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STABILITY OF ISOMETRIES ON BANACH SPACES

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ABSTRACT. Let X and Y be Banach spaces. A mapping $f\colon X \to Y$ is called an ε-isometry if $|\,\|f(x_0) - f(x_1)\| - \|x_0 - x_1\|\,| \leqslant \varepsilon$ for all $x_0, x_1 \in X$. It is shown that there exist constants A and B such that if $f\colon X \to Y$ is a surjective ε-isometry, then $\|f((x_0 + x_1)/2) - (f(x_0) + f(x_1))/2\| \leqslant A(\varepsilon\|x_0 - x_1\|)^{1/2} + B\varepsilon$ for all $x_0, x_1 \in X$. This, together with a result of Peter M. Gruber, is used to show that if $f\colon X \to Y$ is a surjective ε-isometry, then there exists a surjective isometry $I\colon X \to Y$ for which $\|f(x) - I(x)\| \leqslant 5\varepsilon$, thus answering a question of Hyers and Ulam about the stability of isometries on Banach spaces.

Throughout, X, Y and Z denote real Banach spaces. A mapping $f\colon X \to Y$ is called an *ε-isometry* if $|\,\|f(x_0) - f(x_1)\| - \|x_0 - x_1\|\,| \leqslant \varepsilon$ for all $x_0, x_1 \in X$. Hyers and Ulam [3] formulated the stability problem for isometries, that is, the question as to whether for each pair of Banach spaces X and Y there exists a constant $K = K(X, Y)$ such that for each surjective ε-isometry $f\colon X \to Y$ there exists an isometry $I\colon X \to Y$ for which $\|f(x) - I(x)\| \leqslant K\varepsilon$ for all $x \in X$. This problem has been solved in a number of special cases (see [**1**] and [**2**] for a summary of such results), and Gruber [**2**, Theorem 1] went very far towards a general solution by showing that if $f\colon X \to Y$ is a surjective ε-isometry and $I\colon X \to Y$ is an isometry for which $I(0) = f(0)$ and for which $\|f(x) - I(x)\|/\|x\| \to 0$ uniformly as $\|x\| \to \infty$, then I is surjective and $\|f(x) - I(x)\| \leqslant 5\varepsilon$ for all $x \in X$. In what follows we will show that such an isometry always exists so that the answer to the question of Hyers and Ulam is affirmative with $K(X, Y) = 5$ for all X and Y. We do this by establishing that:

There exist constants A and B such that if $f\colon X \to Y$ is a surjective ε-isometry, then

$$\|f((x_0 + x_1)/2) - (f(x_0) + f(x_1))/2\| \leqslant A(\varepsilon\|x_0 - x_1\|)^{1/2} + B\varepsilon \tag{1}$$

for all $x_0, x_1 \in X$.

(We show this with $A = 10$ and $B = 20$, but the specific values of A and B are of no consequence.)

To see that (1) indeed proves the existence of the isometry I of Gruber's result we may assume without loss of generality that $f(0) = 0$. Applying (1) with $x_0 = 2^{n+1}x$

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and $x_1 = 0$ and dividing by 2^n we have

$$\|2^{-n}f(2^n x) - 2^{-n-1}f(2^{n+1}x)\| \leqslant 2^{-n/2}A(2\varepsilon\|x\|)^{1/2} + 2^{-n}B\varepsilon. \tag{2}$$

