

Stability of Cohesive Crack Model: Part I—Energy Principles

Z. P. Bažant

Walter Murphy Professor of Civil Engineering and Material Science, Fellow ASME.

Yuan-Neng Li

Research Assistant Professor, Department of Civil Engineering, Northwestern University, Evanston, IL 60208

The paper deals with a cohesive crack model in which the cohesive (crack-bridging) stress is a specified decreasing function of the crack-opening displacement. Under the assumption that no part of the crack undergoes unloading, the complementary energy and potential energy of an elastic structure which has a cohesive crack and is loaded by a flexible elastic frame is formulated using continuous influence functions representing compliances or stiffnesses relating various points along the crack. By variational analysis, in which the derivatives of the compliance or stiffness functions with respect to the crack length are related to the crack-tip stress intensity factors due to various unit loads, it is shown that the minimizing conditions reduce to the usual compatibility or equilibrium equations for the cohesive cracks. The variational equations obtained can be used as a basis for approximate solutions. Furthermore, the conditions of stability loss of a structure with a growing cohesive crack are obtained from the condition of vanishing of the second variation of the complementary energy or the potential energy. They have the form of a homogeneous Fredholm integral equation for the derivatives of the cohesive stresses or crack opening displacements with respect to the crack length. Loadings with displacement control, load control, or through a flexible loading frame are considered. Extension to the analysis of size effect on the maximum load or maximum displacement are left to a subsequent companion paper.

1 Introduction

Quasi-brittle materials, such as concrete, ice (especially sea ice), rocks, ceramics, and certain composites, exhibit a large fracture process zone in which the material undergoes progressive softening damage. The process zone may be approximated by the cohesive crack model, in which the fracture process zone is represented by crack-bridging tensile stresses (cohesive stresses) which decrease with crack opening. The basic concept originated with the work of Barenblatt (1962) and Dugdale (1960), who introduced two different versions of the cohesive crack model. While Dugdale considered the cohesive stresses to be constant in order to simulate plastic behavior of metals near the crack tip, Barenblatt, in an effort to model the reduction of interatomic bond forces, introduced cohesion as a gradual softening process. Barenblatt studied equilibrium but not stability and solved only problems in which the cohesive zone is infinitesimal and the distribution of the cohesive stresses can be assumed a priori, independent of the solution of the crack opening profile. Dugdale, on the other hand, considered a cohesive zone of finite length, in the context of plastic yielding. The problem of finding the peak load for the Dugdale model was later solved by Bilby, Cottrell, and Swinden (1963), who introduced for plastic cohesive stresses a critical crack-opening displacement at which the cohesive (crack-bridging) stress drops to zero.

Later studies have led to further diversification of the cohesive stress models. Some generalizations of Barenblatt's model

do not have a uniquely defined stress-displacement law. Instead, the stress distribution along the cohesive portion of the crack is assumed a priori, with a process zone which may or may not be infinitesimal. Within this category, the models of Willis (1967), Smith (1974), and Reinhardt (1985) deserve mention. In another common type of cohesive crack models, there is a stress-displacement law but the effective energy release rate is calculated according to Rice's (1968) equation which was originally developed for small-scale yielding only. Nevertheless, this equation has often been extended beyond the small-scale yielding conditions, to situations in which the cohesive zone (or damage zone) is not negligible compared to the characteristic size of the structure (e.g., Suo et al., 1992). Although one basic characteristic of the cohesive crack models introduced by Barenblatt and Dugdale is that the stress intensity factor at the crack tip is zero, some recent modified forms of the cohesive crack model admit a positive stress intensity factor at the crack tip (Foote et al., 1986; Rice, 1992). There are also differences in the definition of stress-displacement law. Normally the cohesive stress is finite as soon as a crack starts to open, being equal to the tensile strength of the material, and subsequently the stress declines. But in some recent models, the cohesive stress starts to increase from zero as the crack begins to open, and only later softening takes place. This formulation, which is not considered in the present paper, has recently been used by Needleman (1990) and Tvergaard and Hutchinson (1992) in studies of the interface between plastic materials.

This study deals with the normal case of cohesive crack model, different from that in the aforementioned studies. The basic characteristics are (1) the length of the cohesive zone is finite, i.e., not negligible compared to the structure dimensions; (2) the stress intensity factor at the crack tip is zero (which means the stress at the tip is finite); (3) the cohesive stress depends on the crack-opening displacement according to a specified softening law; and (4) the material surrounding the crack behaves in a linearly elastic manner. Such a model (under the name "fictitious crack model") was first introduced for con-

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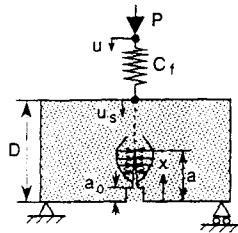


Fig. 1 Elastic structure with a cohesive crack, loaded through a spring

crete by Hillerborg, Mod  r, and Petersson (1976) and extended by Petersson (1981). An equivalent crack band model was proposed by Ba  ant (1976, 1982, 1984), Ba  ant and Cedolin (1991), and Ba  ant and Oh (1983). All these studies were limited to the discrete finite element formulation. Although such a formulation is sufficient for numerical analysis of cracked structures, it is not well suited for characterizing the basic mathematical properties of the cohesive crack model. The aforementioned works were also limited to equilibrium analysis, but recently Li and Liang (1992, 1993) and Li and Ba  ant (1994) introduced a potential energy for the cohesive crack model and showed that the equilibrium conditions are the conditions of stationarity of this potential. However, the elastic strain energy of the structure and the energy of the cohesive stresses were treated in a combined manner, which does not reveal the basic mathematical properties and minimum principle.

This paper introduces continuous influence (Green's) functions along the crack in order to represent the elastic behavior of the structure separately from the cohesive stresses. The complementary energy and the potential energy of a general structure with cohesive crack is formulated and it is shown that the integral equations characterizing compatibility or equilibrium between the structure and the cohesive crack follow according to the well-known principles of minimum complementary energy or potential energy. The main objective of the paper is to present the variational derivation of these integral equations which reveals interesting relations to linear elastic fracture mechanics. The variational equations obtained could be used as a basis for approximate solutions. Finally, the conditions of stability of structures with cohesive cracks under load or displacement control and loading through a flexible frame are derived from the energy functionals in a general form. Further extensions to the analysis of size effect on the maximum load under load control and the maximum deflection under displacement control are relegated to a subsequent companion paper, and applications to sea ice will appear later.

2 Basic Energy Variables

Consider an elastic structure (or specimen) with a cohesive crack of length a (Fig. 1). For the sake of generality, we consider the structure to be loaded through an elastic loading frame of elastic compliance C_f , which is equivalent to a spring coupled in series, as in Fig. 1 (since the columns of a testing machine are placed parallel to the test specimen, a novice might think this is a parallel coupling, but it is a series coupling because the forces transmitted by the machine and by the specimen are equal and their displacements are additive). The special case $C_f = 0$ equivalent to a dead load applied directly on the structure. The problem can be formulated in terms of either stiffnesses or compliances. We first study the latter. In that case the basic variables are the forces, and the thermodynamic potential is the complementary energy Π^* , representing the Gibbs free energy in the case of isothermal conditions, or the enthalpy in the case of isentropic (or adiabatic) conditions.

According to the first law of thermodynamics (balance of energy), Π^* is additive, i.e., represents for sum of the comple-

mentary energies of all the parts of the structure-load system. Therefore,

$$\Pi^* = \Pi_s^* + \Pi_f^* + \Pi_c^*; \quad \Pi_s^* = U^* + \Pi_c^* \quad (1)$$

$$\Pi_c^* = \int_{a_0}^a \Gamma^*[\sigma(x)] dx \quad (2)$$

where a_0 = length of notch or initial traction-free crack; Π_s^* , Π_f^* , Π_c^* , U^* , Π_c^* = complementary energies of the structure (with the crack), the loading frame, the loading (Fig. 2(a)), the elastic structure (without crack), and the cohesive crack; and $\Gamma^*[\sigma(x)]$ = density of complementary energy of the crack at point of length coordinate x (Fig. 1). The elastic structure, as well as the loading frame, is assumed to be internally in equilibrium and internally compatible (which means their internal degrees of freedom are condensed out). Obviously, $\Pi_f^* = C_f P^2/2$ where C_f = compliance of the loading frame.

