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STABILITY OF IMPULSIVE HOPFIELD NEURAL NETWORKS WITH MARKOVIAN SWITCHING AND TIME–VARYING DELAYS

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The paper is concerned with stability analysis for a class of impulsive Hopfield neural networks with Markovian jumping parameters and time-varying delays. The jumping parameters considered here are generated from a continuous-time discrete-state homogenous Markov process. By employing a Lyapunov functional approach, new delay-dependent stochastic stability criteria are obtained in terms of linear matrix inequalities (LMIs). The proposed criteria can be easily checked by using some standard numerical packages such as the Matlab LMI Toolbox. A numerical example is provided to show that the proposed results significantly improve the allowable upper bounds of delays over some results existing in the literature.

Keywords: Hopfield neural networks, Markovian jumping, stochastic stability, Lyapunov function, impulses.

1. Introduction

In recent years, the study of stochastic Hopfield neural networks have been extensively intensified, since it has been widely used to model many of the phenomena arising in areas such as signal processing, pattern recognition, static image processing, associative memory, especially for solving some difficult optimization problems (Cichocki and Unbehauen, 1993; Haykin, 1998). One of the important and interesting problems in the analysis of stochastic Hopfield neural networks is their stability. In the implementation of networks, time delays exist due to the finite switching speed of amplifiers and transmission of signals in the network community, which may lead to oscillation, chaos and instability. Consequently, stability analysis of stochastic neural networks with time delays has attracted many researchers and some results related to this problem have been reported in the literature (Balasubramaniam and Rakkiyappan, 2009; Balasubramaniam *et al*., 2009; Li *et al*., 2008; Singh, 2007; Zhou and Wan, 2008).

Markovian jump systems are a special class of hybrid systems with two different states. The first one refers to the mode which is described by a continuous-time finite-state Markovian process, and the second one refers to the state which is represented by a system of differential equations. Jump or switching systems have the advantage of modeling dynamic systems to abrupt variation in their structures, such as component failures or repairs, sudden environmental disturbance, changing subsystem interconnections, operating at different points of a nonlinear plant.

Neural networks with Markovian jumping parameters and time delay have received much attention (Mao, 2002; Shi *et al*., 2003; Wang *et al*., 2006; Yuan and Lygeros, 2005; Zhang and Wang, 2008). In the work of Li *et al*. (2008), the problem of delay-dependent robust stability of uncertain Hopfield neural networks with Marko-

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vian jumping parameters delays is investigated. Sufficient conditions are derived by Lou and Cui (2009) to guarantee the stochastic stability for a class of delayed neural networks of neutral type with Markovian jump parameters.

Moreover, many physical systems also undergo abrupt changes at certain moments due to instantaneous perturbations, which leads to impulsive effects. Neural networks are often subject to impulsive perturbations that, in turn, affect dynamical behaviors of the systems. Therefore, it is necessary to consider the impulsive effect when investigating the stability of neural networks. The stability of neural networks with impulses and time delays has received much attention (Li *et al.*, 2009; Rakkiyappan *et al.*, 2010; Song and Wang, 2008; Song and Zhang, 2008). However, neural networks with Markovian jumping parameters and impulses have received little attention in spite of their practical importance (Dong *et al.*, 2009). To the best of our knowledge, up to now, the stability analysis problem of time varying delayed Hopfield neural network with Markovian jumping parameters and impulses has not appeared in the literature and this motivates our present work. The main aim of this paper is to study the stochastic stability for a class of time varying delayed Hopfield neural networks with Markovian jumping parameters and impulses by constructing a suitable Lyapunov–Krasovskii functional. The stability conditions are formulated in terms of LMIs and can be easily solved by using the Matlab LMI Control Toolbox. Further, a numerical example is given to show the stability criteria obtained in this paper are less conservative than some existing results.

2. Problem formulation

Notation. The notation in this paper is of standard form. The superscript 'T' stands for matrix transposition; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $P > 0$ means that P is real symmetric and positive definite; I and 0 represents the identity and zero matrices, respectively. The symbol '$*$' within a matrix represents the symmetric term of the matrix. Furthermore, $\mathrm{diag}\{\cdot\}$ denotes a block-diagonal matrix and $\mathrm{E}\{\cdot\}$ represents the mathematical expectation operator.

Consider the following Hopfield neural networks with impulses and a time-varying delay:

$$\begin{aligned}\dot{x}(t) &= -Ax(t) + Bf(x(t-h(t)))\\ &\quad + D\int_{t-\tau(t)}^{t} f(x(s))\,\mathrm{d}s + U, \quad t \neq t_k,\\ x(t_k) &= C_k x(t_k^-), \quad t = t_k,\end{aligned} \tag{1}$$

for $t > 0$ and $k = 1, 2, \dots$, where

$$x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T \in \mathbb{R}^n$$

is the state vector associated with the n neurons at time t;

$$\begin{aligned}&f(x(t-h(t)))\\ &= \Big[f_1(x(t-h(t))), f_2(x(t-h(t))), \dots, f_n(x(t-h(t)))\Big]^T\end{aligned}$$

