Stability of Multidimensional Shocks

Guy Métivier
IRMAR Université de Rennes I
35042 Rennes Cedex, France

April 2, 2003

Contents

This series of lecture is devoted to the study of shock waves for systems of multidimensional conservation laws. In sharp contrast with one dimensional problems, in higher space dimension there are no general existence theorem of solutions which allows discontinuities. Our goal is to study the existence and the stability of the simplest pattern of a single wave front Σ separating two states u^+ and u^- which depend smoothly on the space-time variables x. For example, our analysis applies to perturbations of planar shocks. They are special solutions given by constant states separated by a planar front. Given a multidimensional perturbation of the initial data or a small wave impinging on the front, we study the following stability problem. Is there a local solution with the same wave pattern? Similarly, a natural problem is to investigate the multidimensional stability of one dimensional shock fronts. However, the analysis applies to much more general situation and the main subject is the study of curved fronts.

The equations lead to a free boundary transmission problem, where the transmission equations are the Rankine-Hugoniot conditions. The analysis of this system was performed by A.Majda [Maj 1], [Maj 2] in the early eighties. Introducing the equation of the front as an unknown, and performing a change of variables, the problem is reduced to the construction of smooth solutions of a nonlinear boundary value problem. For such problems, a general theory can be developed. To explain the different steps, recall the construction of smooth local solutions of the Cauchy problem. One first linearizes the equations and study the well-posedness of the linearized problem. This question is studied in L^2 first and the answer is positive when the linearized equations are strictly hyperbolic or symmetric hyperbolic. Next, one shows that the linearized equation are well posed in Sobolev spaces H^s with s large enough. Finally, using an iterative scheme, one solves the nonlinear Cauchy problem.

For initial boundary value problems, one follows the same path. One studies first the well posedness of the linearized equations in L^2. This is the classical analysis made by O.Kreiss ([Kr], see also [Ch-Pi]) which originally applies to linear problem with C^∞ coefficients, but which can be extended to equations with H^s coefficients. The algebraic condition which replaces the hyperbolicity assumption used in the Cauchy problem, is the so-called uniform Lopatinski condition. It is obtained by freezing the coefficients and performing a plane wave analysis. When this condition is satisfied, the linearized equation are also well posed in H^s, and using an iterative scheme one solves the nonlinear equations. However, the proofs are much more technical than for the Cauchy problem. The uniform stability condition is satisfied in many physical examples such as Euler's equations of gas dynamics. However, there are interesting cases where it is not satisfied, for instance for multidimensional scalar conservation laws. This shows that the uniform stability condition, which is necessary for the validity of maximal estimates, is only sufficient for the existence of solutions of the nonlinear problem. In particular, this leaves open the question of existence of weakly stable shocks. The study of weak shocks is a first step in this direction.

• In section 1, we introduce the equations, the linearized equations and the constant coefficient frozen equations. A classical plane wave analysis, links the well posedness of constant coefficient boundary value problems to the nonexistence of exponentially growing modes. This leads to the uniform Lopatinski condition or Kreiss' condition. Applied to our context, this leads to the notion of uniformly stable shock fronts introduced by Majda.

• In section 2, we prove that when the uniform stability condition is satisfied, the solutions of the linearized equations satisfy a maximal L^2 estimate. Our proof is technically different from Majda's proof and we show that the constants which are present in the estimate depend only on the Lipschitz norm of the coefficient. This work was achieved in [Mo].

• Section 3 and 4 are devoted to the construction of solutions of the nonlinear boundary value problem, proving that the uniform stability condition implies the local existence of shock front solutions of the expected form. We recover Majda's results with several improvements. In particular, we show that the wave pattern persists as long as the solution remains Lipschitzean on both side of the front.

• In section 5, we indicate how the estimates break down when the strength of the shock tends to zero. However, weaker estimates, independent of the strength, are still valid. They are used in [Fr-Me] to prove the existence of weak shocks on domains independent of the strength.

1 The uniform stability condition

In [Maj 1], A.Majda introduces the notion of uniformly stable shock fronts. In this section we present the analysis which leads to this definition.

1.1 Piecewise continuous solutions

In space time dimension $1 + n$, consider a $N \times N$ system of conservation laws

$$\sum_{j=0}^{n} \partial_j \, f_j(u) \; = \; 0 \, . \tag{1.1.1}$$

The space time variables are denoted by $x = (x_0, \ldots, x_n)$. We also use the notations $t = x_0$, $x' = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. The flux functions f_j are C^∞. For smooth solutions, (1.1.1) read

$$\sum_{j=0}^{n} A_j(u) \partial_j u \; = \; 0 \, , \quad A_j(u) := f_j'(u) \, . \tag{1.1.2}$$

This system is assumed to be symmetric hyperbolic, meaning that

Assumption 1.1.1. *There is a smooth matrix $S(u)$ such that all the matrices $S(u)A_j(u)$ are symmetric and $S(u)A_0(u)$ is definite positive.*

This assumption is satisfied when there exists a smooth strictly convex entropy.

A piecewise smooth functions u, with discontinuities across manifold Σ, is a weak solution of (1.1) if and only if the equation holds on each side of Σ and the traces on Σ satisfy the *Rankine Hugoniot jump conditions*

$$\sum_{j=0}^{n} \nu_j \, [f_j(u)] \; = \; 0 \, , \tag{1.1.3}$$

where ν is the space time conormal direction to Σ.

Example 1. *Euler's equations of gas dynamics.*

An example which appears to be fundamental in the applications is the system of gas dynamics in Eulerian coordinates :

$$\left\{ \begin{array}{l} \partial_t \rho + \mathrm{div}(\rho v) = 0, \\ \partial_t(\rho v) + \mathrm{div}(\rho v \otimes v) + \mathrm{grad} p = 0, \\ \partial_t(\rho E) \; + \; \mathrm{div}(\rho E v + p v) \; = \; 0 \, , \end{array} \right. \tag{1.1.4}$$

where ρ is the density, v the velocity, p the pressure, $E := \frac{1}{2}|v|^2 + e$ the specific (i.e. per unit of mass) total energy and e the specific internal energy. The variables ρ, p and e are related through the equation of state

$$p = \Pi(\rho, e) \tag{1.1.5}$$

For a polytropic ideal gas the equation of state is given by

$$p = (\gamma - 1)\, \rho\, e\,, \quad \gamma \text{ constant}\,, \quad \gamma > 1\,. \tag{1.1.6}$$

Recall that the specific entropy S is defined by

$$de = TdS + \frac{p}{\rho^2}\, d\rho$$

where $T = T(\rho, e)$ is the temperature. Instead of (ρ, e) one often takes (p, S) or (ρ, S) as unknowns. In the latter case, the equation of state, reads

$$p = P(\rho, S)\,. \tag{1.1.7}$$

For example, for a polytropic ideal gas, $p = \rho^\gamma e^{cS}$ where c is a constant.

For smooth solutions (1.1.4) is equivalent to

$$\begin{cases} \partial_t \rho + v \cdot \nabla \rho + \rho\, \mathrm{div} v = 0\,, \\ \rho \partial_t v + \rho\, v \cdot \nabla v + \mathrm{grad} p = 0\,, \\ \partial_t S + v \cdot \nabla S = 0\,, \end{cases} \tag{1.1.8}$$

The *isentropic* Euler's equations are

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho v) = 0\,, \\ \partial_t(\rho v) + \mathrm{div}(\rho v \otimes v) + \mathrm{grad} p = 0\,. \end{cases} \tag{1.1.9}$$

In this case, the equation of state reduces to $p = P(\rho)$. In particular, when $P(\rho) = P(\rho, S_0)$, (ρ, v) is a *smooth* solution of (1.1.9) if and only if (ρ, v, S_0) is a smooth solution of (1.1.4) with constant entropy S_0. This is *false* for discontinuous solutions. The Rankine Hugoniot for (1.1.4) imply that $[S] \neq 0$ across discontinuities. Thus if (ρ, v) is a discontinuous weak solution of (1.1.9), (ρ, v, S_0) cannot satisfy the Rankine Hugoniot conditions for (1.1.4).

Example 2. *Planar discontinuities.*

The simplest example of piecewise continuous function is

$$u(x) = \begin{cases} u^- & \text{when} \quad \nu x < 0\,, \\ u^+ & \text{when} \quad \nu x > 0\,, \end{cases} \tag{1.1.10}$$

where u^+ and u^- in $\mathbb{R}^N$ and $\nu = (\nu_0, \dots, \nu_d) \in \mathbb{R}^{1+n} \backslash \{0\}$. Then u is a weak solution if and only if the Rankine Hugoniot conditions are satisfied.

We now recall Lax's analysis of Rankine-Hugoniot equations for small jumps (cf [Lax]). Introducing

$$A_j(u^+, u^-) := \int_0^1 A_j(u^- + s[u])\, ds \tag{1.1.11}$$

the equations (1.1.3) read

$$\Big(\sum_j \nu_j A_j(u^+, u^-)\Big)[u] \;=\; 0. \tag{1.1.12}$$

Thus the non trivial solutions ($[u] \neq 0$) are such that $-\nu_0$ is an eigenvalue and $[u]$ an eigenvector of $\sum_{j\geq 1} \nu_j A_0^{-1} A_j(u^+, u^-)$. Thus, for $[u]$ small, $-\nu_0$ must be close to an eigenvalue of

$$A(u, \nu') \;:=\; A_0^{-1}(u) \sum_{j=1}^{n} \nu_j A_j(u)\,. \tag{1.1.13}$$

Note that Assumption 1.1.1 implies that all the eigenvalues of $A(u, \nu')$ are real.