In the case of compliance formulation, the basic variables are the forces (e.g., Ba  ant and Cedolin, 1991, Sec. 10.1). So, if we want to obtain the load-point displacement u (displacement of the loading frame at the point where force P is applied), we must consider u to be constant and load P as variable (Fig. 2(c)). Thus, the complementary energy of the loading $\Pi_f^* = -W^* = -Pu$ (representing the complementary work W^* of the constant displacement u on the varying force P , which is equal to the area to the left of the vertical line in Fig. 2(c)).

The complementary energy Π^* as a potential exists, of course, if and only if the complementary energy potential exists for each part of the structure. For the crack, Π_c^* exists if we assume that no part of the crack ever undergoes unloading (i.e., the crack opens monotonically at all points x) and that the crack-opening displacement $w(x)$ at x (half the crack width) is a function of the cohesive (crack-bridging) stress $\sigma(x)$, i.e.,

$$w(x) = g[\sigma(x)] \quad (3)$$

(Fig. 3). This is the same as if the faces of the crack were held together by nonlinear continuously distributed springs following the law in Eq. (3). This law is softening (i.e., g is a decreasing function), as illustrated by the curve in Fig. 3.

The density of the potential energy of the cohesive crack is

$$\Gamma(w) = \int_0^w f(w') dw' \quad (4)$$

(Fig. 3), where $\sigma = f(\sigma)$ defines the given softening law and is inverse to function $w = g[\sigma(x)]$. The density of complementary energy is $\Pi^* = \sigma w - \Gamma$, that is

$$\Gamma^*(\sigma) = \int_{f_i}^{\sigma} g(\sigma') d\sigma' \quad (5)$$

(Fig. 3), where $w = g(\sigma)$ defines the softening law by function g that is inverse to function f ; f_i = tensile strength, which determines the σ -value at which the cohesive crack begins to open.

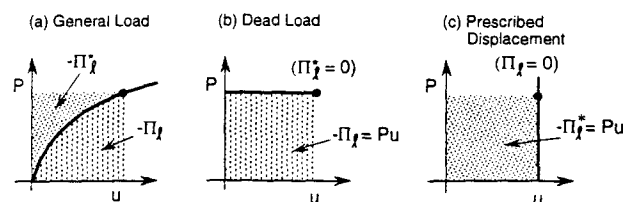


Fig. 2 Potential energy Π , and complementary energy Π^* of loading

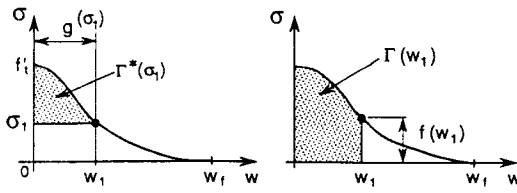


Fig. 3 Softening law relating crack-bridging (cohesive) stress σ and crack-opening displacement w

3 Complementary Energy of Structure With a Cohesive Crack

The complementary strain energy of the elastic structure, U^* , is a function of the load P as well as the surface tractions on the crack faces, which are equal to $-\sigma(x)$. Let C^{PP} = load-point compliance of the structure, $C^{\sigma\sigma}(x, x')$ = influence function representing a crack compliance function, such that the unit surface traction $\sigma(x) = \delta(x - x')$ applied at x' causes at x the opening $w = C^{\sigma\sigma}(x, x')$; $C^{\sigma P}(x)$ = influence function representing a cross-compliance function = crack-opening displacement at x caused by unit load P . Evidently, $C^{\sigma\sigma}(x, x') = C^{\sigma\sigma}(x', x)$, and $C^{\sigma P}(x)$ is equal to $C^{P\sigma}(x)$, defined so that a unit surface traction $\sigma = \delta(x - x')$ causes the load-point displacement $u = C^{P\sigma}(x)$. Expressing U^* in terms of the compliances and combining all the expressions for the complementary energies in (1) and (2), we have

$$\begin{aligned} \Pi^* = & \Pi^*[\sigma(x), a, P] = \int_{a_0}^a \Gamma^*(x) dx \\ & + \frac{1}{2} \int_{a_0}^a \int_{a_0}^a C^{\sigma\sigma}(x, x') \sigma(x) \sigma(x') dx' dx \\ & - P \int_{a_0}^a C^{\sigma P}(x) \sigma(x) dx + \frac{1}{2} C^{PP} P^2 + \frac{1}{2} C_f P^2 - Pu \quad (6) \end{aligned}$$

The reason for the first minus sign is that in our notation the positive directions of $\sigma(x)$ and $w(x)$ are opposite (the vector of w points toward the crack face, while the vector of σ points away from the crack face).

Equation (6) represents a function of P and $\sigma(x)$, giving the complementary energy for all the equilibrium states of the structure which are generally not compatible with the crack opening $w(x)$ and do not have the correct crack length a . The compatible crack opening is obtained by minimizing Π^* with respect to $\sigma(x)$, and the energetically correct crack length a is obtained by minimizing Π^* with respect to a . Although the equations that must result from this minimization are basically known, it will be instructive to carry out the variational procedure of minimization. Besides, the variational equations obtained by this procedure may be of interest for approximate solutions.

4 Compatibility and Minimization of Complementary Energy

A necessary condition of minimum of Π^* is that the first variation $\delta\Pi^* = 0$. We have

$$\begin{aligned} \delta\Pi^* = & \left\{ (C^{PP} + C_f)P - u - \int_{a_0}^a C^{\sigma P}(x) \sigma(x) dx \right\} \delta P \\ & + \int_{a_0}^a \left\{ g[\sigma(x)] \right. \\ & \left. + \int_{a_0}^a C^{\sigma\sigma}(x, x') \sigma(x') dx' - C^{\sigma P}(x)P \right\} \delta\sigma(x) dx \end{aligned}$$

$$\begin{aligned} & + \left\{ \frac{1}{2} \int_{a_0}^a \int_{a_0}^a \frac{\partial C^{\sigma\sigma}(x, x')}{\partial a} \sigma(x) \sigma(x') dx' dx \right. \\ & - P \int_{a_0}^a \frac{\partial C^{\sigma P}}{\partial a} \sigma(x) dx + \frac{1}{2} \frac{\partial C^{PP}}{\partial a} P^2 \\ & + \int_{a_0}^a C^{\sigma\sigma}(x, a) \sigma(x) \sigma(a) dx \\ & \left. - PC^{\sigma P}(a) \sigma(a) + \Gamma^*[\sigma(a)] \right\} \delta a = 0. \quad (7) \end{aligned}$$

This may be simplified by noting that we may substitute $C^{\sigma\sigma}(x, a) = C^{\sigma P}(a) = 0$ because $w = 0$ at the crack tip, and $\Gamma^*[\sigma(a)] = \Gamma^*(f_i) = 0$.