denotes the activation function; $U = [U_1, U_2, \dots, U_n]^T$ is the constant external input vector; the matrix $A = \mathrm{diag}(a_1, a_2, \dots, a_n)$ has positive entries $a_i > 0$; $h(t)$ and $\tau(t)$ denote time-varying delays; the matrices $B = [b_{ij}]_{n\times n}$ and $D = [d_{ij}]_{n\times n}$ represent the delayed connections weight matrix and the connection weight matrix, respectively; $x(t_k) = C_k x(t_k^-)$ is the impulse at moment t_k, the fixed moment of time t_k satisfies $t_1 < t_2 < \dots, \lim_{k\to+\infty} t_k = +\infty$ and $x(t^-) = \lim_{s\to t^-} x(s)$; C_k is a constant real matrix at moments of time t_k.

Let $PC([-\rho, 0], \mathbb{R}^n)$ denote the set of piecewise right continuous functions $\phi : [-\rho, 0] \to \mathbb{R}^n$ with the sup-norm $|\phi| = \sup_{-\rho\leq s\leq 0}\|\phi(s)\|$. For given t_0, and $\phi \in PC([-\rho, 0], \mathbb{R}^n)$, the initial condition of the system (1) is described as $x(t_0+t) = \phi(t)$, for $t \in [-\rho, 0]$, $\phi \in PC([-\rho, 0], \mathbb{R}^n)$, $\rho \in \max\{h, \tau\}$.

Throughout this paper, we assume that the following conditions hold:

(i) The neuron activation function $f(\cdot)$ is continuous and bounded on $\mathbb{R}$ and satisfies the following inequality:

$$0 \leq \frac{f_q(s_1) - f_q(s_2)}{s_1 - s_2} \leq l_q, \quad q = 1, 2, \dots, n, \quad s_1, s_2 \in \mathbb{R}, \quad s_1 \neq s_2.$$

(ii) The time-varying delay $h(t)$ satisfies

$$0 \leq r_1 \leq h(t) \leq r_2, \quad \dot{h}(t) \leq \mu, \tag{2}$$

where r_1, r_2 are constants. Furthermore, the bounded function $\tau(t)$ represents the distributed delay of systems with $0 \leq \tau(t) \leq \bar{\tau}$, $\bar{\tau}$ is a constant.

Let $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T \in \mathbb{R}^n$ be the equilibrium point of Eqn. (1). For simplicity, we can shift the equilibrium x^* to the origin by letting $y(t) = x(t) - x^*$ and $\psi(t) = x(t_0 + t) - x^*$. Then the system (1) can be transformed into the following one:

$$\begin{aligned}\dot{y}(t) &= -Ay(t) + Bg(y(t-h(t)))\\ &\quad + D\int_{t-\tau(t)}^{t} g(y(s))\,\mathrm{d}s, \quad t \neq t_k,\\ y(t_k) &= C_k y(t_k^-), \quad t = t_k,\\ y(t_0 + t) &= \psi(t), \quad t \in [-\rho, 0],\end{aligned} \tag{3}$$

where

$$y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T$$

is the state vector of the transformed system. It follows from Assumption (i) that the transformed neuron activation function satisfies

$$g_j(0) = 0, \quad 0 \le \frac{g_q(y_q)}{y_q} \le l_q, \quad \forall\, y_q \ne 0, \quad q = 1, 2, \dots, n. \tag{4}$$

Now, based on the model (3), we are in a position to introduce Hopfield neural networks with Markovian jumping parameters. Let $\{r(t), t \ge 0\}$ be a right-continuous Markov process on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, P)$ taking values in a finite state space $\mathcal{S} = \{1, 2, \dots, N\}$ with generator $\Pi = (\pi_{ij})_{N\times N}$ given by

$$p\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \pi_{ij}\Delta + o(\Delta) & \text{if } i \ne j, \\ 1 + \pi_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$ and $\lim_{\Delta\to 0} o(\Delta)/\Delta$, π_{ij} is the transition rate from i to j if $i \ne j$, and

$$\pi_{ii} = -\sum_{j\ne i} \pi_{ij}.$$

In this paper, we consider the following time varying delayed Hopfield neural networks with Markovian jumping parameters and impulses, which are actually a modification of (3):

$$\begin{aligned} \dot{y}(t) &= -A(r(t))y(t) + B(r(t))g(y(t-h(t))) \\ &\quad + D(r(t)) \int_{t-\tau(t)}^{t} g(y(s))\,\mathrm{d}s, \quad t \ne t_k \\ y(t_k) &= C_k(r(t))y(t_k^-), \quad t = t_k, \\ y(t_0 + t) &= \psi(t), \quad t \in [-\rho, 0], \quad r(0) = r_0, \end{aligned} \tag{5}$$

where $r_0 \in \mathbb{S}$ is the mode of the continuous state. For simplicity, we write $r(t) = i$, while $A(r(t)), B(r(t))$ and $D(r(t))$ are denoted as A_i, B_i and D_i, respectively.