Consider the Rankine-Hugoniot equations for u^+ and u^- in a neighborhood of $\underline{u}$ and ν close to $\underline{\nu}$, assuming that $-\underline{\nu}_0$ is a *simple* eigenvalue of $A(\underline{u}, \underline{\nu}')$. This implies that for (u, ν') in a neighborhood of $(\underline{u}, \underline{\nu}')$ there is a unique simple eigenvalue $\lambda(u, \nu')$ of $A(u, \nu')$ such that $-\underline{\nu}_0 = \lambda(\underline{u}, \underline{\nu}')$. It depends smoothly on (u, ν') and one can choose a right eigenvector $r(u, \nu')$ which depends also smoothly on (u, ν'). Rotating the axis we can assume that $\underline{\nu}' = (0, \ldots, 0, 1)$ and by homogeneity, restrict our attention to solutions such that $\nu_n = 1$. In this case, the equation of the tangent space to the front is

$$x_n = \sigma t \,+ \sum_{j=1}^{n-1} \eta_j x_j \tag{1.1.14}$$

and the jump equations are

$$[f_n(u)] \;=\; \sigma[f_0(u)] \,+\, \sum_{j=0}^{n-1} \,\eta_j [f_j(u)]\,. \tag{1.1.15}$$

We look for solutions $u^{\pm}$ close to $\underline{u}$, η close to 0 and σ close to $\underline{\sigma} = -\nu_0$. The set of trivial solutions is $\mathcal{R}_0 := \{u^+ = u^-\}$.

Proposition 1.1.2. ([Lax]) *Assume that* λ *is a simple eigenvalue of* $A(\underline{u},\underline{\nu}')$ *and* $\underline{\sigma} = \lambda(\underline{u},\underline{\nu}')$ *where* $\underline{\nu}' = (0,\dots,0,1)$. *On a neighborhood of* $(\underline{u},\underline{u},\underline{\sigma},0)$, *the set of solutions of* (1.1.15) *is the union of* $\mathcal{R}_0$ *and a smooth manifold* $\mathcal{R}$ *of dimension* $N+n$. *Moreover, on a neighborhood of* $(\underline{u},0,0) \in \mathbb{R}^N \times \mathbb{R}^{n-1} \times \mathbb{R}$, *there are smooth functions* U *and* Σ *such that* $\mathcal{R}$ *is given by*

$$\begin{cases} u^+ = U(u^-,\eta,s) = u^- \ + s\, r^- + O(s^2)\ , \\ \sigma = \Sigma(u^-,\eta,s) \ = \ \lambda^- \ + \dfrac{1}{2}\, s\, r^- \cdot \nabla_u \lambda^- \ + \ O(s^2)\,, \end{cases} \tag{1.1.16}$$

where r^-, λ^- *and* $\nabla\lambda^-$ *are the functions evaluated at* $(u^-,\underline{\nu}')$.

1.2 Change of coordinates

An important difficulty in the construction of piecewise smooth solutions, is that the front Σ which carries the discontinuities is unknown. A classical idea in free boundary problems is to introduce the equation of the front as one of the unknowns and use a change of variables to reduce the problem to a fixed domain.

Rotating the axes if necessary, suppose that Σ is the manifold of equation $x_n = \varphi(t,y)$. Suppose that Φ satisfies

$$\Phi(t,y,0) \ = \ \varphi(t,y)\,, \quad \partial_n \Phi > 0\,. \tag{1.2.1}$$

Consider the following change of variables

$$\tilde{x} \ \mapsto \ x := \ (\tilde{x}_0,\dots,\tilde{x}_{n-1},\Phi(\tilde{x}))\,. \tag{1.2.2}$$

It maps $\{\tilde{x}_n = 0\}$ onto Σ and the half spaces $\{\pm\tilde{x}_n > 0\}$ to the two sides $\{\pm(x_n - \varphi(t,y)) > 0\}$ of Σ.

Suppose that u is a piecewise C^1 function with jumps only on Σ. Introduce

$$\tilde{u}(\tilde{x}) \ = \ u(x) \tag{1.2.3}$$

and denote by $\tilde{u}^\pm$ the restriction of $\tilde{u}$ to the half space $\{\pm\tilde{x}_n > 0\}$. Then u is a weak solution of (1.1.1) if and only if $\tilde{u}$ satisfies

$$\begin{cases} L(\tilde{u}^\pm,\nabla\Phi)\tilde{u} = 0\,, & \text{on } \pm\tilde{x}_n > 0\,, \\ B(\tilde{u}^+,\tilde{u}^-,\nabla\varphi) = 0\,, & \text{on } \tilde{x}_n = 0 \end{cases} \tag{1.2.4}$$

where

$$
\begin{aligned}
&L(u,\nabla\Phi)v := \sum_{j=0}^{n-1} A_j(u)\partial_j v + \widetilde{A}_n(u,\nabla\Phi)\partial_n v\,,\\
&\widetilde{A}_n(u,\xi) := \frac{1}{\xi_n}\Big(A_n(u) - \sum_{j=0}^{n-1}\xi_j A_j(u)\Big) := \frac{1}{\xi_n}A_n^\sharp(u,\xi')\,,\\
&B(u^+,u^-,\nabla\varphi) := \sum_{j=0}^{n-1}[f_j(u)]\partial_j\varphi - [f_n(u)]
\end{aligned}
\tag{1.2.6}
$$

Note that $\nabla\Phi$ is an $n+1$ dimensional vector while $\nabla\varphi$ is n dimensional.

Remark. The equations (1.2.4) *are not sufficient to determine* the unknowns $\tilde u$ and Φ because there are no equations for Φ in $\{\pm\tilde x_n > 0\}$. This reflects that the change of variables (1.2.2) is only required to map Σ to $\{\tilde x_n = 0\}$ and is arbitrary outside Σ. To obtain a well posed problem there are two strategies :

1) add equations for Φ in the interior $\tilde x_n \neq 0$,

2) make an explicit link between Φ and its trace $\varphi := \Phi_{|\tilde x_n=0}$. For instance, in [Maj 1, 2], the choice is

$$
\Phi(\tilde x) := \tilde x_n + \varphi(\tilde x_0,\dots,\tilde x_{n-1})\,. \tag{1.2.7}
$$

Equivalently, this means that the front Σ is given by $x_n = \varphi(x_0,\dots,x_{n-1})$.

We refer to §4 for a discussion of the different possibilities.

Example. Consider a piecewise constant solution u (1.1.10), with jump on $\Sigma = \{\nu\cdot x = 0\}$, where $\nu_n \neq 0$. The change of variables (1.2.2) with $\Phi(\tilde x) = \xi\cdot\tilde x$ maps $\{\tilde x_n = 0\}$ to Σ if and only if $\xi' = -\nu'/\nu_n$. This yields solutions of (1.2.4) :

$$
\tilde u = \begin{cases} u^- & \text{for } \tilde x_n < 0 \\ u^+ & \text{for } \tilde x_n > 0 \end{cases}\quad,\qquad \Phi(\tilde x) = \xi\cdot\tilde x \tag{1.2.8}
$$

Only the boundary condition has to be checked and it is equivalent to (1.1.3).

In particular, when the equation of the front Σ is written in the form (1.1.14) i.e. when $\Sigma = \{x_n = \sigma t + \eta\cdot y\}$ with $y := (x_1,\dots,x_{n-1})$, the solutions (1.1.10) of the original equations correspond to solutions

$$
\tilde u = \begin{cases} u^- & \text{for } \tilde x_n < 0 \\ u^+ & \text{for } \tilde x_n > 0 \end{cases}\quad,\qquad \Phi(\tilde x) = \tilde x_n + \sigma t + \eta\cdot y \tag{1.2.9}
$$

of (1.2.4).

1.3 The linearized equations

To analyze the equations (1.2.4) a natural step is to study the structure of the linearized equations. Consider a family $\tilde{u}_s = \tilde{u} + sv$, $\Phi_s = \Phi + s\Psi$. The linearized operators are given by

$$
\tag{1.3.1}
\begin{cases}
L'(\tilde{u}, \Phi)(v, \Psi) \, := \, \dfrac{d}{ds} L(\tilde{u}_s, \Phi_s)\tilde{u}_s{}_{|s=0} \, , \\[2mm]
B'(\tilde{u}^+, u^-, \varphi)(v^+, v^-, \psi) \, := \, \dfrac{d}{ds} B(\tilde{u}_s^+, \tilde{u}_s^-, \nabla\varphi_s)_{|s=0}
\end{cases}
$$

We have denoted by φ_s [resp. ψ] the trace of Φ_s [resp. Ψ] on $\{x_n = 0\}$.

In the computation of these operators, one has to take derivatives of the coefficients with respect to u. So, introduce the zero-th order operator

$$
\tag{1.3.2}
E(u, \Phi)v := \sum_{j=1}^{n-1} v \cdot \nabla_u A_j(u) \partial_j u \, + \, v \cdot \nabla_u \widetilde{A}_n(u, \nabla\Phi) \, \partial_n u \, .
$$

Introduce next the following notations :

$$
\tag{1.3.3}
\begin{aligned}
& b(u^+, u^-) \cdot \nabla\varphi \, := \, \sum_{j=0}^{n-1} [f_j(u)] \partial_j \varphi \, , \\
& M(u^+, u^-, \nabla\varphi)(v^+, v^-) := A_n^\sharp(u^-, \nabla\varphi)v^- - A_n^\sharp(u^+, \nabla\varphi)v^+ \, .
\end{aligned}
$$

where $A_n^\sharp(u, \xi')$ is defined at (1.2.6).

