STABILITY OF DIFFERENTIAL SYSTEMS WITH IMPULSIVE PERTURBATIONS IN TERMS OF TWO MEASURES

S. Leela
Department of Mathematics
SUNY College at Geneseo, Geneseo, N. Y. 14454

and

Department of Mathematics
University of Texas at Arlington
Arlington, Texas 76019

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by

S. Leela*

1. *Introduction*

The study of differential systems of the form

$$Dx = f(t,x) + G(t,x)d\sigma \tag{1.1}$$

where $D\sigma$ denotes the distributional derivative of σ, a function of bounded variation (that is, differential systems with impulsive perturbations, also called measure differential equations), is both interesting and important because most models for biological neural nets, pulse frequency modulation systems, automatic control problems with impulsive inputs and many physical processes are best described by such equations [1-3,8,10,12,13]. Since the solutions of (1.1) are discontinuous (that is, functions of bounded variation), the investigation of the stability properties of (1.1) by the usual techniques of perturbation theory and differential inequalities offers many difficulties. However, in [3,8] some stability results have been obtained under certain assumptions which may be regarded restrictive. See also [1,10,13].

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The advantages of studying stability properties by means of two different measures and the generality and unification which result by such an approach are known [6,9,11]. In [11], necessary and sufficient conditions for the (h_0,h) - stability criteria are discussed in the context of a general dynamical system. In [6], a Massera-type converse theorem is proved for the (h_0,h) - uniform asymptotic stability of an unperturbed system and the result is then applied to obtain the (h_0,h) - total stability of the perturbed system.

In this paper, we investigate the (h_0,h) - stability of the measure differential equation (1.1). Some direct theorems [See Sec. 5] giving sufficient conditions for the (h_0,h)-stability of (1.1) are proved by employing (i) a Lyapunov function which satisfies certain properties relative to the two measures of stability h_0 and h, namely, h-positive definiteness and h_0-decrescentness; and (ii) a comparison theorem involving a right continuous function having a countable number of singularities. In view of the generality one achieves by the approach of using two measures [See Sec. 4] it is clear that our results are general, flexible and most suitable for many applications.

2. *Preliminaries*

Let R^n be the n-Euclidean space with norm $||x|| = \sum_{i=1}^{n} |x_i|$, for $x \in R^n$ and M denote the set of all $n \times m$ matrices of real numbers with the norm $||G|| = \sum_{i=1}^{n} \sum_{j=1}^{m} |G_{ij}|$, where $(G_{ij}) = G \in M$. Let $BV(R_+,R^m)$ denote the set of all vector valued functions from R_+ into

R^m, whose components are scalar functions of bounded variation on R_+, the non-negative real line.

We shall consider the differential system

$$(2.1) \qquad x' = f(t,x), \quad x(t_0) = x_0 \qquad \left(' = \frac{d}{dt}\right)$$

and the measure differential equation

$$(2.2) \qquad Dx = f(t,x) + G(t,x)D\sigma, \quad x(t_0) = x_0,$$

where (i) the functions f and G are defined on $R_+ \times E$ (E being an open set in R^n) with values in R^n and M respectively; (ii) σ is a right continuous function and $\sigma \in BV(R_+,R^m)$; and (iii) Dx, $D\sigma$ denote the distributional derivatives of functions x,σ respectively.

<u>*Definition 2.1.*</u> A function $y(\cdot) = y(\cdot,t\ ,x_0)$ is said to be a solution of eq. (2.2) on R_+ if $y(\cdot)$ is a right continuous function, belonging to $BV(R_+,E)$ and the distributional derivative of $y(\cdot)$ on (t_0,T), $T \in R_+$, satisfies the eq. (2.2).

It is known [3] that $y(\cdot)$ is a solution of (2.2) through (t_0,x_0) if and only if it satisfies the integral equation

$$(2.3) \qquad y(t) = x_0 + \int_{t_0}^{t} f(s,y(s))ds + \int_{t_0}^{t} G(s,y(s))d\sigma(s), \quad t \geq t_0 \geq 0;$$

where for each $y(\cdot) \in BV(R_+,E)$, $f(t,y(t))$ is Lebesgue integrable and $G(t,y(t))$ is integrable with respect to Lebesgue-Stieltjes measure $d\sigma$. The existence and uniqueness of solutions of (2.2) have been discussed in [1,3] and some stability results for eq. (2.2) are proved in [3,8].