Let us first give the following lemmas and definitions which will be used in the proofs of our main results.

Lemma 1. (Gu *et al.*, 2003) *Let $a, b \in \mathbb{R}^n$, P be a positive definite matrix. Then $2a^Tb < a^TP^{-1}a + b^TPb$.*

Lemma 2. (Gu *et al.*, 2003) *For any positive definite matrix $W > 0$, two scalars $b > a$, and a vector function $\omega : [a, b] \to \mathbb{R}^n$, such that the integrations concerned are well defined, the following inequality holds:*

$$\Big(\int_a^b \omega(s)\,\mathrm{d}s\Big)^T W \Big(\int_a^b \omega(s)\,\mathrm{d}s\Big) < (b-a)\int_a^b \omega^T(s)W\omega(s)\,\mathrm{d}s.$$

Definition 1. The system (5) is said to be *stochastically stable* when $U = 0$, for any finite $\psi(t) \in \mathbb{R}^n$ defined on $[-\rho, 0]$ and $r(0) \in \mathbb{S}$, the following condition is satisfied:

$$\lim_{t\to\infty} \mathrm{E}\Big\{\int_0^t y^T(s)y(s)\,\mathrm{d}s | \psi, r_0\Big\} < \infty.$$

Definition 2. (Zhang and Sun, 2005) The function $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^+$ *belongs to class* v_0 if

(i) the function V is continuous on each of the sets $[t_{k-1}, t_k) \times \mathbb{R}^n$ and for all $t \ge t_0$, $V(t, 0) \equiv 0$;

(ii) $V(t, x)$ is locally Lipschitzian in $x \in \mathbb{R}^n$;

(iii) for each $k = 1, 2, \dots,$ there exist finite limits

$$\lim_{(t,q)\to(t_k^-,x)} V(t, q) = V(t_k^-, x),$$
$$\lim_{(t,q)\to(t_k^+,x)} V(t, q) = V(t_k^+, x),$$

with $V(t_k^+, x) = V(t_k, x)$.

3. Stochastic stability results

In this section, we will derive conditions for the stochastic stability of delayed Hopfield neural networks with Markovian jumping parameters and impulsive effects.

Theorem 1. *Consider the neural network system (5) satisfying Assumptions (i) and (ii). Given scalars $r_2 > r_1 \ge 0$, μ and $\bar\tau > 0$, the system (5) is stochastically stable if there exist positive definite matrices $P_i > 0, Q_1, Q_2, Q_3, Q_4 > 0, R_1, R_2 > 0, S_1, S_2 > 0$ and diagonal matrices $T_j = \mathrm{diag}\{t_{1j}, t_{2j}, \dots, t_{nj}\} \ge 0, (j = 1, 2)$, such that the following LMIs hold:*

$$C_{ik}^T P_j C_{ik} - P_i < 0, \tag{6}$$

for $i = 1, 2, \dots, s$, and $k = 1, 2, \dots,$ along with (7), where

$$\begin{aligned} \Phi_1 &= -P_iA_i - A_i^TP_i + \sum_{j=1}^{s}\pi_{ij}P_j + Q_1 + Q_2 + Q_3 \\ &\quad + r_1^2R_1 + (r_2 - r_1)^2R_2, \\ \Phi_2 &= Q_4 + \bar\tau^2S_1 + 2\bar\tau^2D_i^TS_2D_i - 2T_2, \\ \Phi_3 &= -(1-\mu)Q_4 - 2T_1, \\ \Phi_4 &= -S_1 - 2D_i^TS_2D_i. \end{aligned}$$

Proof. In order to prove the stability result, we construct the following Lyapunov–Krasovkii functional

$$V(t, y(t), r(t) = i) = V_1 + V_2 + V_3 + V_4 + V_5, \tag{8}$$

$$\Upsilon_i = \begin{bmatrix} \Phi_1 & 0 & 0 & 0 & L_2^T T_2 & P_i B_i & 0 & 0 & P_i D_i \\ * & -(1-\mu)Q_1 & 0 & 0 & 0 & L_1^T T_1 & 0 & 0 & 0 \\ * & * & -Q_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -Q_3 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Phi_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Phi_3 & 0 & 0 & 0 \\ * & * & * & * & * & * & -R_1 & 0 & 0 \\ * & * & * & * & * & * & * & -R_2 & 0 \\ * & * & * & * & * & * & * & * & \Phi_4 \end{bmatrix} < 0, \tag{7}$$

