J.DIFFERENTIAL GEOMETRY
Vol. 43, No.2 March, 1996

STABILITY OF COMPLEX VECTOR BUNDLES

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0. Introduction

The notion of stability plays a central role in complex and algebraic geometry.

It was introduced by D. Mumford [5] and F. Takemoto [10] for the study of the moduli space of holomorphic vector bundles; S. Kobayshi and M. Lübke found that for irreducible bundles the existence of a Hermitian-Einstein metric is a sufficient condition for stability, and a major achievement of the theory has consisted in the work of M. Narashimhan and C. Seshadri for algebraic curves, S. Donaldson in the case of algebraic manifolds, K. Uhlenbeck and S.T. Yau for general Kähler manifolds (easily extended to regularized Hermitian n-manifolds, i.e., whose Kähler form η satisfies $\partial\overline{\partial}\eta^{n-1} = 0$) proving the existence of a Hermitian-Einstein connection on stable holomorphic vector bundles ([6], [1], [12]). Further generalization to Higgs bundles can be found in [2] and [9].

These results have made the tools of differential geometry available to complex and algebraic geometry, leading to several important applications, e.g., a much more extensive comprehension of Bogomolov-Gieseker type inequalities and the characterization of flat vector bundles. On the other hand, a general theory of the existence of holomorphic structures on complex bundles is far from being understood, and therefore it is very natural to try to extend the differential geometric characterization of stability to complex bundles with an unnecessarily integrable almost complex structure.

The first main result of the present paper is the following.

Theorem 0.1. *Assume a complex vector bundle over a compact almost Hermitian regularized manifold is equipped with a stable almost complex structure. Then it admits a Hermitian-Einstein connection.*

The notion of stability which we consider is the following: we require that $\mu(F) < \mu(E)$ holds for any J-holomorfic subbundle $F \subset E$ which

Received December 10, 1993.

is regular outside a set of Hausdorff codimension at least four and can be extended across the singular set along any local J-holomorphic curve not contained in the singular set. We conjecture that this notion of stability which is equivalent to the one obtained by considering F ranging on J-holomorphic subbundles of $p^{-1}(E)$ where $p : \tilde{M} \longrightarrow M$ is a modification. In the case where the base manifold is complex and E is a holomorphic bundle, this follows from the results of [12]. We expect to prove that this is also the case where the base manifold is two complex dimensional. The plan of th paper is the following: in sections 1 and 2 we extensively investigate the notion of bundle almost complex structure (**bacs**); in section 3, looking for the best bacs, we decompose the Yang-Mills functional YM, obtaining that critical points are characterized by the condition $2\overline{\partial}_{\omega}^{*}\Omega_{\omega}^{0,2} + 2A^{*}\Omega_{\omega}^{2,0} - i\overline{\partial}_{\omega}H_{\omega} = 0$ and, moreover, $YM(\omega) \geq \epsilon(E)$ (a topological constant) with equality if and only if $\Omega_{\omega}^{0,2} = 0$ and $H_{\omega} = 0$; therefore the Hermite-Einstein condition $H_{\omega} = 0$ arises naturally in the search of minima for YM. In section 4 we define stable **bacs** and we start the proof of our main theorem: by utilizing an improved version of Uhlenbeck-Yau and Simpson's techniques, we fix a Hermitian structure h, consider the evolution equation $h_t^{-1}\frac{d}{dt}h_t = -H_t$, and show that the solution converges to a Hermite-Einstein structure, unless a flag of weakly J-holomorphic subbundles is produced, one of which contradicts the stability assumption. The proof here follows from the arguments in [9], but several modifications are needed, due to the nonintegrability of the base manifold. The end of the proof depends on the regularity results for weakly J-holomorphic subbundles. We obtain this as a consequence of a regularity theory for weakly J-holomorphic map developped in section 5. In particular we prove the following.

Theorem 0.2. *Let $(M, J_M, g), (N, J_N, h)$ be two almost Hermitian manifolds with $\dim_R M = 2n$, and assume there exists a bounded closed 2-form α on N such that $\alpha^{1,1} > 0$ uniformly. Let $\sigma : M \longrightarrow N$ be a L_1^2-weakly (J_M, J_N)-holomorphic map. Then there exists a closed subset $S \subset M$ with $\mathcal{H}_{2n-4}(S) < +\infty$, such that σ is smooth on M/S; moreover, for any $x \epsilon S$, any local J-holomorphic curve K through x not contained in S, $\sigma_{|K-\{x\}}$ extends smoothly to K.*

The proof of Theorem 0.2 uses some ideas from [8]. Note that (N, J_N, h, α) is a tamed symplectic manifold in the terminology of [3]. We also prove that, if the target manifold has no rational curves, then a L_1^2-weakly J-holomorphic map is regular. In case $\dim_{BR} M = 4$, we expect to prove in a forthcoming paper that there exists a modification $\tilde{M}$ of M, obtained by blowing up M succesively at isolated points, such that σ can be extended to $\tilde{M}$ to be a smooth J-holomorphic map.

The difficultes with higher dimensional cases are:

1. how to prove that the singular set is a J-invariant subvariety in a suitable sense;

2. how to prove Hironaka's theorem for resolution of singularities in the nonintegrable case.

The first author is partially supported by Italian murst, and the second author by a grant from NSF and an Alfred Sloan fellowship.

1. Bundle almost complex structures

In this paragraph we gather some basic definitions and facts about bundle almost complex structures. Let (M, J_M) be a n-dimensional almost complex manifold.

Definition 1.1. A complex vector bundle $(E, \hat{J})$ of (complex) **rank** r over M is a real vector bundle E of **rank** $2r$ over M equipped with a section $\hat{J}$ of $\mathrm{End}(E)$ such that $\hat{J}^2 = -id_E$.

Given a complex vector bundle E of **rank** r, we can consider the principal $GL(r, \mathbb{C})$-bundle $C(E)$ of complex linear frames on E; thus

$E = C(E) \times_{GL(r,\mathbb{C})} \mathbb{R}^{2r}$, where $GL(r, \mathbb{C})$ acts on $\mathbb{R}^{2r}$ via the standard real representation $\rho : GL(r, \mathbb{C}) \to GL(2r, \mathbb{R})$;

Definition 1.2. A bundle almost complex structure (bacs) on $C(E)$ is an almost complex structure J on $C(E)$ such that:

(1) the bundle protection $\pi : C(E) \to M$ is (J, J_M)-holomorphic;
(2) J induces the standard integrable almost complex structure J_S on the fibres;
(3) $GL(r, \mathbb{C})$ acts J-holomorphically on $C(E)$.

$\mathcal{B}(C(E))$ will denote the set of bacs on $C(E)$.

We can define

$$\mathcal{T}^{p,q}(C(E)) := L^{-1}(\wedge^{p,q}(E)),$$

where $L : \mathcal{T}^*(C(E)) \to \wedge^*(E)$ is the standard isomorphism between tensorial $\mathbb{R}^{2r}$-valued forms on $C(E)$ and E-valued forms on M (cf. [4]), therefore we have

$$\mathcal{T}^r(C(E)) = \bigoplus_{p+q=r} \mathcal{T}^{p,q}(C(E)). \tag{1.2.1}$$

It is easy to check that, if a bacs is assigned on $C(E)$, then (1.2.1) corresponds precisely to the induced decomposition.

Let $\hat{\mathcal{H}}(C(E))$ be the set of all linear differential operators $\overline{\partial}_{C(E)}$: $\mathcal{T}^{p,q}(C(E)) \to \mathcal{T}^{p,q+1}(C(E))$ satisfying the following $\overline{\partial}$-Leibnitz rule: for every $f \;\varepsilon\; C^\infty(M), \alpha \;\varepsilon\; \mathcal{T}^{p,q}(C(E))$

$$\overline{\partial}_{C(E)}\pi^*(f)\alpha = \pi^*(\overline{\partial}_M f) \wedge \alpha + \pi^*(f)\overline{\partial}_{C(E)}\alpha.$$

We have the following.

Proposition 1.3. *Given a $J \;\varepsilon\; \mathcal{B}(C(E))$ the induced operator $\overline{\partial}_J$ maps $\mathcal{T}^{p,q}(C(E))$ into $\mathcal{T}^{p,q+1}(C(E))$ and so, in particular, it belongs to $\mathcal{H}(C(E))$; vice versa, given $\overline{\partial}_{C(E)} \;\varepsilon\; \hat{\mathcal{H}}(C(E))$, there exists a unigue $J \;\varepsilon\; \mathcal{B}(C(E))$ such that $\overline{\partial}_J = \overline{\partial}_{C(E)}$. Then the map $J \mapsto \overline{\partial}_J$ is a bijection between $\mathcal{B}(C(E))$ and $\hat{\mathcal{H}}(C(E))$.*

Proof. Assume $J \;\varepsilon\; \mathcal{B}(C(E))$ is given; since $\mathcal{T}^{p,q}(C(E))$ is locally generated by elements of the form $\pi^*(\alpha) \otimes f$ for $\alpha \;\varepsilon\; \wedge^{p,q}(M)$ and $f \;\varepsilon\; \mathcal{T}^0(C(E))$, it is enough to show that

$$\overline{\partial}_J : \mathcal{T}^0(C(E)) \to \mathcal{T}^{0,1}(C(E)),$$

which follows immediately from the fact that $f \;\varepsilon\; \mathcal{T}^0(C(E))$ is holomorphic when restricted to the fibres. Vice versa, assume $\overline{\partial} = \overline{\partial}_{C(E)} \;\varepsilon\; \hat{\mathcal{H}}(E)$ is given. Then an almost complex structure J on $C(E)$ is uniquely defined by means of the relations

$$\text{for every } f \;\varepsilon\; \mathcal{T}^0(C(E)) \quad df(J(X)) = i(2\overline{\partial}f - df)(X).$$

It is easy to check that $J \;\varepsilon\; \mathcal{B}(C(E))$ and, by construction $\overline{\partial}_J = \overline{\partial}_{C(E)}$.

Now we have the following.

Lemma 1.4. *Let $J \;\varepsilon\; \mathcal{B}(C(E))$ and $\omega \;\varepsilon\; \mathcal{C}(C(E))$, where $\mathcal{C}(C(E))$ denotes the space of connection 1-forms on $C(E)$. Then*

$$\omega^{(0,1)} \;\varepsilon\; \mathcal{T}^{0,1}(C(E)), gl(r,\mathbb{C}), ad)$$

and consequently

$$\omega^{(1,0)} \;\varepsilon\; \mathcal{C}(C(E)).$$

Proof. Let $u \;\varepsilon\; C(E), X \;\varepsilon\; T_uC(E), a \;\varepsilon\; GL(r,\mathbb{C})$. Then we have

$$\begin{aligned}(R_a)^*(\omega^{(0,1)})[u](X) =& \omega^{(0,1)}[ua]((R_a)_*(X)) \\ =& \frac{1}{2}\omega[ua]((R_a)_*(X) + iJ(R_a)_*(X)) \\ =& \frac{1}{2}\omega[ua]((R_a)_*(X) + i(R_a)_*(JX)) \\ =& \frac{1}{2}ad(a^{-1})(\omega[u](X) + i\omega[u](JX)) \\ =& ad(a^{-1})\omega^{(0,1)}[u](X).\end{aligned}$$

Moreover, if $Y \; \varepsilon \; gl(r, C)$, then

$$\omega^{(0,1)}[u](Y^*) = \frac{1}{2}\omega[u](Y^* + iJ(Y^*)) = \frac{1}{2}\omega[u](Y^* - Y^*) = 0.$$

Consequently, we have

Proposition 1.5. *Given $\omega \; \varepsilon \; \mathcal{C}(C(E))$, there exists unique $J \; \varepsilon \; \mathcal{B}(C(E))$ for which ω is of type $(1,0)$.*

Proof. Let $u \; \varepsilon \; C(E)$ and $X \; \varepsilon \; T_u C(E)$, and write $X = X^{(h)} + X^{(r)}$ according to ω. Then define J as follows:

$$J[u](X) = ((\pi^{-1})_* \circ J_M \circ \pi_*)[u](X^{(h)}) + J_S[u](X^{(v)}).$$

It is clear that $J \; \varepsilon \; \mathcal{B}(C(E))$ and ω is of type (1,0) with respect to it; the uniqueness is obvious.

Therefore, we have just constructed a map $\chi : \mathcal{C}(C(E)) \longrightarrow \mathcal{B}(C(E))$; this is not injective but is surjective because of Lemma 1.4.

Definition 1.6. Given $J \; \varepsilon \; \mathcal{B}(C(E))$, we set

$$\mathcal{C}_J^{1,0}(C(E)) := \chi^{-1}(J);$$

i.e., $\mathcal{C}_J^{1,0}(C(E))$ is the set of all connection 1-forms in $C(E)$ that are of type (1,0) with respect to J.

By means of the previous result, we can easily prove the following statement, which has nothing to do with connections:

Proposition 1.7. *Let $J \; \varepsilon \; \mathcal{B}(C(E))$. Then its Nijenhuis tensor $N(J)$ is horizontal. Moreover, if J_M is integrable, then $N(J)$ is vertical-valued.*

Proof. Let $\omega \; \varepsilon \; \mathcal{C}_J^{1,0}(C(E))$.

a. If both X and Y are vertical, then

$$N(J)(X,Y) = N(J_S)(X,Y) = 0,$$

because J_S is integrable.

b. If X is vertical and Y is horizontal, then we can assume $X = A^*$ for $A \; \varepsilon \; gl(r, \mathbb{C})$ and $Y = \hat{Z}$ (horizontal lifting) for $Z \; \varepsilon \; \mathcal{H}(M)$. Since clearly $JY = (J_M Z)\hat{}$, we have

$$\begin{aligned} N(J)(X,Y) &= [JX, JY] - [X,Y] - J[JX,Y] - J[X,JY] \\ &= [(iA)^*, (J_M Z)\hat{}] - [A^*, \hat{Z}] - J[(iA)^*, \hat{Z}] - J[A^*, (J_M Z)\hat{}] \\ &= 0, \end{aligned}$$

this proves that $N(J)$ is horizontal. Moreover, if $X = \hat{Z}$ and $Y = \hat{W}$ for $Z, W \varepsilon \mathcal{H}(M)$, then, since $[\hat{Z}, \hat{W}] = [Z, W]\hat{} + [\hat{Z}, \hat{W}]^{(v)}$,

$$\begin{aligned} &N(J)(X,Y) \\ &= [(J_M Z)\hat{}, (J_M W)\hat{}] - [\hat{Z}, \hat{W}] - J[(J_M Z)\hat{}, \hat{W}] - J[\hat{Z}, (J_M W)\hat{}] \\ &= (N(J_M)(Z,W))\hat{} + \text{vert}. \end{aligned}$$

We have now

Proposition 1.8. *Let $J \varepsilon \mathcal{B}(C(E))$ and let $\omega \varepsilon \mathcal{C}_J^{1,0}(C(E))$. Then*

$$(D_\omega)^{0,1} = \overline{\partial}_J, \tag{1.8.1}$$

and consequently

$$D_\omega : \mathcal{T}^0(C(E)) \longrightarrow \mathcal{T}^1(C(E))$$

splits as

$$D_\omega = \partial_\omega + \overline{\partial}_J, \tag{1.8.2}$$

where $\partial_\omega := (D_\omega)^{1,0}$. More generally, we have that

$$D_\omega : \mathcal{T}^{p,q}(C(E)) \longrightarrow \mathcal{T}^{p+q+1}(C(E))$$

decomposes as

$$D_\omega = \partial_\omega + \overline{\partial}_J + A,$$

where $A : \mathcal{T}^{p,q}(C(E)) \longrightarrow \mathcal{T}^{p+2,q-1}(C(E)) \oplus \mathcal{T}^{p-1,q+2}(C(E))$ is a zero order operator, depending only on $N(J_M)$, and vanishing identically when J_M is integrable. In particular, if $\alpha \varepsilon \mathcal{T}^1(C(E))$, then $A(\alpha)(X,Y) = \alpha(N(J)(X,Y)) = \alpha(Z)$, where $\pi_*(Z) = N(J_M)(\pi_*(X), \pi_*(Y))$.