It is now necessary to relate the compliance derivatives to the energy release rate and the stress intensity factors. The energy release rate is $\partial\Pi^*/\partial a = K^2/E'$ where $E' = \text{Young's modulus } E \text{ for the case of plane stress or } E/(1 - \nu^2) \text{ for the case of plane strain, with } \nu = \text{Poisson ratio. According to the principle of superposition, the total stress intensity factor is}$

$$K = K_P + K_\sigma = Pk_P + \int_{a_0}^a k_\sigma(x) \sigma(x) dx \quad (8)$$

where K_P = stress intensity factor due to load P alone and K_σ = stress intensity factor due to surface tractions $-\sigma(x)$; $k_\sigma(x)$, k_P = stress intensity factor at the crack tip ($x = a$) caused by a unit surface traction, i.e., stress $-\sigma = \delta(x - x')$, or by a unit load $P = 1$. With these notations, we have

$$\begin{aligned} \frac{\partial C^{\sigma\sigma}(x, x')}{\partial a} &= \frac{\partial}{\partial a} \frac{\partial^2 \Pi^*}{\partial \sigma(x) \partial \sigma(x')} = \frac{\partial^2}{\partial \sigma(x) \partial \sigma(x')} \frac{\partial \Pi^*}{\partial a} \\ &= \frac{1}{E'} \frac{\partial^2}{\partial \sigma(x) \partial \sigma(x')} \left[Pk_P + \int_{a_0}^a k_\sigma(x) \sigma(x) dx \right]^2 \\ &= \frac{2}{E'} k_\sigma(x) k_\sigma(x') \quad (9a) \end{aligned}$$

$$\begin{aligned} \frac{\partial C^{\sigma P}(x)}{\partial a} &= -\frac{1}{E'} \frac{\partial^2}{\partial \sigma(x) \partial P} \left[Pk_P + \int_{a_0}^a k_\sigma(x) \sigma(x) dx \right]^2 \\ &= -\frac{2}{E'} k_\sigma(x) k_P \quad (9b) \end{aligned}$$

$$\frac{\partial C^{PP}}{\partial a} = \frac{2}{E'} k_P^2. \quad (9c)$$

Equations (9a, b) represent a continuous generalization of a similar well-known relation for concentrated loads given, e.g., by Tada et al. (1985).

Equation (7) must be satisfied for any variations δP , $\delta\sigma(x)$ and δa (such that $\delta\sigma \geq 0$). This requires that the expressions in brackets $\{ \dots \}$ vanish. With the aforementioned substitutions, the vanishing of the variations with respect to δP , $\delta\sigma(x)$ and δa requires that

$$\frac{\partial \Pi^*}{\partial P} = - \int_{a_0}^a C^{\sigma P}(x) \sigma(x) dx + (C^{PP} + C_f)P - u = 0 \quad (10)$$

$$\begin{aligned} \frac{\partial \Pi^*}{\partial \sigma(x)} &= \int_{a_0}^a C^{\sigma\sigma}(x, x') \sigma(x') dx' \\ &\quad - C^{\sigma P}(x)P + g[\sigma(x)] = 0 \quad (11) \end{aligned}$$

$$\begin{aligned} E' \frac{\partial \Pi^*}{\partial a} &= \int_{a_0}^a \int_{a_0}^a k_\sigma(x) k_\sigma(x') \sigma(x) \sigma(x') dx' dx \\ &\quad + 2Pk_P \int_{a_0}^a k_\sigma(x) \sigma(x) dx + P^2 k_P^2 = 0. \quad (12) \end{aligned}$$

Noting that Eq. (12) may be written as

$$E' \frac{\partial \Pi^*}{\partial a} = \left[P k_p + \int_{a_0}^a k_\sigma(x) \sigma(x) dx \right]^2 = 0, \quad (13)$$

we obtain the relation $K = K_p + K_\sigma = 0$.

Equation (11) represents the crack compatibility condition, i.e., the condition that the crack openings calculated from the deformation of the elastic body match the openings obtained from the given softening law $g(\sigma)$. Equation (10) gives the load-point deflection and, in the sense of the complementary energy approach, has the meaning of the condition of compatibility of the elastic deformation of the structure with the given value of u (it is not an equilibrium condition). Equation (8) with $K = 0$, called the zero- K condition, is the basic condition of any cohesive crack model, proposed by Barenblatt (1962) and Dugdale (1960). This condition means that the stress at the crack tip must be finite, i.e., that there is no singularity. From the variational viewpoint, the condition $K = 0$ is a condition of energy rate balance between the structure and the crack. If K were positive, the rate of energy released from the structure would be larger than that dissipated in the cohesive crack (and then the propagation would be dynamic). If K were negative, propagation would be impossible.

To calculate structure displacement u , we substitute $u = u_s + C_f P$ into (10), considering P to be positive when it puts the loading spring (Fig. 1) into compression. Then, solving for P , we get

$$u_s = C^{PP} P - \int_{a_0}^a C^{P\sigma}(x) \sigma(x) dx. \quad (14)$$

5 Stability Loss in Terms of Compliance

According to the second law of thermodynamics, stability requires that the second variation $\delta^2 \Pi^*$ be a positive definite functional of $\sigma(x)$, P and a (e.g., Bažant and Cedolin, 1991, Sec. 10.1). The limit of stability occurs when $\delta^2 \Pi^* = 0$ for some variation $\delta \sigma(x)$, δP and δa . The variation of u cannot be considered arbitrary because displacements cannot be the variables in the complementary energy functional. Therefore, we consider loading under displacement control conditions, which is, of course, required by the form of the work term in our expression for Π^* . $\delta^2 \Pi^* = 0$ means that

$$\delta(\delta_P \Pi^*) = 0, \quad \delta(\delta_\sigma \Pi^*) = 0, \quad \delta(\delta_a \Pi^*) = 0 \quad (15a, b, c)$$

for some variation. The conditions of stationary Π^* , which are necessary for $\min \Pi^*$, are represented by the vanishing of (10), (11), and (12); they must be imposed only after variations δ are taken. According to the expressions for $\delta_P \Pi^*$, $\delta_\sigma \Pi^*$, and $\delta_a \Pi^*$ implied by (10), (11), and (12), with (13), Eq. (15a, b, c) read:

$$\begin{aligned} -\delta(\delta_P \Pi^*) = & \left[\int_{a_0}^a C^{P\sigma}(x) \delta \sigma(x) dx \right. \\ & + \int_{a_0}^a \frac{\partial C^{P\sigma}(x)}{\partial a} \sigma(x) dx \delta a + C^{P\sigma}(a) \sigma(a) \delta a \\ & \left. - (C^{PP} + C_f) \delta P - \frac{\partial C^{PP}}{\partial a} P \delta a \right] \delta P = 0 \quad (16a) \end{aligned}$$

$$\begin{aligned} -\delta(\delta_\sigma \Pi^*) = & \int_{a_0}^a \left[\int_{a_0}^a C^{\sigma\sigma}(x, x') \delta \sigma(x') dx' \right. \\ & + \int_{a_0}^a \frac{\partial C^{\sigma\sigma}(x, x')}{\partial a} \sigma(x') dx' \delta a - C^{\sigma\sigma}(x, a) \sigma(a) \delta a \\ & \left. - \frac{\partial C^{\sigma P}(x)}{\partial a} P \delta a - C^{\sigma P}(x) \delta P \right] \delta \sigma(x) dx = 0 \quad (16b) \end{aligned}$$

$$+ \frac{dg[\sigma(x)]}{d\sigma} \delta \sigma(x) \delta \sigma(x) dx = 0 \quad (16b)$$

$$E' \delta(\delta_a \Pi^*) = 2(K_p + K_\sigma)(\delta K_p + \delta K_\sigma) \delta a = 0. \quad (16c)$$

The last equation is automatically satisfied because $K_p + K_\sigma = 0$ at $\min \Pi^*$. The other equations may again be simplified using Eq. (9a, b, c) and, noting that,

$$C^{\sigma\sigma}(x, a) = C^{P\sigma}(a) = 0, k_p P = K_p,$$

$$- \int_{a_0}^a k_\sigma(x) \sigma(x) dx = K_\sigma$$

we get

$$\begin{aligned} -\delta(\delta_P \Pi^*) = & \left[\int_{a_0}^a C^{P\sigma}(x) \delta \sigma(x) dx \right. \\ & \left. - \frac{2}{E'} k_p (K_p + K_\sigma) \delta a - (C^{PP} + C_f) \delta P \right] \delta P = 0 \quad (17a) \end{aligned}$$

$$\begin{aligned} \delta(\delta_\sigma \Pi^*) = & \int_{a_0}^a \left[\int_{a_0}^a C^{\sigma\sigma}(x, x') \delta \sigma(x') dx' \right. \\ & + \frac{2}{E'} k_\sigma(x) (K_p + K_\sigma) \delta a - C^{\sigma P}(x) \delta P \\ & \left. + \frac{dg[\sigma(x)]}{d\sigma} \delta \sigma(x) \right] \delta \sigma(x) dx = 0. \quad (17b) \end{aligned}$$