where

$$\begin{aligned} V_1 &= y^T(t) P_i y(t), \\ V_2 &= \int_{t-h(t)}^{t} y^T(s) Q_1 y(s)\,\mathrm{d}s \\ &\quad + \int_{t-r_1}^{t} y^T(s) Q_2 y(s)\,\mathrm{d}s + \int_{t-r_2}^{t} y^T(s) Q_3 y(s)\,\mathrm{d}s \\ &\quad + \int_{t-h(t)}^{t} g^T(y(s)) Q_4 g(y(s))\,\mathrm{d}s, \\ V_3 &= r_1 \int_{t-r_1}^{t} [s-(t-r_1)] y^T(s) R_1 y(s)\,\mathrm{d}s, \\ V_4 &= (r_2 - r_1) \int_{t-r_2}^{t-r_1} [s-(t-r_2)] y^T(s) R_2 y(s)\,\mathrm{d}s, \\ V_5 &= \bar{\tau} \int_{-\bar{\tau}}^{0} \int_{t+\sigma}^{t} g^T(y(s)) S_1 g(y(s))\,\mathrm{d}s\ \mathrm{d}\sigma \\ &\quad + 2\bar{\tau} \int_{-\bar{\tau}}^{0} \int_{t+\sigma}^{t} g^T(y(s)) D_i^T S_2 D_i g(y(s))\,\mathrm{d}s\ \mathrm{d}\sigma. \end{aligned} \tag{9}$$

