalgebra $\mathfrak{l}$ of $\mathfrak{u}$ which satisfies

(i) $\mathfrak{l}$ is isomorphic to $\mathfrak{su}(2r)$,
(ii) $\mathfrak{l}$ is a regular subalgebra of $\mathfrak{u}$ of index 1,
(iii) $\mathfrak{g}$ is a canonically embedded Lie subalgebra of $\mathfrak{l}$.

The orthogonal complement $\mathfrak{g}^{\perp}$ is decomposed as $\mathfrak{g}^{\perp}=\mathfrak{g}_0^{\perp}\oplus\mathfrak{l}^{\perp}$, where $\mathfrak{g}_0^{\perp}$ is the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{l}$ and $\mathfrak{l}^{\perp}$ is the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{u}$.

We can easily see that $(\mathfrak{g}_0{}^{\perp})^C$ is $V(\bar{\omega}_2)$. Let $\rho_1, \cdots, \rho_{2r}$ denote the fundamental weights of $\mathfrak{l}$. We denote by $W(\rho)$ the complex irreducible $\mathfrak{l}$-module with highest weight ρ. Now we decompose $(\mathfrak{l}^{\perp})^C$ as a $\mathfrak{g}$-module. Take an irreducible component $W(\rho)$ of $(\mathfrak{l}^{\perp})^C$ as an $\mathfrak{l}$-module. We know the possibility of ρ by Proposition 3.2. For each possible ρ, we decompose $W(\rho)$ as a $\mathfrak{g}$-module. We have only to consider the case $\rho=\rho_1, \rho_2, \rho_{2r-2}, \rho_{2r-1}$. For the other cases r must be less than or equal to 4. We can easily see

$$W(\rho_1)=V(\bar{\omega}_1),$$
$$W(\rho_{2r-1})=V(\bar{\omega}_1),$$
$$W(\rho_2)=V(\bar{\omega}_2)\oplus V(0),$$
$$W(\rho_{2r-2})=V(\bar{\omega}_2)\oplus V(0).$$

Thus we have the Proposition. Q. E. D.

3.2 *Proof of Theorem B.*

Case 1. $\mathfrak{g}=\mathfrak{su}(r+1),\ r\geqq 1$.

By Theorem 2.2, we can calculate the eigenvalues of the Casimir operator with respect to the canonical inner product in $\mathfrak{g}$,

$$a_{\bar{\omega}_i}=-(r+2)i(r+i-1)/(r+1).$$

Remember that $a_{\bar{\omega}_1}=a_{\bar{\omega}_r}>a_{\bar{\omega}_2}=a_{\bar{\omega}_{r-1}}>\cdots$. By examining the eigenvalues, we determine the set of pairs $(\bar{\omega}, \bar{\omega}')$ such that

$$a_{\bar{\omega}}+a_{\bar{\omega}'}>a_{\bar{\omega}_j} \tag{3.2}$$

for each $\bar{\omega}_j$ given in Proposition 3.2. If $j=3$ $(r\leqq 8)$, $r-2$ $(r\leqq 8)$ or 4 $(r=7)$, then the set of pairs $(\bar{\omega}, \bar{\omega}')$ are

$$(\bar{\omega}_1, \bar{\omega}_1), (\bar{\omega}_1, \bar{\omega}_r), (\bar{\omega}_r, \bar{\omega}_1), (\bar{\omega}_r, \bar{\omega}_r),$$

otherwise such a pair does not exist. On the other hand we have

$$V(\bar{\omega}_1)\otimes V(\bar{\omega}_1)=V(2\bar{\omega}_1)\oplus V(\bar{\omega}_2),$$
$$V(\bar{\omega}_1)\otimes V(\bar{\omega}_r)=V(\bar{\omega}_1+\bar{\omega}_r)\oplus V(0),$$
$$V(\bar{\omega}_r)\otimes V(\bar{\omega}_r)=V(2\bar{\omega}_r)\oplus V(\bar{\omega}_{r-1}).$$

Thus by Theorem 2.1, G is stable as a totally geodesic submanifold in U.

For the other cases we can argue in a similar fashion. So we list

(i) the eigenvalues of the Casimir operator,

(ii) the set of pairs with (3.2),

(iii) the decomposition of the tensor product $V(\bar{\omega})\otimes V(\bar{\omega}')$, for the pair $(\bar{\omega}, \bar{\omega}')$ given in (ii).

Case 2. $\mathfrak{g}=\mathfrak{so}(2r+1)$, $r\geqq 2$.

(i) $a_{\bar{\omega}_i}=-i(2r+1-i)$, $1\leqq i\leqq r-1$,
$a_{\bar{\omega}_r}=-r(2r+1)/4$,

(ii) if $j=r=8$, then the set of pairs $(\bar{\omega}, \bar{\omega}')$ are $(\bar{\omega}_1, \bar{\omega}_1)$, otherwise such a pair does not exist.