Proposition 1.3.1. *Introducing* $V := v - \dfrac{\partial_n u}{\partial_n \Phi} \, \Psi$, *one has*

(1.3.4)

$$
\begin{aligned}
& L'(u, \Phi)(v, \Psi) \, = \, (L(u, \Phi) + E(u, \Phi)) \, V \, + \, \frac{\Psi}{\partial_n \Phi} \partial_n (L(u, \Phi)u) \, , \\
& B'(u^+, u^-, \varphi)(v^+, v^-, \psi) \, = \, b(u^+, u^-)\nabla\psi + M(u^+, u^-, \nabla\varphi)(v^+, v^-) \, .
\end{aligned}
$$

Proof. The second equation is immediate. The first identity is obtained by direct computations (see [Al]). It is better understood if one goes back to the original coordinates. Consider a family $(\tilde{u}_s, \Phi_s)$ as indicated before (1.3.1). Introduce the function u_s which corresponds to $\tilde{u}_s$ through the change of variables Φ_s. It satisfies

$$
\tilde{u}_s(\tilde{x}) = u_s(\tilde{x}', \Phi_x(\tilde{x})) = u_s\big(\tilde{x}', \Phi(\tilde{x}) + s\Psi(\tilde{x})\big) \, ,
$$

where $\tilde{x}' := (x_0, \ldots, x_{n-1})$. With $\dot{u} := \frac{d}{ds} u_{s|s=0}$, one has

$$v(\tilde{x}) \;=\; \dot{u}(\tilde{x}', \Phi(\tilde{x})) \;+\; \Psi(\tilde{x})\, \partial_n u(\tilde{x}', \Phi(\tilde{x})) \;=\dot{u}(\tilde{x}', \Phi(\tilde{x})) \;+\; \frac{\Psi(\tilde{x})}{\partial_n \Phi(\tilde{x})}\, \partial_n \tilde{u}(\tilde{x})\,.$$

Thus

$$V(\tilde{x}) = \dot{u}(\tilde{x}', \Phi(\tilde{x}))\,. \tag{1.3.5}$$

Similarly, introduce $\tilde{f} := L(\tilde{u}, \Phi)\tilde{u}$, $\tilde{f}_s := L(\tilde{u}_s, \Phi_s)\tilde{u}_s$, $f = L(u)u$ and $f_s = L(u_s)u_s$ where $L(u) := \sum A_j(u)\partial_j$. Then

$$\tilde{f}_s(\tilde{x}) = f_s\big(\tilde{x}', \Phi(\tilde{x}) + s\Psi(\tilde{x})\big)$$

and

$$L'(\tilde{u}, \Phi)(v, \Psi)\tilde{()} \;= \dot{f}(\tilde{x}', \Phi(\tilde{x})) \;-\; \frac{\Psi(\tilde{x})}{\partial_n \Phi(\tilde{x})}\, \partial_n \tilde{f}(\tilde{x}).$$

In the original variable, one has

$$\dot{f} = L(u)\dot{u} \;+\; \sum_j (\nabla_u A_j(u) \cdot \dot{u})\partial_j u$$

and with (1.3.5) this implies that

$$\dot{f}(\tilde{x}', \Phi(\tilde{x})) \;=\; \{L(\tilde{u}, \Phi) + E(\tilde{u}, \Phi)\})V(\tilde{x})\,.$$

The proposition follows. □

The linearized equations of (1.2.4) at $(u^\pm, \Phi)$ (we drop the $\sim$) are

$$\begin{cases} L'(u^\pm, \Phi)(v^\pm, \Psi) \;=\; f^\pm & \text{on } \pm\tilde{x}_n > 0\,, \\ B'(u^+, u^-, \varphi)(v^+, v^-, \psi) \;=\; g & \text{on } \tilde{x}_n = 0\,. \end{cases} \tag{1.3.6}$$

This is a first order boundary value problem in $(V^\pm, \Psi)$.

The principal part of the linearized equations is obtained dropping out the zero-th order terms in $(V^\pm, \Psi)$ in the interior equations, and the zero-th order terms in ψ in the boundary condition. It reads

$$\begin{cases} L(u^\pm, \Phi)V^\pm, =\; f^\pm & \text{on } \pm\tilde{x}_n > 0\,, \\ B'(u^+, u^-, \varphi)(V^+, V^-, \psi) \;=\; g & \text{on } \tilde{x}_n = 0\,. \end{cases} \tag{1.3.7}$$

Example. *The linearized equations around a planar discontinuity* Consider a piecewise constant solution (1.2.8). The linearized equations read

$$\left\{\begin{array}{ll} L(u^\pm,\xi)v^\pm = f^\pm\,, & \text{on } \pm\tilde{x}_n > 0\,, \\ b(u^+,u^-)\nabla\varphi + M(u^+,u^-,\xi')(v^+,v^-) = g\,, & \text{on } \tilde{x}_n = 0\,. \end{array}\right. \tag{1.3.8}$$

The unknowns are $v^\pm$ on $\pm\tilde{x}_n > 0$ and φ on $\tilde{x}_n = 0$. This reminds that the linearized equations determine the first order variation of u and only the trace of Φ.

Remark. The equations (1.3.8) have constant coefficients. They serve as a model for the analysis of (1.3.7). Indeed, freezing the coefficients of (1.3.7) at $\tilde{x}_0$ yields (1.3.7) with $u^\pm = u^\pm(x_0)$ and $\xi = \nabla\Phi(x_0)$.

1.4 The uniform Lopatinski condition

The transmission problems (1.3.6) or (1.3.7) are changed into conventional boundary value problems in a half-space, through changing x_n to $-x_n$ in the equation for V^-. With the notations $b_j = [f_j(u)]$, $A_j^\pm = A_j(u^\pm)$ for $j \in \{0,\dots,n-1\}$, $\widetilde{A}_n^\pm = \widetilde{A}_n(u^\pm,\nabla\Phi)$, $M^\pm = \partial_n\Phi\,\widetilde{A}_n^\pm = A_n^\sharp(u^\pm,\nabla\varphi)$ and

$$\mathcal{A}_j := \begin{bmatrix} A_j^+ & 0 \\ 0 & A_j^- \end{bmatrix}, \quad \mathcal{A}_n := \begin{bmatrix} \widetilde{A}_n^+ & 0 \\ 0 & -\widetilde{A}_n^- \end{bmatrix}, \tag{1.4.1}$$

$$v := \begin{bmatrix} v^+ \\ v^- \end{bmatrix}, \quad f := \begin{bmatrix} f^+ \\ f^- \end{bmatrix}, \quad Mv := M^-v^- \,-\, M^+v^+ \tag{1.4.2}$$

the problem (1.3.8) can be written

$$\left\{\begin{array}{ll} \mathcal{L}v := \displaystyle\sum_{j=0}^{n} \mathcal{A}_j\partial_j v \,=\, f\,, & \text{on } x_n > 0\,, \\ b\cdot\nabla\psi \,+\, M\,v \,=\, g\,, & \text{on } x_n = 0\,. \end{array}\right. \tag{1.4.3}$$

The coefficients $\mathcal{A}_j, b_j$ and M depend on $(u^\pm,\nabla\Phi)$.

Our goal is to study the mixed initial boundary value problem for (1.4.3). The classical analysis of O.Kreiss ([Kr]) concerns equations similar to (1.4.3) with boundary conditions $Bv = g$. Thus the novelty is to follow the influence of the term $b\nabla\varphi$ in the boundary conditions.

Consider first **the constant coefficient case** (1.3.8), or equivalently, freeze the values of $(u^+, u^-, \nabla\Phi)$ at a given point. We perform a **plane wave** analysis to motivate the introduction of Lopatinski conditions.

Suppose that f and g are given and vanish in the past $\{t < 0\}$. If the initial boundary value problem for (1.4.3) is well posed (in any reasonable sense), there is a unique solution (v, ψ) which vanishes also in $\{t < 0\}$. If f, g, v and ψ grow at most like $e^{\gamma_0 t}$ as $t \to +\infty$, one can perform a Fourier-Laplace transformation of the equations. Introduce

$$\widehat{v}(\hat\tau, \eta, x_n) = \int e^{-i\hat\tau t + \eta y}\, v(t, y, x_n) dt\, dy\,. \tag{1.4.4}$$

It is well defined for $(\hat\tau, \eta) \in \mathbb{C} \times \mathbb{R}^{n-1}$ and $\operatorname{Im} \hat\tau < -\gamma_0$. Equivalently, $\widehat{v}(\hat\tau, \eta)$ is the usual Fourier transform of $e^{-\gamma t} v$ evaluated at $(\tau, \eta) \in \mathbb{R}^n$ when $\hat\tau = \tau - i\gamma$. In particular, the Laplace-Fourier transform (1.4.4) is well defined for $\operatorname{Im} \hat\tau \leq -\gamma_0$ when $v \in e^{\gamma_0 t} L^2$. The equation (1.4.3) it then equivalent to

$$\begin{cases} \mathcal{A}_n \partial_n \widehat{v} \,+\, i\mathcal{P}(\hat\tau, \eta)\widehat{v} \,=\, \widehat{f}\,, & \qquad x_n > 0\,, \\ ib(\hat\tau, \eta)\widehat{\psi} + M\widehat{v}_{|x_n = 0} = \widehat{g}\,, \end{cases} \tag{1.4.5}$$

with

$$\mathcal{P}(\hat\tau, \eta) := \hat\tau \mathcal{A}_0 + \sum_{j=1}^{n-1} \eta_j \mathcal{A}_j\,, \quad b(\hat\tau, \eta) := \hat\tau b_0 + \sum_{j=1}^{n-1} \eta_j b_j\,. \tag{1.4.6}$$

To solve (1.4.3) in spaces $e^{\gamma t} L^2$, $\gamma \geq \gamma_0$, the idea is to require that (1.4.5) is well posed for $\operatorname{Im} \hat\tau < -\gamma_0$. Because $\mathcal{P}$ and b are homogeneous functions of $(\hat\tau, \eta)$, the restriction on $\operatorname{Im} \hat\tau$ reduces to $\operatorname{Im} \hat t < 0$. We refer to [Kr] [He] for the precise statement of necessary conditions.