Here again we may set $K_p + K_\sigma = 0$. Eliminating δP from the last two equations, we obtain

$$- \frac{dg[\sigma(x)]}{d\sigma} v(x) = \int_{a_0}^a \bar{C}^{\sigma\sigma}(x, x') v(x') dx' \quad (18)$$

in which we introduced the notations $\delta \sigma(x)/\delta a = v(x)$ and

$$\bar{C}^{\sigma\sigma}(x, x') = C^{\sigma\sigma}(x, x') - \frac{C^{\sigma P}(x) C^{\sigma P}(x')}{C^{PP} + C_f}. \quad (19)$$

Equation (18) is a homogeneous Fredholm integral equation for function $v(x)$. It is linear if function $g(\sigma)$ is linear. It characterizes the loss of stability under displacement control with a flexible loading frame. Note that $\bar{C}^{\sigma\sigma}(x, x')$ represents the crack compliances of the system consisting of the structure and the loading frame combined.

The special case $C_f = 0$ is equivalent to loading the structure under displacement control at the point of transmission of load P into the structure (rather than at the loading frame). In this case, the stability limit determines the maximum deflection of the structure, corresponding to the onset of snapback instability. Of course, the condition of maximum deflection could have been obtained more directly, simply by considering a structure without any loading frame from the outset ($C_f = 0$).

The special case $C_f \rightarrow \infty$ (with $u \rightarrow \infty$), for which $\bar{C}^{\sigma\sigma}(x, x') = C^{\sigma\sigma}(x, x')$, is equivalent to loading the structure under conditions of load control (dead load). In that case, the structure and the crack become unstable at maximum load, and so Eq. (18) becomes the condition of maximum load, which is obtained from Eq. (14).

The condition of maximum load ($C_f \rightarrow \infty$) could have been, of course, also obtained more directly, namely by considering the structure without any loading frame, loaded directly by P . In that case, we would have had to discard the term $-Pu$ from Eq. (6) defining Π^* . This term represents the complementary energy for load P varying at constant displacement u . For displacement u varying at constant P (i.e., dead load), the complementary energy of load is zero (Fig. 2(b)).

6 Potential Energy of Structure With Cohesive Crack

Second, we study the stiffness formulation. In that case the thermodynamic potential is the potential energy Π , representing the Helmholtz free energy in the case of isothermal conditions or the total energy in the case of isotropic (adiabatic) conditions. According to the first law of thermodynamics,

$$\Pi = \Pi_s + \Pi_f + \Pi_l, \quad \Pi_s = U + \Pi \quad (20)$$

where

$$\Pi = \int_{a_0}^a \Gamma[w(x)]dx, \quad \Gamma(w) = \int_0^w f[w]dw. \quad (21)$$

The notations Π_s , Π_f , Π_l , and U are analogous to Π_s^* , Π_f^* , Π_l^* , and U^* (U = strain energy of structure, $-\Pi_l = W$ = work of load). Obviously, $\Pi_f = R_f(u - u_s)^2/2$ where $R_f = 1/C_f$ = stiffness of the loading frame, and u_s = displacement of structure at point of contact with the loading frame.

In the case of stiffness formulation, the basic variables are the displacements. If we consider load-control, then P is fixed (dead load) and u is variable, in which case the potential energy of the loading is $\Pi_l = -W = -Pu$. This happens to be the same expression as before but W represents the work of load P on displacement u , which is the area below the horizontal line in Fig. 1 (b). However, we will be more interested in displacement control, in which case $\Pi_l = 0$, because u is fixed ($du = 0$) and P is varied (in which case $dW = Pdu = 0$); Fig. 2(c).

7 Equilibrium and Minimization of Potential Energy

Under our assumptions, the potential exists separately for the load, the frame, the structure, and the cohesive crack. Therefore, it must also exist for the entire system, as given by Eq. (21). The potential energy of the elastic structure (i.e., strain energy) is a function of u_s and $w(x)$ (all the internal displacements of the structure are assumed to be condensed out). Let $R^{ww}(x, x')$ = influence function = crack stiffness function representing the stress at x caused by a unit displacement $w = \delta(x - x')$; $R^{uw}(x)$ = influence function (or stiffness function) representing the stress at x caused by a unit displacement $u_s = 1$; and R^{uu} = load-point stiffness representing the force at the point of contact with the loading frame caused by $u_s = 1$. The potential energy expression for the case of displacement control may now be rewritten as

$$\Pi = \int_{a_0}^a \Gamma[w(x)]dx + \frac{1}{2} \int_{a_0}^a \int_{a_0}^a R^{ww}(x, x')w(x)w(x')dx'dx - u_s \int_{a_0}^a R^{uw}(x)w(x)dx + \frac{1}{2} R^{uu} u_s^2 + \frac{1}{2} R_f (u - u_s)^2 \quad (22)$$

(Note that $\Pi_l = 0$ because the load varies at prescribed displacement.) The first variation of this expression is

$$\begin{aligned} \delta\Pi = & \left\{ (R^{uu} + R_f)u_s - R_f u - \int_{a_0}^a R^{uw}(x)w(x)dx \right\} \delta u_s \\ & + \int_{a_0}^a \left\{ f[w(x)] + \int_{a_0}^a R^{ww}(x, x') \right. \\ & \times w(x')dx' - u_s R^{uw}(x) \left. \right\} \delta w(x)dx \\ & + \left\{ \frac{1}{2} \int_{a_0}^a \int_{a_0}^a \frac{\partial R^{ww}(x, x')}{\partial a} w(x)w(x')dx'dx \right. \\ & \left. - u_s \int_{a_0}^a \frac{\partial R^{uw}(x)}{\partial a} w(x)dx + \frac{1}{2} \frac{\partial R^{uu}}{\partial a} u_s^2 \right\} \delta a = 0. \end{aligned} \quad (23)$$

$$+ \int_{a_0}^a R^{ww}(x, a)w(a)dx - u_s R^{uw}(a)w(a) + \Gamma[w(a)] \left. \right\} \delta a = 0. \quad (23)$$

Now we note that $w(a) = \Gamma(0) = 0$ and, in analogy to Eq. (9a, b, c), we could prove that

$$\begin{aligned} \frac{\partial R^{ww}(x, x')}{\partial a} &= -\frac{2}{E'} k_w(x)k_w(x'), \\ \frac{\partial R^{uw}(x)}{\partial a} &= \frac{2}{E'} k_w(x)k_u, \quad \frac{\partial R^{uu}}{\partial a} = -\frac{2}{E'} k_u^2 \end{aligned} \quad (24)$$

where $k_w(x)$ and k_u are the stress intensity factors at $x = a$ caused by unit displacement $w(x') = \delta(x' - x)$ or by unit displacement $u_s = 1$. The condition of minimum potential energy requires that the expression in the brackets $\{\dots\}$ multiplied by δu_s , $\delta w(x)$, and δa vanish. This yields three equations:

$$u_s = \frac{1}{R^{uu} + R_f} \left[R_f u + \int_{a_0}^a R^{uw}(x)w(x)dx \right] \quad (25)$$

$$f[w(x)] = - \int_{a_0}^a R^{ww}(x, x')w(x')dx' + R^{uw}(x)u_s \quad (26)$$

$$\begin{aligned} - \frac{\partial \Pi}{\partial a} &= \frac{1}{E'} \int_{a_0}^a \int_{a_0}^a k_w(x)k_w(x')w(x)w(x')dx'dx \\ &+ \frac{u_s}{E'} \int_{a_0}^a k_w(x)k_u dx + \frac{1}{E'} k_u^2 u_s^2 \\ &= \frac{1}{E'} \left[k_u u_s + \int_{a_0}^a k_w(x)w(x)dx \right]^2 = 0. \end{aligned}$$

The last equation may be rewritten as

$$\begin{aligned} K &= K_u + K_w = 0, \quad K_u = k_u u_s, \\ K_w &= \int_{a_0}^a k_w(x)w(x)dx. \end{aligned} \quad (28a, b, c)$$

This means that the total stress intensity factor K caused by load-point displacement and crack opening must vanish. This is the same well-known condition as we obtained before, but expressed in terms of the displacement-caused stress intensity factors k_u and k_w .

