Stability of Kronecker products of irreducible characters of the symmetric group

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Abstract

F. Murnaghan observed a long time ago that the computation of the decompositon of the Kronecker product $\chi^{(n-a,\lambda_2,\dots)} \otimes \chi^{(n-b,\mu_2,\dots)}$ of two irreducible characters of the symmetric group into irreducibles depends only on $\overline{\lambda} = (\lambda_2, \dots)$ and $\overline{\mu} = (\mu_2, \dots)$, *but not* on n. In this note we prove a similar result: given three partitions λ, μ, ν of n we obtain a lower bound on n, depending on $\overline{\lambda}$, $\overline{\mu}$, $\overline{\nu}$, for the stability of the multiplicity $c(\lambda, \mu, \nu)$ of χ^ν in $\chi^\lambda \otimes \chi^\mu$. Our proof is purely combinatorial. It uses a description of the $c(\lambda, \mu, \nu)$'s in terms of signed special rim hook tabloids and Littlewood-Richardson multitableaux.

1 Introduction.

Let χ^λ denote the irreducible complex character of the symmetric group $\mathsf{S}(n)$ corresponding to the partition λ. For any three partitions λ, μ, ν of n we denote by

$$c(\lambda, \mu, \nu) := \langle \chi^\lambda \otimes \chi^\mu, \chi^\nu \rangle \tag{1}$$

the multiplicity of χ^ν in the Kronecker product $\chi^\lambda \otimes \chi^\mu$.

F. Murnaghan observed in [6] that the computation of the decompositon of the Kronecker product $\chi^{(n-a,\lambda_2,\dots)} \otimes \chi^{(n-b,\mu_2,\dots)}$ into irreducibles depends only on $\overline{\lambda} = (\lambda_2, \dots)$ and $\overline{\mu} = (\mu_2, \dots)$, *but not* on n. He gave fifty eight formulas for decompositions of Kronecker products corresponding to the simplest choices of $\overline{\lambda}$ and $\overline{\mu}$. In fact, his formulas are *valid for arbitrary* n only if one follows some rules to restore and discard disordered partitions appearing in them, see comment on [6, p.762]. In this note we prove a similar result: given three partitions λ, μ, ν of n we obtain a lower bound on n, depending on λ, μ, ν, for the stability of the coefficients $c(\lambda, \mu, \nu)$.

More precisely. Let $\overline{\lambda} = (\lambda_2, \dots, \lambda_p)$, $\overline{\mu} = (\mu_2, \dots, \mu_q)$, $\overline{\nu} = (\nu_2, \dots, \nu_r)$ be partitions of positive integers a, b, c respectively. For each $n \geq a + \lambda_2$ we consider the partition of n, $\lambda(n) := (n - a, \lambda_2, \dots, \lambda_p)$. Similarly we define $\mu(n)$, and $\nu(n)$. Then we have

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Main Theorem. *If* $\overline{\nu}$ *has one part and* $\overline{\lambda} = \overline{\mu}$*, let* $m = \max\{\lambda_2 + a + c, 2c\}$*; otherwise let* $m = \max\{\lambda_2 + a + c - 1, \mu_2 + b + c - 1, 2c\}$*. Then for all* $n \geq m$

$$c\,(\lambda(n), \mu(n), \nu(n)) = c\,(\lambda(m), \mu(m), \nu(m))\,.$$

We note that m is not symmetric on λ, μ, ν, but $c(\lambda, \mu, \nu)$ is. Therefore we may have three different choices for m and we choose the smallest of the three. For example, consider partitions (1), $(2,1)$, $(2,1)$. If we set $\overline{\lambda} = (1)$, $\overline{\mu} = (2,1)$, $\overline{\nu} = (2,1)$, then $a = 1$, $b = 3$, $c = 3$ and $m = \max\{4, 7, 6\} = 7$. However, if we set $\overline{\lambda} = (2,1)$, $\overline{\mu} = (2,1)$, $\overline{\nu} = (1)$, then $a = 3$, $b = 3$, $c = 1$ and $m = \max\{6, 2\} = 6$, and we get a sharper lower bound. This is the best possible, since $c\,((3,2,1),(3,2,1),(5,1)) = 2$ and $c\,((2,2,1),(2,2,1),(4,1)) = 1$.