(iii) $V(\bar{\omega}_1)\otimes V(\bar{\omega}_1)=V(2\bar{\omega}_1)\oplus V(\bar{\omega}_2)\oplus V(0)$.

Thus by Theorem 2.1, G is stable as a totally geodesic submanifold in U.

Case 3. $\mathfrak{g}=\mathfrak{sp}(r)$, $r\geqq 3$.

(i) $a_{\bar{\omega}_i}=-i(2r+2-i)$, $1\leqq i\leqq r$,

(ii) if $(j, r)=(3,3)$, $(3,4)$ or $(4,4)$, then the set of pairs $(\bar{\omega}, \bar{\omega}')$ are $(\bar{\omega}_1, \bar{\omega}_1)$, otherwise such a pair does not exist.

(iii) $V(\bar{\omega}_1)\otimes V(\bar{\omega}_1)=V(2\bar{\omega}_1)\oplus V(\bar{\omega}_2)\oplus V(0)$.

Thus by Theorem 2.1, G is stable as a totally geodesic submanifold in U

Case 4. $\mathfrak{g}=\mathfrak{so}(2r)$, $r\geqq 4$.

(i) $a_{\bar{\omega}_i}=-ir(2-i)$, $1\leqq i\leqq r-2$,
$a_{\bar{\omega}_{r-1}}=a_{\bar{\omega}_r}=-r(2r+1)/4$,

(ii) such a pair does not exist.

Thus by Theorem 2.1, G is stable as a totally geodesic submanifold in U.

Case 5. $\mathfrak{g}$ *is of exceptional type.*

If $\mathfrak{g}$ is of exceptional type, then the eigenvalue of the Casimir operator for λ given in Proposition 3.1 is the largest except zero. Thus by Theorem 2.1, G is stable as a totally geodesic submanifold in U.

3.3 *Examples and remarks.*

Now we give some examples of simple connected closed Lie subgroups G of index 1 in compact connected simple Lie groups and give the decomposition of $(\mathfrak{g}^{\perp})^C$.

Example 3.6. (1) Let U be the special unitary group $SU(r+s+1)$ and $G=\{\text{Diagonal}(A, I_s): A\in SU(r+1)\}$. Then the index of G in U is equal to 1. If $r\geqq 2$, the G-irreducible decomposition of $(\mathfrak{g}^{\perp})^C$ is as follows:

$$(\mathfrak{g}^{\perp})^C=\underbrace{(V(\bar{\omega}_1)\oplus V(\bar{\omega}_r))\oplus\cdots\oplus(V(\bar{\omega}_1)\oplus V(\bar{\omega}_r))}_{s}\oplus\underbrace{V(0)\oplus\cdots\oplus V(0)}_{s^2}.$$

(2) Let U be the special orthogonal group $SO(2r+2)$ and embed $G=SU(r+1)$ as a subgroup in a standard way. Then the index of G in U is equal to 1. If

$r\geqq 4$, the G-irreducible decomposition of $(\mathfrak{g}^{\perp})^{C}$ is as follows:

$$(\mathfrak{g}^{\perp})^{C}=V(\bar{\omega}_2)\oplus V(\bar{\omega}_{r-1})\oplus V(0).$$

(3) Let U be the compact connected simple exceptional Lie group E_8. Then U has $G=SU(q)/\boldsymbol{Z}_3$ as a subgroup of index 1. The G-irreducible decomposition of $(\mathfrak{g}^{\perp})^{C}$ is as follows (see [M-P, p. 305]):

$$(\mathfrak{g}^{\perp})^{C}=V(\bar{\omega}_3)\oplus V(\bar{\omega}_6).$$

Remark 3.7. For each dominant integral weight λ appeared in Proposition 3.2, there exist a compact connected simple Lie group U and its closed connected subgroup G with the following:

(i) the index of G in U is equal to 1.
(ii) G is locally isomorphic to $SU(r+1)$,
(iii) $V(\lambda)$ is a G-irreducible component of $(\mathfrak{g}^{\perp})^{C}$.

We give further examples of pairs of compact connected simple Lie groups U and their closed connected subgroups G of index 1. We omit the G-irreducible decompositions of $(\mathfrak{g}^{\perp})^{C}$.

Example 3.8. (1) $SO(N)\supset SO(n)$.
(2) $Sp(N)\supset Sp(r)$.
(3) $SU(N)\supset(SU(2r)\supset)Sp(r)$.
(4) $SO(8)\supset Spin(7)$.
(5) $SO(7)\supset G_2$.
(6) $F_4\supset Sp(3)$.

Remark 3.9. (1) Complex simple Lie subalgebras of index 1 in classical complex simple Lie algebras were classified in [Tas2]. By the classification, such a subalgebra corresponds to one of the subgroups given in Example 3.6 (1), (2) and Example 3.8 (1)-(5).

