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# Stability of Impulsive Stochastic Delay Systems with Markovian Switched Delay Effects

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**Abstract:** In this paper, we investigate the $p$th moment exponential stability of impulsive stochastic delay systems with Markovian switched delay effects. The model we consider here is rather different from the models in the existing literature. In particular, the delay is a Markov chain, which is quite different from the traditional deterministic delay. By using the Markov chain theory, stochastic analysis theory, Razumikhin technology and the Lyaponov method, we derive a criterion of $p$th moment exponential stability for the suggested system. Finally, an example is provided to illustrate the effectiveness of the obtained result.

**Keywords:** Markovian switched delay; impulsive stochastic delay system; moment exponential stability; Lyapunov approach; Razumikhin technique

**MSC:** 93D05; 93D23; 93E03; 93E15

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## 1. Introduction

Stochastic systems with Markovian switching is an important class of stochastic hybrid systems. In the real world, they are widely used in many fields such as automatic control, aircraft and air traffic control and electrics. So, the research on stochastic systems with Markovian switching has attracted an increasing amount of attention. Just as studying the stability for stochastic systems is important, the stability for stochastic systems with Markovian switching is also a significant issue. In the past several decades, a large number of works on this topic have been reported in the literature. For example, Mao in [1] considered the stability of stochastic differential equations with Markovian switching. Later, he generalized the above result to stochastic functional differential equations [2] and stochastic delay interval systems [3]. Recently, Ref. [4] considered the mean-square stability for a class of singular stochastic systems with Markovian switching. Ref. [5] derived the stability in distribution of stochastic differential delay equations with Markovian switching based on the pure probability method. By using the multiple Lyapunov functions method, Ref. [6] considered the asymptotic stability of stochastic differential equations with Markovian switching , which generalized the result in [7]. Ref. [8] applied a mode-dependent parameter approach to give a sufficient condition for the finite-time stability of Itô's stochastic systems with Markovian switching. Refs. [9–14] considered the stability and the related questions for the Markovian jump linear systems by using different approaches. In [15,16], the authors obtained some properties for stochastic systems with Markovian switching, and then they also applied these properties to study the networked control systems with delays. Ref. [17] established the exponential-$m$ stability for stochastic switched systems. Refs. [18–21] considered the stability for stochastic systems with semi-Markovian switching. Ref. [22] obtained the $p$th moment exponential stability for stochastic pantograph systems with Markovian switching, which is an important class of stochastic hybrid systems with unbounded delays. In addition, the "dwell time" method has been used to investigate switching systems. In [23], Hespanha and Morse proved that a switched system is exponentially stable if all the subsystems are exponentially stable and its average dwell time is large enough. From then on, a large number of stability criteria have been reported by using “dwell time” method. For example, Ref. [24] studied the stochastic stability of continuous-time systems with random switching signals by using the Lyapunov approach, the LMI (i.e., linear matrix inequalities, see [25]) technique and the “dwell time” method. Ref. [26] used the “dwell time” and the non-convolution type multiple Lyapunov functionals to derive the almost sure exponential stability for switched delay systems with nonlinear stochastic perturbations. Ref. [27] applied the average dwell time approach to obtain the stability results for neutral stochastic switching delay systems. By using the average dwell time method and Lyapunov–Krasovskii functional theory, an $H_\infty$ control problem for network-based stochastic systems with two additive delays was studied in [28,29]. The are also some other methods used in the stability analysis for switching systems. For example, Wu et al. in [30] applied Itô's formula and Dynkin's formula to investigate the stability on stochastic systems with state-dependent switching. Ref. [31] obtained some stability results for slowly switched systems by using the multiple discontinuous Lyapunov function approach. By using the comparison principle and the multiple Lyapunov functions method, Ref. [32] studied the stability of deterministic and stochastic switched systems. For other results with respect to the stability of stochastic differential equations with Markovian switching, please refer to Refs. [33–36] and references therein. For a survey of stability for stochastic hybrid systems, please refer to [37].

In most of the existing works, the switching signals are deterministic functions. However, in the real world, the delay switching is not usually determined. In other words, the delay switching may be random. Therefore, it is interesting and challenging to investigate the stability of randomly switched delay systems. To the best of our knowledge, there have been only a few results on this issue. For example, Ref. [38] studied the moment exponential stability of random delay systems with two-time-scale Markovian switching and the main tool based on the theory of two-time-scale Markov chains. However, the noise disturbance in [38] was ignored.

In addition, impulsive stochastic systems are also important systems in control engineering. There is also some important literature in the stability for impulsive stochastic systems. For example, Refs. [39,40] considered the exponential stability for impulsive stochastic delay differential systems. Ref. [41] established the exponential stability for neutral impulsive stochastic delay differential systems. Ref. [42] designed the impulsive controller for stochastic recurrent neural networks. Ref. [43] studied the stability for impulsive stochastic differential equations driven by G-Brownian motion. For the other stability analysis for impulsive stochastic systems, please refer to [44–46] and references therein.

Inspired by the above discussion, we can see that there is still no result for the $p$th moment exponential stability for impulsive stochastic functional differential equations with Markovian switched delay effects. Thus, in this paper, we will focus on this question. The systems combine the characteristics of the continuous-time systems and discrete-time systems, which leads the stability analysis for such systems being more complicated than in the case of the pure continuous-time systems or discrete-time systems. By applying Markov chain theory, stochastic analysis theory and the Razumikhin technique, we establish a criterion of $p$th moment exponential stability. Finally, an example is provided to verify the efficiency of the obtained result.

The rest of the paper is organized as follows. In Section 2, we introduce the model and some preliminaries. The main result and its proof will be presented in Section 3. An illustrative example is provided in Section 4. Conclusions are drawn in Section 5.

**Notation 1.** *Throughout this paper, we use the following notations. Let* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ *be a complete probability space with a natural filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *satisfying the usual condition (i.e., it is right continuous, and* $\mathcal{F}_0$ *contains all* $\mathbb{P}$*-null sets). For* $r > 0$*, the symbol* $PC([-r,0],\mathbb{R}^n)$ *denotes the family of piecewise right continuous function* $\varphi$ *from* $[-r,0]$ *to* $\mathbb{R}^n$ *with the norm* $\|\varphi\| = \sup_{-r\leq u\leq 0}|\varphi(u)|$*. We use* $PL^p_{\mathcal{F}_0}([-r,0],\mathbb{R}^n)$ *to denote the family of all* $\mathcal{F}_0$*-measurable,* $PC([-r,0],\mathbb{R}^n)$*-valued random variables satisfying* $\sup_{-r\leq u\leq 0}\mathbb{E}|\varphi(u)|^p < \infty$*. We use* $\mathbb{E}[\cdot]$ *to denote the correspondent expectation* *operator with respect to the probability measure* $\mathbb{P}$. *Let* $B_t = B(t) = (B_1(t), B_2(t), \cdots, B_m(t))^T$ *be an m-dimensional Brownian motion defined on a complete probability space.*