When $t = t_k$, we have

$$\begin{aligned} &V(t_k, y, j) - V(t_k^-, y, i) \\ &= y^T(t_k) P_j y(t_k) - y^T(t_k^-) P_i y(t_k^-) \\ &\quad + \int_{t_k - h(t_k)}^{t_k} y^T(s) Q_1 y(s)\,\mathrm{d}s \\ &\quad - \int_{t_k^- - h(t_k^-)}^{t_k^-} y^T(s) Q_1 y(s)\,\mathrm{d}s \\ &\quad + \int_{t_k - r_1}^{t_k} y^T(s) Q_2 y(s)\,\mathrm{d}s \\ &\quad - \int_{t_k^- - r_1}^{t_k^-} y^T(s) Q_2 y(s)\,\mathrm{d}s \\ &\quad + \int_{t_k - r_2}^{t_k} y^T(s) Q_3 y(s)\,\mathrm{d}s - \int_{t_k^- - r_2}^{t_k^-} y^T(s) Q_3 y(s)\,\mathrm{d}s \\ &\quad + \int_{t_k - h(t_k)}^{t_k} g^T(y(s)) Q_4 g(y(s))\,\mathrm{d}s \\ &\quad - \int_{t_k^- - h(t_k^-)}^{t_k^-} g^T(y(s)) Q_4 g(y(s))\,\mathrm{d}s \\ &\quad + r_1 \int_{t_k - r_1}^{t_k} [s-(t-r_1)] y^T(s) R_1 y(s)\,\mathrm{d}s \\ &\quad - r_1 \int_{t_k^- - r_1}^{t_k^-} [s-(t-r_1)] y^T(s) R_1 y(s)\,\mathrm{d}s \\ &\quad + (r_2 - r_1) \int_{t_k - r_2}^{t_k - r_1} [s-(t-r_2)] y^T(s) R_2 y(s)\,\mathrm{d}s \\ &\quad - (r_2 - r_1) \int_{t_k^- - r_2}^{t_k^- - r_1} [s-(t-r_2)] y^T(s) R_2 y(s)\,\mathrm{d}s \\ &\quad + \bar{\tau}\Big[\int_{-\bar{\tau}}^{0} \int_{t_k+\sigma}^{t_k} g^T(y(s)) S_1 g(y(s))\,\mathrm{d}s\ d\sigma \\ &\quad - \int_{-\bar{\tau}}^{0} \int_{t_k^-+\sigma}^{t_k^-} g^T(y(s)) S_1 g(y(s))\,\mathrm{d}s\ d\sigma\Big] \\ &\quad + 2\bar{\tau}\Big[\int_{-\bar{\tau}}^{0} \int_{t_k+\sigma}^{t_k} g^T(y(s)) D_j^T S_2 D_j g(y(s))\,\mathrm{d}s\ d\sigma \\ &\quad - \int_{-\bar{\tau}}^{0} \int_{t_k^-+\sigma}^{t_k^-} g^T(y(s)) D_i^T S_2 D_i g(y(s))\,\mathrm{d}s\ d\sigma\Big] \\ &= y^T(t_k^-) C_{ik}^T P_j C_{ik} y(t_k^-) - y^T(t_k^-) P_i y(t_k^-) \\ &\quad + \int_{t_k - h(t_k)}^{t_k^-} y^T(s) Q_1 y(s)\,\mathrm{d}s + \int_{t_k^-}^{t_k} y^T(s) Q_1 y(s)\,\mathrm{d}s \\ &\quad - \int_{t_k^- - h(t_k^-)}^{t_k^-} y^T(s) Q_1 y(s) ds \\ &\quad + \int_{t_k - r_1}^{t_k^-} y^T(s) Q_2 y(s)\,\mathrm{d}s + \int_{t_k^-}^{t_k} y^T(s) Q_2 y(s)\,\mathrm{d}s \\ &\quad - \int_{t_k^- - r_1}^{t_k^-} y^T(s) Q_2 y(s)\,\mathrm{d}s + \int_{t_k - r_2}^{t_k^-} y^T(s) Q_3 y(s)\,\mathrm{d}s \\ &\quad + \int_{t_k^-}^{t_k} y^T(s) Q_3 y(s)\,\mathrm{d}s \\ &\quad - \int_{t_k^- - r_2}^{t_k^-} y^T(s) Q_3 y(s)\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+\int_{t_k-h(t_k)}^{t_k^-} g^T(y(s))Q_4 g(y(s))\,\mathrm{d}s\\
&+\int_{t_k^-}^{t_k} g^T(y(s))Q_4 g(y(s))\,\mathrm{d}s\\
&-\int_{t_k^- -h(t_k^-)}^{t_k^-} g^T(y(s))Q_4 g(y(s))\,\mathrm{d}s\\
&+r_1\int_{t_k-r_1}^{t_k^-}[s-(t-r_1)]y^T(s)R_1y(s)\,\mathrm{d}s\\
&+r_1\int_{t_k^-}^{t_k}[s-(t-r_1)]y^T(s)R_1y(s)\,\mathrm{d}s\\
&-r_1\int_{t_k^- -r_1}^{t_k^-}[s-(t-r_1)]y^T(s)R_1y(s)\,\mathrm{d}s\\
&+(r_2-r_1)\int_{t_k-r_2}^{t_k^- -r_1}[s-(t-r_2)]y^T(s)R_2y(s)\,\mathrm{d}s\\
&+(r_2-r_1)\int_{t_k^- -r_1}^{t_k-r_1}[s-(t-r_2)]y^T(s)R_2y(s)\,\mathrm{d}s\\
&-(r_2-r_1)\int_{t_k^- -r_2}^{t_k^- -r_1}[s-(t-r_2)]y^T(s)R_2y(s)\,\mathrm{d}s\\
&+\bar\tau\Big[\int_{-\bar\tau}^{0}\int_{t_k+\sigma}^{t_k^-} g^T(y(s))S_1g(y(s))\,\mathrm{d}s\ \mathrm{d}\sigma\\
&+\int_{-\bar\tau}^{0}\int_{t_k^-}^{t_k} g^T(y(s))S_1g(y(s))\,\mathrm{d}s\ \mathrm{d}\sigma\\
&-\int_{-\bar\tau}^{0}\int_{t_k^-+\sigma}^{t_k^-} g^T(y(s))S_1g(y(s))\,\mathrm{d}s\ \mathrm{d}\sigma\Big]\\
&+2\bar\tau\Big[\int_{-\bar\tau}^{0}\int_{t_k+\sigma}^{t_k^-} g^T(y(s))D_j^TS_2D_jg(y(s))\,\mathrm{d}s\ \mathrm{d}\sigma\\
&+\int_{-\bar\tau}^{0}\int_{t_k^-}^{t_k} g^T(y(s))D_j^TS_2D_jg(y(s))\,\mathrm{d}s\ \mathrm{d}\sigma\\
&-\int_{-\bar\tau}^{0}\int_{t_k^-+\sigma}^{t_k^-} g^T(y(s))D_i^TS_2D_ig(y(s))\,\mathrm{d}s\ \mathrm{d}\sigma\Big]\\
=\;& y^T(t_k^-)(C_{ik}^TP_jC_{ik}-P_i)y(t_k^-)\\
&+\int_{t_k^-}^{t_k} y^T(s)Q_1y(s)\,\mathrm{d}s+\int_{t_k^-}^{t_k} y^T(s)Q_2y(s)\,\mathrm{d}s\\
&+\int_{t_k^-}^{t_k} y^T(s)Q_3y(s)\,\mathrm{d}s+\int_{t_k^-}^{t_k} g^T(y(s))Q_4g(y(s))\,\mathrm{d}s\\
&+r_1\int_{t_k^-}^{t_k}[s-(t-r_1)]y^T(s)R_1y(s)\,\mathrm{d}s\\
&+(r_2-r_1)\int_{t_k^- -r_1}^{t_k-r_1}[s-(t-r_2)]y^T(s)R_2y(s)\,\mathrm{d}s\\
&+\bar\tau\int_{-\bar\tau}^{0}\int_{t_k^-}^{t_k} g^T(y(s))S_1g(y(s))\,\mathrm{d}s\ \mathrm{d}\sigma\\
&+2\bar\tau\int_{-\bar\tau}^{0}\int_{t_k^-}^{t_k} g^T(y(s))D_j^TS_2D_jg(y(s))\,\mathrm{d}s\ \mathrm{d}\sigma.
\end{aligned}
$$

Since C_{ik} are constant matrices, the terms involving positive-definite constant matrices $Q_1, Q_2, Q_3, Q_4, R_1, R_2, S_1, S_2$ will be equal to zero and hence

$$
V(t_k, y, j) - V(t_k^-, y, i) < 0. \tag{10}
$$

Let $\mathfrak{F}(\cdot)$ be the weak infinitesimal generator of the process $\{y(t), r(t), t \geq 0\}$ for the system (5) at the point $\{t, y(t), r(t)\}$ given by

$$
\begin{aligned}
&\mathfrak{F}\{V(t,y(t),r(t))\}\\
&\quad=\frac{\partial V}{\partial t}+\dot y^T(t)\frac{\partial V}{\partial y}\Big|_{r(t)=i}+\sum_{j=1}^{s}\pi_{ij}V(t,y(t),i,j).
\end{aligned}
$$