(2) Let λ be a dominant integral weight appeared in Proposition 3.3, 3.4 or 3.5 except the cases that $\lambda=\bar{\omega}_8$ for $\mathfrak{g}=\mathfrak{so}(17)$ and $\lambda=\bar{\omega}_3$ for $\mathfrak{g}=\mathfrak{sp}(4)$. There exist a compact connected simple Lie group U and its closed connected subgroup G with the following:

(i) the index of G in U is equal to 1,
(ii) the Lie algebra of G is isomorphic to $\mathfrak{g}$,
(iii) $V(\lambda)$ is a G-irreducible component of $(\mathfrak{g}^{\perp})^{C}$.

It is easily seen that the assumption on G in Theorem B is weakened as follows:

THEOREM B′. *Let U be a compact connected simple Lie group with a bi-invariant Riemannian metric. A connected semisimple closed Lie subgroup G all of whose simple factors are of index 1 is a stable minimal submanifold.*

By Theorem B', we conclude that the subgroup $G=\{\text{Diagonal}(A, B): A\in SO(p), B\in SO(q)\}$ of $SO(p+q)$, $p, q\geqq 4$, is a stable minimal submanifold.

4. Stability of 3-dimensional simple subgroups.

We shall give a necessary and sufficient condition that a connected 3-dimensional simple Lie subgroup in a compact connected simple Lie group is stable.

A compact connected 3-dimensional simple Lie group is isomorphic to one of $SU(2)$ and $SO(3)$ and its Lie algebra is always isomorphic to $\mathfrak{so}(3)$. We state our results separately for $SU(2)$ and $SO(3)$.

THEOREM C. *Let G be a simple Lie subgroup which is isomorphic to $SU(2)$ in a compact connected simple Lie group U with a bi-invariant Riemannian metric. Then, G is stable if and only if G is of index* 1. *If G is stable, then $n(G)=n_K(G)$.*

In order to state Theorem for G which is isomorphic to $SO(3)$, we fix some notation. We choose a basis $\{H, E, F\}$ for a 3-dimensional compact simple Lie algebra $\mathfrak{g}$ with

$$[H, E]=2F,\quad [H, F]=-2E,\quad [E, F]=H.$$

With respect to the canonical inner product $\langle\,,\rangle_0$ on $\mathfrak{g}$, $\{H/\sqrt{2}, E, F\}$ is an orthonormal basis of $\mathfrak{g}$.

Let G be a 3-dimensional connected simple Lie subgroup in a compact connected simple Lie group U of rank r. Then the Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathfrak{so}(3)$, hence we can take a basis $\{H, E, F\}$ for the Lie algebra $\mathfrak{g}$ of G as above. Let $\langle\,,\rangle$ denote the canonical inner product on the Lie algebra $\mathfrak{u}$ of U. Let $\mathfrak{t}$ be a maximal Abelian subalgebra in $\mathfrak{u}$ such that $H\in\mathfrak{t}$. With respect to a suitable ordering, we may assume $\langle\beta_j, H\rangle\geqq 0$ for the fundamental root system $\{\beta_1, \cdots, \beta_r\}$ of $\Sigma(U)$. The Dynkin diagram of $\{\beta_1, \cdots, \beta_r\}$ marked with the non-negative integer $\langle\beta_j, H\rangle$ at the j-th vertex is called the *characteristic diagram* of G. The characteristic diagram determines the conjugacy class of G (see [D, Theorem 8.2] and [Tas2, Proposition 3.3]). Let $\beta_0=\sum_{j=1}^{r} n_j\beta_j$ be the highest root of $\Sigma(U)$.

THEOREM D. *Let G be a simple Lie subgroup which is isomorphic to $SO(3)$ in a compact connected simple Lie group U with a bi-invariant Riemannian metric. Then, G is stable if and only if there exists k, $1\leqq k\leqq r$, such that*

$$n_k=1,\quad \langle\beta_j, H\rangle=2\delta_{jk},\quad 1\leqq j\leqq r.$$

If G is stable, then $n(G)=n_K(G)$.

Let λ be a weight of a (complex) G-module. Since $\mathfrak{g}$ is of rank 1, λ is determined by its (integral) value $\langle\lambda, H\rangle_0$. On the other hand an integer n

determines an integral weight $nH/2$ of G. For the sake of brevity, we simply denote the weight $nH/2$ by n. Let $V(n)$ be the irreducible (complex) G-module with the highest weight $n\geqq 0$. Then the weight space decomposition of $V(n)$ is as follows:

$$V(n)=\sum_{k=0}^{n} V(n)_{n-2k}, \qquad \dim V(n)_{n-2k}=1. \tag{4.1}$$

By (4.1) and counting the multiplicities of weights, we have the well-known theorem of Clebsh-Gordan.

$$V(n)\otimes V(m)=\sum_{j=0}^{\mathrm{Min}(n,m)} V(|n-m|+2j). \tag{4.2}$$

Let j be the index of G in U. By the definition of the index,

$$\langle X, Y\rangle=j\langle X, Y\rangle_0, \qquad \text{for} \quad X, Y\in \mathfrak{g}. \tag{4.3}$$