## 2. Preliminaries

Let $\tau(t)$ be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a finite state space $\mathcal{S} = \{r_1, r_2, \cdots, r_m\}$ with generator $Q = (q_{ij})_{m\times m}$ given by:

$$\mathbb{P}(\tau(t+\Delta t) = r_j | \tau(t) = r_i) = \begin{cases} q_{ij}\Delta t + o(\Delta t) & if\ i \neq j \\ 1 + q_{ii}\Delta t + o(\Delta t) & if\ i = j \end{cases}$$

where $\Delta t > 0$. Here, $q_{ij} \geq 0$ is the transition rate from $r_i$ to $r_j$ if $r_i \neq r_j$, while $q_{ii} = -\sum_{j\neq i} q_{ij} = -q_i$. Define $\xi_0 = 0$, $\xi_k = \inf\{t > \xi_{k-1}; \tau(t) \neq \tau(\xi_{k-1})\}$ and $r(n) = \tau(\xi_n)$. From Markov chain theory, we know that $\{r(n), n \in Z\}$ is a Markov chain (called the embedded chain of $\tau(t)$). Its transition probability is $r_{ij}^{(1)} = (1 - \delta_{ij})\frac{q_{ij}}{q_i}$, where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$. We use $R = (R_{ij})$ to denote the transition probability matrix of embedded chain $\{r(n), n \in Z\}$. In this paper, we assume that the Markov chain $\{\tau(t), t \geq 0\}$ is independent of the Brownian motion $\{B(t), t \geq 0\}$. For the sake of simplicity, we denote $q_i \doteq q_{r_i}$. We denote $\lambda = \max\{q_i, i \in \mathcal{S}\}$, $v = \max_i\{1 - \frac{q_i}{\lambda}, i \in \mathcal{S}\}$ and $q = \max_{i,j} q_{ij}$.

**Assumption 1.** *In this paper, we always assume that* $v, q and \lambda$ *satisfy* $v \leq \frac{q}{\lambda}$.

We will consider the following impulsive stochastic delay differential equation with Markovian switched delay effects:

$$\begin{cases} dx(t) = f(t, x(t - \tau(t)), x(t))dt + g(t, x(t - \tau(t)), x(t))dB_t, t \neq t_k, \\ x(t_k) = I_k(t_k^- - \tau(t_k^-), x(t_k^- - \tau(t_k^-))), k = 1, 2, \cdots, t = t_k, \\ x_0 = \varphi, \end{cases} \tag{1}$$

where $\varphi = \{\varphi(u) : u \in [-r, 0]\} \in PL^p_{\mathcal{F}_0}([-r, 0], \mathbb{R}^n)$, $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are all Borel measurable. $r = \max\{r_1, r_2, \cdots, r_m\}$. $x(t_k^-)$ and $x(t_k^+)$ denote the left and right limits at $t_k$, respectively. Here, the impulse instants $\{t_k\}_{k=1}^\infty$ are all deterministic. $\Delta x(t_k) = x(t_k) - x(t_k^-)$, i.e., $x(t)$ is right-continuous at the impulse moment. We assume $f$, $g$ and $I_k$ satisfy the Lips conditions and the linear growth condition (see also [44]) in order to guarantee the existence and uniqueness of solutions $x(t)$ for system (1). We also assume that $f(t, 0, 0) = 0$, $g(t, 0, 0) = 0$ and $I_k(t, 0) = 0$ for all $k = 1, 2, \cdots$, which implies that the trivial solution of system (1) exists.

Now, we will provide a real example to show the usefulness of the model (1).

**Example 1** (Electronic control systems)**.** *On the road, if the vehicle flow reaches A, the red light is on. If the vehicle flow is below A, the green light is on. We define a two states of Markov chains* $\{\tau(t), t \geq 0\}$ *with generator:*

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix},$$

$\{the\ vehicle\ flow\ exceed\ A\} = \{the\ red\ light\ is\ on\} = \{\tau(t) = r_1\}$. $\{the\ vehicle\ flow\ notexceed\ A\} = \{the\ green\ light\ is\ on\} = \{\tau(t) = r_2\}$. *The red light or green light sign will transfer to the electronic control system. For different sign lights, the sign transport delay is different. Moreover, the electronic control system is perturbed by noise, and the impulsive sign emerge in some determined instants. If the red light (or green light) is on, the electronic control system can be described by the following two systems, respectively:*

$$\begin{cases} dx(t) = f(t, x(t), x(t - r_i))dt + g(t, x(t), x(t - r_i))dB_t, \\ x(t_k) = I_k(t_k^- - r_i, x(t_k^- - r_i)), \quad k = 1, 2, \cdots, \quad t = t_k, \end{cases}$$

*where $i = 1, 2$. However, the vehicle flow is random, so the electronic control system randomly switched during above two systems, and the switching law is governed by the Markov chain $\tau(t)$. In other words, we can describe the electronic control system by Equation (1). A natural question is how to consider its stability.*

**Definition 1.** *The trivial solution of system (1) is said to be pth moment exponentially stable if there is a pair of positive constants C and $\beta$, such that:*

$$\mathbb{E}|x(t)|^p \leq C\mathbb{E}\|\varphi\|^p e^{-\beta t} \quad on \quad t \geq 0$$

*for all $\varphi \in PL^p_{\mathcal{F}_0}([-r, 0], \mathbb{R}^n)$.*

**Remark 1.** *When $p = 2$, it is said to be mean square exponentially stable.*

In order to use Lyapunov's method, we need the following definition.

**Definition 2.** *The function $V = V(t, x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ belongs to class $\Psi$ if it is continuously differentiable once in t and twice in x.*

Now, we define an operator $\mathcal{L}V$ from $\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ by:

$$\mathcal{L}V(t, x(t - r_i), x(t)) = V_t(t, x(t)) + V_x(t, x(t)) f(t, x(t - r_i), x(t)) + \frac{1}{2} trace[g^T(t) V_{xx}(t, x(t)) g(t)],$$

where $g(t) = g(t, x(t - r_i), x(t))$, $V_t(t, x(t)) = \frac{\partial V(t,x(t))}{\partial t}$, $V_x(t, x(t)) = \left(\frac{\partial V(t,x(t))}{\partial x_1}, \frac{\partial V(t,x(t))}{\partial x_2}, \cdots, \frac{\partial V(t,x(t))}{\partial x_n}\right)$ and $V_{xx}(t, x(t)) = \left(\frac{\partial^2 V(t,x(t))}{\partial x_i \partial x_j}\right)_{n \times n}$.

Before stating our main result in this paper, we need the following two lemmas, which are useful in the proof of the main result. The first lemma can be found in any monograph on Markov chain theory (see e.g., [47]).