Proof. Given $\alpha \varepsilon \mathcal{T}^{p,q}(C(E))$, we have:

$$D_\omega^{0,1}\alpha = (D_\omega \alpha)^{p,q+1} = (d\alpha)^{p,q+1} + (\omega \wedge \alpha)^{p,q+1} = (d\alpha)^{p,q+1} = \overline{\partial}_J \alpha.$$

Moreover, if $\alpha = \pi^*(\gamma) \otimes f$, with $\gamma \varepsilon \Lambda^{p,q}(M)$ and $f \varepsilon \mathcal{T}^0(C(E))$, then

$$D_\omega \alpha = \pi^*(d\gamma) \otimes f + (-1)^{p+q} \pi^*(\gamma) \wedge D_\omega f.$$

Since $d\gamma = \partial_M \gamma + \overline{\partial}_M \gamma + A_M(\gamma)$, taking into account the fact that π is (J, J_M)-holomorphic we have

$$\begin{aligned} D_\omega \alpha = {} & \pi^*(\partial_M \gamma + \overline{\partial}_M \gamma + A_M(\gamma)) \otimes f + (-1)^{p+q} \pi^*(\gamma) \wedge (\partial_\omega f + \overline{\partial}_J f) \\ = {} & \pi^*(\partial_M \gamma) \otimes f + (-1)^{p+q} \pi^*(\gamma) \wedge (\partial_\omega f) \\ & + \pi^*(\overline{\partial}_M \gamma) \otimes f + (-1)^{p+q} \pi^*(\gamma) \wedge (\overline{\partial}_J f) \\ & + \pi^*(A_M(\gamma)) \otimes f = \partial_\omega \alpha + \overline{\partial}_J \alpha + A(\alpha). \end{aligned}$$

Note also for given $\omega \varepsilon \mathcal{C}(C(E))$ and $\alpha \varepsilon \mathcal{T}^0(C(E)), gl(r, \mathbb{C}), ad)$,

$$\overline{\partial}_{\chi(\omega+\alpha)} = \overline{\partial}_{\chi(\omega)} + \alpha^{(0,1)}. \tag{1.8.3}$$

The following is another important consequence of the previous results:

Proposition 1.9. *Let* $J \in C_J^{1,0}(C(E))$. *Then*

$$\Omega_\omega^{1,1} = \overline{\partial}_J \omega, \tag{1.9.1}$$

$$\Omega_\omega^{0,2} = \frac{1}{4}\omega \circ N(J). \tag{1.9.2}$$

Proof. We have:

$$\Omega_\omega^{1,1} = (D_\omega)^{1,1} = (d\omega)^{1,1} = \overline{\partial}_J \omega,$$

and

$$\Omega_\omega^{0,2} = (D_\omega)^{0,2} = (d\omega)^{0,2}.$$

If X and Y are horizontal, then

$$\begin{aligned}
&\Omega_\omega^{0,2}(X,Y)\\
&= (d\omega)^{0,2}(X,Y) = \frac{1}{4} d\omega(X+iJX, Y+iJY)\\
&= \frac{1}{4}(d\omega(X,Y) - d\omega(JX,JY) + i d\omega(JX,Y) + i d\omega(X,JY))\\
&= -\frac{1}{4}(\omega([X,Y]) - \omega([JX,JY]) + i\omega([JX,Y]) + i\omega([X,JY])\\
&= \frac{1}{4}(\omega([JX,JY]) - \omega([X,Y]) - \omega(J[JX,Y]) - \omega(J[X,JY])\\
&= \frac{1}{4}(\omega(N(J)(X,Y)).
\end{aligned}$$

Combining Propositions 1.8 and 1.9 gives immediately

Corollary 1.10. *Let* $J \in \mathcal{B}(C(E))$. *Then*

(a) *If J_M is integrable, then $\Omega_\omega^{0,2}$ is independent of the choice of $\omega \in C_J^{1,0}(C(E))$.*

(b) *If J_M is integrable, then $\Omega_\omega^{0,2} = 0$ for every $\omega \in C_J^{1,0}(C(E))$.*

(c) *If J_M is integrable, then J is integrable if and only if $\Omega_\omega^{0,2} = 0$ for $\omega \in C_J^{1,0}(C(E))$.*

We have seen in Proposition 1.3 that bacs on $C(E)$ are in one-to-one correspondence with elements of $\hat{\mathcal{H}}(C(E))$; $\hat{\mathcal{H}}(C(E))$ is also in one-to-one correspondence with the set $\hat{\mathcal{H}}(E)$ of linear differential operators $\overline{\partial}_E : \wedge^{p,q}(E) \longrightarrow \wedge^{p,q+1}(E)$, satisfying the following $\overline{\partial}$-Leibnitz rule: for every $f \in C^\infty(M), \alpha \in \wedge^{p,q}(E)$,

$$\overline{\partial}_E f\alpha = \overline{\partial}_M f \wedge \alpha + f\overline{\partial}_E \alpha.$$

This correspondence is obviously given by $\overline{\partial}_E = L \circ \overline{\partial}_{C(E))} \circ L^{-1}$ and, given an $(1,0)$-connection form ω, we have the splitting $\nabla = \partial_\nabla + \overline{\partial}_E$ of the induced exterior covariant differential operator, exactly as in Proposition 1.7.

The discription of bacs's through the elements of $\hat{\mathcal{H}}(E)$ makes especially easy to perform some functorial constructions; in fact we have:

Proposition 1.11.

(a) *Assume* a bacs *is given on* $C(E)$. *Then* a bacs *is induced on* $C(E^*)$.

(b) *Assume* bacs*'s are given on* $C(E_1)$ *and* $C(E_2)$. *Then* bacs*'s are induced in* $C(E_1 \oplus E_2)$ *and* $C(E_1 \otimes E_2)$.

Proof.

(a) Let $\overline{\partial}_E : \wedge^{p,q}(E) \longrightarrow \wedge^{p,q+1}(E)$ be the linear operator associated with the given bacs. Then define $\overline{\partial}_{E^*} : \wedge^{p,q}(E^*) \longrightarrow \wedge^{p,q+1}(E^*)$ by means of the relation

$$\overline{\partial}_M < \tau^*, \sigma > = < \overline{\partial}_{E^*}\tau^*, \sigma > + < \tau^*, \overline{\partial}_E \sigma > .$$

(b) Just set

$$\overline{\partial}_{E_1 \oplus E_2} := \overline{\partial}_{E_1} \oplus \overline{\partial}_{E_2}$$

and

$$\overline{\partial}_{E_1 \otimes E_2} := \overline{\partial}_{E_1} \otimes \overline{\partial}_{E_2},$$

where, of course, $(\overline{\partial}_{E_1} \otimes \overline{\partial}_{E_2})(\sigma \otimes \tau) = \overline{\partial}_{E_1}\sigma \otimes \tau + \sigma \otimes \overline{\partial}_{E_2}\tau$.

We need the following four definitions.

Definition 1.12. Let $J \ \varepsilon \ \mathcal{B}(C(E))$. Then a section σ of E is said to be J-holomorphic if it satisfies $\overline{\partial}_E \sigma = 0$; this of course, is equivalent to say that, if $f := L^{-1}(\sigma) \ \varepsilon \ \mathcal{T}^0(C(E))$, then $\overline{\partial}_J f = 0$.

Definition 1.13. Assume bacs's assigned on $C(E_1)$ and $C(E_2)$; a bundle morphism $\phi : E_1 \longrightarrow E_2$ is said to be J-holomorphic if $\overline{\partial}_{E_1^* \otimes E_2} \phi = 0$.

Definition 1.14. Assume $r = p + q$ and let $F \subset E$ be a real vector bundle of rank $2p$. We say that F is a complex subbundle of (complex) rank p of the complex bundle $(E, \hat{J})$ if $\hat{J}_{|F}$ is a section of End(F); it is clear that, in this case, the quotient bundle E/F has an induced structure of complex vector bundle.

Definition 1.15. Let $J \ \varepsilon \ \mathcal{B}(C(E))$. Then a complex subbundle $F \subset E$ is said to be a J-holomorphic subbundle if $\overline{\partial}_E$ maps $\wedge^{p,q}(F)$ into $\wedge^{p,q+1}(F)$.

Remark 1.16. If we consider the complex Grassmann manifold of complex p-planes in $\mathbb{C}^r$,

$$Gr_p(\mathbb{C}^r) := GL(r, \mathbb{C})/L_{q,p}(C),$$

then a complex subbundle $F \subset E$ of rank p corresponds to a section of the complex Grassmann bundle

$$Gr_p(E) := C(E)/L_{q,p}(C) = C(E) \times_{GL(r,\mathbb{C})} Gr_p(\mathbb{C}^r).$$

Moreover, if $J \ \varepsilon \ \mathcal{B}(C(E))$ is given, then an acs is induced on $G_r^{\mathbb{C}}(E)$; it is easy to check that J-holomorphic subbundles correspond to J-holomorphic sections of $G_r^{\mathbb{C}}(E)$.

Note also that, if F is a J-holomorphic subbundle, then

(a) J induces bacs's both on $C(F)$ and $C(E/F)$;

(b) $\overline{\partial}_{E_{|\Lambda^{p,q}(F)}} = \overline{\partial}_F$.

We have now the following two results (the proof of which is essentially strightforward)

Proposition 1.17. *Let $J \ \varepsilon \ \mathcal{B}(C(E))$, and let $F \subset E$ be a J-holomorphic subbundle; then the inclusion map $i : F \longrightarrow E$ is J-holomorphic.*

Proposition 1.18. *Assume* bacs*'s are assigned on $C(E_1)$ and $C(E_2)$, and let $\phi : E_1 \longrightarrow E_2$ be a J-holomorphic bundle morphism with constant rank. Then* kerϕ *and* Im ϕ *are J-holomorphic subbundle and, consequently, $C($*coker$\phi)$ *is equipped with a bacs.*

Definition 1.21. Let $J \ \varepsilon \ \mathcal{B}(C(E))$; then E is said to be J-simple if any J-holomorphic endomorphism is the form λid_E, with $\lambda \ \varepsilon \ C^\infty(M)$ (and therefore satisfying $\overline{\partial}_M \lambda = 0$).

An important fact is inclosed in

Proposition 1.21. *Let $J \ \varepsilon \ \mathcal{B}(C(E))$; then, generically*

(a) *there are no local J-holomorphic sections of E;*

(b) *E is simple;*

(c) *there are no local J-holomorphic subbundles of E.*

2. Hermitian structures

Let now $(E, \hat{J})$ be a complex bundle of rank r over M.

Let $J \ \varepsilon \ \mathcal{B}(C(E))$, assume a Hermitian structure h is assigned on E and let $U_h(E)$ be the principal $U(r)$-bundle of h-unitary frames on E; we have the following fundamental result:

Proposition 2.1. *There exists a unique connection on $U_h(E)$ such that its connection 1-form, when extended to a connection form on $C(E)$ is of type* $(1,0)$ *(in other words $C_J^{1,0}(C(E)) \cap C(U_h(E))$ consists of a single element); this connection is called the canonical Hermitian connection.*

Proof. Let $\hat{h} : C(E) \longrightarrow GL(r, \mathbb{C})$ be defined as

$$\hat{h}(u) := \overline{p}^{-1}(u^*(h));$$

(i.e., if $u = \{\sigma_1, \cdots, \sigma_r, \hat{J}\sigma_1, \cdots, \hat{J}\sigma_r\}$, then $\hat{h}(u) = (h(\sigma_j, \sigma_k) - ih(\sigma_j, \hat{J}\sigma_k))_{1\leq j,k\leq r})$. Then

(1) $U_h(E) = \{u \ \varepsilon \ C(E) \mid \hat{h}(u) = I\}$,

(2) for every $u \ \varepsilon \ C(E), a \ \varepsilon \ GL(r, \mathbb{C})$ we have $\hat{h}(ua) =^t \bar{a}\hat{h}(u)a$, and consequently:

(a) $(\hat{h} \circ R_a)_*[u] =^t \bar{a}\hat{h}_*[u]a$,

(b) if $X \ \varepsilon \ gl(r, \mathbb{C})$ then $\hat{h}_*[u](X^*) =^t \overline{X}\hat{h}(u) + \hat{h}(u)X$.

Set

$$\omega_h := \hat{h}^{-1}\partial_J\hat{h}. \tag{2.1.1}$$

It is easy to check that $\omega_h \ \varepsilon \ C_J^{1,0}(C(E))$; clearly ω_h reduces to an element of $\mathcal{C}(U_h(E))$: in fact, if $u \ \varepsilon \ U_h(E)$, then

$$\ker \omega_h[u] = T_uU_h(E) \cap JT_uU_h(E); \tag{2.1.2}$$

the uniqueness follows from the fact that

$$\mathcal{T}^1(U_h(E), u(r)) \cap \mathcal{T}^{1,0}(C(E), gl(r, \mathbb{C})) = \{0\}, \tag{2.1.3}$$

which is an easy consequence of the relation $u(r) \cap iu(r) = \{0\}$.

Therefore we have:

Corollory 2.2. *There is a one-to-one correspondence between the set $\mathcal{B}(C(E))$ of* bacs *on $C(E)$ and the affine space $\mathcal{C}(U_h(E))$ of connection on $U_h(E)$.*

In order to simplify our notation, from now on we will identify h and $\hat{h}$.

The following proposition describes the behaviour of the canonical Hermitian connection when the Hermitian structure changes.

Proposition 2.3. *Let k be another Hermitian structure on E and let $g := h^{-1}k$; then*

(1)

$$\omega_k = \omega_h + g^{-1}\partial_{\omega_h}g. \tag{2.3.1}$$

Therefore, if J_M is integrable, then

(2)

$$\Omega^{0,2}_{\omega_k} = \Omega^{0,2}_{\omega_h} \tag{2.3.2}$$

and the $(0,2)$-component of the curvature form is independent of the Hermitian structure, i.e.,

(3)

$$\Omega^{2,0}_{\omega_k} = g^{-1}\Omega^{2,0}_{\omega_h}g. \tag{2.3.3}$$

Proof.

(1) We have

$$\omega_k = k^{-1}\partial_J k = (hg)^{-1}\partial_J(hg) = g^{-1}h^{-1}[(\partial_J h)g + h(\partial_J g)]$$
$$= g^{-1}\omega_h g + g^{-1}(\partial_{\omega_h} g - [\omega_h, g]) = \omega_h + g^{-1}\partial_{\omega_h} g.,$$

(2) which follows directly from (1.9.2.) or the $(0,2)$-component of the relation

$$\Omega_{\omega_k} = \Omega_{\omega_h} + D_{\omega_h}(g^{-1}\partial_{\omega_h} g) + \frac{1}{2}[g^{-1}\partial_{\omega_h} g, g^{-1}\partial_{\omega_h} g]. \tag{2.3.4}$$

(3) Taking the $(2,0)$-component of (2.3.4) yields

$$\Omega^{2,0}_{\omega_k} = \Omega^{2,0}_{\omega_h} + \partial_{\omega_h}(g^{-1}\partial_{\omega_h} g) + g^{-1}\partial_{\omega_h} g \wedge g^{-1}\partial_{\omega_h} g$$
$$= \Omega^{2,0}_{\omega_h} - g^{-1}\partial_{\omega_h} g \wedge g^{-1}\partial_{\omega_h} g + g^{-1}\partial^2_{\omega_h} g + g^{-1}\partial_{\omega_h} g \wedge g^{-1}\partial_{\omega_h} g$$
$$= \Omega^{2,0}_{\omega_h} + g^{-1}[\Omega^{2,0}_{\omega_h}, g] = g^{-1}\Omega^{2,0}_{\omega_h} g.$$

Let $(E, \hat{J}, h)$ be a complex vector bundle of rank $r = p + q$ equipped with a Hermitiann structure, and let $F \subset E$ be a complex subbundle of rank p. Then $S := F^{\perp}$ is a complex subbundle of E with rank $S = q$ and $E = F \oplus S$, and $U_h(F) + U_h(S)$ is a $U(p) \times U(q)$-reduction of $U_h(E)$ with embedding

$$i : U_h(F) + U_h(S) \longrightarrow U_h(E).$$

Let $f_F : U_h(F) + U_h(S) \longrightarrow U_h(F)$ and $f_S : U_h(F) + U_h(S) \longrightarrow U_h(S)$ be the natural maps, and let $\omega \;\varepsilon\; \mathcal{C}(U_h(E))$. Then $i^*(\omega)$ splits as

$$i^*(\omega) = \hat{\omega} + \alpha,$$

where

$$\hat{\omega} \;\varepsilon\; \mathcal{C}(U_h(F) + U_h(S)),$$

with

$$\hat{\omega} = f_F^*(\hat{\omega}_F) + f_S^*(\hat{\omega}_S) = \begin{bmatrix} f_F^*(\hat{\omega}_F) & 0 \\ 0 & f_S^*(\hat{\omega}_S) \end{bmatrix}$$

for $\hat{\omega}_F \;\varepsilon\; \mathcal{C}(U_h(F))$ and $\hat{\omega}_S \;\varepsilon\; \mathcal{C}(U_h(S))$, and

$$\alpha = \begin{bmatrix} 0 & -{}^t\overline{\sigma} \\ \sigma & 0 \end{bmatrix}$$

for

$$\sigma \;\varepsilon\; \mathcal{T}^1(U_h(F) + U_h(S), M_{q,p}(\mathbb{C}), \rho),$$

where $\rho : U(p) \times U(q) \longrightarrow \mathrm{Aut}(M_{q,p}(\mathbb{C}))$ is given by $\rho(A,B)(X) := BXA^{-1}$. Therefore, on $U_h(F) + U_h(S)$,

$$\Omega_\omega = \begin{bmatrix} f_F^*(\Omega_{\hat{\omega}_F}) - \frac{1}{2}[{}^t\sigma, \sigma] & -D_{\hat{\omega}}^t \overline{\sigma} \\ D_{\hat{\omega}}\sigma & f_S^*(\Omega_{\hat{\omega}_S}) - \frac{1}{2}[\sigma, {}^t\sigma] \end{bmatrix}. \tag{2.3.5}$$

(2.3.5) is called the Hermitian Codazzi-Mainardi equation, and σ the second fundumental form of F in E. Therefore $-{}^t\sigma$ is the second fundumental form of S in E.