Next, we study (1.4.5) when $\widehat{f} = 0$. The solutions are sums of functions of the form $h(x_n) e^{i\xi_n x_n}$, with

$$\det\Big(\xi_n \mathcal{A}_n \,+ \mathcal{P}(\hat\tau, \eta)\Big) \,=0\,. \tag{1.4.7}$$

and h a polynomial of degree depending on the order of ξ_n as a root of (1.4.7). Note that the hyperbolicity Assumption 1.1.1 implies that when $\operatorname{Im} \hat\tau < 0$, the equation (1.4.7) has no real roots ξ_n. Introduce $\mathbb{E}(\xi_n, \hat\tau, \eta) \subset \mathbb{C}^{2N}$ the space of the boundary values $h(0)$ for all the solutions $h(x_n) e^{i\xi_n x_n}$. It is a generalized eigenspace. Introduce the space of boundary values of L^2 *integrable* solutions

$$\mathbb{E}_+(\hat\tau, \eta) \,=\, \bigoplus_{\operatorname{Im} \xi_n > 0} \mathbb{E}(\xi_n, \hat\tau, \eta)\,. \tag{1.4.8}$$

Therefore, for a fixed $(\hat\tau,\eta)$ with $\operatorname{Im}\hat t<0$, the equation (1.4.5) with $\widehat f=0$, reduces to the equation

$$(1.4.9) \qquad (\ell,h)\in\mathbb{C}\times E_+(\hat\tau,\eta)\,,\quad ib(\hat\tau,\eta)\ell+Mh=\widehat g(\hat\tau,\eta)\,.$$

The basic requirement for stability, is that for all fixed $(\hat\tau,\eta)$ with $\gamma>0$, the equations (1.4.9) has a unique solution. The uniform Lopatinski condition is much stronger (see [Kr], [Ch-Pi]). The following definition was introduced by A.Majda ([Maj 1]).

Definition 1.4.1. *The problem* (1.4.5) *is weakly stable when*

i) The matrix $\mathcal{A}_n$ is invertible,

ii) For all $(\hat\tau,\eta)\in\mathbb{C}\times\mathbb{R}^{n-1}$ with $\operatorname{Im}\hat\tau=-\gamma<0$, the operator $\mathcal{B}_+(\hat\tau,\eta)\,:\,(\ell,h)\mapsto ib(\hat\tau,\eta)\ell+Mh$ from $\mathbb{C}\times\mathbb{E}_+(\hat\tau,\eta)$ to $\mathbb{C}^N$ is an isomorphism.

It is uniformly stable when the mappings $(\mathcal{B}_+(\hat\tau,\eta))^{-1}$ are uniformly bounded for $|\hat\tau|^2+|\eta|^2=1$.

Definition 1.4.2. *The linearized equations* (1.3.7) *are weakly stable [resp. uniformly stable] on $\Omega\subset\mathbb{R}^{n+1}$ if for all $\underline{x}\in\Omega$, the problem* (1.4.5) *obtained by freezing the coefficients at $\underline{x}$ is stable [resp uniformly stable].*

Remark 1. The invertibility condition i) rules out characteristic shocks and contact discontinuities. It is related to the validity of Lax's shock conditions as we show in the next section.

Remark 2. The definition of $\mathbb{E}_+(\hat\tau,\eta)$ extends by continuity to $\gamma=0$ provided that $(\tau,\eta)\neq 0$. Thus, the uniform stability condition is equivalent to

iii) for all $(\hat\tau,\eta)\in\mathbb{C}\times\mathbb{R}^{n-1}$ with $\operatorname{Im}\hat\tau\leq 0$ and $(\hat\tau,\eta)\neq 0$, $\mathcal{B}_+(\hat\tau,\eta)$ is an isomorphism from $\mathbb{C}\times\mathbb{E}(\hat\tau,\eta)$ to $\mathbb{C}^N$.

This holds if and only if

iv) there is a constant $c>0$, such that for all $(\hat\tau,\eta)\in\mathbb{C}\times\mathbb{R}^{n-1}$ with $\operatorname{Im}\hat\tau<0$ and $|\hat\tau|^2+|\eta|^2=1$, one has $\dim\mathbb{E}_+(\hat\tau,\eta)=N-1$ and

$$(1.4.10) \qquad \forall(k,h)\in\mathbb{C}\times\mathbb{E}(\hat\tau,\eta)\,,\quad |ib(\hat\tau,\eta)k+Mh|\,\geq\,c(|k|+|h|)\,.$$

Therefore, if the linearized equations are uniformly stable at $\underline{x}$, they remain uniformly stable for x in a neighborhood of $\underline{x}$.

Proposition 1.4.3. *Suppose that the problem* (1.4.5) *is weakly stable. Then, for all* $(\hat\tau,\eta)\in\mathbb{C}\times\mathbb{R}^{n-1}$ *with* $(\hat\tau,\eta)\neq 0$ *and* $\operatorname{Im}\hat\tau<0$ *one has* $\dim\mathbb{E}_+(\hat\tau,\eta) = N-1$ *and* $b(\hat\tau,\eta)\neq 0$.

When the problem is uniformly stable, then $b(\tau,\eta)\neq 0$ *for all* $(\tau,\eta)\in\mathbb{R}^n\backslash\{0\}$ *and the vectors* b_j *for* $j\in\{0,\dots,n-1\}$ *are linearly independent.*

Proof. For the mapping $\mathcal{B}_+(\hat\tau,\eta)$ to be one to one it is necessary that $b(\hat\tau,\eta)\neq 0$ and $\dim\mathbb{E}_+(\hat\tau,\eta)\leq N-1$. In this case, it is onto if and only if $\dim\mathbb{E}_+(\hat\tau,\eta) = N-1$ and

$$\mathbb{C}^N = \mathbb{C}b(\hat\tau,\eta)\oplus M\mathbb{E}_+(\hat\tau,\eta)\,. \tag{1.4.11}$$

In particular, the solution of $\mathcal{B}_+(\hat\tau,\eta)^{-1}(k,h) = i\,b(\hat\tau,\eta)$ is $k=1, h=0$. Therefore, using that b is homogeneous, the uniform stability condition implies that there is a constant $c>0$ such that

$$\forall(\tau,\eta,\gamma)\in\mathbb{R}^n\times]0,\infty[\,,\quad |b(\tau-i\gamma,\eta)|\geq c\,(|\tau|+|\eta|+\gamma)\,. \tag{1.4.12}$$

This extends to $\gamma=0$ and proves that the vectors $b_j\in\mathbb{R}^N$ are linearly independent. $\square$

Remark. Proposition 1.4.3 shows that the uniform stability condition requires $n\leq N$. In particular, multidimensional shocks for a single conservation law are never uniformly stable. This indicates that the uniform stability condition is sufficient but not necessary for the existence of multidimensional shocks.

1.5 Uniformly stable shocks

Definition 1.5.1. *Consider a piecewise* C^1 *weak solution of* (1.1.1) *with jumps only on a* C^1 *manifold* Σ *of equation* $x_n=\varphi(t,y)$. *It is a weakly stable shock [resp. uniformly stable shock] at* $\underline{x}\in\Sigma$, *if the linearized equations* (1.3.7) *with frozen coefficients at* $\underline{\widetilde{x}}$ *is stable [resp. uniformly stable].*

First, we show that the stability conditions imply Lax's shock conditions. The hyperbolicity assumption implies that for all $\nu'\in\mathbb{R}^n$ the matrix $A(u,\nu')$ defined in (1.1.13) has only real eigenvalues $\lambda_1(u,\nu')\leq\cdots\leq\lambda_n(u,\nu')$. Consider a discontinuity at $\underline{x}$, with front $\Sigma:=\{\nu'\cdot(x'-\underline{x}')=\sigma(t-\underline{t})\}$. Let $u^\pm$ denote the values of u at $\underline{x}$ from each side of Σ. $\underline{x}$. Lax's shock conditions are satisfied when there exists an integer $k\in\{1,\dots,N\}$ such that

$$\lambda_{k-1}(u^-,\xi')\,<\sigma\,<\,\lambda_k(u^-,\xi')\,,\quad \lambda_k(u^+,\xi')\,<\sigma\,<\,\lambda_{k+1}(u^-,\xi')\,. \tag{1.5.1}$$

When $k=1$ [resp $k=N$] the first inequality on the left [resp the last on the right] is ignored.

Proposition 1.5.2. *A weakly stable shock in the sense of Definition* 1.5.1 *satisfies Lax's shock conditions.*

Proof. In the new variables $\underline{x}$ corresponds to $\underline{\tilde{x}}$ with $\underline{\tilde{x}}_n = 0$. The equation of the front Σ is $x_n = \Phi(x_0, \dots, x_{n-1}, 0)$ and the tangent plane at $\underline{x}$ is $\xi' \cdot x' = \sigma t$ with $\sigma = \partial_t \Phi(\underline{\tilde{x}})$, $\xi' = (-\partial_y \Phi(\underline{\tilde{x}}), 1)$. Introduce $\kappa := \partial_n \Phi(\underline{\tilde{x}})$. Then, at $\underline{\tilde{x}}$,

$$\begin{aligned}\widetilde{A}_n^{\pm} := \widetilde{A}_n(u^{\pm}(\underline{\tilde{x}}), \nabla\Phi(\underline{\tilde{x}})) &= \frac{1}{\kappa}\Big(\sum_{j=1}^{n} \xi_j A_j(u^{\pm}) \,-\, \sigma A_0(u^{\pm})\Big) \\ &= \frac{1}{\kappa} A_0(u^{\pm})\big(A(u^{\pm}, \xi') - \sigma Id\big)\,.\end{aligned} \tag{1.5.2}$$

In particular, the eigenvalues of $D^{\pm} := A_0^{-1}(u^{\pm})\,\widetilde{A}_n(u^{\pm}, \nabla\Phi)$ are

$$\mu_k^{\pm} = \frac{1}{\kappa}\left(\lambda_k(u^{\pm}, \xi') \,-\, \sigma\right). \tag{1.5.3}$$

Note that Assumption 1.1.1 implies that the eigenvalues of $D^{\pm}$ are semi-simple and therefore $D^{\pm}$ can be diagonalized.