We also note that the theorem does not always produce the best lower bound. For the partitions $(3,2)$, $(2,2,1)$, $(2,2)$ the lower bound given by the theorem is 11. However, using SYMMETRICA [4], we obtained

$$\begin{aligned} c((4,3,2),(4,2,2,1),(5,2,2)) &= 12 \\ c((5,3,2),(5,2,2,1),(6,2,2)) &= 16 \\ c((6,3,2),(6,2,2,1),(7,2,2)) &= 16, \end{aligned}$$

which shows that the best lower bound is 10.

The rest of this note is devoted to the proof of the theorem.

2 Notation, definitions and known results

In this section we fix the notation and record some definitions and results that will be used in the proof of the theorem.

Let λ be a partition of n, in symbols $\lambda \vdash n$. We denote by $|\lambda|$ the sum of its parts, and by λ' its conjugate. We say that μ is contained in λ, in symbols $\mu \subseteq \lambda$, if $\mu_i \leq \lambda_i$ for all i. We use the notation $\lambda \trianglerighteq \mu$ to indicate that λ is greater or equal than μ in the dominance order. We denote by $\mathcal{P}(n)$ the *diagram lattice*, that is the set of partitions of n together with the dominance order, see [1, 3, 5, 7].

Let H be a subgroup of a group G. If χ is a character of H we denote by $\mathrm{Ind}_H^G(\chi)$ the induction character of χ. For any vector $\pi = (\pi_1, \ldots, \pi_t)$ of non-negative integers such that $\pi_1 + \cdots + \pi_t = n$, let $\mathsf{S}(\pi)$ denote a Young subgroup of $\mathsf{S}(n)$ corresponding to π.

We denote by χ^λ the irreducible character of $\mathsf{S}(n)$ associated to λ, and by $\phi^\lambda = \mathrm{Ind}_{\mathsf{S}(\lambda)}^{\mathsf{S}(n)}(1_\lambda)$ the permutation character associated to λ. They are related by the Young's rule

$$\phi^\mu = \sum_{\lambda \trianglerighteq \mu} K_{\lambda\mu}\, \chi^\lambda, \tag{2}$$

where $K_{\lambda\mu}$ is a Kostka number, that is, the number of semistandard tableaux of shape λ and content μ, see [3, 2.8.5], [7, §2.11].

We will deal with two kinds of products of characters: Let l, m be non-negative integers, and let $n = l + m$. Let χ_1 be a character of $\mathsf{S}(l)$, χ_2 be a character of $\mathsf{S}(m)$, then

(i) $\chi_1 \times \chi_2$ denotes the character of $\mathsf{S}(l) \times \mathsf{S}(m)$ given by $\chi_1 \times \chi_2(\sigma, \tau) = \chi_1(\sigma)\chi_2(\tau)$.

(ii) $\chi_1 \otimes \chi_2$ denotes, if $l = m$, the Kronecker product of χ_1 and χ_2, that is, the character of $\mathsf{S}(l)$ defined by $\chi_1 \otimes \chi_2(\sigma) = \chi_1(\sigma)\chi_2(\sigma)$.