It is immediate to check that, if $s \; \varepsilon \; \wedge^0(F)$, then $\nabla_\omega s$ decomposes according to the splitting $\wedge^1(E) = \wedge^1(F) \oplus \wedge^1(S)$ as

$$\nabla_\omega s = \nabla_{\hat{\omega}_F} s + L(\sigma)s. \tag{2.3.6}$$

Now we have

Proposition 2.4. *Let $(E, \hat{J}, h)$ be a Hermitian bundle, let $F \subset E$ be a complex subbundle, and let $J \; \varepsilon \; \mathcal{B}(C(E))$. Then the following facts are equivalent:*

(a) *F is a J-holomorphic subbundle,*
(b) *the orthogonal projection $\theta_F : E \longrightarrow F$ satisfies*

$$(I - \theta_F) \circ \overline{\partial}_{E^* \otimes E} \theta_F = 0, \tag{2.4.1}$$

(c) *the second fundumental form σ od F in E with respect to the canonical Hermitian connection is of type $(1,0)$ (in the sense that $L(\sigma) \; \varepsilon \; \Lambda^{1,0}(Hom(F, F^\perp))$.*

Proof. $(a) \Leftrightarrow (b)$: Let $t \; \varepsilon \; \Lambda^0(E)$. Therefore $\theta_F \circ t \; \varepsilon \; \Lambda^0(F)$. Since

$$\overline{\partial}_E(\theta_F \circ t) = (\overline{\partial}_{E^* \otimes E} \theta_F)(t) + \theta_F(\overline{\partial}_E t),$$

we obtain

$$(I - \theta_F) \circ \overline{\partial}_E(\theta_F \circ t) = (I - \theta_F) \circ \overline{\partial}_{E^* \otimes E} \theta_F(t),$$

and therefore

$$\overline{\partial}_E \quad \text{maps} \quad \wedge^0(F) \quad \text{into} \quad \wedge^{0,1}(F) \Leftrightarrow (I - \theta_F) \circ \overline{\partial}_{E^* \otimes E} \theta_F = 0.$$

$(a) \Leftrightarrow (c)$: Let $s \; \varepsilon \; \Lambda^0(F)$. From (2.3.6), in particular, it follows

$$\overline{\partial}_E s = (\nabla_{\hat{\omega}_F} s)^{0,1} + (L(\sigma)(s))^{0,1},$$

so that

$$(L(\sigma)(s))^{0,1} = 0 \Leftrightarrow \overline{\partial}_E s \; \varepsilon \; \Lambda^{0,1}(F).$$

Remark 2.5. If $p_F := L^{-1}(\theta_F)$, then (2.4.1) is equivalent to

$$\partial_\omega p_F (I - p_F) = 0. \tag{2.5.1}$$

Let $(E, \hat{J}, h)$ be a Hermitian vector bundle of rank r over M, let $s, t \;\varepsilon\; \mathcal{T}^0(C(E), \mathbb{C}^r)$, and set

$$(2.5.2) \qquad < s, t :=>^t \bar{s}\hat{h}t.$$

Then

$$\begin{aligned} < s, t > (ua) &=^t \overline{s(ua)}\hat{h}(ua)t(ua) \\ &=^t \overline{a^{-1}s(u)}^t \bar{a}\hat{h}(u)aa^{-1}t(u) \\ &=< s, t > (u), \end{aligned}$$

$< s, t > (u)$ is a well defineded function on M, and from the very definition of $\hat{h}$ it follows that

$$(2.5.3) \qquad < s, t >= h(L(s), L(t)).$$

More generally, we obtain

Definition 2.6. Assume M is equipped with a Riemannian structure g; then extend $<,>$ to $\mathcal{T}^p(C(E), \mathbb{C}^r)$ in the following way:

If $\phi, \psi \;\varepsilon\; \mathcal{T}^p(C(E), \mathbb{C}^r)$ are of the form $\phi = \pi^*(\mu) \otimes s, \psi = \pi^*(\nu) \otimes t$, for $\mu, \nu \;\varepsilon\; \Lambda^p(M), s, t \;\varepsilon\; \mathcal{T}^0(C(E), \mathbb{C}^r)$, then set

$$< \phi, \psi >:= g(\mu, \nu) < s, t >,$$

and extend to the general case by the Hermitian bilinearity.

Again $<,>$ is a well defined function on M, and for every $\phi, \psi \;\varepsilon\; \mathcal{T}^p(C(E), \mathbb{C}^r)$ we have

$$(2.6.1) \qquad < \phi, \psi >= (g \otimes h)(L(\phi), L(\psi)).$$

Let now $s \;\varepsilon\; \mathcal{T}^0(C(E), gl(r, \mathbb{C}), ad)$, and define $s^\#$ by the relation

$$(2.6.2) \qquad s^\#(u) := \hat{h}^{-1}(u)\, {}^t\overline{s(u)}\hat{h}(u).$$

It is easy to check that $s^\# \;\varepsilon\; \mathcal{T}^0(C(E), gl(r, \mathbb{C}), ad)$. If $t \;\varepsilon\; \mathcal{T}^0(C(E), gl(r, \mathbb{C}), ad)$, then set

$$(2.6.3) \qquad < s, t >= \text{tr } st^\#.$$

Again the following hold:

(a) $<,>$ is well defined function on M.

(b) Whenever M is equipped with a Riemannian structure, $<,>$ can be extended to

$$\mathcal{T}^p(C(E), gl(r, \mathbb{C}), ad).$$

(c) A relation analogous to (2.6.1) holds.

(d) If $\alpha \;\varepsilon\; \mathcal{T}^p(C(E), gl(r, \mathbb{C}), ad)$, then $\alpha^\#$ is obviously defined by the relation

(2.6.4) $$\alpha^{\#}(X_1, \cdots, X_p); = (\alpha(X_1, \cdots, X_p))^{\#}.$$

Note that $\alpha \; \varepsilon \; \mathcal{T}^{p,q} \Leftrightarrow \alpha^{\#} \; \varepsilon \; \mathcal{T}^{q,p}$; in particular,

(2.6.5) $$\Omega^{2,0}_{\omega_h} = -(\Omega^{0,2}_{\omega_h})^{\#}$$

for a Hermitian structure h on E.

3. Yang-Mills functional, Donaldson's Lagrangian and the Hermite-Einstein condition

From now on, let (M, J_M, g) be a compact n-dimensional almost Hermitian manifold whose Kähler form η satisfies $d\eta^{n-1} = 0$.

Let $(E, \hat{J}, h)$ be a Hermitian vector bundle of rank r over M. Given $\omega \; \varepsilon \; \mathcal{C}(U_h(E))$, we set:

$$K^{1,1}_{\omega} := \Lambda\Omega^{1,1}_{\omega} \quad \text{(contraction with } \eta),$$
$$\sigma_{\omega} := tr\, K^{1,1}_{\omega},$$
$$\deg(E) := \int_M c_1(E) \wedge \eta^{n-1} = \frac{1}{2\pi n}\int_M \sigma\eta^n,$$
$$\mu(E) := \frac{1}{r}\deg(E),$$
$$k(E) = \frac{2\pi n\mu(E)}{n! Vol_g(M)}.$$

$H_{\omega} := K^{1,1}_{\omega} - ik(E)I$; H_{ω} is called the Hermite-Yang-Mills curvature.

We have the following decomposition result for the Yang-Mills functional (cf. also [11]).

Proposition 3.1. *Let $YM : \mathcal{C}(U_h(E)) \longrightarrow \mathbb{R}^+$ be the Yang-Mills functional, i.e.,*

$$YM(\omega) := \frac{1}{2}\int_M |\, \Omega_{\omega} \,|^2 \, d\mu(g);$$

then

(3.1.1) $$YM(\omega) = \epsilon(E) + 2\int_M |\, \Omega^{0,2}_{\omega} \,|^2 \, d\mu(g) + \frac{1}{2}\int_M |\, H_{\omega} \,|^2 \, d\mu(g),$$

where

$$\epsilon(E) := 2\pi^2 n(n-1)\int_M (2c_2(E) - c_1^2(E)) \wedge \eta^{n-2} + \frac{r}{2}\int_M k^2(E) d\mu(g).$$

Proof. Given $\omega \; \varepsilon \; \mathcal{C}(U_h(E))$, we have

$$\Omega_{\omega} = \Omega^{1,1}_{\omega} + (\Omega^{2,0}_{\omega} + \Omega^{0,2}_{\omega}) \text{ and } |\, \Omega_{\omega} \,|^2 = |\, \Omega^{1,1}_{\omega} \,|^2 + 2 \,|\, \Omega^{0,2}_{\omega} \,|^2 .$$

Let p_1 and p_2 be the first two Chern's polinomials. Then

$$\begin{aligned}&(2c_2(E) - c_1^2(E)) \wedge \eta^{n-2}\\ &=(2p_2(\Omega_\omega) - p_1(\Omega_\omega) \wedge p_1(\Omega_\omega)) \wedge \eta^{n-2}\\ &=(2p_2(\Omega_\omega) - p_1(\Omega_\omega) \wedge p_1(\Omega_\omega))^{2,2} \wedge \eta^{n-2}\\ &=(2p_2(\Omega_\omega^{1,1}) - p_1(\Omega_\omega^{1,1}) \wedge p_1(\Omega_\omega^{1,1})) \wedge \eta^{n-2}\\ &\quad + 2(2p_2(\Omega_\omega^{2,0} \wedge \Omega_\omega^{0,2}) - p_1(\Omega_\omega^{2,0}) \wedge p_1(\Omega_\omega^{0,2})) \wedge \eta^{n-2}.\end{aligned}$$

now

$$\begin{aligned}&(2p_2(\Omega_\omega^{1,1}) - p_1(\Omega_\omega^{1,1}) \wedge p_1(\Omega_\omega^{1,1})) \wedge \eta^{n-2}\\ &\quad = (4\pi^2 n(n-1))^{-1}(| \Omega_\omega^{1,1} |^2 - | K_\omega^{1,1} |^2)\eta^n,\end{aligned}$$

$$2p_2(\Omega_\omega^{2,0} \wedge \Omega_\omega^{0,2}) - p_1(\Omega_\omega^{2,0}) \wedge p_1(\Omega_\omega^{0,2}) = \frac{1}{4\pi^2} \sum_{j,k=1}^{r} (\Omega_\omega^{0,2})_{jk} \wedge (\Omega_\omega^{0,2})_{kj}$$

and

$$(\Omega_\omega^{2,0})_{jk} = \Omega_{\alpha\beta jk}^{2,0}\theta_\alpha \wedge \theta_\beta \,\text{and}(\Omega_\omega^{0,2})_{kj} = \Omega_{\overline{\alpha}\overline{\beta}kj}\theta_{\overline{\alpha}} \wedge \theta_{\overline{\beta}},$$

where $\{\theta_1, \cdots, \theta_n\}$ is an orthonormal coframe in M such that $\eta = i\sum_{\alpha=1}^{n} \theta_\alpha \wedge \theta_{\overline{\alpha}}$. We have also

$$\Omega_{\overline{\alpha}\overline{\beta}kj}^{0,2} = -\overline{\Omega_{\alpha\beta jk}^{2,0}}.$$

Therefore

$$\begin{aligned}&n(n-1)\sum(\Omega_\omega^{0,2})_{jk} \wedge (\Omega_\omega^{0,2})_{kj} \wedge \eta^{n-2}\\ &\quad = n(n-1)\sum \Omega_{\alpha\gamma jk}^{2,0}\Omega_{\overline{\beta}\overline{\gamma}kj}^{0,2}\theta_\alpha \wedge \theta_\gamma \wedge \theta_{\overline{\beta}} \wedge \theta_{\overline{\gamma}} \wedge \eta^{n-2}\\ &\quad = \frac{1}{2}\sum(\Omega_{\alpha\gamma jk}^{2,0}\Omega_{\overline{\alpha}\overline{\gamma}kj}^{0,2} - \Omega_{\alpha\gamma jk}^{2,0}\Omega_{\overline{\gamma}\overline{\alpha}kj}^{0,2})\eta^n\\ &\quad = \sum(\Omega_{\alpha\gamma jk}^{2,0}\Omega_{\overline{\alpha}\overline{\gamma}kj}^{0,2})\eta^n = 2 \mid \Omega_\omega^{0,2} \mid^2 \eta^n,\end{aligned}$$

$$\begin{aligned}&(2p_2(\Omega_\omega^{2,0} \wedge \Omega_\omega^{0,2}) - p_1(\Omega_\omega^{2,0}) \wedge p_1(\Omega_\omega^{0,2})) \wedge \eta^{n-2}\\ &\quad = -(4\pi^2 n(n-1))^{-1} \mid \Omega_\omega^{0,2} \mid^2 \eta^n,\end{aligned}$$

$$\begin{aligned}&(2c_2(E) - c_1^2(E)) \wedge \eta^{n-2}\\ &\quad = (4\pi^2 n(n-1))^{-1}(| \Omega_\omega^{1,1} |^2 - | K_\omega^{1,1} |^2 - 2 \mid \Omega_\omega^{0,2} \mid^2)\eta^n,\end{aligned}$$

or

$$\begin{aligned}&| \Omega_\omega^{1,1} |^2 \eta^n\\ &\quad = 4\pi^2 n(n-1)(2c_2(E) - c_1^2(E)) \wedge \eta^{n-2} + (2 \mid \Omega_\omega^{0,2} \mid^2 + \mid K_\omega^{1,1} \mid^2)\eta^n.\end{aligned}$$

Moreover

$$K_\omega^{1,1} = H_\omega + ik(E)I,$$

and therefore

$$\int_M |K_\omega^{1,1}|^2 \, d\mu(g) = \int_M |H_\omega|^2 \, d\mu(g) + r \int_M k^2(E) d\mu(g).$$

Hence the proof of (3.1.1) is complete.

The following two propositions are immediate consequences of Proposition 3.1.

Corollary 3.2. *For every $\omega \in C(U_h(E))$ we have*

$$YM(\omega) \geq \epsilon(E),$$

which the equality holds if and only if

$$\Omega_\omega^{0,2} = 0 \qquad \text{and } H_\omega = 0.$$

Corollary 3.3. *If $H_\omega = 0$, then*

$$\int_M (2rc_2(E) - (r-1)c_1^2(E)) \wedge \eta^{n-2} + \frac{1}{\pi^2 n(n-1)} \int_M (r \,|\, \Omega_\omega^{0,2} \,|^2 - |\operatorname{tr} \Omega_\omega^{0,2}|^2) d\mu(g) \geq 0,$$

with equality iff

$$\Omega_\omega^{11} = i \operatorname{tr} \Omega_\omega^{11} I.$$

We have also

Proposition 3.4. *$\omega \in C(U_h(E))$ is a critical point for the Yang-Mills functional if and only if it satisfies*

$$2\overline{\partial}_\omega^* \Omega_\omega^{0,2} - 2A^* \Omega_\omega^{2,0} - \overline{\partial}_\omega H_\omega = 0. \tag{3.4.1}$$

In particular, this is the case if ω is a critical point both for

$$\omega \mapsto \int_M |\Omega_\omega^{0,2}|^2 \, d\mu(g)$$

and

$$\omega \mapsto \int_M |H_\omega|^2 \, d\mu(g).$$

Proof. Consider in $C(U_h(E))$ a curve $t \mapsto \omega_t = \omega + \alpha_t$ with $\alpha_0 = 0$ and $v = \frac{d}{dt}\alpha_{t|_{t=0}}$. Then

$$\Omega_{\omega_t} = \Omega_\omega^{0,2} + t(\overline{\partial}_\omega v^{0,1} + Av^{1,0}) + o(t)$$

and

$$H_{\omega_t} = H_\omega + t(i\overline{\partial}_\omega^* v^{0,1} - i\partial_\omega^* v^{1,0}) + o(t).$$

Consequently, from Proposition 3.1 it follows directly that

$$\frac{d}{dt} YM(\omega_t)_{|t=0} = \int_M < D^*_\omega \Omega_\omega, v > d\mu(g)$$
$$= \int_M 2Re(< 2\overline{\partial}^*_\omega \Omega^{0,2}_\omega + 2A^* \Omega^{0,2}_\omega - i\overline{\partial}_\omega H_\omega, v^{0,1} > d\mu(g),$$

and so ω is a critical point if and only if it satisfies (3.4.1).