Let X_β be a root vector of $\mathfrak{u}^C$ corresponding to a root $\beta\in\Sigma(U)$. Then by its definition $[H, X_\beta]=\sqrt{-1}\langle\beta, H\rangle X_\beta$. Thus X_β is a weight vector of the G-module $\mathfrak{u}^C$ corresponding to the weight $\langle\beta, H\rangle=\langle j\beta, H\rangle_0$. Therefore the set of weights of G-module $\mathfrak{u}^C$ is given as follows:

$$W(\mathfrak{u}^C)=\{\langle\beta, H\rangle : \beta\in\Sigma(U)\cup\{0\}\}. \tag{4.4}$$

For an integer k, we put

$$\Gamma_k=\{\beta\in\Sigma(U)\cup\{0\} : \langle\beta, H\rangle=k\}.$$

Then the weight space of $\mathfrak{u}^C$ corresponding to the weight k is given by the following:

$$(\mathfrak{u}^C)_k=\sum_{\beta\in\Gamma_k} \mathfrak{u}^C_\beta.$$

Since $\mathfrak{g}^C=V(2)$ is an irreducible component of $\mathfrak{u}^C$, we have $2\in W(\mathfrak{u}^C)$ and $\langle\beta_0, H\rangle \geqq 2$, for $\langle\beta_0, H\rangle$ is the highest weight in $W(\mathfrak{u}^C)$. Define a basis $\{H, X_+, X_-\}$ of $\mathfrak{g}^C$ by

$$X_+=(E-\sqrt{-1}F)/2, \qquad X_-=(E+\sqrt{-1}F)/2.$$

Then $X_+\in(\mathfrak{u}^C)_2$, $X_-\in(\mathfrak{u}^C)_{-2}$ and we can put

$$X_+=\sum_{\beta\in\Gamma_2} X_\beta, \qquad X_\beta\in\mathfrak{u}^C_\beta,$$

$$X_-=\sum_{\beta\in\Gamma_2} X_{-\beta}, \qquad X_{-\beta}\in\mathfrak{u}^C_{-\beta}.$$

Since $H=[E, F]=-2\sqrt{-1}[X_+, X_-]$, we have

$$H\in\sum_{\beta\in\Gamma_2} \boldsymbol{R}\beta. \tag{4.5}$$

By (2.1), the eigenvalue a_n of the Casimir operator on $V(n)$ of $\mathfrak{g}$ with respect to $\langle\, ,\rangle_0$ is given as follows:

$$a_n = -n(n+2)/2. \tag{4.6}$$

Since U is a simple Lie group, we have only to show the Theorem C and D with respect to the invariant Riemannian metric on U induced by $\langle\, ,\rangle/j$. By (4.3), the induced Riemannian metric on G coincides with the invariant Riemannian metric induced by $\langle\, ,\rangle_0$. We remember that

$$\mathcal{D}(SU(2)) = \{V(n) : n \in \boldsymbol{Z},\ n \geqq 0\}$$

$$\mathcal{D}(SO(3)) = \{V(2n) : n \in \boldsymbol{Z},\ n \geqq 0\}\,.$$

Proof of Theorem C. First we prove that if $\sum_{j=2}^{\infty} \#(\Gamma_j) \geqq 2$, then G is unstable. In fact, under the assumption there exists an n $(n \geqq 2)$, such that $V(n) \subset (\mathfrak{g}^{\perp})^C$. By (4.2) and (4.6),

$$a_1 + a_{n-1} > a_n\,,$$

$$V(1) \otimes V(n-1) = V(n) \oplus \cdots\,.$$

Thus by (i) of Theorem 2.1, we conclude that G is unstable. We can easily see that the converse is also true.

We consider the case that G is stable. As we remarked before, $\#(\Gamma_2) \geqq 1$. Thus if G is stable, then $\#(\Gamma_2)=1$, $\#(\Gamma_3)=\#(\Gamma_4)=\cdots=0$, and Γ_2 consists of β_0. By Schwarz' inequality and the definition of index, we have

$$2 = \langle \beta_0, H\rangle \leqq \sqrt{j\langle \beta_0, \beta_0\rangle \langle H, H\rangle_0} = 2\sqrt{j}\,.$$

The equality holds, since β_0 and H are proportional by (4.5). Namely we have $j=1$. Thus, combined with Theorem A, the former half of Theorem C is proved.