In the sequal, let $x(t) = x(t,t_0,x_0)$ and $y(t) = y(t,t_0,x_0)$ denote the solutions of equations (2.1) and (2.2) respectively. We shall assume that the solutions of (2.1) and (2.2) exist and are unique for $t \geq t_0$. For our study of eq. (2.2) it is convenient to introduce the following classes of functions [4,7]:

$K = \{\phi \in C[R_+,R_+] : \phi(s)$ is strictly increasing in s and $\phi(0) = 0\}$,

$L = \{q \in C[R_+,R_+] : q(s)$ is strictly decreasing in s and $\lim_{s\to\infty} q(s) = 0\}$,

$A = \{a \in C[R_+ \times [0,\rho),R_+] : a(t,s)$ is decreasing in t for each s and increasing in s for each t such that $\lim_{\substack{t\to\infty\\ s\to\infty}} a(t,s) = 0\}$,

where $C[A,B]$ is the class of continuous functions from A into B. For any $h \in C[R_+ \times E,R_+]$ and $\rho > 0$, let us set

$$P(h,\rho) = \{(t,x) \in R_+ \times E : h(t,x) < \rho\}.$$

We shall list the following hypotheses for the convenience of later reference:

(H_1) $f \in C[R_+ \times E,R^n]$ and is Lipschitzian in x for a constant $\bar{M} > 0$.

(H_2) $|h(t,x) - h(t,y)| \leq \hat{L}||x - y||$, (t,x), $(t,y) \in R_+ \times E$.

(H_3) $G(t,x)$ is defined on $R_+ \times E$, measurable in t for each x and continuous in x for each t such that for $(t,x) \in P(h,\rho)$,

$$||G(t,x)|| \leq g(t)\psi(h(t,x)),$$

where $\psi \in K$ and $g(t)$ is a dv_σ-integrable function (v_σ denotes the total variation function of σ).

(H$_4$) $\limsup_{\tau\to 0^+} \frac{1}{\tau}\int_t^{t+\tau} g(s)dv_\sigma(s) = w(t)$.

(H$_5$) The discontinuities of σ occur at isolated points, $\{t_k\}$, $t_1 < t_2 < \ldots < t_k < \ldots$, such that

$$||\sigma(t_k) - \sigma(t_k^-)|| \le \mu_k,$$

where μ_k are constants.

The following comparison theorem [5] plays an important role in our study of eq. (2.2).

THEOREM 2.1. Assume that

(i) $m(t) \ge 0$ is a right continuous function on R_+, with isolated singularities at t_k, $k = 1,2,\ldots$, $t_k > t_0$, such that $|m(t_k) - m(t_k^-)| \le \lambda_k$, where $\sum_{k=1}^{\infty} \lambda_k$ is convergent;

(ii) $\hat{g} : R_+ \times R_+ \to R$ satisfies Caratheodory type condition, $g(t,u)$ is nondecreasing in u for each t and

$$Dm(t) \le \hat{g}(t,m(t)), \quad t \in [t_i, t_{i+1}), \quad i = 0,1,2,\ldots,$$

where $Dm(t)$ is any Dini derivative of $m(t)$.

Then, $m(t_0) \le u_0$ implies that

$$m(t) \le r(t,t_0, u_0 + \sum_{k=1}^{\infty} \lambda_k), \quad t \ge t_0,$$

where $r(t,t_0,u_0)$ is the maximal solution of

$$u' = \hat{g}(t,u), \; u(t_0) = u_0 \ge 0, \quad t_0 \in R_+,$$

existing on $[t_0,\infty)$.

3. *Auxiliary Lemmas*

In order to prove some auxiliary results which are useful in discussing the stability of eq. (2.2), we need to employ a Lyapunov function $V(t,x)$ satisfying the following properties:

(1°) $V \in C[R_+ \times P(h,\rho),R_+]$ and $V(t,x)$ is locally Lipschitzian in x for a constant $\bar{L} > 0$;

(2°) V is h-positive definite, i.e.,

$$V(t,x) \geq b(h(t,x)), \ (t,x) \in P(h,\rho), \quad b \in K.$$

The following lemmas 3.1 and 3.2 are concerned with getting a differential inequality with respect to the eq. (2.2), in terms of the Lyapunov function V.