To calculate load P , we substitute $u = u_s + (P/R_f)$ into (25), and solve the equation for P :

$$P = R^{uu} u_s - \int_{a_0}^a R^{uw}(x)w(x)dx. \quad (29)$$

Equation (26) gives the equilibrium value of displacement of the structure at the point of loading by the frame. Equations (27) or (28) give the condition of equilibrium (static) propagation, at which the energy release and energy dissipation rates are equal.

8 Stability Loss in Terms of Stiffness

Let us now consider stability at prescribed displacement u . According to the Lagrange-Dirichlet theorem (which is a consequence of the second law of thermodynamics under the hypothesis of a tangentially equivalent elastic structure, Bažant and Cedolin, 1991, Sec. 10.1), stability requires that $\delta^2\Pi$ be positive definite. The limit of stability is reached when $\delta^2\Pi = 0$ becomes possible. To simplify the calculations, let us eliminate displacement u_s by substituting Eq. (26) into (27) and (28). This yields

$$f[w(x)] = - \int_{a_0}^a \bar{R}^{ww}(x, x') w(x') dx' + \bar{R}^{uw}(x) u \quad (30)$$

$$\bar{K}_u + \bar{K}_w = 0, \quad \bar{K}_u = \bar{k}_u u, \quad \bar{K}_w = \int_{a_0}^a \bar{k}_w(x) w(x) dx. \quad (31)$$

Here we introduced the notations

$$\bar{R}^{ww}(x, x') = R^{ww}(x, x') - \frac{R^{uw}(x) R^{uw}(x')}{R^{uu} + R_f}, \quad (32)$$

$$\bar{R}^{uw}(x) = \frac{R_f}{R^{uu} + R_f} R^{uw}(x) \quad (32)$$

$$\bar{k}_w = k_w(x) + \frac{R^{uw}(x) k_u}{R^{uu} + R_f}, \quad \bar{k}_u = \frac{R_f k_u}{R^{uu} + R_f} \quad (33)$$

which have the meaning of stiffnesses and unit stress intensity factors of the system of the structure and the loading frame combined. The stability limit is reached when $\delta^2 \Pi = 0$ for some variation $\delta w(x)$ and δa . This means that

$$\begin{aligned} \delta(\delta_w \Pi) &= \int_{a_0}^a \frac{df[w(x)]}{dw} [\delta w(x)]^2 dx \\ &+ \int_{a_0}^a \int_{a_0}^a \bar{R}^{ww}(x, x') \delta w(x) \delta w(x') dx' dx \\ &+ \frac{\bar{K}_u + \bar{K}_w}{E'} \int_{a_0}^a \bar{k}_w(x) \delta w(x) dx \delta a = 0 \quad (34a) \end{aligned}$$

$$\begin{aligned} \delta(\delta_a \Pi) &= \frac{\bar{K}_u + \bar{K}_w}{E'} \int_{a_0}^a \bar{k}_w(x) \delta w(x) dx \delta a \\ &+ \frac{\bar{K}_u + \bar{K}_w}{E'} \frac{\partial(\bar{K}_u + \bar{K}_w)}{\partial a} (\delta a)^2 = 0. \quad (34b) \end{aligned}$$

We find that Eq. (34b) is automatically satisfied because $\bar{K}_u + \bar{K}_w = 0$. Furthermore, denoting $\mu(x) = \delta w(x)/\delta a$, we get from Eq. (34a) the condition for the loss of stability under displacement control occurs:

$$- \frac{df[w(x)]}{dw} \mu(x) = \int_{a_0}^a \bar{R}^{ww}(x, x') \mu(x') dx'. \quad (35)$$

Similar to Eq. (18), this is again a homogeneous Fredholm integral equation for function $\mu(x)$. It is linear if function $f(w)$ is linear.

The special case for $R_f \rightarrow \infty$ (with $u = u_s$) represents direct displacement control of the structure (no flexible loading frame). In that case Eq. (35) decides the maximum deflection of the structure, i.e., the onset of snapback instability.

The special case $R_f \rightarrow 0$ (with $u \rightarrow \infty$) represents loading of the structure under load control. Of course, the case of load control can also be obtained directly. To that end, one must set $R_f = 0$, $u_s = u$ and add to Eq. (22) the term $\Pi_l = -Pu$ representing the potential energy of dead load.

9 Conclusion

Under the assumption of no unloading anywhere within the crack, the cohesive crack model can be formulated in terms of minimization of either the complementary energy or the potential energy of the system. Using the relation of the compliance or stiffness derivatives with respect to the crack length to the unit stress intensity factors, the minimum condition yields the usual compatibility or equilibrium relations for the opening dis-

placements and the cohesive (crack-bridging) stresses in the cohesive crack, and the condition of zero stress intensity factor at the crack tip. The energy formulation also provides the conditions for the loss of stability of a structure with a growing cohesive crack. They have the form of a homogeneous Fredholm integral equation for the derivative of the cohesive stresses or crack opening displacements with respect to the crack length. The variational equations obtained can be used for formulation or approximate solutions.

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Stability of Cohesive Crack Model: Part II—Eigenvalue Analysis of Size Effect on Strength and Ductility of Structures

Z. P. Bažant

Walter Murphy Professor of Civil Engineering and Material Science, Fellow ASME.

Yuan-Neng Li

Research Assistant Professor.

Department of Civil Engineering, Northwestern University, Evanston, IL 60208

The preceding paper is extended to the analysis of size effect on strength and ductility of structures. For the case of geometrically similar structures of different sizes, the criterion of stability limit is transformed to an eigenvalue problem for a homogeneous Fredholm integral equation, with the structure size as the eigenvalue. Under the assumption of a linear softening stress-displacement relation for the cohesive crack, the eigenvalue problem is linear. The maximum load of structure under load control, as well as the maximum deflection under displacement control (which characterizes ductility of the structure), can be solved explicitly in terms of the eigenfunction of the aforementioned integral equation.

1 Introduction

As explained in the preceding paper (Bažant and Li, 1995), the cohesive crack model is a nonlinear theory of fracture mechanics in which the condition of stability limit is expressed in terms of the singularity condition of the second variation of the energy potential with respect to cohesive stresses or crack-opening displacements. Although the criterion of stability limit can also be formulated in terms of energy variation with respect to the crack length, the resulting equation is not very useful, since the energy release rate in the cohesive crack model depends on the cohesive stresses or crack-opening displacements.

For a given structure, the criterion of stability limit leads to a highly nonlinear equation for crack length. However, when a class of geometrically similar structures of different sizes is considered and the relative crack length is given, the criterion of stability limit can be treated as an equation for the structure size at which the stability limit occurs at the given relative crack length. In this manner, the criterion of the stability limit is transformed into an eigenvalue problem, with the structure size as the eigenvalue. In the special case of linear softening, the eigenvalue problem is linear. It can be solved independently of the solution of the cohesive crack model. Furthermore, the corresponding maximum value of the load or loading parameter can be expressed explicitly in terms of the eigenfunction. In this way, the size effect curve can be obtained readily, without having to calculate the load-deflection curves for structures of various sizes.