Now we prove the latter half. As we have proved, G is stable if and only if each irreducible component of $(\mathfrak{g}^{\perp})^C$ is equivalent to $V(1)$ or $V(0)$. Let m [resp. n] be the multiplicity of $V(1)$ [resp. $V(0)$] in $(\mathfrak{g}^{\perp})^C$. Note that m is even: $m=2m'$. Then, by (ii) of Theorem 2.1, we have

$$\begin{aligned} n(G) &= m \sum_{\substack{a_\lambda + a_\mu = -3/2 \\ \lambda, \mu \in \mathcal{D}(G)}} \dim \mathrm{Hom}_G(V(\lambda)\otimes V(\mu), V(1)) \dim (V(\lambda)\otimes V(\mu)) \\ &\quad + n \sum_{\substack{a_\lambda + a_\mu = 0 \\ \lambda, \mu \in \mathcal{D}(G)}} \dim \mathrm{Hom}_G(V(\lambda)\otimes V(\mu), V(0)) \dim (V(\lambda)\otimes V(\mu)) \\ &= 4m + n\,. \end{aligned}$$

On the other hand, by (iii) of Theorem 2.1, we also have

$$n_K(G) = 4m + n\,. \qquad \text{Q.E.D.}$$

The Proof of Theorem D. Remember that each weight of a G-module is an even integer. By a similar manner to that of the proof of Theorem C, we can prove that G is unstable if and only if $\sum_{j=4}^{\infty} \#(\Gamma_j) \geqq 1$.

We consider the case that G is stable. In this case we have $\langle \beta_0, H \rangle = 2$. Since a weight $\langle \beta_j, H \rangle$ is equal to 0 or 2, $2 = \langle \beta_0, H \rangle = \sum_{j=1}^{r} n_j$, $\langle \beta_j H \rangle$ implies that there exists an integer k such that $n_k = 1$, and $\langle \beta_j, H \rangle = 2\delta_{jk}$. Conversely, if the condition is satisfied we have $\langle \beta, H \rangle = 0$, 2 or -2 for any $\beta \in \Sigma(U)$. Thus the former half of Theorem D is proved.

The latter half is proved by a similar manner to the latter half of Theorem C. Q. E. D.

5. Classification of stable 3-dimensional simple subgroups.

Now we determine all stable 3-dimensional simple subgroups which satisfy the condition in Theorem D in each compact simple Lie group.

In the case that the ambient group U is of classical type we imbedd $\mathfrak{u}^C$ in $\mathfrak{sl}(N, \boldsymbol{C})$. We denote by ε_i the complex $N \times N$-matrix of which (i, i)-component is equal to $\sqrt{-1}$ and all of the other components are equal to 0. Put

$$\mathfrak{h} = \left\{ \sum_{i=1}^{N} t_i \varepsilon_i : t_i \in \boldsymbol{R},\ t_1 + \cdots + t_N = 0 \right\}.$$

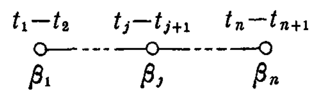
Case 1. $\mathfrak{u} = \mathfrak{su}(n+1)$, $n \geqq 1$. In this case case $\mathfrak{h}^C$ is a Cartan subalgebra of $\mathfrak{su}(n+1)^C = \mathfrak{sl}(n+1, \boldsymbol{C})$. Let $\mathfrak{g}$ be a 3-dimensional simple subalgebra in $\mathfrak{su}(n+1)$. We may assume

$$H = \sum_{i=1}^{N} t_i \varepsilon_i, \qquad t_1 \geqq t_2 \geqq \cdots \geqq t_{n+1}.$$

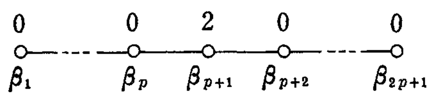
Note that $\{t_1, t_2, \cdots, t_{n+1}\}$ is the set of all weights of $\mathfrak{g}$ acting on $\boldsymbol{C}^{n+1}$. Since

$$\beta_j = \varepsilon_j - \varepsilon_{j+1}, \qquad 1 \leqq j \leqq n,$$

is a system of fundamental roots, the characteristic diagram of $\mathfrak{g}$ is as follows:



By (4.1) $t_i = -t_{n+2-i}$, hence the characteristic diagram of $\mathfrak{g}$ is symmetrical. Since $\beta_0 = \beta_1 + \cdots + \beta_n$, the diagram for $n = 2p+1$, $p \geqq 1$:



is a unique one satisfying the condition in Theorem D. Thus we get $t_1=\cdots=t_{p+1}=1$ and $t_{p+2}=\cdots=t_{2p+2}=-1$. The corresponding subgroup $\tilde{G}$ in $SU(2p+2)$ is

$$\tilde{G}=\{\text{Diagonal}(\underbrace{A, \cdots, A}_{p+1});\ A\in SU(2)\}.$$

Since $\tilde{G}$ is isomorphic to $SU(2)$ and its index is $p+1$, $\tilde{G}$ is unstable by Theorem C.

Each Lie group which is locally isomorphic to $SU(2p+2)$ is of the form $SU(2p+2)/D$ for some subgroup D of the center of $SU(2p+2)$. If a subgroup D of the center of $SU(2p+2)$ contains -1, $G=\tilde{G}/\{\pm 1\}\cong SO(3)$ is stable in $U=SU(2p+2)/D$.