If T is a tableau (a skew diagram filled with positive integers) there is a word $w(T)$ associated to T given by reading the numbers of T from right to left, in succesive rows, starting with the top row. Let $\pi = (\pi_1, \ldots, \pi_t)$ be a vector of positive integers such that $\pi_1 + \cdots + \pi_t = n$. Let $\rho(i) \vdash \pi_i$, $1 \leq i \leq t$. A sequence $T = (T_1, \ldots, T_t)$ of tableaux is called a Littlewood-Richardson *multitableau* of *shape* λ, *content* $(\rho(1), \ldots, \rho(t))$ and *type* π if

(i) There exists a sequence of partitions

$$0 = \lambda(0) \subset \lambda(1) \subset \cdots \subset \lambda(t) = \lambda$$

such that $|\lambda(i)/\lambda(i-1)| = \pi_i$ for all $1 \leq i \leq t$, and

(ii) for all $1 \leq i \leq t$, T_i is a semistandard tableau of shape $\lambda(i)/\lambda(i-1)$ and content $\rho(i)$ such that $w(T_i)$ is a lattice permutation, see [3, 2.8.13], [5, I.9], [7, §4.9].

For each partition λ of n let $c^{\lambda}_{(\rho(1),\ldots,\rho(t))}$ denote the number of Littlewood-Richardson multitableaux of shape λ and content $(\rho(1), \ldots, \rho(t))$. It follows by induction from the Littlewood-Richardson rule that

$$\mathrm{Ind}_{\mathsf{S}(\pi)}^{\mathsf{S}(n)} \left(\chi^{\rho(1)} \times \cdots \times \chi^{\rho(t)}\right) = \sum_{\lambda \vdash n} c^{\lambda}_{(\rho(1),\ldots,\rho(t))} \chi^{\lambda}.$$

Let

$$lr(\lambda, \mu; \pi) := \langle \chi^{\lambda} \otimes \chi^{\mu}, \phi^{\pi} \rangle, \tag{3}$$

then it follows from the Frobenius reciprocity theorem that

$$lr(\lambda, \mu; \pi) = \sum_{\rho(1) \vdash \pi_1, \ldots, \rho(t) \vdash \pi_t} c^{\lambda}_{(\rho(1),\ldots,\rho(t))}\, c^{\mu}_{(\rho(1),\ldots,\rho(t))}.$$

That is, $lr(\lambda, \mu; \pi)$ is the number of pairs (S, T) of Littlewood-Richardson multitableaux of shape (λ, μ), same content, and type π.

Let $K_n = (K_{\lambda\mu})$ be the Kostka matrix with rows and columns arranged in reverse lexicographical order, and let $K_n^{-1} = (K^{(-1)}_{\lambda\mu})$ denote its inverse, see [5, I.6.5]. Then it follows from the Young rule (2) that

$$\chi^{\nu} = \sum_{\pi \trianglerighteq \nu} K^{(-1)}_{\pi\nu} \phi^{\pi}. \tag{4}$$

Therefore from (1), (4) and (3) we obtain

2.1 Proposition

$$c(\lambda,\mu,\nu) = \sum_{\pi \trianglerighteq \nu} K^{(-1)}_{\pi\,\nu}\, lr(\lambda,\mu;\pi).$$

□

This formula gives, together with a result of Eğecioğlu and Remmel [2] (see Theorem 3.2), a combinatorial description of the numbers $c(\lambda,\mu,\nu)$. We will use it to get the stability of $c(\lambda,\mu,\nu)$ from the stability of $K^{(-1)}_{\pi\,\nu}$ and $lr(\lambda,\mu;\pi)$.

3 Proof of the main theorem

Let $\mathcal{P}(n)$ denote the diagram lattice, that is, the lattice of partitions of n ordered under the dominance order, see [1, 3, 5, 7]. For each partition ν of n, let I_ν denote the interval $\{\pi \vdash n \mid \nu \trianglelefteq \pi \trianglelefteq (n)\}$ in $\mathcal{P}(n)$.