The eigenvalue problem of the cohesive crack model was studied by Li and Hong (1992), Li and Liang (1993) and Li and Bažant (1993). However, only the peak-load solution was discussed in these previous papers. In the present paper, the

influence functions are used to formulate the condition of stability limit of a structure with a cohesive crack in the form of a homogenous Fredholm integral equation. The peak load, as well as the maximum displacement (which corresponds to snap-back instability), is obtained. In addition, the cases of a structure loaded through a spring coupled in series (i.e., the case of a soft loading device) and a structure restrained by a spring coupled in parallel are analyzed. Finally, some computational techniques are discussed and a numerical example of the size effect curves for maximum deflection is given.

2 Dimensionless Process Zone Equations

We consider a two-dimensional structure of a unit thickness and introduce the following dimensionless variables:

$$\bar{C} = CE, \quad \alpha = \frac{a}{D}, \quad \xi = \frac{x}{D}, \quad \sigma = \frac{\sigma}{f_t},$$

$$\bar{w} = \frac{w}{w_c}, \quad \bar{P} = \frac{P}{Df_t}, \quad \bar{D} = \frac{D}{2L_0} \quad (1)$$

where $L_0 = EG_f/f_t^2$ = characteristic size of the process zone, f_t = direct tensile strength of the material, and w_c = threshold value of the crack-opening displacement. All the notations from the preceding paper (Bažant and Li, 1994) are retained. To simplify notations in the following text, we will drop the bars, with the understanding that all the variables are dimensionless unless specified otherwise.

For a generic elastic structure, the crack-opening displacement w , the load-line displacement u , the load P , and the crack-bridging stress σ must satisfy the compatibility equations:

$$\frac{w(\xi)}{D} = - \int_{\alpha_0}^{\alpha} C^{\sigma\sigma}(\xi, \xi') \sigma(\xi') d\xi' + C^{\sigma P}(\xi) P \quad (2)$$

$$\frac{u}{D} = - \int_{\alpha_0}^{\alpha} C^{P\sigma}(\xi') \sigma(\xi') d\xi' + C^{PP} P \quad (3)$$

which represent the special case of Eqs. (11) and (10) or (14) of the preceding paper for $C_f = 0$; $C^{\sigma\sigma}(\xi, \xi')$, $C^{\sigma P}(\xi)$, $C^{P\sigma}(\xi)$, C^{PP} are dimensionless compliance influence functions (Green's functions). The lower integration limit α_0 is the relative length

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of the initial traction-free crack (notch); α is the total relative crack length which includes both the process zone (crack-bridging zone) and the stress-free crack. The problem can also be formulated as equilibrium conditions written in terms of stiffness influence functions:

$$D\sigma(\xi) = \int_{\alpha_0}^{\alpha} R^{ww}(\xi, \xi') w(\xi') d\xi' + R^{wu}(\xi) u \quad (4)$$

$$DP = \int_{\alpha_0}^{\alpha} R^{uw}(\xi) w(\xi) d\xi + R^{uu} u. \quad (5)$$

These equations represent the special case of Eqs. (26) and (25). Equation (4) for prescribed load P ensues by solving u from Eq. (5) and substituting it into Eq. (4). The dimensionless stiffness functions are here defined with a unit value of Young's modulus.

In the cohesive crack model, the cohesive stress σ is related to the crack-opening displacement w by the stress-displacement relation, which can be described by either of the following forms

$$w = g(\sigma), \quad \sigma = f(w). \quad (6a, b)$$

Substituting (6a) into (2), we obtain what we call the crack compatibility equation in terms of compliance functions:

$$\frac{1}{D} g[\sigma(\xi)] = - \int_{\alpha_0}^{\alpha} C^{\sigma\sigma}(\xi, \xi') \sigma(\xi') d\xi' + C^{\sigma P}(\xi) P \quad (7)$$

Substituting (6b) into (4), we obtain the crack equilibrium equation in terms of stiffness functions:

$$Df[w(\xi)] = - \int_{\alpha_0}^{\alpha} R^{ww}(\xi, \xi') w(\xi') d\xi' + R^{wu}(\xi) u. \quad (8)$$

3 Peak-Load Solution by the Condition of Structural Stability Limit

As established in Bažant and Li (1995), the singularity condition for the compliance formulation under load control can be expressed as the condition of finding a nonzero solution $v(\xi)$ of the following homogenous equation:

$$D \int_{\alpha_0}^{\alpha} C^{\sigma\sigma}(\xi', \xi) v(\xi') d\xi' = - \frac{dg[\sigma(\xi)]}{d\sigma} v(\xi). \quad (9)$$

Since we are considering geometrically similar structures only, (9) can be regarded as an eigenvalue problem if the relative crack length α is given. The dimensionless quantity D plays the role of an eigenvalue. In the actual calculation, the singularity condition should be solved simultaneously with the basic equations to obtain the nominal strength as the maximum load parameter and the corresponding size for a given relative crack length. Calculation of size effect curves in this manner is very efficient. A discussion of the discrete form of the present formulation has been given by Li and Bažant (1994).

In the following, we restrict attention to the case of linear softening, which is defined as

$$w = g(\sigma) = 1 - \sigma, \quad \sigma = f(w) = 1 - w. \quad (10)$$

Since for linear softening $dg/d\sigma = -1$, the eigenvalue is no longer coupled with the basic equations of the cohesive crack. The eigenvalue problem can now be written as

$$D \int_{\alpha_0}^{\alpha} C^{\sigma\sigma}(\xi', \xi) v(\xi') d\xi' = v(\xi). \quad (11)$$

If the relative crack length is specified and geometrically similar structures are considered, Eq. (11) represents a linear homogeneous Fredholm integral equation (Tricomi, 1957) for function

$v(\xi)$, with size D as the eigenvalue. The size D for which the given α corresponds to the maximum load is the largest eigenvalue of (11). This approach, proposed by Li and Bažant (1994), makes it possible to avoid solving the load-deflection curves for various sizes D . It represents an efficient method of calculating the size effect curve.

The dimensionless crack compatibility equation can be written as

$$\frac{1 - \sigma(\xi)}{D} = - \int_{\alpha_0}^{\alpha} C^{\sigma\sigma}(\xi, \xi') \sigma(\xi') d\xi' + C^{\sigma P}(\xi) P. \quad (12)$$

Multiplying this with the eigenfunction $v(\xi)$ and then integrating with respect to ξ , we obtain

$$\int_{\alpha_0}^{\alpha} \left[\frac{1}{D} - C^{\sigma P}(\xi) P \right] v(\xi) d\xi = \int_{\alpha_0}^{\alpha} \sigma(\xi) \int_{\alpha_0}^{\alpha} \left[\frac{\delta(\xi - \xi')}{D} - C^{\sigma\sigma}(\xi, \xi') \right] v(\xi') d\xi' d\xi. \quad (13)$$

If the singularity condition is satisfied, then the applied load is at its maximum. This maximum value is found to be

$$P = \frac{1}{D} \frac{\int_{\alpha_0}^{\alpha} v(\xi) d\xi}{\int_{\alpha_0}^{\alpha} C^{\sigma P}(\xi) v(\xi) d\xi}. \quad (14)$$

An equivalent expression for the peak load was obtained by Li and Hong (1992), and by Li and Bažant (1994). The eigenvalue problem (11) and the peak load solution (14) provide a powerful set of equations for solving the size-effect curve of the cohesive crack model directly, without any need to solve the load-deflection curve from the basic equations.