**Lemma 1.** *The transition probability $P_{ij}(t)$ of the Markov chain $\{\tau(t), t \geq 0\}$ can be calculated by:*

$$P_{ij}(t) = \sum_{n=0}^{\infty} \tilde{P}_{ij}^{(n)} \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \tag{2}$$

*where $\tilde{P}_{ij}^{(n)}$ is the i,jth component of the nth power of the matrix $\tilde{P}$, which is defined as follows:*

$$\tilde{P}_{ij} = \begin{cases} 1 - \frac{q_i}{\lambda}, & if \ \ j = i, \\ \frac{q_i}{\lambda} R_{ij}, & if \ \ j \neq i. \end{cases}$$

**Lemma 2.** *The transition probability $P_{ij}(t)$ has the following estimate:*

$$P_{ij}(t) \leq \delta_{ij} e^{-\lambda t} + \sum_{n=1}^{\infty} \frac{q}{\lambda} \left(\frac{q}{\lambda}(m - 1) + 2v\right)^{n-1} \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

**Proof.** First, when $k = 1$, from Assumption 1, we know that $\tilde{P}_{ij}^{(1)} \leq \frac{q}{\lambda}$. Next, we will prove that for any $r_i$ and $r_j$ and $k \geq 2$, $\tilde{P}_{ij}^{(k)} \leq \frac{q}{\lambda}(\frac{q}{\lambda}(m - 1) + \frac{k}{k-1}v)^{k-1}$. If $k = 2$, the conclusion is obvious. We assume that the conclusion holds for $k = n - 1$. Then, we have:

$$
\begin{aligned}
\tilde{P}_{ij}^{(n)} &= \sum_s \tilde{P}_{is}^{(n-1)} \tilde{P}_{sj}^{(1)} \\
&\leq \frac{q}{\lambda} v \Big( \frac{q}{\lambda}(m-1) + \frac{n-1}{n-2} v \Big)^{n-2} + (m-1) \Big( \frac{q}{\lambda} \Big)^2 \Big( \frac{q}{\lambda}(m-1) + \frac{n-1}{n-2} v \Big)^{n-2} \\
&= \frac{q}{\lambda} \Big( \frac{q}{\lambda}(m-1) + \frac{n-1}{n-2} v \Big)^{n-2} \Big( \frac{q}{\lambda}(m-1) + v \Big) \\
&= \frac{q}{\lambda} \Big( \frac{q}{\lambda}(m-1) + \frac{n-1}{n-2} v \Big)^{n-1} \frac{\frac{q}{\lambda}(m-1) + v}{\frac{q}{\lambda}(m-1) + \frac{n-1}{n-2} v} \\
&\leq \frac{q}{\lambda} \Big( \frac{q}{\lambda}(m-1) + \frac{n-1}{n-2} v \Big)^{n-1} \\
&\leq \frac{q}{\lambda} \Big( \frac{q}{\lambda}(m-1) + \frac{n}{n-1} v \Big)^{n-1}.
\end{aligned}
$$

Substituting the above inequality into (2), we obtain:

$$
\begin{aligned}
P_{ij}(t) &= \sum_{n=0}^{\infty} \tilde{P}_{ij}^{(n)} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\
&= \tilde{P}_{ij}^{(0)} e^{-\lambda t} + \sum_{n=1}^{\infty} \tilde{P}_{ij}^{(n)} \frac{(\lambda t)^n}{n!} e^{-\lambda t}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \delta_{ij} e^{-\lambda t} + \left[ \frac{q}{\lambda} + \sum_{n=2}^{\infty} \frac{q}{\lambda} \Big( \frac{q}{\lambda}(m-1) + \frac{n}{n-1} v \Big)^{n-1} \right] \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\
&\leq \delta_{ij} e^{-\lambda t} + \sum_{n=1}^{\infty} \frac{q}{\lambda} \Big( \frac{q}{\lambda}(m-1) + 2v \Big)^{n-1} \frac{(\lambda t)^n}{n!} e^{-\lambda t}.
\end{aligned}
$$

□

## 3. Main Results

In this section, we will use the Markov chain theory, stochastic analysis theory and Lyapunov's method to obtain a criterion for $p$th moment exponential stability of system (1).

**Theorem 1.** *Let $q, m, v, \lambda, \gamma, c, c_1, c_2, p, \theta, \rho, \beta$, $M \geq \frac{c_2}{c_1} e^{(\gamma+\lambda-2v\lambda-\theta-q(m-1))r} \geq 1$ be all positive numbers. If there exists a function $V \in \Psi$ such that the following conditions hold:*

*(1) For all $x \in \mathbb{R}^n$:*

$$c_1 |x|^p \leq V(t,x) \leq c_2 |x|^p,$$

*(2) For all $k \in N$:*

$$\mathbb{E}V(t_k, x(t_k)) \leq \rho \mathbb{E}V(t_k^- - \tau(t_k^-), x(t_k^- - \tau(t_k^-))),$$

*(3) For all $k \in N$ and $t \in [t_{k-1}, t_k)$:*

$$\max_{i \in \mathcal{S}} \left[ e^{-\lambda\delta} \mathbb{E}\mathcal{L}V(t, x(t-r_i), x(t)) + \sum_{j=1}^{m} \frac{e^{(q(m-1)+2v\lambda)\eta} - 1}{m - 1 + \frac{2v\lambda}{q}} e^{-\lambda\delta} \mathbb{E}\mathcal{L}V(t, x(t-r_j), x(t)) \right] \leq c\mathbb{E}V(t, x(t))$$

*if $\mathbb{E}V(t+u, x(t+u)) \leq \beta \mathbb{E}V(t, x(t))$ for all $u \in [-r, 0]$ and $\beta \geq Me^{(\gamma+\lambda-2v\lambda-\theta-q(m-1))r}$, where $\eta = \max_k\{t_k - t_{k-1}\} < \infty$ and $\delta = \min_k\{t_k - t_{k-1}\} > 0$.*

*(4) $\theta + q(m-1) + 2v\lambda - \lambda - \gamma < 0$,*

*(5) $\rho \sup_i e^{-(\theta+q(m-1)+2v\lambda-\lambda-\gamma)r_i} \leq \rho \frac{\sum_{i=1}^{m} e^{-(\theta+q(m-1)+2v\lambda-\lambda-\gamma)r_i}}{m-1+\frac{2v\lambda}{q}} \leq e^{(\lambda-q(m-1)-2v\lambda)\eta}$,*

*(6) $\log \frac{c_1}{c_2} + [c + \lambda + \gamma - 2v\lambda - \theta - q(m-1)]\eta < [\lambda + \gamma - 2v\lambda - \theta - q(m-1)]r.$*

*Then, system (1) is p-moment exponentially stable.*

**Proof.** First, we assume that $\mathbb{E}\|\varphi\|^p \neq 0$. We will prove that for any $k$ and $t \in [t_k, t_{k+1})$:

$$\mathbb{E}V(t, x(t)) \leq c_2 \mathbb{E}\|\varphi\|^p e^{(\theta + q(m-1) + 2v\lambda - \lambda - \gamma)t}. \tag{3}$$