3.1 Lemma. *Let $n \geq 2c$. Then the intervals $I_{\nu(n)}$ and $I_{\nu(2c)}$ are isomorphic as posets.*

Proof. For $\pi = (\pi_1,\ldots,\pi_t) \in I_{\nu(n)}$ we define $\widetilde{\pi} := (\pi_1 - (n-2c), \pi_2, \ldots, \pi_t)$. It follows from the inequality $\pi \trianglerighteq \nu(n)$ that $\widetilde{\pi}$ is in $I_{\nu(2c)}$. One can then easily verify that the map $\pi \mapsto \widetilde{\pi}$ is a poset isomorphism from $I_{\nu(n)}$ to $I_{\nu(2c)}$. □

In fact $2c$ is the best lower bound: Choose $\overline{\nu}$ be any partition of c with more than one part. Then $\nu(2c)$ and $\nu(2c-1)$ are well defined partitions, but $I_{\nu(2c)} \ncong I_{\nu(2c-1)}$, because the partition $(c,c) \in I_{\nu(2c)}$ has no corresponding partition in $I_{\nu(2c-1)}$.

Next we prove a stability property for the numbers $K^{(-1)}_{\lambda\,\mu}$. For this we use a combinatorial interpretation of these numbers due to Eğecioğlu and Remmel [2]. Recall that a *special rim hook tabloid* T of shape μ and type λ is a filling of the Ferrers' diagram of μ with rim hooks of sizes $\{\lambda_1,\ldots,\lambda_p\}$ such that each rim hook is *special*, that means, each rim hook has at least one box in the first column. The *sign* of a rim hook H is $(-1)^{\mathrm{ht}(H)-1}$, where ht(H) denotes, as usual, the height of the rim hook. And the sign of T is defined as the product of the signs of the rim hooks of T, see [2, Section 2], [5, Ex. I.6.4] for details. Then

3.2 Theorem. *(Eğecioğlu, Remmel [2])*

$$K^{(-1)}_{\lambda\,\mu} = \sum_{T} \mathrm{sign}(T),$$

where the sum is over all special rim hook tabloids of type λ and shape μ.

From this we get the following two corollaries

3.3 Corollary. *Let* $\overline{\nu} = (\nu_2, \ldots, \nu_r) \vdash c$, *and* $n \geq 2c$. *Then for all* $\alpha(n)$, $\beta(n)$ *in* $I_{\nu(n)}$ *one has*

$$K^{(-1)}_{\alpha(n)\,\beta(n)} = K^{(-1)}_{\alpha(2c)\,\beta(2c)}.$$

Proof. A sign preserving bijection between the set of special rim hook tabloids T of type $\alpha(2c)$ and shape $\beta(2c)$ and the set of special rim hook tabloids $\widehat{T}$ of type $\alpha(n)$ and shape $\beta(n)$ is established in the following way: Let H be the rim hook in T which contains the last box from the first row. Then H is of maximal length among the rim hooks in T. Let $\widehat{H}$ be the rim hook obtained from H by adding $n-2c$ boxes at the end of the first row, and let $\widehat{T}$ be obtained from T by substituting H by $\widehat{H}$. Since H is of maximal length, then $\widehat{T}$ is a rim hook tabloid of type $\alpha(n)$. Clearly it has shape $\beta(n)$ and $\mathrm{sign}(T) = \mathrm{sign}(\widehat{T})$. □

Another proof follows from [5, Ex. I.6.3].

3.4 Corollary. *Let* $\overline{\nu} = (\nu_1, \ldots, \nu_r)$, $\overline{\pi} = (\pi_2, \ldots, \pi_t) \vdash c$, *and suppose* $r > 2$. *Then*

$$K^{(-1)}_{\pi(2c)\,\nu(2c)} = K^{(-1)}_{\overline{\pi}\,\overline{\nu}}.$$

□

Since the sum of the entries of any column of the inverse Kostka matrix (with the obvious exception of the first one) is zero, then it follows

3.5 Corollary. *Let* $m = 2c$, *and suppose* $r > 2$. *Then*

$$\sum_{\pi(m) \trianglerighteq \nu(m),\ \pi(m)_1 = c} K^{(-1)}_{\pi(m)\,\nu(m)} = 0.$$

□

Let denote $\mathsf{LR}(\lambda(n), \mu(n); \nu(n))$ the set of pairs (S, T) of Littlewood-Richardson multitableaux of shape $(\lambda(n), \mu(n))$, same content and type $\nu(n)$.