Since

$$f(x) - 2^{-n}f(2^n x) = \sum_{k=0}^{n-1}\left(2^{-k}f(2^k x) - 2^{-k-1}f(2^{k+1}x)\right),$$

the completeness of Y together with (2) implies that $I(x) = \lim_{n\to\infty} 2^{-n}f(2^n x)$ exists for all $x \in X$ and satisfies $I(0) = 0$ and

$$\|f(x) - I(x)\| \leqslant 2(\sqrt{2} - 1)^{-1}A(\varepsilon\|x\|)^{1/2} + 2B\varepsilon \tag{3}$$

so that $\|f(x) - I(x)\|/\|x\| \to 0$ uniformly as $\|x\| \to \infty$. Since

$$\left|\,\|2^{-n}f(2^n x_0) - 2^{-n}f(2^n x_1)\| - \|x_0 - x_1\|\,\right| \leqslant \varepsilon/2^n,$$

it is clear that I is an isometry. We mention, in passing, that an observation of Gruber [**2**, Remark, p. 266] shows that (1) serves to eliminate some of his considerations. Indeed, (1) implies that $I((x_0 + x_1)/2) = (I(x_0) + I(x_1))/2$ so that if we assume as above that $f(0) = 0$, it follows that I is linear and in turn surjective.

To facilitate the proof of (1) we introduce some terminology. If $f\colon X \to Y$, then any mapping $F\colon Y \to X$ for which

$$\|fF(y) - y\| \leqslant \delta \quad \text{for all } y \in Y \tag{4}$$

is said to be a δ-*inverse* of f. Following Bourgin [**1**] f is called δ-*onto* if it has a δ-inverse. Henceforth the term (δ,ε)-*isometry* will refer to a δ-onto ε-isometry. We have:

(5) If $f\colon X \to Y$ is a (δ,ε)-isometry and F is a δ-inverse of f, then F is a $(\delta + \varepsilon, 2\delta + \varepsilon)$-isometry.

To see that F is $(\delta + \varepsilon)$-onto we note that

$$\|Ff(x) - x\| \leqslant \|fFf(x) - f(x)\| + \varepsilon \leqslant \delta + \varepsilon$$

by (4). To see that F is a $(2\delta + \varepsilon)$-isometry, let $y_0, y_1 \in Y$. Then $\|fF(y_i) - y_i\| \leqslant \delta$ $(i = 0, 1)$ by (4) and

$$\left|\,\|fF(y_0) - fF(y_1)\| - \|F(y_0) - F(y_1)\|\,\right| \leqslant \varepsilon$$

since f is an ε-isometry. Hence

$$\left|\,\|F(y_0) - F(y_1)\| - \|y_0 - y_1\|\,\right| \leqslant \left|\,\|fF(y_0) - fF(y_1)\| - \|y_0 - y_1\|\,\right| + \varepsilon \leqslant 2\delta + \varepsilon.$$

We shall also need:

(6) Let $f_1\colon X \to Y$ be a (δ_1,ε_1)-isometry and let $f_2\colon Y \to Z$ be a (δ_2,ε_2)-isometry. Then $f_2 f_1$ is a $(\delta_1 + \delta_2 + \varepsilon_2, \varepsilon_1 + \varepsilon_2)$-isometry.

It is immediate that $f_2 f_1$ is an $(\varepsilon_1 + \varepsilon_2)$-isometry. To see the rest, let F_i be a δ_i-inverse of f_i $(i = 1, 2)$ and let $z \in Z$. Then

$$\begin{aligned}\|f_2 f_1 F_1 F_2(z) - z\| &\leqslant \|f_2 f_1 F_1 F_2(z) - f_2 F_2(z)\| + \delta_2\\ &\leqslant \|f_1 F_1 F_2(z) - F_2(z)\| + \varepsilon_2 + \delta_2\\ &\leqslant \delta_1 + \varepsilon_2 + \delta_2,\end{aligned}$$

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where we have used (4) as it applies to f_2 and f_1 and the fact that f_2 is an ε_2-isometry.