For $t \in [t_{k-1}, t_k)$, taking account of (8), $\mathfrak{F}V$ can be derived as

$$
\begin{aligned}
\mathfrak{F}V_1(t) &= 2y^T(t)P_i\dot y(t)+y^T(t)\sum_{j=1}^{s}\pi_{ij}P_jy(t)\\
&=2y^T(t)P_i[-A_iy(t)+B_ig(y(t-h(t)))\\
&\quad+D_i\int_{t-\tau(t)}^{t} g(y(s))\,\mathrm{d}s]\\
&\quad+y^T(t)\sum_{j=1}^{s}\pi_{ij}P_jy(t)\\
&=y^T(t)[-P_iA_i-A_i^TP_i]y(t)\\
&\quad+2y^T(t)P_iB_ig(y(t-h(t)))\\
&\quad+2y^T(t)P_iD_i\Big(\int_{t-\tau(t)}^{t} g(y(s))\,\mathrm{d}s\Big)\\
&\quad+y^T(t)\sum_{j=1}^{s}\pi_{ij}P_jy(t),
\end{aligned}\tag{11}
$$

$$
\begin{aligned}
\mathfrak{F}V_2(t) &\le y^T(t)Q_1y(t)\\
&\quad-(1-\mu)y^T(t-h(t))Q_1y(t-h(t))\\
&\quad+y^T(t)Q_2y(t)-y^T(t-r_1)Q_2y(t-r_1)\\
&\quad+y^T(t)Q_3y(t)-y^T(t-r_2)Q_3y(t-r_2)\\
&\quad+g^T(y(t))Q_4g(y(t))\\
&\quad-(1-\mu)g^T(y(t-h(t)))Q_4g(y(t-h(t))),
\end{aligned}\tag{12}
$$

$$
\begin{aligned}
\mathfrak{F}V_3(t) &= r_1^2y^T(t)R_1y(t)-r_1\int_{t-r_1}^{t}y^T(s)R_1y(s)\,\mathrm{d}s\\
&\le r_1^2y^T(t)R_1y(t)\\
&\quad-\Big(\int_{t-r_1}^{t}y(s)\,\mathrm{d}s\Big)^TR_1\Big(\int_{t-r_1}^{t}y(s)\,\mathrm{d}s\Big)
\end{aligned}\tag{13}
$$

$$
\begin{aligned}
\mathfrak{F}V_4(t) &= (r_2-r_1)^2y^T(t)R_2y(t)\\
&\quad-(r_2-r_1)\int_{t-r_2}^{t-r_1}y^T(s)R_2y(s)\,\mathrm{d}s
\end{aligned}
$$

$$\leq (r_2 - r_1)^2 y^T(t) R_2 y(t) - \Big(\int_{t-r_2}^{t-r_1} y(s)\,\mathrm{d}s\Big)^T R_2 \Big(\int_{t-r_2}^{t-r_1} y(s)\,\mathrm{d}s\Big), \tag{14}$$

$$\begin{aligned}\mathfrak{F}V_5(t) &= \bar{\tau}^2 g^T(y(t)) S_1 g(y(t)) - \bar{\tau}\int_{t-\bar{\tau}}^{t} g^T(y(s)) S_1 g(y(s))\,\mathrm{d}s \\ &\quad + 2\bar{\tau}^2 g^T(y(t)) D_i^T S_2 D_i g(y(t)) - 2\bar{\tau}\int_{t-\bar{\tau}}^{t} g^T(y(s)) D_i^T S_2 D_i g(y(s))\,\mathrm{d}s \\ &\leq g^T(y(t))[\bar{\tau}^2 S_1 + 2\bar{\tau}^2 D_i^T S_2 D_i] g(y(t)) \\ &\quad - \Big(\int_{t-\bar{\tau}}^{t} g(y(s))\,\mathrm{d}s\Big)^T [S_1 + 2D_i^T S_2 D_i] \times \Big(\int_{t-\bar{\tau}}^{t} g(y(s))\,\mathrm{d}s\Big).\end{aligned} \tag{15}$$

From Assumption (i), we can get the following inequalities:

$$2g^T(y(t-h(t))) T_1 L y(t-h(t)) - 2g^T(y(t-h(t))) T_1 g(y(t-h(t))) \geq 0 \tag{16}$$