Now, we will check that (3) holds for $k = 0$. If (3) is not true, then there exists $t \in [0, t_1)$, s.t. $\mathbb{E}V(t, x(t)) > c_2 \mathbb{E}\|\varphi\|^p e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t}$. Let:

$$t^* = \inf\{t \in [0, t_1) : \mathbb{E}V(t, x(t)) > c_2 \mathbb{E}\|\varphi\|^p e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t}\}$$

and

$$t^{**} = \sup\{t \in [-r, t^*] : \mathbb{E}V(t, x(t)) \leq c_1 \mathbb{E}\|\varphi\|^p e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)(t+r)}\}.$$

Then, for any $t \in [t^{**}, t^*]$ and $u \in [-r, 0]$:

$$\begin{aligned}
& e^{(\gamma+\lambda-2v\lambda-\theta-q(m-1))(t+u)} \mathbb{E}V(t+u, x(t+u)) \\
\leq & c_2 \mathbb{E}\|\varphi\|^p \\
\leq & c_1 M e^{-(\gamma+\lambda-2v\lambda-\theta-q(m-1))r} \mathbb{E}\|\varphi\|^p \\
= & M e^{(\gamma+\lambda-2v\lambda-\theta-q(m-1))t^{**}} \mathbb{E}V(t^{**}, x(t^{**})) \\
\leq & M e^{(\gamma+\lambda-2v\lambda-\theta-q(m-1))t} \mathbb{E}V(t, x(t)),
\end{aligned}$$

which implies that for all $u \in [-r, 0]$:

$$\mathbb{E}V(t+u, x(t+u)) \leq M e^{(\gamma+\lambda-2v\lambda-\theta-q(m-1))r} \mathbb{E}V(t, x(t)) \leq \beta \mathbb{E}V(t, x(t)).$$

By condition (3), we obtain:

$$\begin{aligned}
& \mathbb{E}\mathcal{L}V(t, x(t-\tau(t)), x(t)) \\
= & \sum_{j=1}^{m} \mathbb{P}(\tau(t) = r_j) \mathbb{E}\mathcal{L}V(t, x(t-r_j), x(t)) \\
= & \sum_{i=1}^{m} \sum_{j=1}^{m} \mathbb{P}(\tau(0) = r_i) P_{ij}(t) \mathbb{E}\mathcal{L}V(t, x(t-r_j), x(t)) \\
\leq & \sum_{i=1}^{m} \sum_{j=1}^{m} \mathbb{P}(\tau(0) = r_i) \left( \delta_{ij} e^{-\lambda t} + \sum_{n=1}^{\infty} \frac{q}{\lambda} \left( \frac{q}{\lambda}(m-1) + 2v \right)^{n-1} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \right) \mathbb{E}\mathcal{L}V(t, x(t-r_j), x(t)) \\
= & \sum_{i=1}^{m} \mathbb{P}(\tau(0) = r_i) \left[ e^{-\lambda t} \mathbb{E}\mathcal{L}V(t, x(t-r_i), x(t)) + \sum_{j=1}^{m} \frac{e^{(q(m-1)+2v\lambda)t} - 1}{m - 1 + \frac{2v\lambda}{q}} e^{-\lambda t} \mathbb{E}\mathcal{L}V(t, x(t-r_j), x(t)) \right] \\
\leq & \sum_{i=1}^{m} \mathbb{P}(\tau(0) = r_i) \left[ e^{-\lambda\delta} \mathbb{E}\mathcal{L}V(t, x(t-r_i), x(t)) + \sum_{j=1}^{m} \frac{e^{(q(m-1)+2v\lambda)\eta} - 1}{m - 1 + \frac{2v\lambda}{q}} e^{-\lambda\delta} \mathbb{E}\mathcal{L}V(t, x(t-r_j), x(t)) \right] \\
\leq & c \mathbb{E}V(t, x(t)).
\end{aligned}$$

Using Itô's formula and the standard stopping time technique, we derive:

$$
\begin{aligned}
&\mathbb{E}V(t^*, x(t^*)) \\
\leq &\mathbb{E}V(t^{**}, x(t^{**}))e^{c(t^*-t^{**})} \\
= &e^{-(\gamma+\lambda-2v\lambda-\theta-q(m-1))r}c_1\mathbb{E}\|\varphi\|^p e^{ct^*}e^{-(\gamma+\lambda-2v\lambda-\theta-q(m-1)+c)t^{**}} \\
\leq &e^{-(\gamma+\lambda-2v\lambda-\theta-q(m-1))r}c_1\mathbb{E}\|\varphi\|^p e^{(c+\gamma+\lambda-2v\lambda-\theta-q(m-1))t^*}e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t^*} \\
\leq &e^{(c+\gamma+\lambda-2v\lambda-\theta-q(m-1))\eta}e^{-(\gamma+\lambda-2v\lambda-\theta-q(m-1))r}c_1\mathbb{E}\|\varphi\|^p e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t^*} \\
= &e^{(c+\gamma+\lambda-2v\lambda-\theta-q(m-1))\eta}e^{-(\gamma+\lambda-2v\lambda-\theta-q(m-1))r}\frac{c_1}{c_2}\mathbb{E}V(t^*, x(t^*)).
\end{aligned}
$$

Noting that $\mathbb{E}\|\varphi\|^p \neq 0$, so $\mathbb{E}V(t^*, x(t^*)) \neq 0$. This implies

$$
1 \leq \frac{c_1}{c_2}e^{(c+\gamma+\lambda-2v\lambda-\theta-q(m-1))\eta}e^{-(\gamma+\lambda-2v\lambda-\theta-q(m-1))r},
$$

which contradict condition (6). So, (3) holds for $k = 0$.

Assume that (3) holds for $k \leq l-1$, $k \in \mathbb{N}$. We will use the mathematical inductive method to prove that (3) holds for $k = l$. To this end, we divide the proof into two steps. Firstly, we will prove that it holds for $t = t_l$. From condition (2), we have:

$$
\begin{aligned}
&\mathbb{E}V(t_l, x(t_l)) \\
\leq &\rho\mathbb{E}V(t_l^- - \tau(t_l^-), x(t_l^- - \tau(t_l^-))) \\
\leq &\rho\sum_{j=1}^m \mathbb{E}V(t_l^- - r_j, x(t_l^- - r_j))\mathbb{P}(\tau(t_l^-) = r_j) \\
\leq &\rho\sum_{j=1}^m c_2\mathbb{E}\|\varphi\|^p e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)(t_l-r_j)}\mathbb{P}(\tau(t_l^-) = r_j) \\
= &\rho\sum_{j=1}^m c_2\mathbb{E}\|\varphi\|^p e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t_{l-1}}e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)(t_l-t_{l-1})}e^{-(\theta+q(m-1)+2v\lambda-\lambda-\gamma)r_j}\mathbb{P}(\tau(t_l^-) = r_j).
\end{aligned}
$$