Lemma 3.1. Assume that the hypotheses (H_1), (H_3) and (H_4) hold. Let there exist a Lyapunov function $V(t,x)$ satisfying properties (1°), (2°) and the differential inequality

(3.1) $$\underset{(2.1)}{D^+V(t,x)} \leq g_1(t,V(t,x)), \ (t,x) \in P(h,\rho),$$

where $\underset{(2.1)}{D^+V(t,x)} = \limsup_{\tau\to 0^+} \frac{1}{\tau}[V(t+\tau, x+\tau f(t,x)) - V(t,x)]$ and $g_1 \in C[R_+ \times R_+,R]$. Then, the differential inequality

(3.2) $$D^+m(t) \leq g_2(t,m(t)),$$

is valid for those t where $m(t) = V(t,y(t,t_0,x_0))$ is defined, $y(t,t_0,x_0)$ being the solution of eq. (2.2) through (t_0,x_0) and

$g_2(t,u) = g_1(t,u) + \bar{L}w(t)\tilde{\psi}(u), \quad \tilde{\psi} \in K.$

Proof. For $(t,x) \in R_+ \times P(h,\rho)$, set

$$m(t) = V(t,y(t,t_0,x_0)).$$

By writing $x = y(t,t_0,x_0)$, we note that $y(t+\tau,t_0,x_0) = y(t+\tau,t,x)$, $\tau > 0$, in view of the uniqueness of solutions of (2.2). Let $x(s,t,x)$ denote the solution of eq. (2.1). Using the definition of $m(t)$ and the Lipschitzian property of V, we get

$$\begin{aligned}
D^+m(t) &= \limsup_{\tau\to0^+} \frac{1}{\tau}\Big[V(t+\tau,y(t+\tau,t_0,x_0)) - v(t,y(t,t_0,x_0))\Big] \\
&= \limsup_{\tau\to0^+} \frac{1}{\tau}\Big[V(t+\tau,y(t+\tau,t,x)) - V(t,x)\Big] \\
&\le \limsup_{\tau\to0^+} \frac{1}{\tau}\Big[V(t+\tau,y(t+\tau,t,x)) - V(t+\tau,x(t+\tau,t,x))\Big] \\
&\quad + \limsup_{\tau\to0^+} \frac{1}{\tau}\Big[V(t+\tau,x(t+\tau,t,x)) - V(t+\tau,x+\tau f(t,x))\Big] \\
&\quad + \limsup_{\tau\to0^+} \frac{1}{\tau}\Big[V(t+\tau,x+\tau f(t,x)) - V(t,x)\Big] \\
&\le \limsup_{\tau\to0^+}\Big[\frac{\bar{L}}{\tau}\,||y(t+\tau,t,x) - x(t+\tau,t,x)||\Big] \\
&\quad + \limsup_{\tau\to0^+}\Big[\frac{\bar{L}}{\tau}\,||x(t+\tau,t,x) - (x+\tau f(t,x))||\Big] + \underset{(2.1)}{D^+V(t,x)}.
\end{aligned}$$

Observe that $\limsup_{\tau\to0^+} \frac{1}{\tau}\,||x(t+\tau,t,x) - (x+\tau f(t,x)||$ approaches zero as $\tau \to 0$. Hence, using hypothesis (H_3), we obtain

$$D^+m(t) \leq \lim_{\tau\to 0^+} \sup \left[\frac{\bar{L}}{\tau}\left\{\int_t^{t+\tau} ||f(s,y(s,t,x)) - f(s,x(s,t,x))||ds + \int_t^{t+\tau} g(s)\psi(h(s,y(s,t,x)))dv_\sigma(s)\right\}\right] + D^+V(t,x). \quad (2.1)$$