and

$$2g^T(y(t)) T_2 L y(t) - 2g^T(y(t)) T_2 g(y(t)) \geq 0. \tag{17}$$

From (11)–(15), we obtain

$$\begin{aligned}&\mathfrak{F}V(t) \\ &\leq y^T(t)[-P_iA_i - A_i^T P_i + \sum_{j=1}^{s}\pi_{ij}P_j + Q_1 \\ &\quad + Q_2 + Q_3 + r_1^2 R_1 + (r_2-r_1)^2 R_2] y(t) \\ &\quad + y^T(t) L^T T_2 g(y(t)) + 2y^T(t) P_i B_i g(y(t)) \\ &\quad + 2y^T(t) P_i D_i g(y(t-h(t))) \\ &\quad + y^T(t-h(t))[-(1-\mu)Q_1] y(t-h(t)) \\ &\quad + y^T(t-h(t)) L^T T_1 g(y(t-h(t))) \\ &\quad - y^T(t-r_1) Q_2 y(t-r_1) \\ &\quad - y^T(t-r_2) Q_3 y(t-r_2) \\ &\quad + g^T(y(t))[Q_4 + \bar{\tau}^2 S_1 + 2\bar{\tau}^2 D_i^T S_2 D_i - 2T_2] g(y(t)) \\ &\quad + g^T(y(t-h(t)))[-(1-\mu)Q_4 - 2T_1] g(y(t-h(t))) \\ &\quad - \Big(\int_{t-r_1}^{t} y(s)\,\mathrm{d}s\Big)^T R_1 \Big(\int_{t-r_1}^{t} y(s)\,\mathrm{d}s\Big) \\ &\quad - \Big(\int_{t-r_2}^{t-r_1} y(s)\,\mathrm{d}s\Big)^T R_2 \Big(\int_{t-r_2}^{t-r_1} y(s)\,\mathrm{d}s\Big) \\ &\quad - \Big(\int_{t-\bar{\tau}}^{t} g(y(s))\,\mathrm{d}s\Big)^T [S_1 + 2D_i^T S_2 D_i] \\ &\quad \times \Big(\int_{t-\bar{\tau}}^{t} g(y(s))\,\mathrm{d}s\Big) \\ &= \zeta^T(t)\, \Upsilon_i\, \zeta(t),\end{aligned} \tag{18}$$

where Υ_i is defined in (7) and

$$\begin{aligned}\zeta(t) = \Big[&y^T(t)\ y^T(t-h(t))\ y^T(t-r_1)\ y^T(t-r_2) \\ &g^T(y(t))\ g^T(y(t-h(t)))\ \Big(\int_{t-r_1}^{t} y(s)\,\mathrm{d}s\Big)^T \\ &\Big(\int_{t-r_2}^{t-r_1} y(s)\,\mathrm{d}s\Big)^T \Big(\int_{t-\bar{\tau}}^{t} g(y(s))\,\mathrm{d}s\Big)^T\Big]^T.\end{aligned}$$

This implies that $\Upsilon_i < 0$. Setting

$$\delta_1 = \min\{\lambda_{\min}(-\Upsilon_i), i \in \mathbb{S}\},$$

we get $\delta_1 > 0$. For any $t \geq h$, we have

$$\mathfrak{F}[V(y(t), i)] \leq -\delta_1 \zeta^T(t)\zeta(t) \leq -\delta_1 y^T(t) y(t).$$

By Dynkin's formula, we get

$$\mathrm{E}\{V(y(t), i)\} - \mathrm{E}\{V(y_0, r_0)\} \leq -\delta_1 \mathrm{E}\Big\{\int_0^t y^T(s) y(s)\,\mathrm{d}s\Big\}$$

and hence

$$\mathrm{E}\Big\{\int_0^t y^T(s) y(s)\,\mathrm{d}s\Big\} \leq \frac{1}{\delta_1}\Big\{V(\psi, r_0) - \mathrm{E}\{V(y(t), i)\}\Big\}. \tag{19}$$

On the other hand, from the definitions of $V_i(y(t), i), (i = 1, 2, 3, 4, 5)$, there exists a scalar $\delta_2 > 0$, such that for any $t \geq 0$ we have

$$\begin{aligned}&\mathrm{E}\{V(y(t), i)\} \\ &= \mathrm{E}\{V_1(y(t), i)\} + \mathrm{E}\{V_2(y(t), i)\} + \mathrm{E}\{V_3(y(t), i)\} \\ &\quad + \mathrm{E}\{V_4(y(t), i)\} + \mathrm{E}\{V_5(y(t), i)\} \\ &\geq \delta_2 \mathrm{E}\{y^T(t) y(t)\},\end{aligned} \tag{20}$$

where $\delta_2 = \min\{\lambda_{\min}(P_i), i \in \mathbb{S}\} > 0$. From (19) and (20) it follows that

$$\mathrm{E}\Big\{y^T(t) y(t)\Big\} \leq -\beta_1 \mathrm{E}\Big\{\int_0^t y^T(s) y(s)\,\mathrm{d}s\Big\} + \beta_2 V(y_0, r_0),$$

where $\beta_1 = \delta_1\delta_2^{-1}$, $\beta_2 = \delta_2^{-1}$. Thus we have

$$\mathrm{E}\Big\{\int_0^t y^T(s) y(s)\,\mathrm{d}s | \psi, r_0\Big\} \leq \beta_1^{-1}\beta_2[1 - \exp(-\beta_1 t)] V(y_0, r_0).$$