Using the Markov property in point $t_{l-1}$, we obtain:

$$
\begin{aligned}
&\mathbb{E}V(t_l, x(t_l)) \\
\leq &\rho\sum_{j=1}^m \mathbb{E}V(t_l - r_i, x(t_l - r_j))\mathbb{P}(\tau(t_l^-) = r_j) \\
\leq &\rho\sum_{j=1}^m c_2\mathbb{E}\|\varphi\|^p e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t_{l-1}}e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)(t_l-t_{l-1})}e^{-(\theta+q(m-1)+2v\lambda-\lambda-\gamma)r_j}\mathbb{P}(\tau(t_l^-) = r_j) \\
= &\rho\sum_{j=1}^m c_2\mathbb{E}\|\varphi\|^p e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t_{l-1}}e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)(t_l-t_{l-1})}e^{-(\theta+q(m-1)+2v\lambda-\lambda-\gamma)r_j} \\
&\times\mathbb{E}(\mathbb{P}(\tau(t_l^-) = r_j)|\mathcal{F}_{t_{l-1}}) \\
= &\rho\sum_{j=1}^m c_2\mathbb{E}\|\varphi\|^p e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t_{l-1}}e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)(t_l-t_{l-1})}e^{-(\theta+q(m-1)+2v\lambda-\lambda-\gamma)r_j} \\
&\times\mathbb{E}(P_{\tau(t_{l-1})j}(t_l - t_{l-1})) \\
= &\rho\sum_{j=1}^m c_2\mathbb{E}\|\varphi\|^p e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t_{l-1}}e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)(t_l-t_{l-1})}e^{-(\theta+q(m-1)+2v\lambda-\lambda-\gamma)r_j} \\
&\times\sum_{i=1}^m P_{ij}(t_l - t_{l-1})\mathbb{P}(\tau(t_{l-1}) = r_i),
\end{aligned}
$$

where the symbol $P_{\tau(t_{l-1})j}(t_l - t_{l-1})$ in the second equality denotes the transition probability from state $\tau(t_{l-1})$ to $j$ during time $t_l - t_{l-1}$. Combining Lemma 2 and condition (5), we have:

$$
\begin{aligned}
&\sum_{i=1}^{m} \mathbb{P}(\tau(t_{l-1}) = r_i)\left( \sum_{j=1}^{m} \rho c_2 \mathbb{E}\|\varphi\|^p e^{-(\theta+q(m-1)+2v\lambda-\lambda-\gamma)r_j} e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)(t_l-t_{l-1})} \right.\\
&\left. \times e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t_{l-1}} P_{ij}(t_l - t_{l-1}) \right)\\
\leq &\sum_{i=1}^{m} \mathbb{P}(\tau(t_{l-1}) = r_i)\left[ \rho c_2 \mathbb{E}\|\varphi\|^p e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t_{l-1}} e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)(t_l-t_{l-1})} \right.\\
&\times e^{-(\theta+q(m-1)+2v\lambda-\lambda-\gamma)r_i} e^{-\lambda(t_l-t_{l-1})} + \left( \sum_{j=1}^{m} e^{-(\theta+q(m-1)+2v\lambda-\lambda-\gamma)r_j} \rho c_2 \mathbb{E}\|\varphi\|^p \right.\\
&\times e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t_{l-1}} e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)(t_l-t_{l-1})}\\
&\left.\left. \times \sum_{n=1}^{\infty} \frac{q}{\lambda}\left( \frac{q}{\lambda}(m-1) + 2v \right)^{n-1} \frac{(\lambda(t_l - t_{l-1}))^n}{n!} e^{-\lambda(t_l-t_{l-1})} \right) \right]\\
\leq &\sum_{i=1}^{m} \mathbb{P}(\tau(t_{l-1}) = r_i)\left[ \rho c_2 \mathbb{E}\|\varphi\|^p e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t_{l-1}} e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)(t_l-t_{l-1})} \right.\\
&\times e^{-\lambda(t_l-t_{l-1})} \frac{\sum_{j=1}^{m} e^{-(\theta+q(m-1)+2v\lambda-\lambda-\gamma)r_j}}{m-1+\frac{2v\lambda}{q}}\\
&+ \left( \rho c_2 \mathbb{E}\|\varphi\|^p \sum_{j=1}^{m} e^{-(\theta+q(m-1)+2v\lambda-\lambda-\gamma)r_j} e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t_{l-1}} e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)(t_l-t_{l-1})} \right.\\
&\left.\left. \times \sum_{n=1}^{\infty} \frac{q}{\lambda}\left( \frac{q}{\lambda}(m-1) + 2v \right)^{n-1} \frac{(\lambda(t_l - t_{l-1}))^n}{n!} e^{-\lambda(t_l-t_{l-1})} \right) \right]\\
= &\sum_{i=1}^{m} \mathbb{P}(\tau(t_{l-1}) = r_i)\left[ \rho c_2 \mathbb{E}\|\varphi\|^p e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t_{l-1}} e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)(t_l-t_{l-1})} \right.\\
&\left. \times \sum_{j=1}^{m} e^{-(\theta+q(m-1)+2v\lambda-\lambda-\gamma)r_j} \sum_{n=0}^{\infty} \frac{q}{\lambda}\left( \frac{q}{\lambda}(m-1) + 2v \right)^{n-1} \frac{(\lambda(t_k - t_{k-1}))^n}{n!} e^{-\lambda(t_k-t_{k-1})} \right]\\
= &e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t_{l-1}} e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)(t_l-t_{l-1})} \rho c_2 \mathbb{E}\|\varphi\|^p \frac{\sum_{j=1}^{m} e^{-(\theta+q(m-1)+2v\lambda-\lambda-\gamma)r_j}}{m-1+\frac{2v\lambda}{q}}\\
&\times e^{(q(m-1)+2v\lambda-\lambda)(t_l-t_{l-1})}\\
\leq &e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t_{l-1}} e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)(t_l-t_{l-1})} \rho c_2 \mathbb{E}\|\varphi\|^p \frac{\sum_{j=1}^{m} e^{-(\theta+q(m-1)+2v\lambda-\lambda-\gamma)r_j}}{m-1+\frac{2v\lambda}{q}}\\
&\times e^{(q(m-1)+2v\lambda-\lambda)\eta}\\
\leq &c_2 \mathbb{E}\|\varphi\|^p e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t_{l-1}} e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)(t_l-t_{l-1})}\\
= &c_2 \mathbb{E}\|\varphi\|^p e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t_l}.
\end{aligned}
$$

This implies that (3) is satisfied for $t = t_l$. Next, we will prove that (3) holds for $t \in (t_l, t_{l+1})$. If (3) is not true, then there exists $t \in (t_l, t_{l+1})$, s.t. $\mathbb{E}V(t, x(t)) > c_2 \mathbb{E}\|\varphi\|^p e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t}$. Let:

$$t^* = \inf\{t \in [t_l, t_{l+1}) : \mathbb{E}V(t, x(t)) > c_2 \mathbb{E}\|\varphi\|^p e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t}\}$$

and

$$t^{**} = \sup\{t \in (t_l, t^*] : \mathbb{E}V(t, x(t)) \leq c_1 \mathbb{E}\|\varphi\|^p e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)(t+r)}\}.$$