Proposition 3.5. *Let $\omega \ \varepsilon \ \mathcal{C}(U_h(E))$ and let $J \ \varepsilon \ \mathcal{B}(C(E))$ be the corresponding bacs. If E is J-simple, then $D_\omega H_\omega = 0$ is equivalent to $H_\omega = 0$.*

Proof. From $D_\omega H_\omega = 0$ it follows that the eigenvalues of H_ω are constant, so that E decomposes J-holomorphically into eigenbundles; by J-simplicity, this decomposition is trivial implying that $H_\omega = 0$ since $\int_M \operatorname{tr} H_\omega \eta^n = 0$.

One of the main purposes of this paper is to characterize those elements of $\mathcal{B}(C(E))$ for which there exists a Hermite-Einstein structure h, i.e., a Hermitian structure satisfying the Hermite-Einstein condition $H_{\omega_h} \equiv 0$. Assume from now on that $\overline{\partial}_M \partial_M \eta^{n-1} = 0$.

We need to introduce some further machineries (cf. [9]).

Let $J \ \varepsilon \ \mathcal{B}(C(E))$, $\mathcal{H}erm(E) :=$ {Hermitian structure on E},, and fix $h \ \varepsilon \ \mathcal{H}erm(E)$.

(1) Let

$$S_h(E) := \{p \ \varepsilon \ \mathcal{T}^0(C(E), gl(r, \mathbb{C}), ad) \mid p = p^\#\}.$$

If $s \ \varepsilon \ S_h(E)$, then for every $x \ \varepsilon \ M$ we can choose $C : \pi^{-1}(x) \longrightarrow GL(r, \mathbb{C})$ such that

$$s(u) = C^{-1}(u) \wedge C(u)$$

with

$$\wedge = \begin{bmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_r \end{bmatrix}, \quad \lambda_1, \cdots, \lambda_r \ \varepsilon \ \mathbb{R},$$

$$C(ua) = C(u)a \quad \text{and} \quad C(u)C^\#(u) = \hat{h}(u).$$

Moreover

$$| s(u)^2 | = \operatorname{tr} \wedge^2;$$

in general, if $p \ \varepsilon \ \mathcal{T}(C(E), gl(r, \mathbb{C}), ad)$, then $\tilde{p}_C(u) :=$ C$(u)p(u)C^{-1}(u)$ depends only on x and $< p, q >= \operatorname{tr} \tilde{p}_C \tilde{q}^*_C$.

(2) **a.** Given $\varphi \in C^\infty(\mathbb{R},\mathbb{R})$ and $s \in S_h(E), x, C$ as before, we set

$$\varphi(s) := C^{-1}\varphi(\Lambda)C$$

where, of course,

$$\varphi(\Lambda) := \begin{bmatrix} \varphi(\lambda_1) & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \varphi(\lambda_r) \end{bmatrix}.$$

b. Given $\Phi \in C^\infty(\mathbb{R}\times\mathbb{R},\mathbb{R}), s \in S_h(E)$ as before and $p \in \mathcal{T}^0(C(E), gl(r,\mathbb{C}), ad)$, we set

$$\Phi[s](p) := C^{-1}\Phi(\Lambda, \tilde{p}_C)C,$$

where

$$\Phi(\Lambda, \tilde{p}_C) := (\Phi(\lambda_j, \lambda_k)(\tilde{p}_C)_{jk}).$$

The following are clear:

i) $\varphi(s)$ and $\Phi[s](p)$ are independent of the choice of C,

ii) $< \Phi[s](p), p > = \sum_{jk=1}^{r} \Phi(\lambda_j, \lambda_k) \mid (\tilde{p}_C)_{jk} \mid^2$.

c. Finally, if $\varphi \in C^\infty(\mathbb{R},\mathbb{R})$, then we set

$$\delta\varphi(\lambda,\mu) = \begin{cases} \frac{\varphi(\lambda)-\varphi(\mu)}{\lambda-\mu}, & \text{if } \lambda \neq \mu, \\ \varphi'(\lambda), & \text{if } \lambda = \mu. \end{cases}$$

A straightforward computation gives:

$$\delta\varphi[s](\overline{\partial}s) = \overline{\partial}\varphi(s).$$

(3) Let $S_h^0(E) := \{s \in S_h(E) \mid \int_M \operatorname{tr} s\eta^n = 0\}$ and define Donaldson's Lagrangian

$$V_h : S_h^0(E) \longrightarrow \mathbb{R} \quad \text{as follows:}$$

$$V_h(s) := \int_M < is, H > \eta^n + \int_M < s, \Lambda\overline{\partial}(\Phi[s](\partial_{\omega_h} s)) > \eta^n,$$

where

$$\Phi(\lambda,\mu) := \phi(\lambda-\mu) \ \text{ with } \ \phi(\lambda) := \frac{e^{-\lambda}+\lambda-1}{\lambda^2} \ \text{ and } \ H = H_{\omega_h}.$$

The basic property of Donaldson's Lagrangian is contained in

Lemma 3.6. *Given $r \in S_h^0(E)$ and $s \in S_{he^r}^0(E)$, we have*

$$V_h(\log e^r e^s) = V_h(r) + V_{he^r}(s); \tag{3.6.1}$$

equivalently, if $h, k, j \in \mathcal{H}erm(E)$ *and* $M(h,k) := V_h(\log h^{-1}k)$, *then*

$$M(h,j) = M(h,k) + M(k,,j). \tag{3.6.2}$$

Proof. First of all we have the following two relations:

$$\frac{\partial^2}{\partial x \partial y} V_{he^{xr}}(ys)_{|(0,0)} = \frac{\partial^2}{\partial x \partial y} V_h(\log e^{xr} e^{ys})_{|(0,0)}, \tag{3.6.3}$$

$$\frac{\partial^2}{\partial x \partial y} V_h((x+y)r)_{|(x,0)} = \frac{\partial^2}{\partial x \partial y} V_{he^{xr}}(yr)_{|(x,0)}; \tag{3.6.4}$$

in fact

$$\frac{\partial^2}{\partial x \partial y} V_{he^{xr}}(ys)_{|(0,0)} = \int_M i(\operatorname{tr} s\Lambda\overline{\partial}\partial_{\omega_h} r)\eta^n,$$

and

$$\begin{aligned}\frac{\partial^2}{\partial x \partial y} V_h(\log e^{xr} e^{ys})_{|(0,0)} &= \frac{1}{2}\int_M i(\operatorname{tr} r\Lambda\overline{\partial}\partial_{\omega_h} s + \operatorname{tr} s\Lambda\overline{\partial}\partial_{\omega_h} r)\eta^n \\ &= \int_M i(\operatorname{tr} s\Lambda\overline{\partial}\partial_{\omega_h} r)\eta^n;\end{aligned}$$

(the last equality follows from the assumption $\overline{\partial}_M \partial_M \eta^{n-1} = 0$). It is clear that by rescalling it is enough to prove (3.6.4) for the case $x = 1$. Of course,

$$\frac{\partial^2}{\partial x \partial y} V_h((x+y)r)_{|(1,0)} = \frac{\partial^2}{\partial x^2} V_h(xr)_{|1}.$$

Since

$$\begin{aligned}\frac{\partial^2}{\partial x^2} V_h(xr)_{|1} &= \int_M \frac{\partial^2}{\partial x^2} < r, \Lambda\overline{\partial}(x^2\Phi[xr](\partial_{\omega_h} r)) > \eta^n \\ &= \int_M < r, \Lambda\overline{\partial}(\Psi[r](\partial_{\omega_h} r)) > \eta^n,\end{aligned}$$

where

$$\Psi(\lambda,\mu) := \frac{\partial^2}{\partial x^2} x^2 \Phi(x\lambda, x\mu) = e^{\mu-\lambda},$$

if $r = C^{-1}\Lambda C$,

$$\begin{aligned}C(\Psi[r](\partial_{\omega_h} r))C^{-1} &= \partial_{\omega_h}\Lambda + e^{-\Lambda}[\Lambda, \partial_{\omega_h} CC^{-1}]e^{\Lambda} \\ &= e^{-\Lambda}(\partial_{\omega_h}\Lambda + [\Lambda, \partial_{\omega_h} CC^{-1}])e^{\Lambda},\end{aligned}$$

and consequently

$$\Psi[r](\partial_{\omega_h} r) = e^{-r}\partial_{\omega_h} r e^{r} = \partial_{\omega_{he^r}} r.$$

Thus

$$\frac{\partial^2}{\partial x^2} V_h(xr)_{|1} = \int_M i(\operatorname{tr} r\Lambda\overline{\partial}\partial_{\omega_{he^r}} r)\eta^n = \frac{\partial^2}{\partial x \partial y} V_{he^{xr}}(yr)_{|(1,0)}.$$

From (3.6.4) it follows that $\frac{\partial}{\partial y}(V_h((x+y)r) - V_{he^{xr}}((x+y)r))_{|y=0} = 0.$

(a) Passing from x to $x + y_0$ gives

$$\frac{\partial}{\partial y}(V_h((x + y_0 + y)r) - V_{he^{(x+y_0)r}}((x + y)r))_{|y=0} = 0.$$

(b) Passing from h to he^{xr} yields

$$\frac{\partial}{\partial y}(V_{he^{xr}}((y_0 + y)r) - V_{he^{(x+y_0)r}}((x + y)r))_{|y=0} = 0.$$

Therefore

$$\frac{\partial}{\partial y}(V_h((x + y)r) - V_{he^r}(yr))_{|y=y_0} = 0,$$

which implies that

$$V_h((x + y)r) = V_h(xr) + V_{he^r}(yr). \tag{3.6.5}$$

Let $f(x, y) := V_h(xr) + V_{he^{xr}}(ys) - V_h(\log e^{xr}e^{ys})$. Then clearly

$$f(0,0) = 0, \quad f(0,y) = 0, \quad f(x,0) = 0.$$

Moreover, (3.6.3) implies that $\frac{\partial^2 f}{\partial x \partial y}(0,0) = 0$ so that

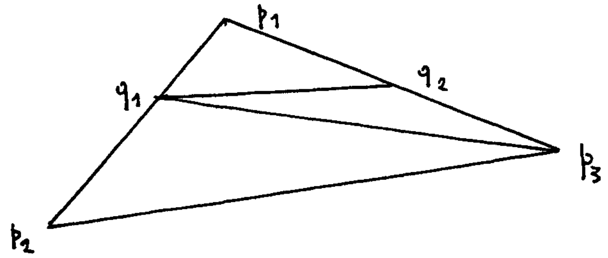
$$f(x,y) = o(|x|^2 + |y|^2). \tag{3.6.6}$$

Consider the triangle $T := \{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathbb{R}^2 \mid 0 \le y \le x \le 1\}$ and let $H : T \longrightarrow \mathcal{H}erm(E)$ be defined as $H\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] := he^{xr}e^{ys}$; finally set $L : T \times T \longrightarrow \mathbb{R}$ as $L(p,q) := M(H(p), H(q))$. Then the following hold:

(a) (3.6.5) is equivalent to saying that, for given $p_1, p_2, p_3 \in T$ on the same line,

$$L(p_1, p_2) + L(p_2, p_3) - L(p_1, p_3) = 0,$$

and consequently, given any $p_1, p_2, p_3 \in T$, if we choose q_1, on the line $\overline{p_1 p_2}$ and q_2 on the line $\overline{p_1 p_3}$,



then

$$\begin{aligned} L(p_1,p_2) + L(p_2,p_3) - L(p_1,p_3) \\ &= L(p_1,q_1) + L(q_1,q_2) - L(p_1,q_2) \\ &\quad + L(q_1,p_2) + L(p_2,p_3) - L(q_1,p_3) \\ &\quad + L(q_1,p_3) - L(q_1,q_2) - L(q_2,p_3), \end{aligned}$$

i.e., we can reduce the problem to smaller triangles.

(b) From (3.6.6) it follows that there exist two positive constants C and K, depending on r and s, such that, for any $\epsilon > 0$, there exists $\delta > 0$ such that, if $p_1, p_2, p_3 \;\varepsilon\; T$ satisfy
   (α): $d(p_j, p_k) < \epsilon, 1 \leq j, k \leq 3,$
   (β): $C^{-1}d(p_1,p_2) \leq d(p_2,p_3) \leq Cd(p_1,p_2),$
   (γ): $d(p_1,p_2) \cdot d(p_2,p_3) \leq C\,\text{area}(p_1,p_2,p_3),$

then

$$L(p_1,p_2) + L(p_2,p_3) - L(p_1,p_3) \leq \epsilon K\,\text{area}(p_1,p_2,p_3).$$

Taking arbitrary small triangular nondegenerate subdivision, we can easily conclude that $f(1,1) = 0$.

As a consequence of Lemma 3.6, we obtain immediately

Corollary 3.7. *$s \;\varepsilon\; S_h^0(E)$ is a critical point for $V_{\hat{h}}$ if and only if $\hat{k} := \hat{h}e^s$ corresponds to a Hermite-Einstein structure.*

We also have

Corollary 3.8. *If E is J-simple, then there exists at most one (up to homotheties) Hermite-Einstein structure on E.*

Proof. Let h, k be two Hermitian structures on E and let $\hat{k} := \hat{h}e^s$; clearly, we can assume $s \;\varepsilon\; S_h^0(E)$. Then set $\hat{h}_t := \hat{h}e^{ts}$, $0 \leq t \leq 1$. A direct computation gives

$$\frac{d^2}{dt^2}V_{\hat{h}}(ts) = \|\bar{\partial}s\|^2_{\hat{h}_t};$$

in particular, it follows that if both h and k satisfy the Hermite- Einstein condition, then $\bar{\partial}s = 0$. q.e.d.

As a general result, let us mention also the following:

Proposition 3.9. *Any J-holomorphic line bundle F admits a Hermite-Einstein structure.*

Proof. Let h be any Hermitian structure on F; then $K^{1,1}_{\omega_h} = i\lambda I$ for $\lambda \;\varepsilon\; C^\infty(M)$, and if $\hat{k} = e^\mu \hat{h}$, then $K^{1,1}_{\omega_k} = K^{1,1}_{\omega_h} + i(\Box\mu)I = i(\lambda + \Box\mu)I$, where, of course $\Box := i\Lambda\partial_M\bar{\partial}_M$.

Now,

$$\int_M K^{1,1}_{\omega_h}\eta^n = k(F)\,\mathrm{Vol}_g(M) \quad \text{and consequently} \quad \int_M (k(F)-\lambda)\eta^n = 0.$$

It is possible to find μ such that $\Box\mu = k(F) - \lambda$ and clearly $K^{1,1}_{\omega_k} = ik(F)I$. q.e.d.

Finally, we have:

Proposition 3.10. *Assume E is equipped with a Hermite-Einstein structure. Then the following hold:*

(1) *If $\partial\,\mathrm{eg}(E) < 0$, then E admits nonon zero J-holomorphic sections.*

(2) *If $\partial\,\mathrm{eg}(E) = 0$, then every J-holomorphic section of E is parallel.*

Proof. Assume $\bar\partial\sigma = 0$. Then

$$\partial_M\bar\partial_M|\sigma|^2\Lambda\eta^{n-1} = (|\partial\sigma|^2 - k(E)|\sigma|^2)\eta^n,$$

and so

$$\int_M (|\partial\sigma|^2 - k(E)|\sigma|^2)\eta^n = 0.$$

Hence the result follows immediately.