Apply Definition 1.4.1 with $\eta = 0$ and $\hat{\tau} = \tau - i\gamma \neq 0$. The interior equation (1.4.5) with $f = 0$ reduces to

$$\begin{cases} D^{+}\partial_n \hat{v}^{+} \,+ i\,\hat{\tau}\hat{v}^{+} \,=\, 0\,, \\ D^{-}\partial_n \hat{v}^{-} \,-\, i\,\hat{\tau}\hat{v}^{-} \,=\, 0\,. \end{cases} \tag{1.5.4}$$

The stability condition requires that $\mathcal{A}_n$ is invertible, thus 0 is not an eigenvalue of $D^{\pm}$. Therefore, there are integers k_+ and k_- in $\{0, \dots, N\}$ such that

$$\mu_{k_{\pm}}^{\pm} < 0 < \mu_{1+k_{\pm}}^{\pm}\,. \tag{1.5.5}$$

The solutions of (1.5.4) are

$$\hat{v}^{\pm}(x_n) = \sum_k h_k e^{i x_n \theta_k^{\pm}}\,, \quad \theta_k^{\pm} \,=\, \mp\frac{\hat{\tau}}{\mu_k^{\pm}}\,, \quad h_k \in \ker(D^{\pm} - \mu_k^{\pm})\,.$$

The exponentials are bounded when $\operatorname{Im}\theta_k^{\pm} = \pm\gamma/\mu_k^{\pm} > 0$. Thus introducing the notation

$$E_+ = \bigoplus_{k > k_+} \ker(D^{+} - \mu_k^{+})\,, \quad E_- = \bigoplus_{k \leq k_-} \ker(D^{-} - \mu_k^{-}) \tag{1.5.6}$$

one has

$$
\mathbb{E}_+(\hat\tau, 0) \ = \{(h^+, h^-) \,|\, h^+ \in E_+ \,, h^- \in E_-\}. \tag{1.5.7}
$$

Proposition 1.4.3 implies that $\dim \mathbb{E}_+(\hat\tau, 0) = N - 1$. With (1.5.6) and (1.5.7) this shows that $k_+ = k_- + 1$. Therefore, (1.5.5) and (1.5.3) imply that the shock conditions (1.5.1) are satisfied with $k = k_+$.

□

Proposition 1.5.3. *In space dimension one, a shock is uniformly stable if and only if the Lax shock conditions are satisfied and, with notations as in* (1.5.6)*,* $\mathbb{C}^N$ *is the direct sum* $\mathbb{C}b_0 \oplus A_0^+ E_+ \oplus A_0^- E_-$*.*

Proof. We use the notations introduced in the proof of the previous proposition. In space dimension one, (1.5.7) shows that the space $\mathbb{E}_+(\hat\tau)$ is independent of $\hat\tau$. Moreover, the uniform stability condition holds if and only if for $|\hat\tau| = 1$ and $\mathrm{Im}\,\hat\tau < 0$, the mapping

$$
(\ell, h^+, h^-) \ \mapsto \ i\hat\tau\ell b_0 \, + \, \kappa\big(\widetilde{A}_n^- h^- - \widetilde{A}_n^+ h^+\big)
$$

is an isomorphism from $\mathbb{C} \times E_+ \times E_-$ to $\mathbb{C}^N$ with uniformly bounded inverse. Since $D^\pm$ is an isomorphism from $E^\pm$ into itself, $\kappa\widetilde{A}_n^\pm = \kappa A_0^\pm D^\pm$ is an isomorphism from $E_\pm$ onto $A_0^\pm E_\pm$. The proposition follows.

□

Example 1. *In space dimension one, entropic weak shock associated to genuinely nonlinear eigenvalues are uniformly stable.*

Denote by $\lambda_j(u)$ the eigenvalues of $A_0^{-1} A_1(u)$. Suppose that for $u \in \mathcal{O} \subset \mathbb{R}^N$, λ_k is simple and *genuinely nonlinear*. Denoting by $r_k(u)$ a right eigenvector, this means that $r_k(u)\nabla_u \lambda_k(u) \neq 0$. Thus;one can choose r_k such that

$$
r_k(u) \cdot \nabla_u \lambda_k(u) = 1 \,. \tag{1.5.8}
$$

Consider (u^+, u^-, σ) satisfying the Rankine Hugoniot condition

$$
[f_1(u)] = \sigma[f_0(u)] \tag{1.5.9}
$$

located on the k-th curve given by Proposition 1.1.2. Thus

$$
u^+ = u^- + s r_k(u^-) + O(s^2) \,, \quad \sigma = \lambda_k(u^-) \, + \frac{1}{2} s + O(s^2) \,. \tag{1.5.10}
$$

The parameter s measures the strength of the shock since $|s| \approx |u^+ - u^-|$, provided that they are small enough.

Because λ_k is simple, one has $\lambda_j < \lambda_k$ for $j < k$ and $\lambda_l > \lambda_k$ for $l > k$. Thus, for s small enough, the conditions (1.5.1) are satisfied if and only if

$$s < 0\,. \tag{1.5.11}$$

This condition is also equivalent to other *entropy* conditions (see [Lax]).

Moreover, E_- is the space generated by the eigenvectors of $A_0^{-1}A_1(u^-)$ associated to the eigenvalues $\lambda_j(u^-)$ with $j < k$ and E_+ is the space generated by the eigenvectors of $A_0^{-1}A_1(u^+)$ associated to the eigenvalues $\lambda_j(u^+)$ with $j > k$. When $u^+ = u^-$, these spaces do not intersect and their sum does not contain $r_k(u)$. This remains true for small values of s and because $[u]$ is almost parallel to $r_k(u^-)$, for s small enough, one has $\mathbb{C}^N = \mathbb{C}[u] \oplus E_+ \oplus E_-$. Because $A_0(u^-) \simeq A_0(u^+)$ and $b_0 := [f_0(u)] \simeq A_0(u^-)[u]$, this implies that for s small enough, $\mathbb{C}^N = \mathbb{C}b_0 \oplus A_0^- E_+ \oplus A_0^+ E_-$. Therefore, Proposition 1.5.2 shows that for s small enough, the shock is uniformly stable if and only if it satisfies the entropy condition (1.5.11).

Example 2. *Multidimensional weak shocks*

We refer to section 5 for a detailed discussion of weak shocks. Under suitable technical assumptions, we show that weak shocks are uniformly stable, provided that they satisfy the entropy condition (1.5.1).

Example 3. *Euler's equations.*

The weak and uniform stability of shock waves for the equations of gas dynamics (1.4) is discussed with great details in [Maj 1]. For simplicity, we restrict here our attention to the isentropic system (1.1.9) and refer to [Maj 1] for the general case.

The eigenvalues of $A(u,\xi')$ are $\xi' \cdot v$ with multiplicity 2 and $\xi' \cdot v + c$ and $\xi' \cdot v - c$ which are simple. The sound speed c is given by

$$c^2 := \frac{dP}{d\rho}(\rho)\,, \tag{1.5.12}$$

where $p = P(\rho)$ is the equation of state. Note that the hyperbolicity assumption means that the right hand side is positive

Rotating the axes, we can assume that the tangent hyperplane to the front is

$$x_3 = \sigma t. \tag{1.5.13}$$

The Rankine Hugoniot conditions are

$$\begin{cases} \sigma[\rho] = [\rho v_3]\,, \\ \sigma[\rho v_j] = [\rho v_3 v_j]\,, \quad j = 1, 2\,, \\ \sigma[\rho v_3] = [\rho v_3^2] + [p]\,. \end{cases} \tag{1.5.14}$$

According to Lax's terminology, the jump is a *contact discontinuity* when

$$[p] = 0\,, \quad \sigma = v_3^+ = v_3^- \,. \tag{1.5.15}$$

It is a *shock* when

$$\begin{cases} [v_2] = [v_3] = 0, \qquad ([v_3])^2 \;=\; -[p]\,[\tau] \\ \sigma = v_3^+ - \dfrac{[v_3]}{[\tau]}\tau^+ \;= v_3^- - \dfrac{[v_3]}{[\tau]}\tau^- \,. \end{cases} \tag{1.5.16}$$

In these equations, $\tau := 1/\rho$. Note that p is a decreasing function of τ so that $[p][\tau] < 0$. We assume that Lax's shock conditions are satisfied. Changing x_3 into $-x_3$ and v_3 into $-v_3$ if necessary, we can assume that (1.5.1) is satisfied with $k = 1$ and $\lambda_1 = v_3 - c(\rho)$. Thus,

$$v_3^- - c^- > \sigma\,, \quad v_3^+ > \sigma > v_3^+ - c^+ \,. \tag{1.5.17}$$

and

$$\frac{c^-}{\tau^-} \;<\; \frac{[v_3]}{[\tau]} \;<\; \frac{c^+}{\tau^+}\,, \tag{1.5.18}$$

or

$$\frac{[v_3]}{[\tau]} > 0\,, \quad -\Big(\frac{dp}{d\tau}\Big)^- \;<\; -\frac{[p]}{[\tau]} < \;-\Big(\frac{dp}{d\tau}\Big)^+ \,. \tag{1.5.19}$$

Introduce the Mach number M

$$M^2 = \frac{[p]/[\tau]}{\Big(dp/d\tau\Big)^+} \tag{1.5.20}$$

Proposition 1.5.4. (Majda, [Maj 1]) *The shock is uniformly stable if and only if the Mach number satisfies*

$$M^2\Big(\frac{\tau^-}{\tau^+} - 1\Big) < 1 \quad \textit{and} \quad M^2 < 1\,.$$

2 The linearized stability

In this section we study the stability of the linearized shock equations. The goal is to prove a *maximal L^2 estimate* for the solutions of the boundary value problem (1.3.6). For C^∞ coefficients, the analysis was performed by Kreiss ([Kr]), who proved that the uniform stability condition implies that the maximal L^2 estimates are satisfied. However, in order to tackle non-linear problems, one has to consider coefficients with limited smoothness. The fundamental result proved in [Maj 1] is to extend Kreiss' analysis to H^s coefficients, s large. The proof is based on the construction of a symmetrizer. The symmetrizer is not local, it is *pseudo-local*, i.e. it depends not only on x but also on the frequencies. This will be shown in the constant coefficient case, where the symmetrizer is a Fourier multiplier. In the variable coefficient case, the natural extension of Fourier multipliers are pseudo-differential operators. When the coefficient have a limited regularity a convenient version of this calculus is the *paradifferential calculus* of J.M Bony. In this section, we use this approach and extend Kreiss' analysis to the case of Lipschitzean coefficients. The interest of this improvement will be clear in section 4. The results were announced in [Mo].