Then, for any $t \in [t^{**}, t^*]$ and $u \in [-r, 0]$, we obtain:

$$\begin{aligned}
&e^{(\gamma+\lambda-2v\lambda-\theta-q(m-1))(t+u)}\mathbb{E}V(t+u, x(t+u)) \\
\leq& c_2\mathbb{E}\|\varphi\|^p \\
<& c_1 M e^{-(\gamma+\lambda-2v\lambda-\theta-q(m-1))r}\mathbb{E}\|\varphi\|^p \\
=& M e^{(\gamma+\lambda-2v\lambda-\theta-q(m-1))t^{**}}\mathbb{E}V(t^{**}, x(t^{**})) \\
\leq& M e^{(\gamma+\lambda-2v\lambda-\theta-q(m-1))t}\mathbb{E}V(t, x(t)),
\end{aligned}$$

which implies that for all $u \in [-r, 0]$:

$$\mathbb{E}V(t+u, x(t+u)) \leq M e^{(\gamma+\lambda-2v\lambda-\theta-q(m-1))r}\mathbb{E}V(t, x(t)) \leq \beta\mathbb{E}V(t, x(t)).$$

By condition (3), we obtain:

$$\begin{aligned}
&\mathbb{E}\mathcal{L}V(t, x(t-\tau(t)), x(t)) \\
=&\sum_{j=1}^{m}\mathbb{P}(\tau(t)=r_j)\mathbb{E}\mathcal{L}V(t, x(t-r_j), x(t)) \\
=&\sum_{i=1}^{m}\sum_{j=1}^{m}\mathbb{P}(\tau(t_l)=r_i)P_{ij}(t-t_l)\mathbb{E}\mathcal{L}V(t, x(t-r_j), x(t)) \\
\leq&\sum_{i=1}^{m}\sum_{j=1}^{m}\mathbb{P}(\tau(t_l)=r_i)\left(\delta_{ij}e^{-\lambda(t-t_l)}+\sum_{n=1}^{\infty}\frac{q}{\lambda}\left(\frac{q}{\lambda}(m-1)+2v\right)^{n-1}\frac{(\lambda(t-t_l))^n}{n!}e^{-\lambda(t-t_l)}\right) \\
&\times\mathbb{E}\mathcal{L}V(t, x(t-r_j), x(t)) \\
=&\sum_{i=1}^{m}\mathbb{P}(\tau(t_l)=r_i)\Bigg[e^{-\lambda(t-t_l)}\mathbb{E}\mathcal{L}V(t, x(t-r_i), x(t))+\sum_{j=1}^{m}\frac{e^{(q(m-1)+2v\lambda)(t-t_l)}-1}{m-1+\frac{2v\lambda}{q}}e^{-\lambda(t-t_l)} \\
&\times\mathbb{E}\mathcal{L}V(t, x(t-r_j), x(t))\Bigg] \\
\leq&\sum_{i=1}^{m}\mathbb{P}(\tau(t_l)=r_i)\left[e^{-\lambda\delta}\mathbb{E}\mathcal{L}V(t, x(t-r_i), x(t))+\sum_{j=1}^{m}\frac{e^{(q(m-1)+2v\lambda)\eta}-1}{m-1+\frac{2v\lambda}{q}}e^{-\lambda\delta}\mathbb{E}\mathcal{L}V(t, x(t-r_j), x(t))\right] \\
\leq& c\mathbb{E}V(t, x(t)).
\end{aligned}$$

For any $t \in [t^{**}, t^*]$, it follows that:

$$\begin{aligned}
&\mathbb{E}V(t^*, x(t^*)) \\
\leq&\mathbb{E}V(t^{**}, x(t^{**}))e^{c(t^*-t^{**})} \\
=&e^{-(\gamma+\lambda-2v\lambda-\theta-q(m-1))r}c_1\mathbb{E}\|\varphi\|^p e^{ct^*}e^{-(\gamma+\lambda-2v\lambda-\theta-q(m-1)+c)t^{**}} \\
\leq&e^{-(\gamma+\lambda-2v\lambda-\theta-q(m-1))r}c_1\mathbb{E}\|\varphi\|^p e^{(c+\gamma+\lambda-2v\lambda-\theta-q(m-1))t^*}e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t^*} \\
\leq&e^{(c+\gamma+\lambda-2v\lambda-\theta-q(m-1))\eta}e^{-(\gamma+\lambda-2v\lambda-\theta-q(m-1))r}c_1\mathbb{E}\|\varphi\|^p e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t^*} \\
=&e^{(c+\gamma+\lambda-2v\lambda-\theta-q(m-1))\eta}e^{-(\gamma+\lambda-2v\lambda-\theta-q(m-1))r}\frac{c_1}{c_2}\mathbb{E}V(t^*, x(t^*)).
\end{aligned}$$

This is a contradiction to condition (6). So, (3) holds for $t \in (t_l, t_{l+1})$. In other words, we have proved that (3) holds for any $k$ and $t \in [t_k, t_{k+1})$. According to condition (1), we have:

$$\mathbb{E}|x(t)|^p \leq \frac{c_2}{c_1}\mathbb{E}\|\varphi\|^p e^{(\theta+q(m-1)+2v\lambda-\lambda-\gamma)t},$$

which verifies that system (1) is $p$th moment exponentially stable. □

**Remark 2.** *The model we consider here combines the characteristics of the continuous-time systems and discrete-time systems. So, the stability analysis of such systems is more complex than the case of the pure continuous time-systems and discrete-time systems. In the proof of Theorem 1, the key* *step is to estimate the transition probability of Markov chain $\tau(t)$. Here, we have applied some classic results in the Markov chain theory. In order to overcome the difficulties that arise from the discrete-time systems, we have used the Markov property of $\tau(t)$. The technique applied here is new, and it is different from those methods used in the existing literature.*

**Remark 3.** *According to Theorem 1, we can see that if a certain subsystem is unstable, and then the switched system remains stable. In fact, from the statement of Theorem 1, we can see that if the parameters (the size of delay, the altitude of impulsive control gain, the impulsive times interval) of every subsystem are given, we can control the parameters $q$, $v$ and $\lambda$ in order to ensure the stability of the switched system.*

## 4. An Example

In this section, we will consider an example to illustrate the validity of our result.