As $t \to \infty$, there exists a scalar $\eta > 0$ such that

$$\lim_{t\to\infty} \mathrm{E}\Big\{ \int_0^t y^T(s)y(s)\,\mathrm{d}s|\psi, r_0 \Big\} \le \beta_1^{-1}\beta_2 V(y_0, r_0) \le \eta \sup_{-\rho\le s\le 0} |\psi(s)|^2.$$

Thus, by Definition 2, the impulsive Hopfield neural network with Markovian switching (5) is stochastically stable. The proof is thus complete. ■

4. Numerical example

Example 1. Consider the stochastic Hopfield neural networks with Markovian jumping parameters and impulses (5):

$$A_1 = \begin{bmatrix} 1.4576 & 0 \\ 0 & 1.3680 \end{bmatrix}, \quad \Pi = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1.7631 & 0 \\ 0 & 0.0253 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -0.9220 & -1.7676 \\ -0.6831 & -2.0429 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -2.8996 & 0.4938 \\ -0.6736 & -1.0183 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.5 & -0.5 \\ 0.2 & 0.7 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.3 & 0.2 \\ -0.5 & 0.4 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.6 \end{bmatrix}.$$

By solving the LMIs in Theorem 1 for positive definite matrices $P_1, P_2, Q_1, Q_2, Q_3, Q_4, R_1, R_2, S_1, S_2$ and diagonal matrices T_1, T_2, it can be verified that the system (5) is stochastically stable and a set of feasible solutions can be obtained as follows:

$$P_1 = \begin{bmatrix} 161.1159 & 31.7466 \\ 31.7466 & 6.4729 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 52.5649 & -68.1129 \\ -68.1129 & 267.9161 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 48.2983 & 11.0789 \\ 11.0789 & 174.3372 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 0.4483 & -0.3879 \\ -0.3879 & 0.5140 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 0.4483 & -0.3879 \\ -0.38797 & 0.5140 \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} 4.8974 & 4.2524 \\ 4.2524 & 4.4817 \end{bmatrix},$$

$$R_1 = \begin{bmatrix} 461.2151 & 0 \\ 0 & 461.2151 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 0.0104 & -0.0085 \\ -0.0085 & 0.0119 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 24.1828 & -12.4256 \\ -12.4256 & 7.3894 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 3.0184 & 0.6498 \\ 0.6498 & 0.8849 \end{bmatrix},$$

$$T_1 = \begin{bmatrix} 1.4261 & 0 \\ 0 & 2.0428 \end{bmatrix} \times 10^3,$$

$$T_2 = \begin{bmatrix} 1.3894 & 0 \\ 0 & 0.5656 \end{bmatrix} \times 10^3.$$

In the work of Zhang and Sun (2005), when $\mu = 0$, the maximum allowable bounds for r_2 and $\bar{\tau}$ are obtained as $r_2 = 0.3$ and $\bar{\tau} = 0.6$. In the paper by Liu *et al.* (2009), the stochastic stability of a delayed Hopfield neural network with Markovian jumpings and constant delays is discussed, but the upper bound of the delay is not taken into account. In this paper, when $\mu = 0$ and $r_1 = 0$, by using Theorem 1, we obtain the maximum allowable upper bounds $r_2 = \bar{\tau} = 6.7568$. Moreover, it is obvious that the upper bound obtained in our paper is better than those found by Liu *et al.* (2009) or Zhang and Sun (2005). The result reveals the stability criteria obtained in this paper are less conservative than some existing results.

If we take the initial values of (5) as $[y_1(s), y_2(s)] = [\cos(s), 0.3\sin(s)]$, $s \in [-2, 0]$. Figure 1 depicts the time response of state variables y_1 and y_2 with and without impulsive effects.

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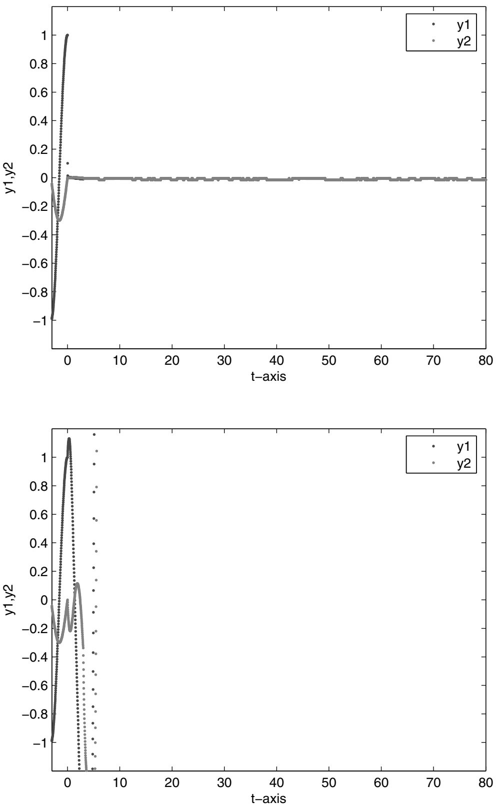

Fig. 1. State response of variables y_1, y_2 for Example 1 with and without impulses.

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