2.1 The basic L^2 estimate

Consider the mixed problem (1.4.3)

$$\left\{\begin{aligned} &\mathcal{L}_a u := \sum_{j=0}^{n} \mathcal{A}_j(a)\partial_j u \;=\; f\,, &&\text{on } x_n > 0\,,\\ &\mathcal{B}_a(\phi, u) := b(a)\cdot\nabla\phi \;+\; M(a)\,u \;=\; g\,, &&\text{on } x_n = 0\,. \end{aligned}\right. \tag{2.1.1}$$

The coefficients $\mathcal{A}_j, b_j$ and M are C^∞ and real functions of variables $a \in \mathcal{U} \subset \mathbb{R}^M$. The function $a(x)$ is given and valued in $\mathcal{U}$.

Assumption 2.1.1. *i) The system $\mathcal{L}$ is hyperbolic symmetric, i.e. there is a smooth matrix valued function $a \mapsto \mathcal{S}(a)$ on $\mathcal{U}$ such that $\mathcal{S}\mathcal{A}_j$ is symmetric for all j and $\mathcal{S}\mathcal{A}_0$ is definite positive.*

ii) For all $a \in \mathcal{U}$, the constant coefficient system $(\mathcal{L}_a, \mathcal{B}_a)$ is uniformly stable, in the sense of Definition 1.4.1.

The basic stability estimate is an L^2 estimate for the solutions of (2.1.1) on $\Omega := \mathbb{R}^n \times [0, \infty[$. We also denote by $\omega = \mathbb{R}^n = \{x_n = 0\}$ the boundary of Ω.

Theorem 2.1.2. *Suppose that Assumption* 2.1.1 *and the block structure Assumption* 2.3.3 *explicited below are satisfied. Fix a constant $K > 0$ and a compact set $\mathcal{K} \subset \mathcal{U}$. Then there are $\gamma_0 > 0$ and C such that for all Lipschitzean function a on Ω valued in $\mathcal{K}$ satisfying $\|a\|_{W^{1,\infty}} \leq K$, for all $\gamma \geq \gamma_0$ and for all $(u, \varphi) \in H^1_\gamma(\Omega) \times H^1_\gamma(\omega)$, the following estimate holds*

$$\begin{aligned} \gamma\| u \|^2_{L^2_\gamma(\Omega)} + \| u_{|x_n=0} \|^2_{L^2_\gamma(\omega)} \, + \, \| \nabla\varphi \|^2_{L^2_\gamma(\omega)} \, \leq \\ C \left(\frac{1}{\gamma} \| \mathcal{L}_a u \|^2_{L^2_\gamma(\Omega)} \, + \| \mathcal{B}_a(\varphi, u) \|^2_{L^2_\gamma(\omega)} \right). \end{aligned} \tag{2.1.4}$$

Here, $L^2_\gamma := e^{\gamma t} L^2$, $H^1_\gamma := e^{\gamma t} H^1$. The norm in L^2_γ is

$$\| u \|^2_{L^2_\gamma(\Omega)} = \int_\Omega e^{-2\gamma t} |u(x)|^2 \, dx = \| e^{-\gamma t} u \|^2_{L^2(\Omega)} \tag{2.1.5}$$

Similarly

$$\| u \|^2_{H^1_\gamma(\Omega)} = \| e^{-\gamma t} u \|^2_{H^1(\Omega)} \tag{2.1.6}$$

Introducing $v := e^{-\gamma t} u$, $\psi := e^{-\gamma t} \varphi$, the equations (2.1.1) are equivalent to

$$\begin{cases} \mathcal{L}^\gamma_a v \, := \, \mathcal{L}_a v + \gamma \mathcal{A}_0 v \, = e^{-\gamma t} f \\ \mathcal{B}^\gamma_a(\psi, v) \, := \, \mathcal{B}_a(\psi, v) + \gamma b_0 \psi \, = e^{-\gamma t} g \, . \end{cases} \tag{2.1.7}$$

By definition,

$$\|u\|_{L^2_\gamma} = \|v\|_0$$

where $\| \cdot \|_0$ denotes the usual norm in L^2. In addition, note that the inequality

$$\gamma \|e^{-\gamma t} \varphi\|_0 \leq \|e^{-\gamma t} \partial_t \varphi\|_0$$

implies that

$$\|\varphi\|_{H^1_\gamma} \approx \|\psi\|^2_{1,\gamma} := \|\nabla \psi\|^2_0 \, + \, \gamma^2 \|\psi\|^2_0 \, . \tag{2.1.8}$$

Therefore, Theorem 2.1.2 is equivalent to

Theorem 2.1.3. *Suppose that Assumptions* 2.1.1 *and* 2.3.3 *are satisfied. Fix a constant $K > 0$ and a compact set $\mathcal{K} \subset \mathcal{U}$. Then there are $\gamma_0 > 0$ and C such that for all Lipschitzean function a on Ω valued in $\mathcal{K}$ satisfying $\|a\|_{W^{1,\infty}} \leq K$, for all $\gamma \geq \gamma_0$ and for all $(v, \psi) \in H^1(\Omega) \times H^1(\omega)$, the following estimate holds*

$$\gamma\| v \|^2_0 \, + \, \| v_{|x_n=0} \|^2_0 \, + \, \| \psi \|^2_{1,\gamma} \, \leq C \left(\frac{1}{\gamma} \| \mathcal{L}^\gamma_a v \|^2_0 \, + \| \mathcal{B}^\gamma_a(\psi, v) \|^2_0 \right). \tag{2.1.9}$$

2.2 The method of symmetrizers

Consider a family $\{\mathcal{H}^\gamma(x_n)\}_{x_n\ge 0,\gamma\ge 1}$ of operators from $H^1(\Omega)$ to $L^2(\Omega)$. Suppose that for all γ, the operators $\mathcal{H}^\gamma(x_n)$ and $(\mathcal{H}^\gamma(x_n))^*$ are bounded from H^1 to L^2 uniformly for $x_n \ge 0$. The relation

$$(\mathcal{H}^\gamma u)(\,\cdot\,,x_n) \;=\; \mathcal{H}^\gamma(x_n)\,u(\,\cdot\,,x_n) \tag{2.2.1}$$

defines bounded operators $\mathcal{H}^\gamma$ from $H^1(\Omega)$ to $L^2(\Omega)$. Below $\mathcal{H}^\gamma$ is the paradifferential version of

$$\mathcal{H}(a,\partial_t+\gamma,\partial_y) := \mathcal{A}_n(a)^{-1}\Big(\sum_{j=0}^{n-1}\mathcal{A}_j(a)\partial_j \;+\;\gamma\mathcal{A}_0(a)\Big)\,. \tag{2.2.2}$$

Consider the equation

$$\left\{\begin{array}{ll} \partial_n v + \mathcal{H}^\gamma v = \,f\,, & \text{on } x_n > 0\,,\\ B^\gamma v = \,g\,, & \text{on } x_n = 0\,. \end{array}\right. \tag{2.2.3}$$

where B^γ is a bounded operator in $L^2(\omega)$.

To prove an energy estimate for (2.2.3), one considers a symmetrizer. It is given by a bounded and Lipschitzean family $\{\mathcal{R}^\gamma(x_n)\}_{x_n\ge 0;\gamma\ge 1}$ of *self adjoint* operators in $L^2(\mathbb{R}^n)$. Using (2.2.1), it defines bounded self adjoint operators $\mathcal{R}^\gamma$ in $L^2(\Omega)$. The starting point is the following identity.

(2.2.4)

$$\begin{aligned}\big(\!\!\big(\mathcal{R}^\gamma(0)v(0),v(0)\big)\!\!\big)_{L^2(\omega)} &- \mathrm{Re}\,\big(\!\!\big((\mathcal{R}^\gamma\mathcal{H}^\gamma + (\mathcal{H}^\gamma)^*\mathcal{R}^\gamma)v,v\big)\!\!\big)_{L^2(\Omega)}\\ &= -2\mathrm{Re}\,\big(\!\!\big(\partial_n v + \mathcal{H}^\gamma v, \mathcal{R}^\gamma v\big)\!\!\big)_{L^2(\Omega)} - \big(\!\!\big([\partial_n,\mathcal{R}^\gamma]v,v\big)\!\!\big)_{L^2(\Omega)}\,.\end{aligned}$$

Here $v(0) = v_{|x_n=0}$ denotes the trace of v on the boundary and $(\!(\cdot,\cdot)\!)$ is the scalar product in L^2. The following result is elementary.