The proof of (1) which we now present is an adaptation of a proof given by Vogt [4] of a generalization of the Mazur-Ulam theorem on isometries. Let $x_0, x_1 \in X$, $y_i = f(x_i)$ $(i = 0, 1)$, $p = (x_0 + x_1)/2$ and $q = (y_0 + y_1)/2$. Until the very end of the proof we shall assume that $y_0 \neq y_1$. Since f is a $(0,\varepsilon)$-isometry, by (5) it has a 0-inverse F which is an $(\varepsilon,\varepsilon)$-isometry and for which $F(y_i) = x_i$ $(i = 0, 1)$. We define sequences $(g_k)_{k\geqslant 0}$ and $(G_k)_{k\geqslant 0}$ of mappings of Y into Y with the following properties:

(7) $\quad g_k$ is a $(4^{k+1}\varepsilon, 4^{k+1}\varepsilon)$-isometry and $g_k(y_i) = y_{1-i} \quad (i = 0, 1)$,

(8) $\quad G_k$ is a $4^{k+1}\varepsilon$-inverse of g_k and $G_k(y_i) = y_{1-i} \quad (i = 0, 1)$.

To begin we let $g_0(y) = f(2p - F(y))$ for $y \in Y$. By (6), g_0 is a $(2\varepsilon, 2\varepsilon)$-isometry and it is clear that it permutes y_0 and y_1. Thus (7) holds for $k = 0$. We let G_0 be any mapping which satisfies (8) for $k = 0$. Next, we let $g_1(y) = G_1(y) = 2q - y$ for $y \in Y$. Obviously (7) and (8) are then satisfied for $k = 1$. Finally, assuming that we have $g_0, \ldots, g_n$ and $G_0, \ldots, G_n$ which satisfy the stipulated conditions, we define $g_{n+1} = g_{n-1}g_nG_{n-1}$. A simple argument based on (5) and (6) shows that g_{n+1} satisfies (7) with $k = n + 1$. G_{n+1} is then taken to be any mapping satisfying (8) with $k = n + 1$.

We next define a sequence $(a_n)_{n\geqslant 1}$ of points of Y recursively by $a_1 = q$ and $a_{n+1} = g_{n-1}(a_n)$ for $n \geqslant 1$. Let $d = \|y_0 - y_1\|/2$. Denoting by $B(y,r)$ the closed ball of radius r and center y, we have that $g_k(B(y_i,r)) \subset B(y_{1-i}, r + 4^{k+1}\varepsilon)$. Since $a_1 \in B(y_0,d) \cap B(y_1,d)$ and $a_n = g_{n-2}g_{n-3}\cdots g_0(a_1)$, successive application of this inclusion with $k = 0, 1, \ldots, n-2$ yields

$$a_n \in B(y_0, d + 4^n\varepsilon) \cap B(y_1, d + 4^n\varepsilon) \subset B(q, d + 4^n\varepsilon).$$

Since the diameter of this last ball is $2(d + 4^n\varepsilon)$ we conclude that

$$\|a_n - a_{n-1}\| \leqslant 2(d + 4^n\varepsilon) \quad \text{for } n \geqslant 2. \tag{9}$$

We now show that for all $y \in Y$ there holds

$$\|g_n(y) - y\| \geqslant 2\|a_n - y\| - 2(4^n - 1)\varepsilon \quad \text{for } n \geqslant 1. \tag{10}$$

Since $g_1(y) = 2q - y$, this is true for $n = 1$. Assuming that it is valid for a given $n \geqslant 1$, we have

$$\begin{aligned}
\|g_{n+1}(y) - y\| &= \|g_{n-1}g_nG_{n-1}(y) - y\| \\
&\geqslant \|g_{n-1}g_nG_{n-1}(y) - g_{n-1}G_{n-1}(y)\| - 4^n\varepsilon \\
&\geqslant \|g_nG_{n-1}(y) - G_{n-1}(y)\| - 2\cdot 4^n\varepsilon \\
&\geqslant 2\|a_n - G_{n-1}(y)\| - (4^{n+1} - 2)\varepsilon \\
&\geqslant 2(\|g_{n-1}(a_n) - g_{n-1}G_{n-1}(y)\| - 4^n\varepsilon) - (4^{n+1} - 2)\varepsilon \\
&\geqslant 2(\|a_{n+1} - y\| - 2\cdot 4^n\varepsilon) - (4^{n+1} - 2)\varepsilon \\
&= 2\|a_{n+1} - y\| - 2(4^{n+1} - 1)\varepsilon,
\end{aligned}$$