3.6 Lemma. *Let* $m = \max\{\lambda_2 + a, \mu_2 + b, \nu_2 + c\}$. *Then for all* $n \geq m$ *there is an injective map*

$$\Phi : \mathsf{LR}(\lambda(m), \mu(m); \nu(m)) \longrightarrow \mathsf{LR}(\lambda(n), \mu(n); \nu(n)).$$

Proof. Let $(S, T) \in \mathsf{LR}(\lambda(m), \mu(m); \nu(m))$. Let $\widehat{S}$ be obtained from S by adding $n-m$ 1's at the end of the first row of S_1, and shifting $n-m$ places to the right the remaining 1's belonging to the tableaux $S_2, \ldots S_r$. Let $\widehat{T}$ be defined in a similar way. Then $(\widehat{S}, \widehat{T})$ belongs to $\mathsf{LR}(\lambda(n), \mu(n); \nu(n))$, and the map $\Phi(S, T) := (\widehat{S}, \widehat{T})$ is injective. □

3.7 Proposition. *Let* $m = \max\{\lambda_2 + a + c - 1, \mu_2 + b + c - 1, \nu_2 + c\}$ *if* $\overline{\lambda} \neq \overline{\mu}$, *and* $m = \max\{\lambda_2 + a + c, \nu_2 + c\}$ *if* $\overline{\lambda} = \overline{\mu}$. *Then for all* $n \geq m$

$$lr(\lambda(n), \mu(n); \nu(n)) = lr(\lambda(m), \mu(m); \nu(m)).$$

Proof. We show that under our hypothesis, we can define a map

$$\Psi : \mathsf{LR}(\lambda(n), \mu(n); \nu(n)) \longrightarrow \mathsf{LR}(\lambda(m), \mu(m); \nu(m))$$

inverse to Φ. Let (S, T) be in $\mathsf{LR}(\lambda(n), \mu(n); \nu(n))$, and let $(\rho(1), \ldots, \rho(r))$ be the common content of S and T. We define $\widetilde{\rho(1)} := (\rho(1)_1 - (n - m), \rho(1)_2, \ldots, \rho(1)_u)$, where u is the length of $\rho(1)$. Note that $\rho(1) \subseteq \lambda(n)$ and that $|\lambda(n)/\rho(1)| = c$. Then, if $\overline{\lambda} = \overline{\mu}$, we have that $\rho(1)_1 \geq \lambda(n)_1 - c = n - a - c \geq n - m + \lambda_2$. And, if $\overline{\lambda} \neq \overline{\mu}$, we have that $\rho(1)_1 \geq \lambda(n)_1 - (c - 1) = n - a - (c - 1) \geq n - m + \lambda_2$. Therefore, in both cases, $\rho(1)_1 - (n - m) \geq \lambda_2 \geq \rho(1)_2$, and $\widetilde{\rho(1)}$ is a partition of $m - c = \nu(m)_1$. Let $\widetilde{S}$ be obtained from S by deleting the first $(n - m)$ 1's in the first row and shifting to the left the remaining numbers $n - m$ places. In this way $\widetilde{S}$ is a multitableau of shape $\lambda(m)$, content $(\widetilde{\rho(1)}, \rho(2), \ldots, \rho(r))$ and type $\nu(m)$. Moreover, since $\rho(1)_1 - (n - m) \geq \lambda_2$, $\widetilde{S}$ is a Littlewood-Richardson multitableau. We define in a similar way a Littlewood-Richardson multitableau $\widetilde{T}$ of shape $\mu(m)$, same content as $\widetilde{S}$ and type $\nu(m)$. It is straightforward to check that the map $(S, T) \mapsto (\widetilde{S}, \widetilde{T})$ yields the inverse of Φ. □