Case 2. $\mathfrak{u}=\mathfrak{so}(2n+1)$, $n\geqq 2$. We imbedd $\mathfrak{so}(2n+1)^C=\mathfrak{so}(2n+1, \boldsymbol{C})$ in $\mathfrak{sl}(2n+1, \boldsymbol{C})$ as follows:

$$\mathfrak{so}(2n+1, \boldsymbol{C})=\left\{\begin{bmatrix} 0 & a & b \\ -{}^t b & X & Y \\ -{}^t a & Z & -{}^t X \end{bmatrix} : \begin{array}{l} {}^tY=-Y,\ {}^tZ=-Z, \\ X, Y, Z\in M_n(\boldsymbol{C}),\ a, b\in \boldsymbol{C}^n \end{array}\right\}$$

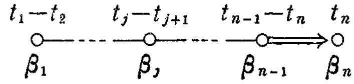
In this case $\mathfrak{h}^C\cap\mathfrak{so}(2n+1, \boldsymbol{C})$ is a Cartan subalgebra of $\mathfrak{so}(2n+1, \boldsymbol{C})$. Let $\mathfrak{g}$ be a 3-dimensional simple subalgebra in $\mathfrak{so}(2n+1)$. We may assume

$$H=\sum_{i=1}^{n} t_i(\varepsilon_{i+1}-\varepsilon_{i+n+1}), \qquad t_1\geqq t_2\geqq\cdots\geqq t_n\geqq 0.$$

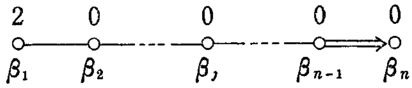
Then $\{0, t_1, \cdots, t_n, -t_1, \cdots, -t_n\}$ is the set of all weights of $\mathfrak{g}$ acting on $\boldsymbol{C}^{2n+1}$.

$$\beta_j=(\varepsilon_{j+1}-\varepsilon_{j+2}-\varepsilon_{j+n+1}+\varepsilon_{j+n+2})/\sqrt{2}, \qquad 1\leqq j\leqq n-1,$$
$$\beta_n=(\varepsilon_{n+1}-\varepsilon_{2n+1})/\sqrt{2}$$

is a system of fundamental roots, the characteristic diagram of $\mathfrak{g}$ is as follows:



Since $\beta_0=\beta_1+2\beta_2+\cdots+2\beta_n$, the diagram:



is a unique one satisfying the condition in Theorem D. Thus we get $t_1=2$ and $t_2=\cdots=t_n=0$. The corresponding subgroup G in $SO(2n+1)$ is

$$G=\{\text{Diagonal}(A, I_{2n-2});\ A\in SO(3)\}.$$

Since G is isomorphic to $SO(3)$, it is stable in $SO(2n+1)$. The index of G is equal to 2.

The corresponding subgroup in $Spin(2n+1)$ is isomorphic to $SU(2)$. Thus, by Theorem C, it is not stable.

Case 3. $\mathfrak{u}=\mathfrak{sp}(n)$, $n\geqq 3$. We imbedd $\mathfrak{sp}(n)^C=\mathfrak{sp}(n, \boldsymbol{C})$ in $\mathfrak{sl}(2n, \boldsymbol{C})$ as follows:

$$\mathfrak{sp}(n, \boldsymbol{C})=\left\{\begin{bmatrix} X & Y \\ Z & -{}^tX \end{bmatrix} : {}^tY=Y,\ {}^tZ=Z,\ X, Y, Z\in M_n(\boldsymbol{C})\right\}$$

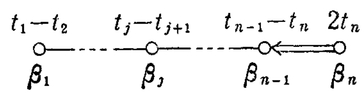
In this case $\mathfrak{h}^C\cap\mathfrak{sp}(n, \boldsymbol{C})$ is a Cartan subalgebra of $\mathfrak{sp}(n, \boldsymbol{C})$. Let $\mathfrak{g}$ be a 3-dimensional simple subalgebra in $\mathfrak{sp}(n)$. We may assume

$$H=\sum_{i=1}^{n} t_i(\varepsilon_i-\varepsilon_{i+n}), \qquad t_1\geqq t_2\geqq \cdots \geqq t_n\geqq 0.$$

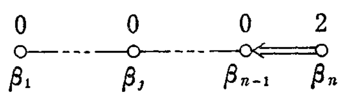
Note that $\{t_1, \cdots, t_n, -t_1, \cdots, -t_n\}$ is the set of all weights of $\mathfrak{g}$ acting on $\boldsymbol{C}^{2n}$. Since

$$\beta_j=(\varepsilon_j-\varepsilon_{j+1}-\varepsilon_{j+n}+\varepsilon_{j+n+1})/2, \qquad 1\leqq j\leqq n-1,$$
$$\beta_n=\varepsilon_n-\varepsilon_{2n}$$

is a system of fundamental roots, the characteristic diagram of $\mathfrak{g}$ is as follows:



Since $\beta_0=2\beta_1+\cdots+2\beta_{n-1}+\beta_n$, the diagram:



is a unique one satisfying the condition in Theorem D. Thus we get $t_1=\cdots=t_n=1$. The corresponding subgroup $\tilde{G}$ in $Sp(n)$ is

$$\tilde{G}=\{\text{Diagonal}(\underbrace{A, \cdots, A}_{n}) : A\in Sp(1)\}.$$

Since $\tilde{G}$ is isomorphic to $SU(2)$ and its index is n, $\tilde{G}$ is unstable by Theorem C.