so that (10) holds for all $n \geqslant 1$ by induction. (Here we have used in order: the definition of g_{n+1}, (4) as applied to g_{n-1} with $\delta = 4^n\varepsilon$, the fact that g_{n-1} is a

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$4^n\varepsilon$-isometry, the inductive hypothesis, the fact that g_{n-1} is a $4^n\varepsilon$-isometry once again, and finally the definition of a_{n+1} together with (4) as applied to g_{n-1}.) The bound (10) implies that $\|a_{n+1} - a_n\| = \|g_{n-1}(a_n) - a_n\| \geqslant 2\|a_n - a_{n-1}\| - 2 \cdot 4^{n-1}\varepsilon$, which by induction gives

$$\|a_n - a_{n-1}\| \geqslant 2^{n-2}\|a_2 - a_1\| - 4^{n-1}\varepsilon.$$

Together with (9) this means that, for $n \geqslant 2$, $\|a_2 - a_1\|$ is bounded above by $2^{2-n}(2d + 2 \cdot 4^n\varepsilon + 4^{n-1}\varepsilon)$ or, equivalently,

$$\|a_2 - a_1\| \leqslant 2(d2^{-n} + 18\varepsilon 2^n) \quad \text{for } n \geqslant 0. \tag{11}$$

We have

$$\begin{aligned}\|a_2 - a_1\| &= \|f(2p - F(q)) - q\| = \|f(2p - F(q)) - fF(q)\| \\ &\geqslant 2\|p - F(q)\| - \varepsilon \geqslant 2(\|f(p) - fF(q)\| - \varepsilon) - \varepsilon \\ &= 2\|f(p) - q\| - 3\varepsilon,\end{aligned}$$

so that by (11)

$$\|f(p) - q\| \leqslant d2^{-n} + 18\varepsilon 2^n + 2\varepsilon \quad \text{for } n \geqslant 0.$$

For the moment we assume that $d > 18\varepsilon$ and let t be such that $d2^{-t} = 18\varepsilon 2^t$; that is, $t = (\log 4)^{-1}\log(d/18\varepsilon) > 0$. If we let n be the greatest integer less than or equal to t, the above bound for $\|f(p) - q\|$ gives

$$\begin{aligned}\|f(p) - q\| &\leqslant 2d2^{-t} + 18\varepsilon 2^t + 2\varepsilon = 3d2^{-t} + 2\varepsilon \\ &= 3(18\varepsilon d)^{1/2} + 2\varepsilon \leqslant 10(\varepsilon\|x_0 - x_1\|)^{1/2} + 2\varepsilon,\end{aligned}$$

since $\|x_0 - x_1\| \geqslant \|y_0 - y_1\| - \varepsilon = 2d - \varepsilon \geqslant 35d/18$. On the other hand, if $d \leqslant 18\varepsilon$ (which covers the case $y_0 = y_1$ that was excluded at the beginning of the proof), then $\|y_0 - y_1\| \leqslant 36\varepsilon$ and so $\|x_0 - x_1\| \leqslant 37\varepsilon$. Thus $\|x_i - p\| \leqslant 19\varepsilon$ and, consequently, $\|y_i - f(p)\| \leqslant 20\varepsilon$ $(i = 0, 1)$. Since $q = (y_0 + y_1)/2$ we have $\|f(p) - q\| \leqslant 20\varepsilon$. Therefore in either case there holds $\|f(p) - q\| \leqslant 10(\varepsilon\|x_0 - x_1\|)^{1/2} + 20\varepsilon$.

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