Let $\psi_\tau(t,x) = \sup_{t\leq s\leq t+\tau} \psi(h(s,y(s,t,x)))$. Since

$\lim_{\tau\to 0^+} \psi_\tau(t,x) = \psi(h(t,x))$, we get, in view of hypotheses (H_1), (H_4) and the inequality (3.1),

$$D^+m(t) \leq \bar{L}w(t)\psi(h(t,x)) + g_1(t,V(t,x)).$$

Moreover, using property (2°) and the definition of $m(t)$, we have

$$D^+m(t) \leq \bar{L}w(t)\tilde{\psi}(m(t)) + g_1(t,m(t)) \equiv g_2(t,m(t)),$$

where $\tilde{\psi}(u) = \psi(b^{-1}(u)) \in K$. We note that (3.2) is satisfied almost everywhere since $m(t) = V(t,y(t,t_0,x_0))$ is a right continuous function with singularities at $\{t_k\}$, $k = 0,1,2,\ldots$. The Lemma is proved.

<u>*Lemma 3.2.*</u> Let the hypotheses (H_1), (H_2), (H_3), and (H_4) hold. Assume that there exists a Lyapunov function $V(t,x)$ satisfying properties (1°), (2°) and the differential inequality

$$(3.3) \qquad D^+_{(2.1)}V(t,x) + C(h(t,x)) \leq g_1(t,V(t,x)), \quad (t,x) \in P(h,\rho),$$

where $C \in K$ such that $C'(u)$ exists, $C' \in K$ and $g_1 \in C[R_+ \times R_+,R]$, $g_1(t,u)$ is nondecreasing in u for each $t \in R_+$. Then, (3.2) holds almost everywhere.

Proof. Setting

$$m(t) = V(t,y(t,t_0,x_0)) + \int_{t_0}^{t} C(h(s,y(s,t_0,x_0)))ds,$$

we have $V(t,y(t,t_0,x_0)) \leq m(t)$. Proceeding as in Lemma 3.1, with $x = y(t,t_0,x_0)$ and using the hypotheses (H_1), (H_3) and (H_4) together with the properties (1°), (2°) of V, we obtain

$$(3.4) \qquad D^+m(t) \underset{(2.1)}{\leq} D^+V(t,x) + \bar{L}\omega(t)\tilde{\psi}(V(t,x))$$

$$+ \limsup_{\tau\to 0^+} \frac{1}{\tau} \left[\int_{t_0}^{t+\tau} C(h(s,y(s,t_0,x_0)))ds - \int_{t_0}^{t} C(h(s,y(s,t_0,x_0)))ds\right].$$

The third term in (3.4) can be written as

$$(3.5) \qquad \limsup_{\tau\to 0^+} \frac{1}{\tau}\left\{\int_t^{t+\tau}\left[C(h(s,y(s))) - C(h(s,x(s)))\right]ds + \int_t^{t+\tau} C(h(s,x(s)))ds\right.$$

where $y(s,t_0,x_0) = y(s,t,x) = y(s)$ and $x(s,t,x) = x(s)$, $s \geq t$. Making use of the properties of the function C and (H_2), it is easy to see that

$$(3.6) \qquad \limsup_{\tau\to 0^+} \frac{1}{\tau}\left\{\int_t^{t+\tau}\left[C(h(s,y(s))) - C(h(s,x(s)))\right]ds\right.$$

$$\leq \limsup_{\tau\to 0^+} \frac{1}{\tau}\int_t^{t+\tau} C'(\xi)\hat{L}\|y(s) - x(s)\|ds$$

which approaches zero as $\tau \to 0$. Here, ξ is such that

$$h(s,y(s)) \leq \xi \leq h(s,x(s)) \quad \text{or} \quad h(s,x(s)) \leq \xi \leq h(s,y(s))$$

and $\hat{L}$ is the Lipschitz constant for the function $h(t,x)$. Now in view of (3.4), (3.5), (3.6) and (2°), we get

$$D^+m(t) \underset{(2.1)}{\leq} D^+V(t,x) + C(h(t,x)) + \bar{L}w(t)\tilde{\psi}(V(t,x)),$$

which, in turn, yields

$$D^+m(t) \leq g_1(t,m(t)) + \bar{L}w(t)\tilde{\psi}(m(t)),$$

because of (3.3), the fact that $V(t,y(t,t_0,x_0)) \leq m(t)$ and the nondecreasing property of $g_1(t,u)$ in u. The proof is complete.