Lemma 2.2.1. *Suppose that there are constants C and c such that for all $v\in H^1(\Omega)$ and all $\gamma\ge 1$*

$$\|\mathcal{R}^\gamma v\|_0 \le C\|v\|_0\,, \tag{2.2.5}$$

$$\|[\partial_n,\mathcal{R}^\gamma]v\|_0 \le C\|v\|_0\,, \tag{2.2.6}$$

$$\mathrm{Re}\,\big(\!\!\big((\mathcal{R}^\gamma\mathcal{H}^\gamma + (\mathcal{H}^\gamma)^*\mathcal{R}^\gamma)v,v\big)\!\!\big)_{L^2(\Omega)} \le -c\gamma\|v\|_0^2\,, \tag{2.2.7}$$

$$\big(\!\!\big(\mathcal{R}^\gamma(0)v(0),v(0)\big)\!\!\big)_{L^2(\mathbb{R}^n)} + C\|B^\gamma v(0)\|^2 \;\ge c\|v(0\|_0^2\,. \tag{2.2.8}$$

Then, there are γ_0 and C_1, which depend only on the constant C and c above, such that for all $v \in H^1(\Omega)$ and all $\gamma \geq \gamma_0$

$$\gamma \| v \|_0^2 + \| v_{|x_n=0} \|_0^2 \leq C_1 \left(\frac{1}{\gamma} \| (\partial_n + \mathcal{H}^\gamma) v \|_0^2 + \| B^\gamma v \|_0^2 \right). \tag{2.2.9}$$

2.3 The constant coefficient case

In this section, we prove Theorem 2.1.3 when the operators have constant coefficients. This serves as an introduction for the general case which follows the same lines.

Consider the equations (2.1.1) or (2.1.7) with frozen coefficients at $a \in \mathcal{K}$. Performing a Fourier transform in (t, y) leads to the equations (1.4.5)

$$\begin{cases} \mathcal{A}_n(a)\partial_n \widehat{v} + i\mathcal{P}(a, \hat{\tau}, \eta)\widehat{v} = \widehat{f}, & x_n > 0, \\ ib(a, \hat{\tau}, \eta)\widehat{\psi} + M(a)\widehat{v}_{|x_n=0} = \widehat{g}, \end{cases} \tag{2.3.1}$$

with $\hat{\tau} = \tau - i\gamma$. We want to prove the energy estimate (2.1.9) for the constant coefficient equation (2.3.1). By Plancherel's theorem, it is sufficient to prove the following estimate

$$\gamma \|\widehat{v}\|_0^2 + |\widehat{v}(0)|^2 + (|\tau|^2 + |\eta|^2 + |\gamma|^2)|\widehat{\psi}|^2 \leq C \left(\frac{1}{\gamma} \|\widehat{f}\|_0^2 + |\widehat{g}|^2 \right), \tag{2.3.2}$$

for all $\widehat{v} \in H^1([0, \infty[)$ and all $(\tau, \eta, \gamma) \in \mathbb{R}^n \times]0, \infty[$, and C independent of v and (τ, η, γ).

The equations (2.3.1) and the estimate (2.3.2) are invariant under the scaling $(\tau', \eta', \gamma') = \rho(\tau, \eta, \gamma)$, $v'(x_n) = \widehat{v}(\rho x_n)$, $f'(x_n) = \rho \widehat{f}(\rho x_n)$, $g' = \widehat{g}$ and $\psi' = \rho^{-1}\widehat{\psi}$. Thus it is sufficient to prove (2.3.2) when (τ, η, γ) belongs to the unit sphere Σ in $\mathbb{R}^{n+1}$ and $\gamma > 0$. We denote by Σ_+ this subset of Σ. To simplify notations, we denote the full set of parameters by

$$z := (a, \tau, \eta, \gamma) \in \mathcal{U} \times \mathbb{R} \times \mathbb{R}^{n-1} \times]0, \infty[\,. \tag{2.3.3}$$

In applications, $\gamma > 0$, but at some places it is important to study the behaviour of matrices and symbols as γ tends to zero and to consider smoth functions up to $\gamma = 0$.

First, one eliminates ψ, using Proposition 1.4.3. If the problems $(\mathcal{L}_a, \mathcal{B}_a)$ are uniformly stable, the vectors $b_0(a), \ldots, b_{n-1}(a)$ are linearly independent and there is a constant $C > 0$ such that for all $(\tau, \eta, \gamma) \in \mathbb{R}^{n+1}$

$$\frac{1}{C}(\gamma + |\tau| + |\eta|) \leq |b(a, \tau, \eta, \gamma)| \leq C(\gamma + |\tau| + |\eta|) \tag{2.3.4}$$

Moreover, by compactness the constant can be chosen independent of $a \in \mathcal{K}$.

In particular $b(z) \neq 0$ and therefore, one can introduce the projector

$$\Pi(z)\, h \;:=\; h \;-\; \frac{(h, b(z))}{|b(z)|^2} b(z) \tag{2.3.5}$$

Thus the boundary condition in (2.3.1) is equivalent to

$$\begin{cases} B(z)\widehat{v}(0) \;= \Pi(z) g \quad \text{with} \quad B(z) := \Pi(z) M(a)\,, \\ i\,|b(z)|^2 \widehat{\psi} \;=\; (b(z), g - Mv(0))\,. \end{cases} \tag{2.3.6}$$

In particular, (2.3.4) implies that

$$(|\tau|^2 + |\eta|^2 + |\gamma|^2)|\widehat{\psi}|^2 \;\leq\; C\big(|\widehat{g}|^2 \;+\; |\widehat{v}(0)|^2\big)\,. \tag{2.3.7}$$

In addition to the boundary symbol $B(z)$ defined above, introduce the following interior symbol related to the operators (2.2.2) and (2.3.1)

$$\begin{aligned} \mathcal{H}(a, \tau, \eta, \gamma) :&= (\tau - i\gamma) Id + \sum_{j=1}^{n-1} \eta_j \mathcal{A}_0(a)^{-1} \mathcal{A}_j(a) \\ &= \mathcal{A}_n^{-1}(a) \mathcal{P}(\tau - i\gamma, \eta)\,. \end{aligned} \tag{2.3.8}$$

With (2.3.7), we see that (2.3.2) follows from the estimate

$$\gamma \|v\|_0^2 \;+\; |v(0)|^2 \;\leq\; C\left(\frac{1}{\gamma}\|(\partial_n + i\mathcal{H}(z))v\|_0^2 + \,|\widehat{B}(z)v(0)|^2\right), \tag{2.3.9}$$

for all $\widehat{v} \in H^1([0, \infty[)$ and all $z \in \mathcal{K} \times \Sigma_+$, with C independent of v and z.

To prove the energy estimate (2.3.8), one looks for a symmetrizer $\mathcal{R}(z)$ with the following properties.

Definition 2.3.1 (Kreiss' symmetrizers). *A symmetrizer is a smooth and bounded family of self adjoint matrices $\mathcal{R}(a, \tau, \eta, \gamma)$ for $z = (a, \tau, \eta, \gamma) \in \mathcal{U} \times \mathbb{R} \times \mathbb{R}^{n-1} \times [0, \infty[$ and $(\tau, \eta, \gamma) \neq 0$, homogeneous of degree zero in (τ, η, γ) and such that for all compact $\mathcal{K} \subset \mathcal{U}$ there are constants $c > 0$ and C such that for all $z \in \mathcal{K} \times \Sigma_+$*

$$\operatorname{Im}(\mathcal{R}(z)\mathcal{H}(z)) \geq c\gamma Id \tag{2.3.10}$$

$$\mathcal{R}(z) \;+\; C\, B(z)^* B(z) \geq c Id \tag{2.3.11}$$

Proposition 2.3.2. *If there exists a Kreiss' symmetrizer, then for all $a \in \mathcal{U}$ there is C such that for the constant coefficient system $(\mathcal{L}_a^\gamma, \mathcal{B}_a^\gamma)$, the energy estimate* (2.1.9) *is satisfied for all $\gamma > 0$*

Proof. It is sufficient to prove (2.3.9). For all $z \in \mathcal{K} \times \Sigma^+$ and $v \in H^1(]0,\infty[)$, one has

$$\begin{aligned}\big(\!\big(\mathcal{R}(z)v(0), v(0)\big)\!\big)_{\mathbb{C}^{2N}} &+ \operatorname{Im}\big(\!\big((\mathcal{R}(z)\mathcal{H}(z) - \mathcal{H}^*(z)\mathcal{R}(z))v, v\big)\!\big)_{L^2(]0,\infty[)} \\ &= -2\mathrm{Re}\,\big(\!\big(\partial_n v + i\mathcal{H}(z)v, \mathcal{R}(z)v\big)\!\big)_{L^2(]0,\infty[)}.\end{aligned}$$

The estimate (2.3.10) and (2.3.11) immediately imply (2.3.9). □

When the uniform Lopatinski condition is satisfied, the existence of symmetrizer is proved in [Kr] (see also [Ch-Pi]) for *strictly hyperbolic* systems. However, many examples of physical systems are not strictly hyperbolic. An example is Euler's system of gas dynamics. To cover this case, Majda introduced the following technical assumption.