4. Stability and existence of Hermite-Einstein structures

Let (M, J_M, g) be a compact n-dimensional almost Hermitian manifold whose Kähler form η satisfies $\partial_M\bar\partial_M\eta^{n-1} = 0$, let $\mathcal{H}_s$ denote the s-dimensional Hausdorff measure let $(E, \hat J)$ be a complex vector bundle of **rank** r over M, and let $J \in \mathcal{B}(C(E))$. Consider the following class of objects: $F \in \mathcal{F}(J)$ if

[1] there exists a closed subset $S \subset M$ with $\mathcal{H}_{2n-4}(S) < +\infty$, such that $F_{|M\setminus S}$ is a J-holomorphic subbundle of $E_{|M\setminus S}$;

[2] for any $x \in S$, and any local J-holomorphic curve K through x not contained in $S, F_{|K-\{x\}}$ extends to K as subbundle.

Note that, by a result of Nijenhuis and Woolfs [7], given any complex tangent vector to M, there exists a local J-holomorphic curve tangent to it .

In the case $n = 2$, we can assume $\mathcal{F}(J)$ to be the class of J-holomorphic bundles F on M for which there exists a J-holomorphic generically immersive map $i : F \longrightarrow E$.

If $F \in \mathcal{F}(J)$, it is easy to see, by slicing and then using Fubini's theorem, that the corresponding section π of $E^* \otimes E$ is in L^2_1, and so it is possible to define $\partial\operatorname{eg}(F)$, according to the Chern-Weil formula, as

$$\partial\operatorname{eg}(F) := \int_M (< i\pi, K^{1,1}_{\omega_h} > -|\bar{\partial}\pi|^2)\eta^n,$$

where h is any Hermitian structure on E. Clearly if F is regular, by Codazzi-Mainardi equations, this definition coincides with the one given at the beginning of Section 3.

We set the following definition.

Definition 4.1. We say that E is J-stable (resp. J-semistable) if, for any $F \varepsilon \mathcal{F}(J)$, with $0 < \operatorname{rank} F < r$, we have:

$$\mu(F) < \mu(E) \quad (\text{ resp.} \quad \mu(F) \leq \mu(E)).$$

We first have

Proposition 4.2. *Assume E is J-simple and admits a Hermite-Einstein structure h. Then E is J-stable.*

Proof. Let $F \varepsilon \mathcal{F}(J)$ with $0 < \operatorname{rank} F = p < r$, and let π be the corresponding section of $E^* \otimes E$. In general, we have

$$\begin{aligned}\int_M < i\pi, K^{1,1}_{\omega_h} > \eta^n &= \int_M < i\pi, H_{\omega_h} > \eta^n + \int_M < \pi, k(E)I > \eta^n \\ &= \int_M < i\pi, H_{\omega_h} > \eta^n + 2\pi pn\mu(E),\end{aligned}$$

and so

$$\mu(F) = \mu(E) + \frac{1}{2\pi pn}\int_M (< i\pi, H_{\omega_h} > -|\bar{\partial}\pi|^2)\eta^n.$$

Consequently, if $H_{\omega_h} = 0$, it follows that $\mu(F) \leq \mu(E)$ where the equality implies that F corresponds to π satisfying $D_{\omega_h}\pi = 0$ so that π is globally regular and $E = F \oplus F^{\perp}$ J-holomorphically, contradicting J-simplicity.

We are now in position to state our main theorem.

Theorem 4.3. *Let (M, J_M, g) be a compact n-dimensional almost Hermitian manifold whose Kähler form η satisfies $\partial_M\bar{\partial}_M\eta^{n-1} = 0$, let $(E, \hat{J})$ be a complex vector bundle of* **rank r** *over M, and let $J \varepsilon \mathcal{B}(C(E))$ such that E is J-stable. Then there exists a unique (up to homotheties) Hermitian structure h on E satisfying the Hermite-Einstetn condition $H_{\omega_h} = 0$.*

Proof. The general lines of the proof are the following. Investigate the existence of a Hermite-Einstein structure via the heat equation, show that the only possible obstruction to the solution is the existence of a weak J-holomorphic subbundle, and then obtain the result by proving the regularity of weakly J-holomorphic maps.

Fix a Hermitian structure h such that $\operatorname{tr} H = 0$ where $H = H_{\omega_h}$.

We consider the evolution equation

$$h_t^{-1}\frac{d}{dt}h_t = -H_t, \tag{4.3.1}$$

which is a parabolic equation. By the standard theory of parabolic equations, there exists a $T > 0$ such that (4.3.1) can be solved for $t \,\varepsilon\, [0,T[$;
let $\{h_t\}_{t\in[0,T[}$ be a solution with $h_0 = h$. Set $g_t := h_0^{-1}h_t$ we have

Lemma 4.4.

(1) *For every* $t \,\varepsilon\, [0,T[$, *we have* $\operatorname{tr}\log g_t = 0$.

(2) $\| H_t \|_\infty$ *is a monotone decreasing function of* t *and thus, in particular, there exists* $C_1 > 0$ *such that for every* $t \,\varepsilon\, [0,T[$, *we have*

$$\| H_t \|_\infty < C_1. \tag{4.4.1}$$

(3) $V_h(\log g_t)$ *is a monotone decreasing function of* t *and thus, in particular, there exists* $C_2 > 0$ *such that for every* $t \,\varepsilon\, [0,T[$, *we have*

$$V_h(\log g_t) < C_2. \tag{4.4.2}$$

Proof. (1) and (2): From $h_t^{-1}\frac{d}{dt}h_t = -H_t$ it follows $\frac{d}{dt}H_t = -\Box_t H_t$ and so

$$\begin{cases} (\Delta + \frac{d}{dt})\operatorname{tr} H_t = 0, \\ (\Delta + \frac{d}{dt})|H_t|^2 = -2|\partial_{\omega_t} H_t|^2 \le 0. \end{cases}$$

Therefore (2) follows directly from the maximum principle;
this gives also $\operatorname{tr} H_t = 0$.
Finally,

$$0 = -\operatorname{tr} H_t = \operatorname{tr} h_t^{-1}\frac{d}{dt}h_t = \frac{d}{dt}\operatorname{tr}\log g_t, \quad \operatorname{tr}\log g_0 = 0.$$

(3) Consider $\frac{d}{dt}V_h(g_t)$. Because of Lemma 3.6, we only need to compute
it for $t = 0$ and so we easily obtain

$$\frac{d}{dt}V_h(g_t) = -\int_M |H_t|^2\eta^n \le 0.$$

Therefore Donaldson's functional is decreasing along the given path.

Corollary 4.5. *There exist constants $K_1 > 0, K_2 > 0$ such that, for every $t \varepsilon [0,T[$, we have:*

$$\| \log g_t \|_\infty \leq K_1 + K_2 \| \log g_t \|_1 . \tag{4.5.1}$$

Proof. The desired estimate follows from the following three facts:

(a) If $f \varepsilon C^\infty(M)$ satisfies $\Delta f \leq k$, then $\| f \|_\infty \leq c(k) \| f \|_1$.
(b) If $g = h^{-1}k$, then we have $\Delta \log \operatorname{tr} g \leq 2(|H_{\omega_k}|^2 + |H_{\omega_h}|^2)$.
(c) If $g \varepsilon S_h(E)$ satisfies $\operatorname{tr}\log g = 0$, then
$|\log g| \leq C|\log \operatorname{tr} g| \leq A + B|\log g|$.

Only (b) deserves some further comments. We start from

$$i\Lambda\overline{\partial}\partial_{\omega_h} g = i\Lambda\overline{\partial}(gg^{-1}\partial_{\omega_h} g) = ig(H_{\omega_k} - H_{\omega_h}) + i\Lambda(\overline{\partial}g \wedge g^{-1}\partial_{\omega_h} g).$$

Therefore

$$\begin{aligned}\Delta \operatorname{tr} g \leq &(|H_{\omega_k}| + |H_{\omega_h}|)\operatorname{tr} g + i\Lambda \operatorname{tr}(\overline{\partial}g \wedge g^{-1}\partial_{\omega_h} g)\\ =&(|H_{\omega_k}| + |H_{\omega_h}|)\operatorname{tr} g - |\overline{\partial}gg^{-1/2}|^2,\end{aligned}$$

i. e.,

$$\Delta \operatorname{tr} g + |\overline{\partial}gg^{-1/2}|^2 \leq (|H_{\omega_k}| + |H_{\omega_h}|)\operatorname{tr} g.$$

Since

$$\begin{aligned}\operatorname{tr} g\Delta \log \operatorname{tr} g &= \Delta \operatorname{tr} g + |\overline{\partial} \operatorname{tr} g|^2(\operatorname{tr} g)^{-1}\\ &= \Delta \operatorname{tr} g + |\operatorname{tr}(\overline{\partial}gg^{-1/2}g^{1/2})|^2|g^{1/2}|^{-2}\\ &\leq \Delta \operatorname{tr} g + |\overline{\partial}gg^{-1/2}|^2,\end{aligned}$$

we obtain (b).

Now, there are two possibilities:

(1) There exists $K > 0$ such that , for every $t \varepsilon [0,T[$,

$$\| \log g_t \|_\infty < K.$$

It follows that $g_t \longrightarrow g$ and g corresponds to a Hermite-Einstein structure.

(2) $\limsup \| \log g_t \|_1 = +\infty$.

Assume we are in case (1). Then $g_t \longrightarrow g_T$. If $T < +\infty$, by the theory of parabolic equations we can extend $\{g_t\}$ to $[0, T + \varepsilon[$ for some

ε and so $T = +\infty$; g_∞ corresponds to a Hermite-Einstein structure; in fact, for any $s \,\varepsilon\, S_h(E)$, we have

$$V_h(s) \geq -\int_M |s||H|\eta^n + \int_M < s, \Lambda\overline{\partial}(\Phi[s](\partial_{\omega_h} s) > \eta^n$$
$$= -\int_M |s||H|\eta^n + \int_M < s, \Phi[s](\partial_{\omega_h} s) > \Lambda\partial\eta^{n-1}$$
$$+ \int_M < \Phi[s](\overline{\partial}s), \overline{\partial}s > \eta^n.$$

Since

$$\left|\int_M < s, \Phi[s](\partial_{\omega_h} s) > \Lambda\partial\eta^{n-1}\right| \leq C' \parallel s \parallel_\infty \parallel \overline{\partial}s \parallel_2^2,$$

from $\parallel \log g_t \parallel_\infty < K$ for every $t \,\varepsilon\, [0, +\infty[$, it follows that $\Phi \geq C > 0$ on the range of the $\log g_t$'s, so that

$$\int_M < \Phi[\log g_t](\overline{\partial}\log g_t),\ \overline{\partial}\log g_t > \eta^n \geq C \parallel \overline{\partial}\log g_t \parallel_2^2 .$$

Therefore, there exists $A > 0$ such that , for every $t \,\varepsilon\, [0, +\infty[$,

$$V_h(\log g_t) \geq -A.$$

Consequently,

(a) from $\int_0^p |H_t|^2 dt = -V_h(\log g_p)$, it follows $\int_0^\infty |H_t|^2 dt < +\infty$ and, in particular, $\lim_{t \to +\infty} \parallel H_t \parallel_2 = 0$;

(b) $\int_M < \Phi[\log g_t](\overline{\partial}\log g_t), \overline{\partial}\log g_t > \eta^n \leq K'$ uniformly on t and so $\parallel \overline{\partial}\log g_t \parallel_2$ and $\parallel \overline{\partial} g_t \parallel_2$ are uniformly bounded.

Thus it follows that , up to subsequences, $g_t \longrightarrow g_\infty$ in L_1^2 and $H_\infty = 0$; the standard elliptic regularity implies that g_∞ is smooth.

Assume, from now on, we are in case (2).

In particular, we can choose $(C_m), C_m \longrightarrow +\infty$ and $(t_m), t_m \longrightarrow +\infty$ such that

$$i) \quad \parallel \log g_{t_m} \parallel_1 \longrightarrow +\infty,$$
$$ii) \quad \parallel \log g_{t_m} \parallel_1 \geq C_m V_h(\log g_{t_m}).$$

Let $\nu_m := \parallel \log g_{t_m} \parallel_1$ and $s_m := \nu_m^{-1} \log g_{t_m}$. Then

$$\parallel s_m \parallel_1 = 1 \quad \text{and} \quad \parallel s_m \parallel_\infty \leq K.$$

Now

$$\begin{aligned} \| \log g_{t_m} \|_1 \geq & C_m V_h(\log g_{t_m}) \\ & \Rightarrow i\nu_m \int_M < s_m, H > \eta^n \\ & \quad + \nu_m^2 \int_M < \Phi[\nu_m s_m](\overline{\partial} s_m), \overline{\partial} s_m > \eta^n \\ & \quad + \nu_m^2 \int_M < s_m, \Phi[\nu_m s_m](\partial_{\omega_h} s_m) > \wedge \partial \eta^{n-1} \\ & \leq C_m^{-1} \nu_m. \end{aligned} \tag{4.5.2}$$

From

$$\nu_m < \Phi[\nu_m s_m](\overline{\partial} s_m), \overline{\partial} s_m > \geq < \Phi[s_m](\overline{\partial} s_m), \overline{\partial} s_m >$$

and the fact that

$$\left| \nu_m \int_M < s_m, \Phi[\nu_m s_m](\partial_{\omega_h} s_m) > \wedge \partial \eta^{n-1} \right| \leq K' \| \overline{\partial} s_m \|_2$$

uniformly, it follows that

$$\int_M < \Phi[s_m](\overline{\partial} s_m), \overline{\partial} s_m > \eta^n \leq K_0 + K' \| \overline{\partial} s_m \| .$$

Since $\Phi \geq C > 0$ on the range of the s_m's, we obtain

$$\| \overline{\partial} s_m \|_2^2 \leq K_1 + K'' \| \overline{\partial} s_m \|, \quad \text{i.e.,} \quad \| \overline{\partial} s_m \|_2 \leq K.$$

Finally, passing to a subsequence, s_m converges weakly to u in L_1^2; clearly, u is nontrivial.

A close examination of the convergence leads to

Proposition 4.6.

$$u = \sum_{k=1}^{q} \lambda_k p_k,$$

where:

(a) $q \geq 2, \quad \lambda_k \, \varepsilon \, \mathbb{R}, \quad 1 \leq k \leq q$ *and* $\lambda_1 < ... < \lambda_q$;

(b) $p_k = p_k^{\#} = p_k^2, \quad 1 \leq k \leq q, \quad p_k p_j = \delta_{jk} p_j, \quad 1 \leq j,k \leq q$ *and* $\sum_{k=1}^{q} p_k = I$;

(c) $\pi_j := \sum_{k=1}^{j} p_k$ *satisfies* $(I - \pi_j)\overline{\partial}\pi_j = 0, \quad 1 \leq j \leq q$;

(d) *for at least one* $j, 1 \leq j \leq q-1$, *we have* $\mu(\pi_j) \geq \mu(E)$.