**Example 2.** *Consider the following 2D stochastic neural network:*

$$\begin{cases} dx(t) = [Ax(t) + Bx(t-\tau(t))]dt + Df(x(t-\tau(t)))dW(t) \\ \Delta x(t_k) = -0.35x(t_k^- - \tau(t_k^-)),\ t = t_k,\quad k = 1,2,\cdots \\ x_0 = \varphi(u) = [0.015, -0.02]^T,\quad u \in [-0.5, 0], \end{cases} \tag{4}$$

*where*

$$A = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.02 \end{bmatrix},\quad B = \begin{bmatrix} -0.015 & -0.001 \\ -0.002 & -0.025 \end{bmatrix},$$
$$D = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.03 \end{bmatrix}$$

*where $\delta = \eta = t_k - t_{k-1} = 0.2$. $f(\cdot) = \tanh(\cdot)$. The Markov process $\{\tau(t),\ t \ge 0\}$ takes values in $S = \{0.3, 0.5\}$ with generator:*

$$Q = \begin{bmatrix} -0.4 & 0.4 \\ 0.3 & -0.3 \end{bmatrix}.$$

*Now, we assert that system (4) is mean square stable. Obviously, $\lambda = 0.4$, $q = 0.4$, $v = 0.25$. Taking $V(t, x(t)) = |x(t)|^2$, then condition (1) of Theorem 1 holds for $p = 2, c_1 = c_2 = 1$. Let $\gamma = 0.4$ and $\theta = 0.1$. From a direct computation, it follows that $\rho = 0.422$. Using Itô's formula, we have:*

$$\begin{aligned} &\mathbb{E}\mathcal{L}V(t, x(t-r_i), x(t)) \\ =&0.04\mathbb{E}|x(t)|^2 + 0.025\mathbb{E}|x(t)|^2 + 0.025\mathbb{E}|x(t-r_i)|^2 + 0.0009\mathbb{E}|x(t-r_i)|^2 \\ \le&0.065\mathbb{E}|x(t)|^2 + (0.026 \times 1.1)\mathbb{E}|x(t)|^2 \\ =&0.0936\mathbb{E}|x(t)|^2, \end{aligned}$$

$c = 0.0936[e^{-\lambda\delta} + \frac{m}{m-1+\frac{2v\lambda}{q}}(e^{(q(m-1)+2v\lambda)\delta} - 1)e^{-\lambda\delta}] = 0.1$. *Thus, conditions (1)–(3) are satisfied.*

*Next, we turn to check that conditions (4)–(6) hold. In fact, $\theta + q(m-1) + 2v\lambda - \lambda - \gamma = -0.1 < 0$. $\sup_i e^{-(\theta+q(m-1)+2v\lambda-\lambda-\gamma)r_i} = e^{0.05} = 1.05 < \frac{\sum_{i=1}^m e^{-(\theta+q(m-1)+2v\lambda-\lambda-\gamma)r_i}}{m-1+\frac{2v\lambda}{q}} = \frac{e^{0.03}+e^{0.05}}{1.5} = 1.38$, $\rho\frac{\sum_{i=1}^m e^{-(\theta+q(m-1)+2v\lambda-\lambda-\gamma)r_i}}{m-1+\frac{2v\lambda}{q}} = 0.422 \times \frac{e^{0.03}+e^{0.05}}{1.5} = 0.422 \times 1.38 = 0.59 < e^{-0.2\times 0.2} = 0.96$. $(c + \lambda + \gamma - \theta - 2v\lambda - q(m-1))\delta = 0.04 < (\lambda + \gamma - \theta - 2v\lambda - q(m-1))r = 0.05$. That is, all the conditions of Theorem 1 hold. Therefore, from Theorem 1, we see that the neural network (4) is mean square exponentially stable.*

## 5. Conclusions

In this paper, we have investigated the $p$th moment exponential stability of impulsive stochastic functional differential equations with Markovian switched delay effects. By using stochastic process theory, stochastic analysis theory, Razumikin technology and the Lyaponov method, a novel sufficient condition is obtained. Different from the previous literature, the model that we study is new and more complex. Moreover, an example is provided to show the efficiency of our result. In addition, it maybe more reasonable that if the impulse instants $\{t_k\}_{k=1}^{\infty}$ are random variables, and how to consider the stability is more challenging. In the future, we will consider this question.

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## References

1. Mao, X. Stability of stochastic differential equations with Markovian switching. *Stoch. Process. Appl.* **1999**, *79*, 45–67. [CrossRef]
2. Mao, X. Stability of stochastic functional differential equations with Markovian switching. *Funct. Differ. Equ.* **1999**, *6*, 375–396.
3. Mao, X. Exponential stability of stochastic delay interval systems with Markovian switching. *IEEE Trans. Autom. Control* **2002**, *47*, 1604–1612.
4. Huang, L.; Mao, X. Stability of singular stochastic systems with Markovian switching. *IEEE Trans. Autom. Control* **2011**, *56*, 424–429. [CrossRef]
5. Du, N.; Dang, N.; Dieu, N. On stability in distribution of stochastic differential delay equations with Markovian switching. *Syst. Control Lett.* **2014**, *65*, 43–49. [CrossRef]
6. Wang, B.; Zhu, Q. Stability analysis of Markov switched stochastic differential equations with both stable and unstable subsystems. *Syst. Control Lett.* **2017**, *105*, 55–61. [CrossRef]
7. Zhu, F.; Han, Z.; Zheng, J. Stability analysis of stochastic differential equations with Markovian switching. *Syst. Control Lett.* **2012**, *61*, 1209–1214. [CrossRef]
8. Yan, Z.; Zhang, W.; Zhang, G. Finite-time stability and stabilization of Ito stochastic systems with Markovian switching: Mode-dependent parameter approach. *IEEE Trans. Autom. Control* **2015**, *60*, 2428–2433. [CrossRef]
9. Aberkane, S. Stochastic stabilization of a class of nonhomogeneous Markovian jump linear systems. *Syst. Control Lett.* **2011**, *60*, 156–160. [CrossRef]
10. Benjelloun, K.; Boukas, E. Mean square stochastic stability of linear time-daley system with Markovian jumping parameters. *IEEE Trans. Autom. Control* **1998**, *43*, 1456–1460. [CrossRef]
11. Bolzern, P.; Colaneri, P.; Nicolao, D. Markov jump linear systems with switching transition rate: Mean square stability with dwell time. *Automatica* **2010**, *46*, 1081–1088. [CrossRef]
12. Fragoso, M.; Costa, O. A unified approach for stochastic and mean square stability of continuous-time linear systems with Markovian jumping parameters and additive disturbances. *SIAM J. Control Optim.* **2005**, *44*, 1165–1191. [CrossRef]
13. Ugrinovskii, V.; Pota, H. Decentralized control of power systems via robust control of uncertain Markov jump parameter systems. *Int. J. Control* **2005**, *78*, 662–677. [CrossRef]
14. Wang, G.; Xu, L. Almost sure stability and stabilization of Markovian jump systems with stochastic switching. *IEEE Trans. Autom. Control* **2022**, *67*, 1529–1536. [CrossRef]
15. Antunes, D.; Hespanha, J.; Silvestre, C. Stochastic hybrid systems with renewal transitions: Moment analysis with application to networked control systems with delays. *SIAM J. Control Optim.* **2013**, *51*, 1481–1499. [CrossRef]
16. Cetinkaya, A.; Ishii, H.; Hayakawa, T. Analysis of stochastic switched systems with application to networked control under jamming attacks. *IEEE Trans. Autom. Control* **2019**, *64*, 2013–2028. [CrossRef]
17. Filipovic, V. Exponential stability of stochastic switched systems. *Trans. Inst. Meas. Control* **2009**, *31*, 205–212. [CrossRef]
18. Schioler, H.; Simonsen, M.; Leth, J. Stochastic stability of systems with semi-Markovian switching. *Automatica* **2014**, *50*, 2961–2964. [CrossRef]
19. Wang, B.; Zhu, Q. Stability analysis of semi-Markov switched stochastic systems. *Automatica* **2018**, *94*, 72–80. [CrossRef]
20. Mu, X.; Hu, Z. Stability analysis for semi-Markovian switched singular stochastic systems. *Automatica* **2020**, *118*, 109014. [CrossRef]
21. Luo, W.; Liu, X.; Yang, J. Stability of stochastic functional differential systems with semi-Markovian switching and Levy noise and its application. *Int. J. Control Autom.* **2020**, *18*, 708–718. [CrossRef]
22. Caraballo, T.; Mchiri, L.; Mohsen, B.; Rhaima, M. $p$th moment exponential stability of neutral stochastic pantograph differential equations with Markovian switching. *Commun. Nonlinear Sci. Numer. Simulat.* **2021**, *102*, 105916. [CrossRef]
23. Hespanha, J.; Morse, A. Stability of switched systems with average dwell time. In Proceedings of the 38th IEEE Conference on Decision and Control, Phoenix, AZ, USA, 7–10 December 1999; pp. 2655–2660