Since the solution $y(t,t_0,x_0)$ of eq. (2.2) is a right continuous function of bounded variation, we need to estimate the jumps of $V(t,y(t,t_0,x_0))$ at the points t_k, $k = 1,2,\ldots$. The next lemma provides us with that estimate.

Lemma 3.3. (A) Let the hypotheses (H_1), (H_3) and (H_5) hold. Assume that there exists a Lyapunov function $V(t,x)$ satisfying property (1°). If $y(t,t_0,x_0)$ is a solution of eq. (2.2), then at the points of discontinuity $\{t_k\}$,

$$(3.7) \qquad |m(t_k) - m(t_k^-)| \leq \lambda_k,$$

where $m(t) = V(t,y(t,t_0,x_0))$ and $\lambda_k = \bar{L}\psi(\rho)g(t_k)\mu_k$.

(B) In addition to the assumptions in (A), let (H_2) also hold. Then, at the points $\{t_k\}$, (3.7) is valid where $m(t) = V(t,y(t,t_0,x_0)) + \int_{t_0}^{t} C(h(s,y(s,t_0,x_0)))ds$.

Proof. Following the arguments in [3; page 153], we obtain

(3.8) $$||y(t_k,t_0,x_0) - y(t_k^-,t_0,x_0)|| = ||G(t_k,y(t_k,t_0,x_0))[\sigma(t_k) - \sigma(t_k^-)]||$$

Setting $m(t) = V(t,y(t,t_0,x_0))$ and using the Lipschitzian property of V, we get

$$|m(t_k) - m(t_k^-)| \leq \bar{L}||y(t_k,t_0,x_0) - y(t_k^-,t_0,x_0)||,$$

which, in view of (3.8) and hypotheses (H_3), (H_5), yields (3.7) and the part A is proved.

However, setting $m(t) = V(t,y(t,t_0,x_0)) + \int_{t_0}^{t} C(h(s,y(s,t_0,x_0)))ds$ and noting that $\left| \int_{t_0}^{t_k} C(h(s,y(s,t_0,x_0)))ds - \int_{t_0}^{t_k^-} C(h(s,y(s,t_0,x_0)))ds \right|$ $= \lim_{\tau\to 0} \int_{t_k-\tau}^{t_k} C(h(s,y(s,t_0,x_0)))ds$ is zero, we obtain

$$|m(t_k) - m(t_k^-)| \leq |V(t_k,y(t_k,t_0,x_0)) - V(t_k^-,y(t_k^-,t_0,x_0))|.$$

Using the Lipschitzian property of V and arguments similar to those in the proof of part (A), it is easy to see that the conclusion of part (B) is valid.

Finally, we merely state a lemma which relates the maximal solutions of

(3.9) $$u' = g_2(t,u), \quad u(t_0) = u_0$$

and

(3.10) $$v' = [-v\, D^+A(t) + g_2(t,vA(t))] \frac{1}{A(t)}, \quad v(t_0) = v_0$$

where $A(t) > 0$ is continuous on R_+ and $A(t) \to \infty$ as $t \to \infty$.

Lemma 3.4. Let $r(t) = r(t,t_0,u_0)$ and $R(t) = R(t,t_0,v_0)$ be the maximal solutions of (3.9) and (3.10) respectively, existing on $[t_0,\infty)$. Then $r(t) = A(t)R(t)$.

The proof of the lemma is a consequence of the variation of parameters method and hence is omitted.

4. (h_0,h) - *Stability Concepts*

Let h_0, $h \in C[R_+ \times E, R_+]$. We shall assume that $\inf\{h_0(t,x) : (t,x) \in R_+ \times E\} = 0$. In this section we shall define the concepts of stability for the differential system (2.1) in terms of h_0 and h and illustrate how this enables us to unify the various concepts of stability seen in the literature. The functions h_0 and h are called the measures of stability. Suppose that $x(t) = x(t,t_0,x_0)$ is any solution of (2.1) existing in $P(h,\rho)$.