Proof. Set $\partial_0 := \partial_{\omega_h}$. Recall that, given a positive definite $g \,\varepsilon\, S_h(E)$, if $g = C^{-1}e^{\Lambda}C$, then:

$$\partial_0 \log g = C^{-1}(\partial_0\Lambda + [\Lambda, P])C, \tag{4.6.1}$$

$$\overline{\partial} \log g = C^{-1}(\overline{\partial}\Lambda + [\Lambda, Q])C, \tag{4.6.2}$$

$$g^{-1}\partial_0 g = C^{-1}(\partial_0\Lambda + P - e^{-\Lambda}Pe^{\Lambda})C, \tag{4.6.3}$$

$$g^{-1}\overline{\partial} g = C^{-1}(\overline{\partial}\Lambda + Q - e^{-\Lambda}Qe^{\Lambda})C \tag{4.6.4}$$

with $P := \partial_0 CC^{-1}$ and $Q := \bar{\partial}CC^{-1}$. Consequently

$$|\partial_0 \log g|^2 = |(\partial_0\Lambda|^2 + |[\Lambda, P]|^2, \tag{4.6.5}$$

$$|g^{-1}\partial_0 g|^2 = |\partial_0\Lambda|^2 + |P - e^{-\Lambda}Pe^{\Lambda}|^2. \tag{4.6.6}$$

moreover

$$|[\Lambda, P]|^2 = \sum_{\alpha<\beta}(\lambda_\alpha - \lambda_\beta)^2(|P_{\alpha\beta}|^2 + |P_{\beta\alpha}|^2), \tag{4.6.7}$$

(4.6.8)

$$|P - e^{-\Lambda}Pe^{\Lambda}|^2 = \sum_{\alpha<\beta}(1 - e^{\lambda_\beta - \lambda_\alpha})|P_{\alpha\beta}|^2 + (1 - e^{\lambda_\alpha - \lambda_\beta})|P_{\beta\alpha}|^2,$$

$$\begin{aligned} < \partial_0 \log g,\ g^{-1}\partial_0 g > &= \sum_{\alpha,\beta}(\lambda_\alpha - \lambda_\beta)(1 - e^{\lambda_\beta - \lambda_\alpha})|P_{\alpha\beta}|^2 + |\partial_0\Lambda|^2 \\ &\geq |\partial_0\Lambda|^2, \end{aligned} \tag{4.6.9}$$

and similar equation for $\overline{\partial}$. Therefore, we obtain that, for any $k \,\varepsilon\, \mathbb{Z}^+$, we have:

$$\partial_0(\log g)^k = C^{-1}(\partial_0\Lambda^k + [\Lambda^k, P])C,$$

$$|\operatorname{tr}\partial_0(\log g)^k| \leq |\log g|^{k-1}|\partial_0\Lambda| \leq |\log g|^{k-1} < \partial_0 \log g,\ g^{-1}\partial_0 g >^{1/2}.$$

It follows immediately that along the heat flow,

$$\int_M |\operatorname{tr}\partial_0(\log g_t)^k|\eta^n$$

(4.6.10)

$$\leq C' \parallel \log g_t \parallel_\infty^{k-1} \left(\int_M < \partial_0 \log g_t, g_t^{-1}\partial_0 g_t > \eta^n\right)^{1/2}.$$

On the other hand, along the heat flow, we have

$$\int_M < \partial_0 \log g_t, g_t^{-1}\partial_0 g_t > \eta^n \leq K \int_M |\log g_t|\eta^n. \tag{4.6.11}$$

In fact,

$$\begin{aligned}\int_M < \partial_0 \log g_t, g_t^{-1}\partial_0 g_t > \eta^n &= \int_M < \log g_t, \partial_0^*(g_t^{-1}\partial_0 g_t) > \eta^n \\ &= \int_M < \log g_t, i\Lambda\overline{\partial}(g_t^{-1}\partial_0 g_t) > \eta^n \\ &= \int_M < \log g_t, H - H_t > \eta^n \\ &\leq K\int_M |\log g_t|\eta^n.\end{aligned}$$

Substituting (4.6.11) in (4.6.10) yields

$$\int_M |\operatorname{tr}\partial_0(\log g_t)^k|\eta^n \leq C'' \parallel \log g_t \parallel_\infty^{k-1} \left(\int_M |\log g_t|\eta^n\right)^{1/2};$$

in particular

$$\int_M |\operatorname{tr}(\partial_0(s_m)^k)|\eta^n \leq C\nu_m^{-1/2} \longrightarrow 0,$$

which implies that

$$\partial_0 \operatorname{tr} u^k = 0,$$

and u has constant eigenvalues. Hence we can write $u = \sum_{k=1}^{q} \lambda_k p_k$, which gives directly (a) and (b). Moreover, if we write $u = D^{-1}\Lambda D$, then $\partial_0 u = D^{-1}[\Lambda, \partial_0 D D^{-1}]D$ and so $< u, \Phi[u](\partial_0 u) >= 0$ because $D\Phi[u](\partial_0 u)D^{-1}$ is zero on the diagonal. Consequently,

$$\begin{aligned}&\lim_{m\to\infty}\int < s_m, \Phi[s_m](\partial_0 s_m) > \wedge\partial\eta^{n-1} \\ &\quad = \lim_{m\to\infty}\int < s_m, \nu_m\Phi[\nu_m s_m](\partial_0 s_m) > \wedge\partial\eta^{n-1} = 0.\end{aligned}$$

Moreover,

$$V_h(u) = \lim_{m\to\infty}\Big(\int_M < is_m, H > \eta^n + \int_M < \Phi[s_m](\overline{\partial} s_m), \overline{\partial} s_m > \eta^n$$
$$+ \int_M < s_m, \nu_m \Phi[\nu_m s_m](\partial_{\omega_h} s_m) > \wedge \partial\eta^{n-1}\Big)$$
$$\leq \lim_{m\to\infty}\Big(\int_M < is_m, H > \eta^n + \int_M < \nu_m \Phi[\nu_m s_m](\overline{\partial} s_m), \overline{\partial} s_m > \eta^n$$
$$+ \int_M < s_m, \nu_m \Phi[\nu_m s_m](\partial_{\omega_h} s_m) > \wedge \partial\eta^{n-1}\Big) \leq 0.$$

In the same manner, if $A \,\varepsilon\, C^\infty(\mathbb{R}\times\mathbb{R}, \mathbb{R})$ satisfyes $A(\lambda,\mu) < (\lambda-\mu)^{-1}$, whenever $\lambda > \mu$, then

$$\int_M < iu, H > \eta^n + \int_M < A[u](\overline{\partial} u), \overline{\partial} u > \eta^n \leq 0. \tag{4.6.12}$$

In fact, for m sufficiently large, on the range of the s_m's we have:

$$\nu_m < \Phi[\nu_m s_m](\overline{\partial} s_m), \overline{\partial} s_m > \ \geq \ < A[s_m](\overline{\partial} s_m), \overline{\partial} s_m >,$$

and so

$$\int_M < iu, H > \eta^n + \int_M < A[u](\overline{\partial} u), \overline{\partial} u > \eta^n$$
$$\leq \lim_{m\to\infty}\Big(\int_M < is_m, H > \eta^n + \int_M < \nu_m \Phi[\nu_m s_m](\overline{\partial} s_m), \overline{\partial} s_m > \eta^n\Big)$$
$$\leq 0.$$

Now let $\pi_j := \sum_{k=1}^{j} p_k$ and, for a suitable $\delta > 0$, let

$$\phi_j(x) = \begin{cases} 1 & \text{if } x \leq \lambda_j + \delta, \\ 0 & \text{if } x \geq \lambda_{j+1} - \delta, \end{cases}$$

$$\psi_j(x) = \begin{cases} 0 & \text{if } x \leq \lambda_j + \delta, \\ 1 & \text{if } x \geq \lambda_{j+1} - \delta. \end{cases}$$

Then, of course, $\phi_j\psi_j \equiv 0$ and $\phi_j(s_m) \longrightarrow \pi_j, \psi_j(s_m) \longrightarrow I - \pi_j$ in L_1^2. We want to show that

$$\int_M |\partial_0 \phi_j(s_m)\psi_j(s_m)|\eta^n \longrightarrow 0. \tag{4.6.13}$$

At first we see that if $\log g_m = C_{(m)}^{-1}\Lambda_{(m)}C_{(m)}$ and $P_{(m)} := \partial_0 C_{(m)} C_{(m)}^{-1}$, then

$$|\partial_0\phi_j(s_m)\psi_j(s_m)|^2 = \sum_{\alpha=1}^{j}\sum_{\beta=j+1}^{r}|P_{\alpha\beta}^{(m)}|^2. \tag{4.6.14}$$

Fom (4.6.11) taking (4.6.9.) into account, we obtain, in particular,

$$\int_M \nu_m^{-1}\sum_{\alpha<\beta}(\lambda_\alpha^{(m)} - \lambda_\beta^{(m)})(1 - e^{\lambda_\beta^{(m)}-\lambda_\alpha^{(m)}})|P_{\alpha\beta}^{(m)}|^2\eta^n \leq K.$$

If $\alpha < \beta$, then clearly $(\lambda_\alpha^{(m)} - \lambda_\beta^{(m)})(1 - e^{\lambda_\beta^{(m)}-\lambda_\alpha^{(m)}}) > (\lambda_\alpha^{(m)} - \lambda_\beta^{(m)})^2$ and so

$$\int_M \nu_m^{-2}\sum_{\alpha<\beta}(\lambda_\alpha^{(m)} - \lambda_\beta^{(m)})^2|P_{\alpha\beta}^{(m)}|^2\eta^n \leq \nu_m^{-1}K.$$

Since $\nu_m^{-1}(\lambda_\alpha^{(m)} - \lambda_\beta^{(m)}) \longrightarrow \lambda_\alpha - \lambda_\beta$, we have immediately (4.6.13.).

(*d*) Write $u = \lambda_q I - \sum_{j=1}^{q-1}(\lambda_{j+1} - \lambda_j)\pi_j$

If $T := \lambda_q \partial\,\mathrm{eg}(E) - \sum_{j=1}^{q-1}(\lambda_{j+1} - \lambda_j)\partial\,\mathrm{eg}\,\pi_j$, then

$$\begin{aligned} T &= \int_M < iu, H > \eta^n + \int_M \sum_{j=1}^{q-1}(\lambda_{j+1} - \lambda_j)|\overline{\partial}\pi_j|^2\eta^n \\ &= \sum_{j=1}^{q-1}(\lambda_{j+1} - \lambda_j)\int_M (|\overline{\partial}\pi_j|^2 - < i\pi_j, H >)\eta^n \\ &= \int_M < iu, H > \eta^n + \int_M < \sum_{j=1}^{q-1}(\lambda_{j+1} - \lambda_j)(\delta\varphi_j)^2[u](\overline{\partial}u), \overline{\partial}u > \eta^n. \end{aligned}$$

Let $A(\lambda,\mu) := \sum_{j=1}^{q-1}(\lambda_{j+1} - \lambda_j)(\delta\varphi_j)^2(\lambda,\mu)$. Then $A(\lambda,\mu) < (\lambda-\mu)^{-1}$ for $\lambda > \mu$, and $T \leq 0$, as a consequence of (4.6.12). Thus

$$\int_M (|\overline{\partial}\pi_j|^2 - < i\pi_j, H >)\eta^n \leq 0$$

for at least one $j, 1 \leq j \leq q-1$. Since

$$\mu(\pi_j) = \mu(E) + (2\pi n \cdot \mathrm{rank}\,\pi_j)^{-1}\int_M (< i\pi_j, H > -|\overline{\partial}\pi_j|^2)\eta^n \geq \mu(E),$$

we can achieve the proof of Theorem 4.3, if we can show that $\pi_j \,\varepsilon\, \mathcal{F}(J)$, contradicting J-stability; this will follow from some general results concerning the regularity of weakly J-holomorphic maps, that will be proved in the next paragraph.

5. Regularity of weakly J-holomorphic maps

In this section we prove Theorem 0.2. Let $(M, J_M, g), (N, J_N, h)$ be two almost Hermitian manifolds with $\dim_{\mathbb{R}} M = 2n$ and assume there exists a closed 2-form α on N such that

(a) $\alpha^{1,1} > 0$;
(b) there exists $K > 1$, such that , for every $x \,\varepsilon\, N$ and every $X, Y \,\varepsilon\, T_x N$, we have

$$K^{-1}|X|_h|Y|_h \leq |\alpha(X,Y)| \leq K|X|_h|Y|_h. \tag{5.0.1}$$

Given $\sigma : M \longrightarrow N$, we can define $\overline{\partial}^0_{J_M}\sigma$ as the section of $\wedge^{0,1}M \otimes \sigma^{-1}(T^{1,0}N)$ given by

$$\overline{\partial}^0_{J_M}\sigma(X) := \frac{1}{4}[d\sigma(X + iJ_M X) - iJ_N d\sigma(X + iJ_M X)].$$

Similarly, we can define $\partial^0_{J_M}\sigma$. Clearly $d\sigma = \partial^0_{J_M}\sigma + \overline{\partial^0_{J_M}\sigma} + \overline{\partial}^0_{J_M}\sigma + \overline{\overline{\partial}^0_{J_M}\sigma}$, and σ is (J_M, J_N)-holomorphic $\Leftrightarrow \overline{\partial}^0_{J_M}\sigma = 0$.

We can embed N into $\mathbb{R}^N$ isometrically. We recall that a L^2_1-weakly (J_M, J_N)-holomorphic map $f : M \longrightarrow N$ is a map for which there exists a sequence $\{f_m\}_{m\in\mathbb{Z}^+}$ of smooth maps $f_m : M \longrightarrow \mathbb{R}^N$ such that the following hold:

(a) both $\{f_m\}_{m\in\mathbb{Z}^+}$ and $\{df_m\}_{m\in\mathbb{Z}^+}$ converge in $L^2(M, \mathbb{R}^N)$;
(b) $\lim_{m\to\infty} f_m = f$ and $f(x) \,\varepsilon\, N$ for a.e. $x \,\varepsilon\, M$;
(c) if we define $df := \lim_{m\to\infty} df_m$, then, for a.e. $x \,\varepsilon\, M, df$ sends $T^{1,0}_x M$ into $T^{1,0}_{f(x)}N$.

For reader's convenience we restate Theorem 0.2.

Theorem 5.1. *Let σ be a L^2_1-weakly (J_M, J_N)-holomorphic map. Then, there exists a closed subset $S \subset M$ with $\mathcal{H}_{2n-4}(S) < +\infty$, such that σ is smooth on $M\backslash S$; moreover, for any $x \,\varepsilon\, S$ and any local J-holomorphic curve K through x not contained in S , $\sigma_{|K-\{x\}}$ extends smoothly to K.*

The proof of Theorem 5.1 is broken into a sequel of steps.

First of all, since the result is of local nature, we can assume $M = B_1(0) := \{x \,\varepsilon\, \mathbb{R}^{2n} | |x| = 1\}$ equipped with the flat metric and an almost complex structure J such that $J(0)$ is the standard structure J_0 and, as $N \subset \mathbb{R}^N$ isometrically, consider $\sigma = (\sigma_1, ..., \sigma_N)$ as an $\mathbb{R}^N$-valued map.

We start with the following lemma.

Lemma 5.2. *In the sense of distributions, we have*

$$\Delta\sigma = O(|\nabla\sigma|^2), \tag{5.2.1}$$

where Δ and ∇ are the ordinary Laplacian and gradient acting on component functions.

Proof. At any $x \in B_1(0)$, consider an orthonormal frame $\{e_1, ..., e_n, e_{n+1}, ..., e_{2n}\}$ with $e_{n+j} = J_M(0)e_j, 1 \leq j \leq n$. Since σ is L_1^2-weakly (J_M, J_N)-holomorphic, for any j, $1 \leq j \leq n$, we have

$$(I - iJ_N)(e_j(\sigma) + ie_{n+j}(\sigma)) = 0. \tag{5.2.2}$$

Taking the covariant derivative in the sense of distributions in $\wedge^{0,1}M \otimes \sigma^{-1}(T^{1,0}N)$, we easily obtain

$$\begin{aligned}
&(I - iJ_N)(e_j - ie_{n+j})(e_j(\sigma) + ie_{n+j}(\sigma)) + O(|\nabla\sigma|^2) = 0\\
&= (I - iJ_N)(e_je_j(\sigma) + e_{n+j}e_{n+j}(\sigma)) + (e_je_{n+j}(\sigma)\\
&\quad - e_{n+j}e_j(\sigma)) + O(|\nabla\sigma|^2)\\
&= (I - iJ_N)(e_je_j(\sigma) + e_{n+j}e_{n+j}(\sigma)) + O(|\nabla\sigma|^2)\\
&= (e_je_j(\sigma) + e_{n+j}e_{n+j}(\sigma)) + O(|\nabla\sigma|^2).
\end{aligned}$$

On the other hand

$$\Delta\sigma = \sum_{j=1}^{n}(e_je_j(\sigma) + e_{n+j}e_{n+j}(\sigma)) + O(|\nabla\sigma|^2).$$

Thus Lemma 5.2. is proved.

Let, as usual, $B_r(x)$ be the ball of center x and radius r in $\mathbb{R}^{2n}$, and set

$$E_r^{(x)}(\sigma) := \int_{B_r(x)} |\nabla\sigma|^2\eta^n, \quad E_r(\sigma) = E_r^{(0)}(\sigma),$$

η being the standard Kähler form in $\mathbb{C}^n$. We have

Lemma 5.3. (Energy Weakly Monotonicity Formula) *There exist $r_0 > 0, C > 0$ and a smooth function $\varepsilon(r) \longrightarrow 0$ with r such that, for every $x \in B_1(0)$ and every $\tau, \rho, 0 < \tau < \rho \leq \min\{r_0, 1 - |x|\}$, we have*

$$\frac{1 + C\rho^2}{\rho^{2n-2}} \int_{B_\rho(x)} |\nabla\sigma|^2\eta^n \geq (1 - \varepsilon(\tau))\frac{1 + C\tau^2}{\tau^{2n-2}} \int_{B_\tau(x)} |\nabla\sigma|^2\eta^n; \tag{5.3.1}$$

in particular, $\lim_{r\to 0} r^{2-2n} \int_{B_r(x)} |\nabla\sigma|^2\eta^n$ exists.