The solution can also be generalized to include the case of multiple (conservative) loads. They can vary arbitrarily but in such a manner that there is no crack closure. Then the relation among the load values at the stability limit of the structure is linear. For instance, when a beam is subjected to combined action of lateral load P and axial load N , as shown in Fig. 1, the crack compatibility equation can be written as

$$\frac{1 - \sigma(\xi)}{D} = - \int_{\alpha_0}^{\alpha} C^{\sigma\sigma}(\xi, \xi') \sigma(\xi') d\xi' + C^{\sigma P}(\xi) P + C^{\sigma N}(\xi) N \quad (15)$$

where the symbols are self-explanatory. Since the loading terms do not enter the criterion of stability limit, the equation for the structural stability limit remains the same. If the condition for the stability limit is satisfied, the relation between these two loads is found to be linear:

$$\frac{P}{P^*} + \frac{N}{N^*} = 1 \quad (16)$$

where the denominators, defined as

$$P^* = \frac{1}{D} \frac{\int_{\alpha_0}^{\alpha} v(\xi) d\xi}{\int_{\alpha_0}^{\alpha} C^{\sigma P}(\xi) v(\xi) d\xi}$$

$$N^* = \frac{1}{D} \frac{\int_{\alpha_0}^{\alpha} v(\xi) d\xi}{\int_{\alpha_0}^{\alpha} C^{\sigma N}(\xi) v(\xi) d\xi}, \quad (17)$$

represent the critical loads when P and N are applied to the structure separately. Equation (16) is the general interaction relation when the structure fails by tensile fracture and the

softening stress-displacement law is linear. A relation of this type was also reported by Li, Müller, and Wörner (1994) in a discrete (matrix) form. Generalization to an arbitrary number of applied loads is self-evident.

When the stress-displacement relation for a cohesive crack is nonlinear, one can use an iterative succession of linear approximations representing tangents of the stress-displacement curve according to the preceding approximation (this approach was formulated for the maximum load in Li and Bažant, 1994).

4 Solution of Maximum Deflection

If the structure is loaded by controlled displacement (i.e., with a rigid grip), the stability limit is reached when there is a snap back in the diagram of load P versus load-line displacement u . The crack equilibrium equation for this case is Eq. (30) of the preceding paper which, in the case of linear softening, yields

$$[1 - w(\xi)]D = - \int_{\alpha_0}^{\alpha} R^{ww}(\xi, \xi') w(\xi') d\xi' + R^{wu}(\xi) u. \quad (18)$$

The dimensionless condition of stability limit may now be written as

$$\frac{1}{D} \int_{\alpha_0}^{\alpha} R^{ww}(\xi, \xi') v(\xi) d\xi = v(\xi'). \quad (19)$$

Since α is constant for geometrically similar structures, (19) is a linear homogeneous Fredholm integral equation for the unknown cohesive stress $v(\xi)$ in the process zone. This represents an eigenvalue problem with $1/D$ as the eigenvalue. Only the smallest eigenvalue $1/D$ represents a stability limit. The maximum deflection, characterizing snap back, is found to be

$$u = \frac{D \int_{\alpha_0}^{\alpha} v(\xi) d\xi}{\int_{\alpha_0}^{\alpha} S^{wu} v(\xi) d\xi}. \quad (20)$$

However, the maximum deflection can also be solved in terms of the compliance functions. To this end, we eliminate the load parameter P from (3) and (7) and obtain the following crack compatibility equation under displacement control:

$$1 - \sigma(\xi) = -D \int_{\alpha_0}^{\alpha} \tilde{C}^{\sigma\sigma}(\xi, \xi') \sigma(\xi') d\xi' + \frac{C^{\sigma P}(\xi)}{C^{PP}} u \quad (21)$$

where

$$\tilde{C}^{\sigma\sigma}(\xi, \xi') = C^{\sigma\sigma}(\xi, \xi') - C^{\sigma P}(\xi) C^{P\sigma}(\xi') \frac{1}{C^{PP}}. \quad (22)$$

The corresponding eigenvalue problem now becomes

$$v(\xi) = D \int_{\alpha_0}^{\alpha} \tilde{C}^{\sigma\sigma}(\xi', \xi) v(\xi') d\xi'. \quad (23)$$

This is equivalent to the eigenvalue problem (19) of stiffness formulation, because the modified compliance function is the inverse of the stiffness function R^{ww} . The maximum deflection can be expressed as

$$u = C^{PP} \frac{\int_{\alpha_0}^{\alpha} v(\xi) d\xi}{\int_{\alpha_0}^{\alpha} C^{\sigma P} v(\xi) d\xi}. \quad (24)$$

The compliance formulation is of course equivalent to the stiffness formulation. In a similar way, we can also express the maximum load in terms of the stiffness influence functions. The details will not be given because the derivation is analogous.

5 Stability Limit of Structure Loaded Through a Spring

If the device that controls loading (e.g., the testing machine) has finite compliance C_f , the device can be represented as a spring connected to the structure in series. In such a connection, the device and the structure share the same force. Denote as u the total deflection that is controlled, which is the sum of the deflection u_s of the structure and the deflection of the device $u - u_s$. Using (3), we can solve load P in terms of u as

$$P = (C^{PP} + C^f)^{-1} \left[\frac{u}{D} - \int_{\alpha_0}^{\alpha} C^{P\sigma}(\xi) \sigma(\xi) d\xi \right] \quad (25)$$

In the dimensionless form, the process zone equation is

$$1 - \sigma(\xi) = -D \int_{\alpha_0}^{\alpha} \tilde{C}^{\sigma\sigma}(\xi, \xi') \sigma(\xi') d\xi' + C^{\sigma P}(\xi) (C^{PP} + C_f)^{-1} u \quad (26)$$

where

$$\tilde{C}^{\sigma\sigma}(\xi, \xi') = C^{\sigma\sigma}(\xi, \xi') - C^{\sigma P}(\xi) (C^{PP} + C_f)^{-1} C^{P\sigma}(\xi'). \quad (27)$$

The form of the eigenvalue problem is the same as (23) except that the modified compliance function is defined by (27). The maximum deflection is found to be

$$u = (C^{PP} + C_f) \frac{\int_{\alpha_0}^{\alpha} v(\xi) d\xi}{\int_{\alpha_0}^{\alpha} C^{\sigma P} v(\xi) d\xi}. \quad (28)$$

This formula reduces to (24) when compliance C_f approaches zero.

On the other hand, if the spring is connected to the structure in parallel, it shares the same deflection with the structure. Denote by P the total load applied to the structure-spring system, which is the sum of the load P_s which acts on the structure and $S_f u$ where $S_f = 1/C_f$. Using (5) we can express u in terms of P as

$$u = \left[\int_{\alpha_0}^{\alpha} R^{uw}(\xi) w(\xi) d\xi + DP \right] (R^{uu} + R_f)^{-1}. \quad (29)$$

Substituting (29) into (8), one obtains the following crack compatibility equation:

$$1 - w(\xi) = - \frac{1}{D} \int_{\alpha_0}^{\alpha} \tilde{R}^{ww}(\xi, \xi') w(\xi') d\xi' + R^{wu}(\xi) P (R^{uu} + R_f)^{-1} \quad (30)$$

where the modified stiffness function is defined as

$$\tilde{R}^{ww}(\xi, \xi') = R^{ww}(\xi, \xi') - R^{wu}(\xi) R^{uw}(\xi') (R^{uu} + R_f)^{-1}. \quad (31)$$

The eigenvalue problem is to find a nonzero eigenfunction satisfying

$$v(\xi) = \frac{1}{D} \int_{\alpha_0}^{\alpha} \tilde{R}^{ww}(\xi, \xi') v(\xi') d\xi'. \quad (32)$$

The maximum total applied load can be calculated from the following equation:

$$P = (R^{uu} + R_f) \frac{\int_{\alpha_0}^{\alpha} v(\xi) d\xi}{\int_{\alpha_0}^{\alpha} R^{wu} v(\xi) d\xi}. \quad (33)$$

Of course when the spring constant of the connected spring

approaches zero, (33) becomes the peak-load solution in the stiffness formulation without a spring.

6 Numerical Implementation

As a numerical example, a three-point bent fracture specimen (Fig. 1) is analyzed. The finite element method is used to obtain the compliance functions in a discretized form (although other methods, such as the boundary element method, might also be suitable). The four-node finite element, which is the simplest, is chosen to discretize one half of the beam. To determine the nodal compliance matrix, the displacement solutions are obtained for one unit load applied successively at each node along the potential crack path or at the load point.