Definition 4.1. The differential system (2.1) is

(i) (h_0,h) - stable, if given $\varepsilon > 0$ and $t_0 \in R_+$, there exists a number $\delta = \delta(t_0,\varepsilon) > 0$ such that $h_0(t_0,x_0) < \delta$ implies $h(t,x(t)) < \varepsilon$, $t \geq t_0$;

(ii) (h_0,h) - uniformly stable, if (i) holds with δ being independent of t_0;

(iii) (h_0,h) - equiasymptotically stable, if (i) holds and for any $\nu > 0$, $t_0 \in R_+$ there exist numbers $\hat{\delta} = \hat{\delta}(t_0) > 0$ and $T = T(t_0,\nu) > 0$ such that $h_0(t_0,x_0) < \hat{\delta}$ implies $h(t,x(t)) < \nu$, $t \geq t_0 + T$;

(iv) (h_0,h) - uniformly asymptotically stable if (ii) holds and the numbers $\hat{\delta}$ and T in (iii) are independent of t_0;

(v) (h_0,h) - unstable if it is not (h_0,h) - stable.

A few choices for the measures h_0 and h as given below in (1)-(5) are enough to indicate the generality of the definition 4.1. It is easy to observe that the above definition reduces to the definition of

(1) the well known Lyapunov stability of the invariant set $\{0\}$, if $h_0(t,x) = h(t,x) = ||x||$;

(2) the partial stability of the invariant set $\{0\}$ if $h_0(t,x) = ||x||$ and $h(t,x) = ||x||_s = \sqrt{x_1^2 + \ldots + x_s^2}$, $s < n$;

(3) the stability of a set $M \subset R^n$, if $h_0(t,x) = h(t,x) = d(x,M)$ where $d(x,M)$ denotes the distance of x from the set M;

(4) the stability of the conditionally invariant set B with respect to the set A if $h_0(t,x) = d(x,A)$ and $h(t,x) = d(x,B)$ where A and B are such that $A \subset B$ and solutions starting in A remain in B;

(5) the stability of the symptotically self invariant set $\{0\}$ if $h(t,x) = ||x||$ and $h_0(t,x) = a(t,||x||)$, $a \in \Lambda$ or $h_0(t,x) = ||x|| + \lambda(t)$, $\lambda \in L$, [4].

Thus, it is obvious how an extremely unified study of stability concepts of different types of invariant sets is possible by employing two measures h_0 and h.

$\tilde{h}_0(t,x) = a(h_0(t,x)) + \sigma(t)$, $\sigma \in L$ such that $\sigma(t_j) = \sum_{k=j}^{\infty} \lambda_k$.

. Let $0 < \varepsilon < \rho$ and $t_0 \in R_+$ be given. Since the null solution of (5.1) is uniformly stable, given $b(\varepsilon) > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that

(5.2) $$u(t,t_0,v_0) < b(\varepsilon), \quad t \geq t_0, \quad \text{whenever} \quad v_0 < \delta.$$

Consider the function

(5.3) $$\tilde{h}_0(t,x) = a(h_0(t,x)) + \sigma(t)$$

and observe that $inf\, \{\tilde{h}_0(t,x) : (t,x) \in R_+ \times E\} = 0$ and $\tilde{h}_0$ is uniformly finer than h. We now claim that with the δ obtained above, the system (2.2) is $(\tilde{h}_0,h)$ - uniformly stable. If this is not true, there exists a solution $y(t) = y(t,t_0,x_0)$ of (2.2) and a $\hat{t} > t_0$ such that

(5.4a) $$\tilde{h}_0(t_0,x_0) < \delta,$$

(5.4b) $$h(\hat{t},y(\hat{t})) = \varepsilon,$$

and $h(t,y(t)) < \varepsilon$, $t \in [t_0,\hat{t})$. Hence, in $[t_0,\hat{t}]$, $h(t,y(t)) \leq \varepsilon < \rho$, and by Lemma 3.1, we have

$$D^+m(t) \leq g_2(t,m(t)), \quad t \in [t_i,t_{i+1}), \quad t_{i+1} \leq \hat{t},$$

$i = 0,1,2,\dots,$ where $m(t) = V(t,y(t))$. Also, by Lemma 3.3(A), $|m(t_k) - m(t_k^-)| \leq \lambda_k$. Now, using Theorem 2.1, we get for $t \in [t_0,\hat{t}]$,

$$V(t,y(t)) \leq r(t,t_0,u_0 + \sum_{k=1}^{\infty} \lambda_k)$$
$$\leq r(t,t_0,u_0 + \sum_{k=0}^{\infty} \lambda_k)$$
$$= r(t,t_0,u_0 + \sigma(t_0)).$$