Proof. If J_0 is the standard structure in $\mathbb{C}^n$ and we set

$$\overline{\partial}^0\sigma(X) := \frac{1}{4}[d\sigma(X + iJ_0X) - iJ_N d\sigma(X + iJ_0X)],$$

then

$$\overline{\partial}^0_J\sigma = \overline{\partial}^0\sigma + \pi\partial^0\sigma + \pi\overline{\partial}^0\sigma, \tag{5.3.2}$$

where, clearly, $\pi\alpha(X) := \alpha((J - J_0)(X))$ and so $\pi(z) = O(|z|)$. For simplicity, we assume $x = 0$. Then, from (5.3.1) it follows that there exists $C_0 > 0$ such that in $B_1(0)$ we have

$$|\overline{\partial}\sigma|^2 \leq C_0|\partial^0\sigma|^2. \tag{5.3.3}$$

First of all, we have that if r is sufficiently small, then there exists B such that, if $0 < \tau < \rho \leq r$, then for a. e. $z \,\varepsilon\, B_\rho(0)\backslash B_\tau(0)$,

(5.3.4)

$$\sigma^*(\alpha) \wedge ((\partial\overline{\partial}\log|z|^2)^{n-1} + C_0((\partial\overline{\partial}\log|z|^2)^{n-2} \wedge \partial\overline{\partial}|z|^2)) \geq 0.$$

In fact, without loss of generality, we can assume that $z = (z_1, 0, ..., 0)$ so that

$$\partial\overline{\partial}\log|z|^2 = \frac{1}{|z|^2}\sum_{j=2}^{n} dz_j \wedge d\bar{z}_j.$$

Consequenltly, a direct computation shows that :

$$\begin{aligned}(a)\quad & \sigma^*(\alpha)\wedge(\partial\overline{\partial}\log|z|^2)^{n-1}\\ &\geq(n-1)!|z|^{2(1-n)}(K^{-1}|\partial^0_{z_1}\sigma|^2\\ &\quad - K|\overline{\partial}^0_{z_1}\sigma|^2 - 2(K-K^{-1})|\partial^0_{z_1}\sigma||\overline{\partial}^0_{z_1}\sigma|)\\ &\geq(n-1)!|z|^{2(1-n)}(K^{-1}|\partial^0_{z_1}\sigma|^2 - K|\overline{\partial}^0_{z_1}\sigma|^2\\ &\quad - 2(K-K^{-1})(C^{-1}|\partial^0_{z_1}\sigma|^2 + C|\overline{\partial}^0_{z_1}\sigma|^2))\\ &=(n-1)!|z|^{2(1-n)}(K^{-1} - 2(K-K^{-1})C^{-1})|\partial^0_{z_1}\sigma|^2\\ &\quad - (K + 2(K-K^{-1})C)|\overline{\partial}^0_{z_1}\sigma|^2),\end{aligned}$$

$$\begin{aligned}(b)\quad & \sigma^*(\alpha)\wedge(\partial\overline{\partial}\log|z|^2)^{n-2}\wedge\partial\overline{\partial}|z|^2\\ &\geq|z|^{2(1-n)}|z|^2(K^{-1}|\partial^0\sigma|^2\\ &\quad - K|\overline{\partial}^0\sigma|^2 - 2(K-K^{-1})|\partial^0\sigma||\overline{\partial}^0\sigma|)\\ &\quad + (n-2)|z|^{2(1-n)}(K^{-1}|\partial^0_{z_1}\sigma|^2 - K|\overline{\partial}^0_{z_1}\sigma|^2\\ &\quad - 2(K-K^{-1})|\partial^0_{z_1}\sigma||\overline{\partial}^0_{z_1}\sigma|).\end{aligned}$$

If r is sufficiently small, from (5.3.3) it follows that

$$\sigma^*(\alpha) \wedge (\partial\overline{\partial} \log |z|^2)^{n-2} \wedge \partial\overline{\partial}|z|^2 \geq C_1 |z|^{2(1-n)}(|z|^2|\partial^0\sigma|^2),$$

so that if $C > 2K(K-K^{-1})$ and $B > (n-1)!C_1^{-1}(K + 2(K-K^{-1})$, then we get (5.3.4). Using (5.3.4) we obtain:

$$\begin{aligned}
0 \leq & \int\limits_{B_\rho(0)\setminus B_\tau(0)} \sigma^*(\alpha) \wedge ((\partial\overline{\partial}\log|z|^2)^{n-1} + C(\partial\overline{\partial}\log|z|^2)^{n-2} \wedge \partial\overline{\partial}|z|^2) \\
= & \int\limits_{bB_\rho(0)} \sigma^*(\alpha) \wedge ((\overline{\partial}\log|z|^2 \wedge \partial\overline{\partial}\log|z|^2)^{n-1} \\
& + C((\overline{\partial}\log|z|^2 \wedge \partial\overline{\partial}\log|z|^2)^{n-2} \wedge \partial\overline{\partial}|z|^2) \\
& - \int\limits_{bB_\tau(0)} \sigma^*(\alpha) \wedge ((\overline{\partial}\log|z|^2 \wedge \partial\overline{\partial}\log|z|^2)^{n-1} \\
& + C((\overline{\partial}\log|z|^2 \wedge \partial\overline{\partial}\log|z|^2)^{n-2} \wedge \partial\overline{\partial}|z|^2) \\
= & \frac{1+C\rho^2}{\rho^{2n-2}} \int\limits_{bB_\rho(0)} \sigma^*(\alpha) \wedge \overline{\partial}|z|^2 \wedge (\partial\overline{\partial}|z|^2)^{n-2} \\
& - \frac{1+C\tau^2}{\tau^{2n-2}} \int\limits_{bB_\tau(0)} \sigma^*(\alpha) \wedge \overline{\partial}|z|^2 \wedge (\partial\overline{\partial}|z|^2)^{n-2} \\
= & \frac{1+C\rho^2}{\rho^{2n-2}} \int\limits_{B_\rho(0)} \sigma^*(\alpha) \wedge (\partial\overline{\partial}|z|^2)^{n-1} \\
& - \frac{1+C\tau^2}{\tau^{2n-2}} \int\limits_{B_\tau(0)} \sigma^*(\alpha) \wedge (\partial\overline{\partial}|z|^2)^{n-1} \\
= & \frac{1+C\rho^2}{\rho^{2n-2}} \int\limits_{B_\rho(0)} (|\partial\sigma|^2 - |\overline{\partial}\sigma|^2)\eta^n - \frac{1+C\tau^2}{\tau^{2n-2}} \int\limits_{B_\tau(0)} (|\partial\sigma|^2 - |\overline{\partial}\sigma|^2)\eta^n.
\end{aligned}$$

Let $\delta(x) := \frac{|\overline{\partial}\sigma|^2}{|\partial\sigma|^2}(x)$Then

$$\frac{1+C\rho^2}{\rho^{2n-2}} \int\limits_{B_\rho(0)} \left(\frac{1-\delta^2}{1+\delta^2}\right)|\nabla\sigma|^2\eta^n \geq \frac{1+C\tau^2}{\tau^{2n-2}} \int\limits_{B_\tau(0)} \left(\frac{1-\delta^2}{1+\delta^2}\right)|\nabla\sigma|^2\eta^n.$$

Finally, in order to obtain the required inequality we choose $\epsilon(r)$ in such a way that $1-\epsilon(r) \leq \inf\limits_{B_r(0)} \frac{1-\delta^2}{1+\delta^2}$.

The next step is given by

Proposition 5.4. *There exists $\epsilon_0 > 0$, independent of σ, such that, for any $x \in B_1(0)$ and any $r < 1 - |x|$, if $r^{2-2n}E_r^{(x)}(\sigma) \leq \epsilon_0$, then σ is Hölder continuous in $B_{r/2}(x)$, and moreover there exist positive constants α and C, independent of σ, x and r, such that for every $z, w \in B_{r/2}(x)$, we have*

$$|\sigma(z) - \sigma(w)| \leq C|z - w|^\alpha.$$

Proof. The proof will follow closely Schoen-Uhlenbeck argument for weakly harmonic maps [8]. Fix $\epsilon_0 > 0$.

(1) Let $\phi \in C_0^\infty(B_1(0))$ such that $\int_{B_1(0)} \phi\eta^n = 1$ and, for $h \in]0, \frac{1}{4}]$, let

$$\sigma^{(h)}(x) := \int_{B_1(0)} \phi(y)\sigma(x - hy)\eta^n(y).$$

Then from the basic estimates of Lemma 5.3, it follows directly that, if $h_0 = \sqrt[4]{\epsilon_0}$, then:

$$\begin{aligned} &(a) \quad \text{for any} \quad h \in]0, h_0] \\ &\qquad \int_{B_{1/2}(0)} |\nabla\sigma^{(h)}|^2\eta^n \leq cE_1(\sigma), \\ &(b) \quad \sup_{x \in B_{1/2}(0)} |\sigma^{(h_0)}(x) - \sigma^{(h_0)}(0)| < ch_0^2. \end{aligned}$$

(2) Let $r = \sqrt[8]{\epsilon_0}$ and assume $\theta \in]\tau, \frac{1}{4}[$. Let $h = h(r)$ be a smooth nonincreasing function satisfying:

$$\begin{cases} h(r) = h_0 & \text{if } r \leq \theta, \\ h(r) = 0 & \text{if } r \geq \theta + \tau, \end{cases} \qquad |h'(r)| \leq 2\tau,$$

and set:

$$\sigma(h(x))(x) := \int_{B_1(0)} \phi^{(h(x))}(x - y)\sigma(y)\eta^n(y),$$

finally,

$$\sigma_{h(x)}(x) := p \circ \sigma^{(h(x))}(x),$$

where $\phi^{(h)}(x) := h^{-2n}\phi(x/h)$ and $p : T(N) \longrightarrow N$ is the smooth nearest point projection map from a tubular neighborhood.

Then we have the following:

$$(c) \quad \sigma_h \in L_1^2(B_{1/2}(0), N);$$
$$(d) \quad \sigma_h = \sigma \quad \text{on} \quad B_{1/2}(0) \backslash B_{\theta+\tau}(0);$$
$$(e) \quad \int_{B_{\theta+\tau}(0)\backslash B_\theta(0)} |\nabla \sigma_h|^2 \eta^n \leq c' \int_{B_{\theta+2\tau}(0)\backslash B_{\theta-\tau}(0)} |\nabla \sigma|^2 \eta^n.$$

(3) Let $v : B_1(0) \longrightarrow \mathbb{R}^N$ be such that

$$\begin{cases} \Delta v = 0 \quad \text{in } B_{1/2}(0), \\ (v - \sigma^{(h_0)})_{|bB_{1/2}(0)} = 0. \end{cases}$$

Then by the previous estimates we obtain immediately:

$$(f) \quad \sup_{B_{1/2}(0)} |v - \sigma^{(h_0)}|^2 \leq c\sqrt{\epsilon_0};$$
$$(g) \quad \sup_{B_{1/4}(0)} |\nabla v|^2 \leq c_1 E_{1/2}(v) \leq c_2 E_1(\sigma).$$

Now, for any $\theta \in]0, \frac{1}{4}[$, we have:

$$\begin{aligned} \theta^{2-2n} E_\theta(\sigma^{(h_0)}) =& \theta^{2-2n} \int_{B_\theta(0)} \left| \nabla(\sigma^{(h_0)} - v) + \nabla v \right|^2 \eta^n \\ \leq& \theta^{2-2n} \int_{B_\theta(0)} \left| \nabla(\sigma^{(h_0)} - v) \right|^2 \eta^n + \theta^{2-2n} \int_{B_\theta(0)} |\nabla v|^2 \eta^n, \end{aligned}$$

which together with (g) iimplies that $\theta^{2-2n} E_\theta(v) \leq c_2 \theta^2 E_1(\sigma)$ and

$$\begin{aligned} \int_{B_\theta(0)} \left| \nabla(\sigma^{(h_0)} - v) \right|^2 \eta^n \leq& \int_{B_{1/2}(0)} \left| \nabla(\sigma^{(h_0)} - v) \right|^2 \eta^n \\ =& - \int_{B_{1/2}(0)} (\sigma^{(h_0)} - v) \Delta \sigma^{(h_0)} \eta^n \\ \leq& c_3 \sqrt[4]{\epsilon_0} \int_{B_{1/2}(0)} \left| \Delta \sigma^{(h_0)} \right| \eta^n. \end{aligned}$$

Now,

$$\Delta \sigma^{(h_0)}(x) = - \int_{\mathbb{R}^{2n}} \phi^{(h_0)}(x - y) \Delta \sigma(y) \eta^n(y).$$

From (5.2.1) it follows directly that

$$\int_{B_{1/2}(0)} \left|\Delta\sigma^{(h_0)}\right|\eta^n \leq c_4 E_1(\sigma),$$

so that for any $\theta \,\varepsilon\,]0, \frac{1}{4}[$,

$$\theta^{2-2n} E_\theta(\sigma_{h_0}) \leq c_5(\theta^{2-2n}\sqrt[4]{\epsilon_0} + \theta^2)E_1(\sigma),$$

since $E_\theta(\sigma_{h_0}) \leq CE_\theta(\sigma^{(h_0)})$. Let $\gamma = \gamma(n) \,\varepsilon\,]0, 2^{-4}[$, $\theta_0 = \varepsilon_0^\gamma$, and $p = [\theta_0(3\tau)^{-1}]$, write

$$[\theta_0, \theta_0 + 3p\tau] = \bigcup_{j=1}^{p} I_j, \quad \text{with} \quad I_j = [\theta_0 + 3(j-1)\tau, \theta_0 + 3j\tau],$$

and note that $p \geq \frac{1}{3\sqrt[16]{\epsilon_0}} - 1$. Since

$$\int_{B_{\theta+3p\tau}(0)\setminus B_\theta(0)} |\nabla\sigma|^2\eta^n = \sum_{j=1}^{p}\int_{r\,\varepsilon\, I_j} |\nabla\sigma|^2\eta^n \leq E_1(\sigma),$$

there exists at least one $j_0, 1 \leq j_0 \leq p$, for which

$$\int_{r\,\varepsilon\, I_{j_0}} |\nabla\sigma|^2\eta^n \leq p^{-1}E_1(\sigma) \leq c_6 \sqrt[16]{\epsilon_0}E_1(\sigma).$$

Let $\theta = \theta_0 + (3(j_0-1)+1)\tau$ and so $I_{j_0} = [\theta - \tau, \theta + 2\tau]$, and let $h = h(r)$ as in (2). Then

$$\sigma_h \,\varepsilon\, L_1^2(B_{1/2}(0), N),$$

$$\sigma_h(x) = \sigma(x) \quad \text{for} \quad |x| \geq \theta + \tau,$$

$$\int_{r\,\varepsilon\,[\theta,\theta+\tau]} |\nabla\sigma_h|^2\eta^n \leq c\int_{r\,\varepsilon\, I_{j_0}} |\nabla\sigma|^2\eta^n,$$

and consequently

$$\int_{r\,\varepsilon\,[\theta,\theta+\tau]} |\nabla\sigma_h|^2\eta^n \leq c_\gamma \sqrt[16]{\epsilon_0}E_1(\sigma).$$

Since σ and σ_h are homotopically equivalent, it follows that $E_{\theta+\tau}(\sigma) \leq C_0 E_{\theta+\tau}(\sigma_h)$, and therefore

$$E_\theta(\sigma) \leq E_{\theta+\tau}(\sigma) \leq c_8 E_{\theta+\tau}(\sigma_h) \leq c_8 E_\theta(\sigma_{h_0}) + c_\gamma \sqrt[16]{\epsilon_0}E_1(\sigma).$$

Thus for $\theta \,\varepsilon\, [\theta_0, 2\theta_0]$,

$$\theta_0^{2-2n}E_\theta(\sigma) \leq c_9(\theta_0^{2-2n}\sqrt[16]{\epsilon_0} + \theta_0^2)E_1(\sigma);$$

in particular,

$$\theta_0^{2-2n} E_\theta(\sigma) \leq c_{10}(\sqrt[16]{\epsilon_0} \varepsilon^{\gamma(2-2n)} + \varepsilon_0^{2\gamma}) E_1(\sigma).$$

If $\gamma = \min\{(32(2n-2))^{-1}, 64^{-1}\}$, then $\theta_0^{2-2n} E_\theta(\sigma) \leq c_{10} \varepsilon_0^{2\gamma} E_1(\sigma)$. Consequently, by choosing ϵ_0 such that $c_{10}\epsilon_0^{2\gamma} \leq \frac{1}{2}$ we obtain the following:

(h) There exist constants $\epsilon_0 > 0$, and $\theta_0 \,\varepsilon\,]0,1[$, independent of σ, such that, if $E_1(\sigma) \leq \varepsilon_0$, then

$$\theta_0^{2-2n} E_\theta(\sigma) \leq \frac{1}{2} E_1(\sigma).$$

(4) Set $\sigma_{\theta_0}(x) := \sigma(\theta_0 x)$. Then by (h) we have

$$E_1(\sigma_{\theta_0}) = \theta_0^{2-2n} E_{\theta_0}(\sigma) \leq \frac{1}{2} E_1(\sigma) \leq \epsilon_0.$$

Iterating the procedureyields for any nonnegative integer k.

$$(\theta_0^k)^{2-2n} E_{\theta_0^k}(\sigma) \leq 2^{-k} E_1(\sigma)$$

Now, given any $r \,\varepsilon\,]0, \theta_0]$, there exists $k > 0$ such that $r \,\varepsilon\, [\theta_0^{k+1}, \theta_0^k]$. If $\alpha = (\log 2)(-2\log\theta_0)^{-1}$, then

$$(\theta_0^k)^{2-2n} E_r(\sigma) \leq (\theta_0^k)^{2\alpha} E_1(\sigma),$$

and therefore

$$r^{2-2n} E_r(\sigma) \leq \theta_0^{-2\alpha} r^{2\alpha} E_1(\sigma).$$

Hence Morrey's lemma completes the proof of Proposition 5.4.

