

STABILITY OF EQUIVARIANT SMOOTH MAPS

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This research announcement is a summary of a paper which will appear elsewhere [5], and which continues the program started in [4].

1. We consider a compact Lie group G and smooth compact G -manifolds X and Y . By $C_G^\infty(X, Y)$, $\text{Diff}_G(X)$, $\text{Diff}_G(Y)$ we denote the C^∞ , G -equivariant mappings $X \rightarrow Y$, respectively, diffeos of X or diffeos of Y .

There is a natural group action

$$\text{Diff}_G(X) \times \text{Diff}_G(Y) \times C_G^\infty(X, Y) \xrightarrow{\Phi} C_G^\infty(X, Y),$$

and for each $f \in C_G^\infty(X, Y)$, we define the corresponding orbit-map

$$\text{Diff}_G(X) \times \text{Diff}_G(Y) \xrightarrow{\Phi_f} C_G^\infty(X, Y).$$

We consider the G -bundles TX, TY, f^*TY and their "invariant sections" $\Gamma^\infty(TX)^G, \Gamma^\infty(TY)^G, \Gamma^\infty(f^*TY)^G$. (These are modules over the corresponding rings of G -invariant functions.)

As in the usual case [3], [6] we have linear mappings

$$\begin{array}{ccc} \Gamma^\infty(TX)^G & \xrightarrow{\beta_f} & \Gamma^\infty(f^*TY)^G \\ & \nearrow \alpha_f & \\ \Gamma^\infty(TY)^G & & \end{array}$$

defined in a natural way.

By definition, f is infinitesimally stable if $\alpha_f + \beta_f$ is surjective.

By definition, f is stable if $\text{Image } \Phi_f$ is a neighbourhood of $f \in C_G^\infty(X, Y)$.

With these definitions we have the

STABILITY THEOREM. *Let $f \in C_G^\infty(X, Y)$ be infinitesimally stable. Then:*

(i) *Whenever Z_1 is the germ of a metrizable or compact topological space, Z_2 the germ of a smooth finite dimensional manifold, and $\psi: Z_1 \times Z_2 \rightarrow C_G^\infty(X, Y)$ a $C^{0,\infty}$ -germ of a map sending the base points to f , there is a germ of a $C^{0,\infty}$ map $\Psi: Z_1 \times Z_2 \rightarrow \text{Diff}_G(X) \times \text{Diff}_G(Y)$ sending the base points to $(\text{id } X) \times (\text{id } Y)$ and such that*

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$$\begin{array}{ccc}
 & \text{Diff}_G(X) \times \text{Diff}_G(Y) & \\
 \Psi \nearrow & & \downarrow \Phi_f \\
 Z_1 \times Z_2 & \xrightarrow{\psi} & C_G^\infty(X, Y)
 \end{array}$$

is commutative.

(ii) There is a neighbourhood $f \in N \subset C_G^\infty(X, Y)$ such that every $f' \in N$ is also infinitesimally stable

(iii) f is stable. \square

The proof relies heavily on the work of J. Mather [2], [3] (which is generalized by this theorem) and of G. Schwarz [7].

2. Let $G \xrightarrow{\psi} \text{Aut}(V)$ be a linear representation of the compact Lie group G . Let $x \in V$ be the current point of V and $R[x]^G \subset R[x]$ be the algebra of G -invariant polynomials. According to a classical theorem of Hilbert [1], [8] we can always choose a finite system $(\rho_1, \dots, \rho_k) = \rho \subset R[x]^G$ of algebra generators of $R[x]^G$. We shall attach to the representation (G, ψ) the number

$$\text{ord}(G, \psi) = \min_{\rho} \left(\max_i \deg \rho_i \right) \in Z^+.$$

Suppose now X is a (not necessarily compact) G -manifold (G compact). By the slice-representation we have a naturally defined function on the space of orbits $X/G \xrightarrow{\text{ord}} Z^+$.

One of the technical ingredients occurring in the context of the stability theorem is the following

SEMICONTINUITY LEMMA. *For every orbit $\{Gx\} \in X/G$ there exists a neighbourhood $\{Gx\} \in W \subset X/G$, such that, for any $\{Gx'\} \in W$, one has*

$$\text{ord}(Gx') \leq \text{ord}(Gx). \quad \square$$

This might be useful in the study of deformations of group actions suggested by Thom.

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