The center of $Sp(n)$ is $\{\pm 1\}$. The corresponding subgroup $G=\tilde{G}/\{\pm 1\}$ in $U=Sp(n)/\{\pm 1\}$, which is isonorphic to $SO(3)$, is stable.

Case 4. $\mathfrak{u}=\mathfrak{so}(2n)$, $n\geqq 4$. We imbedd $\mathfrak{so}(2n)^C=\mathfrak{so}(2n, \boldsymbol{C})$ in $\mathfrak{sl}(2n, \boldsymbol{C})$ as follows:

$$\mathfrak{so}(2n, \boldsymbol{C})=\left\{\begin{bmatrix} X & Y \\ Z & -{}^tX \end{bmatrix} : {}^tY=-Y,\ {}^tZ=-Z,\ X, Y, Z\in M_n(\boldsymbol{C})\right\}$$

In this case $\mathfrak{h}^C\cap\mathfrak{so}(2n, \boldsymbol{C})$ is a Cartan subalgebra of $\mathfrak{so}(2n, \boldsymbol{C})$. Let $\mathfrak{g}$ be a 3-dimensional simple subalgebra in $\mathfrak{so}(2n)$. We may assume

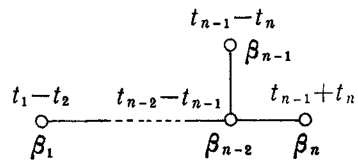
$$H=\sum_{i=1}^{n} t_i(\varepsilon_i-\varepsilon_{i+n}), \qquad t_1\geqq t_2\geqq \cdots \geqq t_{n-1}\geqq|t_n|.$$

Then $\{t_1, \cdots, t_n, -t_1, \cdots, -t_n\}$ is the set of all weights of $\mathfrak{g}$ acting on $\boldsymbol{C}^{2n}$. Since

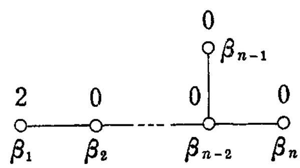
$$\beta_j=(\varepsilon_j-\varepsilon_{j+1}-\varepsilon_{j+n}+\varepsilon_{j+1+n})/\sqrt{2}, \qquad 1\leqq j\leqq n-1,$$

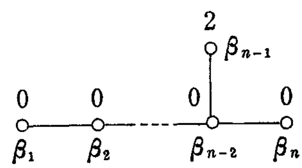
$$\beta_n=(\varepsilon_{n-1}+\varepsilon_n-\varepsilon_{2n-1}-\varepsilon_{2n})/\sqrt{2}.$$

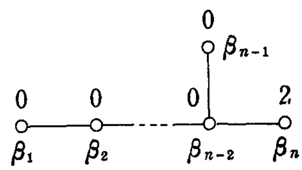
is a system of fundamental roots, the characteristic diagram of $\mathfrak{g}$ is as follows:



Since $\beta_0=\beta_1+2\beta_2+\cdots+2\beta_{n-2}+\beta_{n-1}+\beta_n$, the diagrams satisfying the condition in Theorem D are







Thus $t_1=2$, $t_2=\cdots=t_n=0$ or $t_1=\cdots=t_{n-1}=1$, $t_n=\pm 1$.

(i) If $t_1=2$, $t_2=\cdots=t_n=0$, then the corresponding subgroup in $SO(2n)$ is

$$\{\text{Diagonal}(A, I_{2n-3}) : A\in SO(3)\}$$

and its index is 2. Since the corresponding subgroup $\tilde{G}$ in $Spin(2n)$ is isomorphic to $SU(2)$, $\tilde{G}$ is unstable. Let Z be the center of $Spin(2n)$. Then $\tilde{G}\cap Z=\{\pm 1\}$.

If n is odd, Z is isomorphic to $\boldsymbol{Z}_4$ and the groups which is locally isomorphic to $Spin(2n)$ are $Spin(2n)$, $SO(2n)$ and $Spin(2n)/Z$. Since the corresponding subgroups in $SO(2n)$ and $Spin(2n)/Z$ are isomorphic to $SO(3)$, they are stable.