Corollary 5.5. *Let* $S := \mathrm{Sing}(\sigma)$; *then* $\mathcal{H}_{2n-2}(S \cap B_{1/2}(0)) = 0$.

Proof. Let $x \,\varepsilon\, S$; then, simply by rescaling, we obtain, for any $\lambda < 1 - |x|$,

$$\lambda^{2-2n} \int_{B_{\lambda(x)}} |\nabla\sigma|^2 \eta^n > \varepsilon_0. \tag{5.5.1}$$

Let $\{B_\delta(x_1), \ldots, B_\delta(x_p)\}, p = p(\delta)$, be a maximal family of disjoint balls with $x_1, \ldots, x_p \,\varepsilon\, B_{1/2}(0) \cap S$, then by the maximality we have

$$S \cap B_{1/2}(0) \subset \bigcup_{j=1}^{p} B_{2\delta}(x_j)$$

and, in consequence of (5.5.1),

$$p\delta^{2n-2} < \epsilon_0 \int_{A_\delta} |\nabla\sigma|^2 \eta^n \leq \epsilon_0^{-1} E_1(\sigma), \tag{5.5.2}$$

where $A_\delta := \bigcup_{j=1}^{p} B_\delta(x_j)$. Therefore

$$\mathcal{H}_{2n-2}(S \cap B_{1/2}(0)) \leq C \int_{A_\delta} |\nabla \sigma|^2 \eta^n.$$

From (5.5.2) it follows that $\mathcal{H}_{2n}(A_\delta) \leq C\delta^2 E_1(\sigma)$, and thus, by the dominated convergence,

$$\lim_{\delta \to 0} \int_{A_\delta} |\nabla \sigma|^2 \eta^n = 0.$$

Hence $\mathcal{H}_{2n-2}(S \cap B_{1/2}(0)) = 0$. q.e.d.

Note that σ is C^∞-smooth in $B_1(0) \backslash S$ since σ satisfies (5.2.1).

For further developments, we need

Lemma 5.6. (Energy Comparison) *Let σ be a L_1^2-weakly (J_M, J_N)-holomorphic map and assume $\dim_C M = n > 1$. Then there exists $C_0 > 0$ such that, for $x \,\varepsilon\, B_1(0)$, any $r < 1 - |x|$ and any L_1^2-map $\sigma_0 : B_1(0) \longrightarrow N$ with $\sigma_{0|bB_r(x)} = \sigma_{|bB_r(x)}$, we have:*

$$E_r^{(x)}(\sigma) \leq C_0 E_r^{(x)}(\sigma).$$

Proof. Glue together two copies of $B_r(x)$ in order to obtain a $2n$-dimensional sphere S. Then interpretate σ and σ_0 as $\phi = (\sigma, \sigma_0) \,\varepsilon\, L_1^2(S, \mathbb{R}^N)$, and let $\phi_m \longrightarrow \phi$ be a smooth approximation. Clearly, for every $m \,\varepsilon\, \mathbb{Z}^+$,

$$\int_S \phi_m^*(\alpha) \wedge \eta^{n-1} = 0,$$

and, consequently,

$$\int_{B_r(x)} \sigma^*(\alpha) \wedge \eta^{n-1} - \int_{B_r(x)} \sigma_0^*(\alpha) \wedge \eta^{n-1} = 0,$$

which implies that

$$E_r^{(x)}(\sigma) \leq C_0^{-1} \int_{B_r(x)} \sigma^*(\alpha) \wedge \eta^{n-1} = C_0^{-1} \int_{B_r(x)} \sigma_0^*(\alpha) \wedge \eta^{n-1} \leq C_0 E_r^{(x)}(\sigma).$$

q.e.d.

Now we can quote the following general result [8].

Lemma 5.7. *For any $x \in B_1(0)$, any $r < 1 - |x|$, and any $\tau \in L_1^2(B_r(x), N)$ set*

$$\tau_r^*(x) := \int_{B_r(x)} r^{-2n}\phi(\frac{y-x}{r})\tau(y)\eta^n(y)$$

and

$$W_r^{(x)}(\tau) := \int_{B_r(x)} |\tau - \tau_r^*|^2\eta^n.$$

Let $K \subset N$ be a compact subset. Then there exist δ and q such that, for any $\epsilon \in]0,1[$ and any $\tau \in L_1^2(bB_r(x), K)$ such that

$$r^{4-4n}E_r^{(x)}(\tau)W_r^{(x)}(\tau) \leq \delta\epsilon^q$$

there exists $\hat{\tau} \in L_1^2(B_r(x), N)$ such that:

$$\begin{aligned} &i) \quad \hat{\tau}_{|bB_r(x)} = \tau, \\ &ii) \quad E_r^{(x)}(\hat{\tau}) \leq C(\epsilon r E_r^{(x)}(\tau) + \epsilon^{-q}r^{-1}W_r^{(x)}(\tau)), \\ &iii) \quad W_r^{(x)}(\hat{\tau}) \leq C\epsilon^q r W_r^{(x)}(\tau). \end{aligned}$$

Remark 5.8. As a consequence of Lemma 5.7, we can improve Proposition 5.5 as follows (cf. again [8]): there exists $\epsilon_0 > 0$, independent of σ, such that, for any $x \in B_1(0)$ and any $r < 1 - |x|$, if

(5.8.1) $$r^{-2n}W_r^{(x)}(\sigma) \leq \epsilon_0',$$

then σ is Hölder continuous in $B_{r/2}(x)$, and moreover there exist positive constants α and C, independent of σ, x and r, such that for every $z, w \in B_{r/2}(x)$, we have

$$|\sigma(z) - \sigma(w)| \leq C|z-w|^\alpha.$$

In fact from Lemmas 5.6 and 5.7, we obtain immediately

$$E_r^{(x)}(\sigma) \leq C_0E_r^{(x)}(\hat{\sigma}) \leq C_1(\epsilon r E_r^{(x)} + \epsilon^{-q}r^{-1}W_r^{(x)}(\sigma))$$

and so (5.8.1) leads easily to $r^{2-2n}E_r^{(x)}(\sigma) \leq \epsilon_0$. Finally we use Proposition 5.4.

We are now in a position to prove

Proposition 5.9.

$$\mathcal{H}_{2n-4}(S) < +\infty.$$

Proof. Let $s \in \mathbb{R}, s < 2n-2$, and $\phi^s(S) = \inf\{\sum_j r_j^s | S \subset \bigcup_j B_{r_j}(x_j)\}$.

Assume $\phi^s(S) > 0$. Then a basic density lemma in geometric measure theory ensures that for a.e. $x \varepsilon S$, we have

$$\limsup_{\lambda \to 0} \quad \lambda^{-s}\phi^s(S \cap B_\lambda(x)) \geq C > 0.$$

Therefore it is possible to choose $x_0 = 0 \varepsilon S, \lambda_n \longrightarrow 0$ in such a way that:

(1) $\lim_{n\to\infty} \quad \lambda_n^{-s}\phi^s(S \cap B_{\lambda_n/2}(0)) > 0$,

(2) $\sigma_{\lambda_n} \to \sigma_\infty$ weakly in $L_1^2(B_1(0))$.

Clearly σ_∞ is weakly J-holomorphic with respect to the standard integrable structure of $\mathbb{C}^n$, and moreoverby Remark 5.8, if $S_\infty := \mathrm{Sing}(\sigma_\infty)$, then $\sigma_{\lambda_\infty} \longrightarrow \sigma_\infty$ uniformly on compact subsets of $B_1(0)\backslash S_\infty$. Let $S_n := \mathrm{Sing}(\sigma_{\lambda_n})$. Then

$$\phi^s(S_\infty \cap B_{1/2}(0)) \geq \limsup_{n\to\infty} \phi^s(S_n \cap B_{1/2}(0)). \tag{5.9.1}$$

In fact, for any $\delta > 0$, let $\{B_{r_1}(x_1), ..., B_{r_q}(x_q)\}$ be a covering of $S_\infty \cap B_{1/2}(0)$ by balls satisfying

$$\sum_{j=1}^{q} r_j^s \leq \phi^s(S_\infty \cap B_{1/2}(0)) + \delta.$$

Consider $A := \overline{B_{1/2}(0)} \backslash \bigcup_{j=1}^{q} B_{r_j}(x_j)$. Then for n sufficiently large, σ_{λ_n} is smooth on A. Consequently $S_n \cap B_{1/2}(0) \subset \bigcup_{j=1}^{q} B_{r_j}(x_j)$ and

$$\phi^s(S_n \cap B_{1/2}(0)) \leq \phi^s(S_\infty \cap B_{1/2}(0)) + \delta,$$

which implies (5.9.1). Clearly, $S_n \cap B_{1/2}(0) = \{x/\lambda_n | x \varepsilon S \cap B_{\lambda_n/2}(0)\}$ and so

$$\phi^s(S_n \cap B_{1/2}(0)) = \lambda_n^{-s}\phi^s(S \cap B_{\lambda_n/2}(0)).$$

Therefore, from (1) it follows that $\lim_{n\to\infty} \phi^s(S_n \cap B_{1/2}(0)) > 0$ so that, by (5.9.1)

$$\phi^s(S_\infty \cap B_{1/2}(0)) > 0.$$

Now we have

$$\int_{B_1(0)} \sigma_\infty^*(\alpha) \wedge (\partial\overline{\partial} \log |z|^2)^{n-1} = 0. \tag{5.9.2}$$

In fact,

$$r^{2-2n}\int_{B_r(0)}|\nabla\sigma_\infty|^2\eta^n=\lim_{n\to\infty}r^{2-2n}\int_{B_r(0)}|\nabla\sigma_{\lambda_n}|^2\eta^n$$
$$=\lim_{n\to\infty}(r\lambda^n)^{2-2n}\int_{B_{\lambda_n r}(0)}|\nabla\sigma|^2\eta^n=\text{const}.,$$

and

$$\int_{B_1(0)}\sigma_\infty^*(\alpha)\wedge(\partial\bar\partial\log|z|^2)^{n-1}=\lim_{r\to 0}\int_{B_1(0)\setminus B_r(0)}\sigma_\infty^*(\alpha)\wedge(\partial\bar\partial\log|z|^2)^{n-1}$$
$$=\lim_{r\to 0}\Big(\int_{B_1(0)}|\nabla\sigma_\infty|^2\eta^n-r^{2-2n}\int_{B_1(0)}|\nabla\sigma_\infty|^2\eta^n\Big)$$
$$=0.$$

It follows that σ_∞ is complex homogeneous. In particular, $\frac{\partial\sigma_\infty}{\partial r}=0$ a. e. and thus $\lambda S_\infty\subset S_\infty$ for every λ.

Now, there are two possibilities:

(1) $s\le 0$: then, there is nothing to prove ;

(2) there exists $x_1\ \varepsilon\ bB_{3/4}(0)$ such that

$$\limsup_{\lambda\to 0}\lambda^{-s}\phi^s(S_\infty\cap B_\lambda(x_1))>0.$$

we can choose complex coordinates centered at x_1, in such a way that $\operatorname{Re}z_1$ is radial. By repeating the previous argument, we obtain $\sigma_\infty^{(1)}$ satisfying on $B_{1/2}(x_1)$

$$\begin{cases}\frac{\partial\sigma_\infty^{(1)}}{\partial r}=0,\\[1ex] \frac{\partial\sigma_\infty^{(1)}}{\partial z_1}=\frac{\partial\sigma_\infty^{(1)}}{\partial\bar z_1}=0.\end{cases}$$

By iterating the procedure, eventually we get $m, s\le 2m\le s+2<n$ and σ_∞^m, weakly J-holomorphic, satisfying on $B_{1/2}(x_m)$

$$\begin{cases}\frac{\partial\sigma_\infty^{(m)}}{\partial r}=0,\\[1ex] \frac{\partial\sigma_\infty^{(m)}}{\partial z_j}=\frac{\partial\sigma_\infty^{(1)}}{\partial\bar z_j}=0,\quad 1\le j\le m.\end{cases}$$

If $2m=2n-2$, then $S_\infty^{(n-2)}\supset\{z_n=0\}\cap B_{1/2}(x_{n-2})$ and this is a contradiction because $\mathcal{H}_{2n-2}S_\infty^{(n-2)}=0$. Consequently $2m\le 2n-4$ and finally $\mathcal{H}_{2n-4}(S)<+\infty$.

Since Proposition 5.5 gives, in particular, that σ is smooth along J-holomorphic curves, the proof of Theorem 0.2 is complete.

We can easily deduce from the proof of Theorem 0.2 a special regularity result; consider first the following.

Definition 5.10. A rational curve in an almost complex manifold (N, J_N) is the image of a noncostant $(J_{\mathbb{P}^1(\mathbb{C})}, J_N)$-holomorphic map $\phi : \mathbb{P}(\mathbb{C}) \longrightarrow N$.

Then we have

Corollary 5.11. *Let $(M, J_M, g), (N, J_N, h)$ as in Theorem 0.2. If N has no rational curves, then any L^2_1-weakly (J_M, J_N)-holomorphic map $\sigma : M \longrightarrow N$ is regular.*

Proof. Let $x \,\varepsilon\, M$. As in the proof of Theorem 0.2, we can construct $\sigma_\infty : \mathbb{C}^n \longrightarrow N$ L^2_1-weakly (J_0, J_N)-holomorphic, factor σ_∞ as $\sigma_\infty = \psi \circ \pi$, where $\pi : \mathbb{C}^n - \{0\} \longrightarrow \mathbb{P}^{n-1}(\mathbb{C})$ is the standard projection and $\mathcal{H}_{2n-6}(\operatorname{Sing}\psi) < +\infty$.

Now,

$$J_N \circ \psi_* \circ \pi_* = J_N \circ (\sigma_\infty)_* = (\sigma_\infty)_* \circ J_0$$
$$= \psi_* \circ \pi_* \circ J_0 = \psi_* \circ J_{\mathbf{P}^{n-1}(\mathbf{C})} \circ \pi_*.$$

Since π_* is surjective, $J_N \circ \psi_* = \psi_* \circ J_{\mathbf{P}^{n-1}(\mathbf{C})}$, i.e., ψ is L^2_1 -weakly $(J_{\mathbf{P}^{n-1}(\mathbf{C})}, J_N)$-holomorphic. Let $\mathbb{P}^1(\mathbb{C}) \subset \mathbb{P}^{n-1}(\mathbb{C}) \setminus \operatorname{Sing}\psi$. Then $\psi(\mathbb{P}^1(\mathbb{C}))$ is a rational curve in N unless ψ is constant, i. e., σ is regular at x .

As far as Theorem 0.1 is concerned, we simply note that the L^2_1 -weakly J-holomorphic subbundles, which we constructed in Section 4, correspond to L^2_1-weakly J-holomorphic maps from (M, J_M, g) to some Grassmann bundle $\operatorname{Gr}_p(E)$. Certainly, if $U \subset M$ is a sufficiently small domain, then $\pi^{-1}_{\operatorname{Gr}} \subset \operatorname{Gr}_p(E)$ can be equipped with a tamed Symplectic structure just by approximating the standard Kähler structure on $U \times \operatorname{Gr}_p(\mathbb{C}^r)$. Therefore Theorem 0.2 applies and the proof of Theorem 0.1 is complete.

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