Each column of matrix $C^{\sigma\sigma}$ represents nodal displacements on the crack line when a unit load is applied to one node in the process zone, $C^{\sigma P}$ represents the nodal displacements in the process zone when a unit load is applied at the load point, and C^{PP} represents the load-line displacement when a unit load is applied at the load point. During the calculation, the total relative crack length α is first taken to correspond to the node that is farthest from the crack mouth as allowed by the compliance matrix, and then cracks reaching successively to nodes closer and closer to the crack mouth are considered. In each case the nodal displacements that lie in the uncracked ligament are eliminated. In this way, the dependence of the compliance function on the crack length is reflected by the sizes of the compliance matrices.

Starting with Hillerborg (1976), the zero- K condition has been approximated by the condition that the elastic stress ahead of the cohesive crack tip be equal to the tensile strength. So in our dimensionless definition, $\sigma_{tip} = 1$. In the space of continuous functions, this condition is mathematically equivalent to the condition that the stress intensity factor K at the crack tip be zero (Barenblatt, 1962). After discretization, however, these two conditions are equivalent only approximately. Thus the use of the condition $\sigma_{tip} = 1$ inevitably introduces additional numerical error into the discrete solution. But this small price is quite justifiable, because we do not need the corresponding stress intensity factors, which are not easy to calculate anyway.

However, numerical results (Li and Bažant, 1994) show that, in order to obtain good accuracy for large (dimensionless) structural sizes, it seems important to assume the cohesive stress to vary linearly from node to node in the process zone, rather than a piece-wise constant manner. The assumption of linear variation of cohesive stress between the nodes leads to a tridiagonal matrix connecting the nodal values of cohesive stresses to the cohesive nodal forces (in detail, see Li and Bažant, 1994). Numerically, the differences in the maximum load values calculated by the eigenvalue analysis and by the load-deflection curves are usually in the fifth or sixth digit for linear (or nearly linear) softening laws.

7 Numerical Solution of the Maximum Deflection

Numerical examples for the peak load solution using the eigenvalue approach have been given in previous papers (e.g., Li and Hong, 1992; Li and Bažant, 1994). Therefore, we will discuss only the numerical solution of maximum deflection, which characterizes ductility of a structure. Although the maximum load solution and the maximum deflection solution are

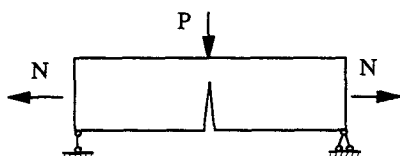


Fig. 1 Beam under combined lateral load and axial load

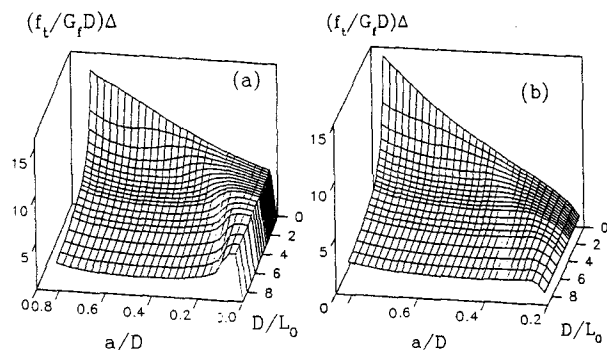


Fig. 2 Dimensionless deflection for (a) $\alpha_0 = 0$ and (b) $\alpha_0 = 0.2$

mathematically similar, there exists one important difference. For three-point-bent beams, the maximum load always exists, no matter how large the relative process zone length $\alpha - \alpha_0$ is, or how small the dimensionless size D/L_0 is. However, for maximum deflections, the situation is different. As shown in Fig. 2, there is no maximum deflection if the relative length α of the cohesive crack is large enough. The smallest dimensionless size D below which there is no snap back will be called the critical size of the structure. The critical size is a function of relative notch depth α_0 as well as the slenderness ratio (span-to-depth ratio of the beam).

The dependence of the critical size on the relative notch depth can also be seen in Fig. 2. Fig. 2(a) gives the deflection for beams without a notch ($\alpha_0 = 0$), and Fig. 2(b) for beams with relative notch depth $\alpha_0 = 0.2$. For $\alpha_0 = 0$, the critical size is found to be approximately 0.43 and for $\alpha_0 = 0.2$ approximately 1.4.

According to Eqs. (23) and (24), we can obtain the size effect curves for maximum deflection for any given relative length α of the cohesive crack. Figure 3 shows the size effect curves for different initial notch ratios. Note that, paradoxically, the curves extend even to the left of the critical sizes (dashed lines); these portions of the curves are of course physically meaningless since there exists no maximum deflection at all. The explanation is that these portions correspond to cases with negative σ , whereas our analytical expressions are valid only when the crack-opening displacement in the process zone is less than the crack-opening threshold w_c (at which the stress is reduced to zero). With careful observation, one finds that, when the condition of stability limit is satisfied, the critical size D is actually the size at which the crack-mouth-opening displacement becomes equal to the threshold w_c . Above the critical size (i.e., on right portions of the curves in Fig. 3), the obtained maximum deflections are exactly what one would obtain if the load-deflection curve were solved by the conventional method, that is, by solving the basic equations step by step for each different cohesive crack lengths.

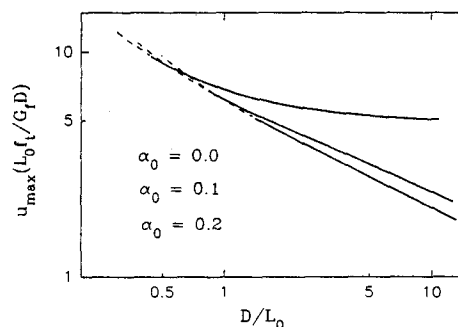


Fig. 3 Size effect curve for maximum deflection

As a check, we select, from the size effect curve, a maximum deflection value with its corresponding relative crack length α and its dimensionless size D . Then we use this dimensionless size as the input and solve the process zone equation together with the crack-tip equation ($\sigma_{tip} = 1$) for different crack lengths. In all the cases examined, the maximum deflection is found to be the same (within the numerical precision of the calculation) and to occur at the same relative crack length.

8 Final Remarks and Conclusions

The cohesive crack model can be effectively analyzed in terms of continuous influence functions. Under the assumption of a linear softening stress-displacement law, the criterion of stability limit, which has been analyzed by Bažant and Li (1995), becomes a linear eigenvalue problem when geometrically similar structures are considered. The peak value of the load parameter can be determined by solving the eigenvalue problem. In this manner, the size effect of the cohesive crack model becomes intimately related to the solutions of the eigenvalue problem. There are some similarities between the eigenvalue problem studied here and the eigenvalue problem for the buckling load of a structure. Both eigenvalue problems are derived from the criterion of structural stability limit. Whereas, in the buckling problem, the eigenvalue is Euler's critical load, in the cohesive crack model the eigenvalue is the structure size for which the loading parameter is maximized at a given relative cohesive crack length. The maximum load or load parameter can be calculated from the eigenfunctions. The following conclusions can be drawn:

- 1 When geometrically similar structures are considered, the criterion of stability limit becomes an eigenvalue problem. The size for which a given relative crack length corresponds to either the maximum load or the maximum displacement is the first eigenvalue of a homogeneous Fredholm integral equation. The size effect curve can thus be calculated efficiently.

- 2 If the softening stress-displacement law for the cohesive crack is linear, the eigenvalue problem becomes linearized and can be solved independently. The critical value of the loading parameter (either the maximum load or the maximum load-

line deflection), can be determined through the eigenfunction obtained.

- 3 Numerical examples of the solution of the maximum deflection as a function of the dimensionless beam depth demonstrate that the maximum deflection solution ceases to be valid if the structure dimension (e.g., beam depth) becomes smaller than a certain critical value. This critical value is characterized by the condition that the crack opening at the stability limit reaches the threshold value at which the cohesive stress gets reduced to zero.

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