If n is even: $n=2m$, Z is isomorphic to $\boldsymbol{Z}_2\times\boldsymbol{Z}_2$. The subgroups of Z are $\{(0, 0)\}$, $\{(0, 0), (1, 0)\}$, $\{(0, 0), (0, 1)\}$, $\{(0, 0), (1, 1)\}$ and Z. The element $(1, 1)\in Z$ corresponds to $-1\in Spin(4m)$. Let U be a Lie group locally isomorphic to $Spin(4m)$ and D be the subgroup of Z such that U is isomorphic to $Spin(4m)/D$. If D is $\{(0, 0), (1, 1)\}$ or Z, then the subgroup corresponding to $\tilde{G}$ in U is isomorphic to $SO(3)$ and is stable. Otherwise, the subgroup corresponding to $\tilde{G}$ in U is isomorphic to $SU(2)$ and is unstable.

(ii) If $t_1=\cdots=t_{n-1}=1$, $t_n=\pm 1$ then n must be even: $n=2m$ and the corresponding subgroup in $SO(4m)$ is

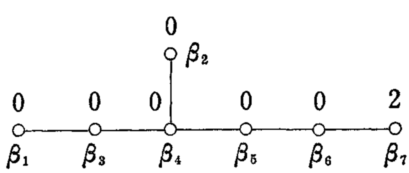
$$\{\text{Diagonal}(\underbrace{A, \cdots, A}_{m}) : A\in Sp(1)\},$$

where we regard $Sp(1)$ as a subgroup of $SO(4)$. The index of it is m. Let $\tilde{G}$ be the corresponding subgroup in $Spin(4m)$ and Z be the center of $Spin(4m)$. Let U be a Lie group which is locally isomorphic to $Spin(4m)$ and D be the subgroup of Z such that U is isomorphic to $Spin(4m)/D$. Since $\tilde{G}/(\tilde{G}\cap Z)$ is isomorphic to $SO(3)$, if D is $(\tilde{G}\cap Z)$ or Z, then the subgroup corresponding to $\tilde{G}$ in U is isomorphic to $SO(3)$ and is stable. Otherwise, the subgroup corresponding to $\tilde{G}$ in U is isomorphic to $SU(2)$ and is unstable.

Case 5. $\mathfrak{u}=\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$. Due to Table 18 in [D], there is no subgroup in E_6 which satisfies the condition in Theorem D.

Due to Table 19 in [D], there is a subgroup $\tilde{G}$ in E_7 corresponding to the following characteristic diagram.

(5.1)



It is isomorphic to $SU(2)$ and its index is 3. Thus $\tilde{G}$ is not stable in E_7. The center Z of E_7 is isomorphic to $\boldsymbol{Z}_2$. Therefore $G=\tilde{G}/Z$ is isomorphic to $SO(3)$ and stable in $E_7/Z=Ad(E_7)$.

There is no coefficient of the highest root of E_8 which is equal to 1.

Case 6. $\mathfrak{u}=\mathfrak{f}_4$. There is no coefficient of the highest root of F_4 which is equal to 1.

Case 7. $\mathfrak{u}=\mathfrak{g}_2$. There is no coefficient of the highest root of G_2 which is equal to 1.

Now we summarize the above argument.

THEOREM E. *All stable 3-dimensional simple subgroups G isomorphic to SO(3) in compact connected simple Lie groups with bi-invariant Riemannian metrics are as follows.*

(1) *Let* $\tilde{G}=\{\text{Diagonal}(\underbrace{A, \cdots, A}_{n}): A \in SU(2)\} \subset SU(2n)$ *and D be a subgroups of the center of* $SU(2n)$ *containing* $\{\pm 1\}$. *Then* $G=\tilde{G}/D$ *is stable in* $SU(2n)/D$. *Its index is equal to n.*

(2) *Let* $\tilde{G}=\{\text{Diagonal}(\underbrace{A, \cdots, A}_{n}): A \in Sp(1)\} \subset Sp(n)$. *Then* $G=\tilde{G}/\{\pm 1\}$ *is stable in* $Ad(Sp(n))=Sp(n)/\{\pm 1\}$. *Its index is equal to n.*

(3) $G=\{\text{Diagonal}(A, I_{n-3}): A \in SO(3)\}$ *is stable in* $SO(n)$. *If n is even*: $n=2m$, *then* $Ad(G)$ *is also stable in* $Ad(SO(2m)=PSO(2m)$. *Their indices are equal to* 2.

(4) *Let Z be the center of* $Spin(4n)$ *and* $\tilde{G}$ *be the subgroup of* $Spin(4n)$ *obtained by pulling back* $\{\text{Diagonal}(\underbrace{A, \cdots, A}_{n}): A \in Sp(1)\}$ *in* $SO(4n)$, *where we regard* $Sp(1)$ *as a subgroup of* $SO(4)$ *in a natural manner. Then* $G=\tilde{G}/\tilde{G}\cap Z$ *is stable in* $Spin(4n)/\tilde{G}\cap Z$ *and* $Spin(4n)/Z=PSO(4n)$. *Their indices are equal to n.*

(5) *Let G be a subgroup of* $Ad(E_7)$ *corresponding to the characteristic diagram* (5.1). *Then it is stable. Its index is equal to* 3.

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