Choosing $u_0 = a(h_0(t_0,x_0))$, we have

$$V(t,y(t)) \leq r(t,t_0,\tilde{h}_0(t_0,x_0)), \quad t \in [t_0,\hat{t}].$$

Hence, using (5.4b) and the h-positive definiteness of V, we obtain

$$b(\varepsilon) = b(h(\hat{t},y(\hat{t})) \leq V(\hat{t},y(\hat{t})) \leq r(\hat{t},t_0,\tilde{h}_0(t_0,x_0)),$$

which, in virtue of (5.3), (5.4a), the choice of $u_0 = a(h_0(t_0,x_0))$ and (5.2), results in a contradiction and the theorem is proved.

Theorem 5.2. Suppose that the assumption (B_1) of Theorem 5.1 holds. Assume further that the hypotheses of Lemma 3.2 and Lemma 3.3(B) are valid. Then, the uniform stability of the null solution of (5.1) implies the $(\tilde{h}_0,h)$ - uniform asymptotic stability of (2.2), where $\tilde{h}_0(t,x)$ is the same function as in Theorem 5.1.

Proof. Let $0 < \varepsilon < \rho$ and $t_0 \in R_+$ be given. Since the null solution of (5.1) is uniformly stable, the $(\tilde{h}_0,h)$ - uniform stability of (2.2) can be proved as in Theorem 5.1. Now, to prove that (2.2) is $(\tilde{h}_0,h)$ - uniformly asymptotically stable, it remains to show that for every $\varepsilon > 0$, there exist numbers $\delta_0 > 0$ and $T = T(\varepsilon) > 0$ such that $h(t,y(t)) < \varepsilon$, $t \geq t_0 + T$ whenever $\tilde{h}_0(t_0,x_0) < \delta_0$.

5. *Main Results*

In this section we give some sufficient conditions for the (h_0,h) - stability of the measure differential equation (2.2) in terms of a Lyapunov function $V \in C[R_+ \times P(h,\rho),R_+]$. For the sake of convenience, let us state the following definitions.

Definition 5.1. The measure h_0 is said to be uniformly finer than the measure h if there exists a constant $\lambda > 0$ and a function $\phi \in K$ such that $h_0(t,x) < \lambda$ implies $h(t,x) < \phi(h_0(t,x))$.

Definition 5.2. The function V is said to be h_0-decrescent if there exists a $\rho_0 > 0$ and a function $a \in K$ such that $h_0(t_0,x_0) < \rho_0$ implies $V(t,x) \leq a(h_0(t,x))$.

We are now in a position to state and prove our main stability results for the system (2.2).

Theorem 5.1. (A_1) Let the hypotheses of Lemma 3.1 and Lemma 3.3(A) hold. (B_1) Assume further that (i) $g_1(t,u)$ is nondecreasing in u for each t and $g_1(t,0) \equiv 0$; (ii) $\sum_{k=1}^{\infty} \lambda_k$ converges; (iii) h_0 is uniformly finer than h; and (iv) $V(t,x)$ is h_0-decrescent in $P(h_0,\rho_0)$, $\rho_0 \in (0,\lambda)$ being such that $\phi(\rho_0) < \rho$. Then, the uniform stability of the null solution of the scalar equation

$$(5.1) \qquad u' = g_2(t,u), \quad u(t_0) = u_0 \geq 0,$$

($g_2(t,u)$ being the function obtained in Lemma 3.1) implies the $(\tilde{h}_0,h)$ - uniform stability of the system (2.2), where

Let $\delta_0 = \delta(\rho)$ where δ is the positive number associated with $(\tilde{h}_0, h)$ - uniform stability of (2.2). Choose $T = T(\varepsilon) = \dfrac{b(\rho)}{C(\delta(\varepsilon))}$.