3.8 Corollary. *Let* m *be defined as in Proposition 3.7. Let* $\pi(m) = (m - e, \pi_2, \ldots, \pi_t)$ *be in* $I_{\nu(m)}$. *Then for all* $n \geq m$

$$lr\left(\lambda(n), \mu(n); \pi(n)\right) = lr\left(\lambda(m), \mu(m); \pi(m)\right).$$

□

The main theorem now follows from Proposition 2.1, Corollaries 3.3 and 3.8, either if $\overline{\lambda} \neq \overline{\mu}$, or if $\overline{\lambda} = \overline{\mu}$ and $r = 2$. It remains to prove it in the case $\overline{\lambda} = \overline{\mu}$ and $r > 2$.

3.9 Lemma. *Let* $m = \max\{\lambda_2 + a + c - 1, 2c\}$, $\pi(m) = (m - c, \pi_2, \ldots, \pi_t)$ *be in* $I_{\nu(m)}$, *and suppose* $r > 2$. *Then for all* $n > m$

$$lr(\lambda(n), \lambda(n); \pi(n)) = lr(\lambda(m), \lambda(m); \pi(m)) + 1.$$

Proof. Let (S, T) be in $\mathsf{LR}(\lambda(n), \lambda(n); \pi(n))$, and let $(\rho(1), \ldots, \rho(t))$ be the common content of S and T. Then, as in the proof of Proposition 3.7, we have that $\rho(1)_1 \geq n - a - c$. If $\rho(1)_1 > n - a - c$, then $\rho(1)_1 \geq n - a - (c - 1) \geq n - m + \lambda_2$. Again, as in the proof of Proposition 3.7, there exists $(\widetilde{S}, \widetilde{T}) \in \mathsf{LR}(\lambda(m), \lambda(m); \pi(m))$, such that $\Phi(\widetilde{S}, \widetilde{T}) = (S, T)$. If $\rho(1)_1 = n - a - c$, then $\lambda(n)/\rho(1) = (c)$. In this situation, there is exactly one Littlewood-Richardson multitableau R of shape $\lambda(n)$ and type $\pi(n)$.

It has content $(\lambda(n)/(c),(\pi_2),\ldots,(\pi_t))$. Therefore the pair (R,R) is the only one in $\mathsf{LR}(\lambda(n),\lambda(n);\pi(n))$ which is not in the image of Φ. The claim follows. □

3.10 Corollary. *Let* $m=\max\{\lambda_2+a+c-1,2c\}$*, and suppose* $r>2$*. Then for all* $n>m$

$$c(\lambda(n),\lambda(n),\nu(n))=c(\lambda(m),\lambda(m),\nu(m)).$$

Proof. By Proposition 2.1 it is enough to prove

$$\sum_{\pi(n)\trianglerighteq\nu(n)} K^{(-1)}_{\pi(n)\,\nu(n)}\, lr(\lambda(n),\lambda(n);\pi(n)) = \sum_{\pi(m)\trianglerighteq\nu(m)} K^{(-1)}_{\pi(m)\,\nu(m)}\, lr(\lambda(m),\lambda(m);\pi(m)).$$

Note that if $\pi(n)=(n-e,\pi_2,\ldots,\pi_t)$ and $e<c$, then by Proposition 3.7

$$lr(\lambda(n),\lambda(n);\pi(n))=lr(\lambda(m),\lambda(m);\pi(m)),$$

and if $e=c$, then by Lemma 3.9

$$lr(\lambda(n),\lambda(n);\pi(n))=lr(\lambda(m),\lambda(m);\pi(m))+1.$$

The claim now follows from Corollaries 3.3 and 3.5. □

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