24. Xiong, J.; Lam, J.; Shu, Z.; Mao, X. Stability analysis of continuous-time switched systems with a random switching signal. *IEEE Trans. Autom. Control* **2014**, *59*, 180–186. [CrossRef]
25. Mao, X.; Yuan, C. *Stochastic Differential Equations with Markovian Switching*; Imperial College Press: London, UK, 2006.
26. Chen, Y.; Zheng, W. Stability analysis and control for switched stochastic delayed systems. *Int. J. Robust Nonlinear Control* **2016**, *26*, 303–328. [CrossRef]
27. Chen, H.; Shi, P.; Lim, C. Stability of neutral stochastic switched time delay systems: An average dwell time approach. *Int. J. Robust Nonlinear Control* **2017**, *27*, 512–532. [CrossRef]
28. Meng, X.; Lam, J.; Gao, H. Network-based $H_\infty$ control for stochastic systems. *Int. J. Robust Nonlinear Control* **2009**, *19*, 295–312. [CrossRef]
29. Zhou, Q.; Shi, P. A new approach to network-based $H_\infty$ control for stochastic systems. *Int. J. Robust Nonlinear Control* **2012**, *22*, 1036–1059. [CrossRef]
30. Wu, Z.; Cui, M.; Shi, P.; Karimi, H. Stability of stochastic nonlinear systems with state-dependent switching. *IEEE Trans. Autom. Control* **2013**, *58*, 1904–1918. [CrossRef]
31. Zhao, X.; Peng, S.; Yin, Y.; Nguang, S. New results on stability of slowly switched systems: A multiple discontinuous Lyapunov function approach. *IEEE Trans. Autom. Control* **2017**, *62*, 3502–3509. [CrossRef]
32. Chatterjee, D.; Liberzon, D. Stability analysis of deterministic and stochastic switched systems via a comparison principle and multiple Lyapunov functions. *SIAM J. Control Optim.* **2006**, *45*, 174–206. [CrossRef]
33. Peng, S.; Zhang, Y. Some new criteria on $p$th moment stability of stochastic functional differential equations with Markovian switching. *IEEE Trans. Autom. Control* **2010**, *55*, 2886–2890. [CrossRef]
34. Huang, L.; Mao, X. On input-to-state stability of stochastic retarded systems with Markovian switching. *IEEE Trans. Autom. Control* **2009**, *54*, 1898–1902. [CrossRef]
35. Yue, D.; Han, Q. Delay-dependent exponential stability of stochastic systems with time-varying delay nonlinearity and Markovian switching. *IEEE Trans. Autom. Control* **2005**, *50*, 217–222.
36. Ding, K.; Zhu, Q. Extended dissipative anti-disturbance control for delayed switched singular semi-Markovian jump systems with multi-disturbance via disturbance observer. *Automatica* **2021**, *128*, 109556. [CrossRef]
37. Teel, A.; Subbaraman, A.; Sferlazza, A. Stability analysis for stochastic hybrid systems: A survey. *Automatica* **2014**, *50*, 2435–2456. [CrossRef]
38. Wu, F.; Yin, G.; Wang, L. Moment exponential stability of random delay systems with two-time-scale Markovian switching. *Nonlinear Anal. RWA* **2012**, *13*, 2476–2490. [CrossRef]
39. Vinodkumar, A.; Senthilkumar, T.; Hariharan, S.; Alzabut, J. Exponential stabilization of fixed and random time impulsive delay differential system with applications. *Math. Biosci. Eng.* **2012**, *13*, 2476–2490. [CrossRef]
40. Rengamannar, K.; Balakrishnan, G.; Palanisamy, M.; Niezabitowski, M. Exponential stability of non-linear stochastic delay differential system with generalized delay-dependent impulsive points. *Appl. Math. Comput.* **2012**, *13*, 2476–2490. [CrossRef]
41. Rengamannar, K.; Palanisamy, M. Exponential stability of non-linear neutral stochastic delay differential system with generalized delay-dependent impulsive points. *J. Franklin Inst.* **2021**, *358*, 5014–5038.
42. Chandrasekar, A.; Rakkiyappan, R. Impulsive controller design for exponential synchronization of delayed stochastic memristor-based recurrent neural networks. *Neurocomputing* **2016**, *173*, 1348–1355. [CrossRef]
43. Hu, L.; Ren, Y.; Sakthivel, R. Stability of square-mean almost automorphic mild solutions to impulsive stochastic differential equations driven by G-Brownian motion. *Int. J. Control* **2020**, *93*, 3016–3025. [CrossRef]
44. Peng, S.; Deng, F. New criteria on $p$th moment input-to-state stability of impulsive stochastic delayed differential systems. *IEEE Trans. Autom. Control* **2017**, *62*, 3573–3579. [CrossRef]
45. Hu, W.; Zhu, Q.; Karimi, H. Some improved Razumikhin stability criteria for impulsive stochastic delay differential systems. *IEEE Trans. Autom. Control* **2019**, *64*, 5207–5213. [CrossRef]
46. Hu, W.; Zhu, Q. Stability criteria for impulsive stochastic functional differential systems with distributed-delay dependent impulsive effects. *IEEE Trans. Syst. Man. Cybern. Syst.* **2021**, *51*, 2027–2032. [CrossRef]
47. Ross, S. *Stochastic Processes*; John Wiley & Sons: New York, NY, USA, 1996.