Suppose that $\tilde{h}_0(t_0,x_0) < \delta_0$ and $h(t,y(t)) \geq \delta(\varepsilon)$, $t \in [t_0,T)$. Then, we have, by virtue of Lemmas 3.2, 3.3(B) and Theorem 2.1,

$$(5.5) \qquad V(t,y(t)) + \int_{t_0}^{t} C(h(s,y(s))ds \leq r(t,t_0,u_0 + \sum_{k=1}^{\infty} \lambda_k),$$

$t \in [t_0, t_0 + T]$. Since $C \in K$, $C(h(s,y(s))) \geq C(\delta(\varepsilon))$, $s \in [t_0,t]$ and $r(t,t_0,u_0 + \sum_{k=1}^{\infty} \lambda_k) \leq r(t,t_0,u_0 + \sum_{k=0}^{\infty} \lambda_k) = r(t,t_0,\tilde{h}_0(t_0,x_0))$ with the choice of $u_0 = a(h_0(t_0,x_0))$, we get from (5.5),

$$(5.6) \qquad V(t,y(t)) \leq r(t,t_0,\tilde{h}_0(t_0,x_0)) - C(\delta(\varepsilon))(t - t_0), \quad t \in [t_0,T].$$

Since the null solution of (5.1) is uniformly stable, we have $r(t,t_0,\tilde{h}_0(t_0,x_0)) < b(\rho)$, $t \geq t_0$ whenever $\tilde{h}_0(t_0,x_0) < \delta_0$. Hence, using the h-positive definiteness of V, the choice of T and (5.6), we have

$$0 < b(\delta(\varepsilon)) \leq b'(h(t_0 + t,y(t_0 + T))) \leq V(t_0 + T,y(t_0 + T))$$

$$\leq b(\rho) - C(\delta(\varepsilon))T = 0.$$

This contradiction shows that there exists a $t_1 \in [t_0,t_0 + T]$ such that $h(t_1,y(t_1)) < \delta(\varepsilon)$. Now, in view of the $(\tilde{h}_0,h)$ - uniform stability of (2.2), it is clear that $h(t,y(t)) < \varepsilon$, $t \geq t_0 + T$, whenever $\tilde{h}_0(t_0,x_0) < \delta_0$. The proof is complete.

THEOREM 5.3. Let the hypotheses of Lemma 3.1 hold with the inequality (3.1) being replaced by

$$(5.7) \qquad \underset{(2.1)}{D^+V(t,x)A(t) + V(t,x)D^+A(t)} \le g_1(t,V(t,x)A(t)),$$

for $(t,x) \in P(h,\rho)$, where $A(t) \ge 1$ is a continuous function on R_+ such that $A(t) \to \infty$ as $t \to \infty$. Suppose further that (B_1) of Theorem 5.1 and hypotheses of Lemma 3.3(A) hold. Then, the stability properties of the null solution of

$$(5.8) \qquad V' = \frac{1}{A(t)}[-vD^+A(t) + g_2(t,vA(t))], \quad v(t_0) = v_0$$

($g_2(t,u)$ is the same function obtained before in Lemma 3.1) imply the corresponding $(\tilde{h}_0,h)$ - stability properties of the system (2.2).

Proof. Letting $v(t,x) = V(t,x)A(t)$, (5.7) can be written as

$$\underset{(2.1)}{D^+v(t,x)} \le g_1(t,v(t,x)), \quad (t,x) \in P(h,\rho).$$

Now, following the proof in Lemma 3.1, we get

$$(5.9) \qquad D^+m(t) \le g_2(t,m(t)),$$

as long as $y(t)$ exists in $P(h,\rho)$, where $m(t) = V(t,y(t))A(t)$. Using the conclusion of Lemma 3.3(A) and applying Theorem 2.1, (5.9) yields as before,

$$m(t) \le r(t,t_0,u_0 + \sum_{k=1}^{\infty} \lambda_k) \le r(t,t_0,\tilde{h}_0(t_0,x_0))$$

$$\equiv r(t), \quad t \ge t_0$$

provided $m(t_0) \leq u_0$. That is,

$$V(t,y(t)) \leq r(t)A^{-1}(t) = R(t),$$

in virtue of Lemma 3.4, where $R(t)$ is the maximal solution of (5.8). Now, arguing as in Theorems 5.1 and 5.2, it is easy to show that the stability properties of the null solution of (5.8) imply the corresponding $(\tilde{h}_0,h)$ - stability properties of (2.2).

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