Assumption 2.3.3 (Block structure condition). *For all $\underline{z} \in \mathcal{U} \times \overline{\Sigma}_+$, there is a neighborhood $\mathcal{O}$ of $\underline{z}$ in $\mathcal{U} \times \mathbb{R}^{n+1}$ and matrices $T(z)$ depending smoothly on $z \in \mathcal{O}$, such that $T(z)^{-1}\mathcal{H}(z)T(z)$ has the following block diagonal structure*

$$T(z)^{-1}\mathcal{H}(z)T(z) = \begin{bmatrix} Q^+(z) & 0 & 0 & \cdots & 0 \\ 0 & Q^-(z) & 0 & \cdots & 0 \\ 0 & 0 & Q_1(z) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix} \tag{2.3.12}$$

with

i) $\operatorname{Im} Q^+(z) := i((Q^+)^* - Q^+)$ *is definite positive and* $\operatorname{Im} Q^-(z)$ *is definite negative,*

ii) for $j \geq 1$, $Q_j(z)$ has real coefficients real when $\gamma = 0$, $Q_j(\underline{z}) = \mu_j Id + N_j$ where N_j is the nilpotent matrix

$$N_j = \begin{bmatrix} 0 & 1 & 0 & \\ 0 & 0 & \ddots & 0 \\ & \ddots & \ddots & 1 \\ & & \cdots & 0 \end{bmatrix} \tag{2.3.13}$$

and the lower left hand corner of $\partial Q_j / \partial\gamma(\underline{z})$ does not vanish.

The size of a block can be zero, meaning that there is no such block in the reduction (2.1.12). When the size of Q_j is one, the condition is that the scalar $Q_j(z)$ is real when $\gamma = 0$ and $\partial_\gamma Q_j \neq 0$.

Remark. When $\underline{\gamma} > 0$, the hyperbolicity assumption implies that the eigenvalues of $\mathcal{H}(\underline{z})$ are non-real. This remains true on a neighborhood of $\underline{z}$ and (2.3.11) holds with only the two blocks Q^+ and Q^-. Thus the block structure assumption has to be checked only near points $\underline{z}$ with $\underline{\gamma} = 0$.

This assumption is satisfied when the system is strictly hyperbolic ([Kr], [Ra], [Ch-Pi]). It is also satisfied by several nonstriclty hyperbolic systems and in particular by the Euler's equations of gas dynamics ([Maj 1]). This is contained in the next result [1].

Proposition 2.3.4. *Consider a symmetric hyperbolic system* (1.1.1) *Suppose that the eigenvalues $\lambda_k(u,\xi')$ of $A(u,\xi')$ have constant multiplicity and that for all k, either λ_k is simple or $\lambda_k(u,\xi') = \mathbf{v}_k(a)\cdot\xi'$ is linear in ξ'. Consider the linearized shock problem* (1.4.3) *and assume that $\mathcal{A}_n$ is not characteristic. Then, the block structure assumption is satisfied.*

Proof. The operator $\mathcal{L}_a$ has a diagonal form

$$\mathcal{L}_a = \begin{bmatrix} L_a^+ & 0 \\ 0 & L_a^- \end{bmatrix}$$

Therefore, it is sufficient to prove the block structure condition for each block $L_a^\pm$. We make the proof for L^+ and for simplicity we forget to mention explicitly the $+$ sign. Thus we consider

$$L := A_0\partial_t + \sum_{j=1}^{n-1} A_j\partial_j + \widetilde{A}_n\partial_n \tag{2.3.14}$$

with

$$\widetilde{A}_n = \frac{1}{\kappa}\Big(A_n - \sum_{j=1}^{n-1}\beta_j A_j - \sigma A_0\Big)\,. \tag{2.3.15}$$

In addition (σ,β,κ) is the frozen value of $\nabla\Phi$ (cf the definitions in (1.2.6)). The parameters, which we have not written explicitly are $a = (u,\sigma,\beta,\kappa)$. Changing ξ_n to $\kappa\xi_n$, there is no restriction to assume that $\kappa = 1$. Then, the roots of

$$\det\Big(\tau Id + \sum_{j=1}^{n-1}\xi_j A_0 A_j + \xi_n\widetilde{A}_n\Big) \tag{2.3.16}$$

[1] It is now proved that the block structure condition is satisfied by all hyperbolic systems with constant mulitplicity [Mét4].

are

$$\tau = \sigma - \lambda_k(\eta - \beta\xi_n, \xi_n) \tag{2.3.17}$$

The matrix $\mathcal{H}$ is also block diagonal, and the block which corresponds to L^+ is

$$H(\tau - i\gamma, \eta) \; = \; (\widetilde{A}_n)^{-1}\Big((\tau - i\gamma)A_0 + \sum_{j=1}^{n-1} \eta_j A_j\Big) \tag{2.3.18}$$

We prove that H satisfies the block structure condition. It is sufficient to consider $\underline{z}$ such that $\underline{\gamma} = 0$.

We can perform a first block diagonal reduction (2.3.12) of $H(z)$ such that the eigenvalues of Q^+ [resp. Q^-] have positive [resp. negative] imaginary part, $Q_j(\underline{z})$ has only one real eigenvalue, denoted by $\underline{\mu}_j$ and the $\underline{\mu}_j$ are pairewise distinct. Thus, for all j, there is a unique k such that

$$\underline{\tau} = \underline{\sigma}\underline{\mu}_j - \lambda_k(\underline{\eta} - \underline{\beta}\,\underline{\mu}_j, \underline{\mu}_j)\,. \tag{2.3.19}$$

a) If λ_k is a simple eigenvalue, then Kreiss' construction applies (see also [Ra], [Ch-Pi]). One can reduce $Q_j(\underline{z})$ to the Jordan form (2.3.13) and find a conjugate matrix $T_j^{-1}(z)Q_j(z)T_j(z)$ having the properties listed in *ii*) of Assumption 2.3.3.

b) If $\lambda_k(\eta, \xi_n) = \mathbf{b}\eta + \mathbf{a}\xi_n$ is a multiple eigenvalue, then (2.3.19) reads

$$\underline{\tau} = -\mathbf{b}\underline{\eta} - (\mathbf{a} - \mathbf{b}\underline{\beta} - \sigma)\underline{\mu}_j\,. \tag{2.3.20}$$

Note that $\widetilde{\mathbf{a}} := (\mathbf{a} - \mathbf{b}\underline{\beta} - \sigma) \neq 0$ since we have assumed that the boundary is not characteristic for L, i.e. that zero is not an eigenvalues of $\widetilde{A}_n$. The constant multiplicity assumption implies that the eigenprojector $\Pi_k(\eta, \xi_n)$ associated to depends smoothly on the parameters and analytically on (η, ξ_n). In addition, the identity

$$\Big((\mathbf{b}\eta + \mathbf{a}\xi_n)A_0 + \sum_{j=1}^{n-1} \eta_j A_j + \xi_n A_n\Big)\Pi_k(\eta, \xi_n) = 0$$

extends analytically to a neighborhood of the real domain. It implies that

$$\Big((\mathbf{b}\eta + \widetilde{\mathbf{a}}\xi_n)A_0 + \sum_{j=1}^{n-1} \eta_j A_j + \xi_n \widetilde{A}_n\Big)\Pi_k(\eta - \beta\xi_n, \xi_n) = 0$$

Introducing
$$\Pi_k^\sharp(\hat\tau,\eta) := \Pi(\eta-\xi_n,\xi_n)_{|\xi_n=(\hat\tau-\mathbf{b}\eta)/\widetilde{\mathbf{a}}}$$
we see that
$$\Big(\hat\tau A_0 + \sum_{j=1}^{n-1}\eta_j A_j + \frac{\hat\tau-\mathbf{b}\eta}{\widetilde{\mathbf{a}}}\widetilde{A}_n\Big)\Pi_k^\sharp(\hat\tau,\eta) = 0 \tag{2.3.21}$$
With (2.3.20), this shows that for $(\hat\tau,\eta)$ close to $(\underline{\tau},\underline{\eta})$ $\mu := -(\hat\tau-\mathbf{b}\eta)/\widetilde{\mathbf{a}}$ is an eigenvalue of constant multiplicity of $\mathcal{H}(\hat\tau,\eta)$. Thus, one can further reduce the block Q_j to
$$Q_j = -\frac{\hat\tau-\mathbf{b}\eta}{\widetilde{\mathbf{a}}}\,Id\,.$$
The final form (2.3.12) is achieved considering the diagonal elements of Q_j as matrices of dimension one with entry $q_j := -(\hat\tau-\mathbf{b}\eta)/\widetilde{\mathbf{a}}$. Because q_j is real when $\operatorname{Im}\hat\tau = 0$ and $\partial_\gamma q_j = -1/\widetilde{\mathbf{a}} \neq 0$ the conditions in *ii*) of Assumption 2.3.3 are satisfied.

$\square$

Theorem 2.3.5 (Kreiss). ([Kr], [Ch-Pi]) *When Assumptions* 2.1.1 *and* 2.3.2 *are satisfied, there exist Kreiss' symmetrizers.*

With Proposition 2.3.2, it implies the energy estimate in Theorem 1.2.3 for constant coefficient operators.

2.4 The paradifferential calculus with parameter

In this section, we introduce the symbolic calculus which will be used in the proof of Theorem 1.2.3. We first introduce a tangential paradifferential calculus with parameters, which combines the Bony-Meyer's calculus and the introduction of a large parameter (see [Mo] [Met 2]).

In this section we work in $\mathbb{R}^n$. In the applications, this will be the hyperplane $x_n = constant$. The time variable does not play any particular role, and risking confusion we denote by x the variable in $\mathbb{R}^n$. The parameter γ is looked as an auxiliary variable.

2.4.1 The Littlewood-Paley decomposition

Introduce $\chi \in C_0^\infty(\mathbb{R}^n\times\mathbb{R})$, such that $0\le\chi\le 1$ and
$$\chi(\xi,\gamma) = 1 \quad \text{for } \gamma^2+|\xi|^2 \le 1\,, \quad \chi(\xi,\gamma) = 0 \quad \text{for } \gamma^2+|\xi|^2 \ge 2\,. \tag{2.4.1}$$

For $k \in \mathbb{N}$, introduce $\chi_k(\xi, \gamma) := \chi(2^{-k}\xi, 2^{-k}\gamma)$, $\widetilde{\chi}_k^\gamma(x)$ its inverse Fourier transform with respect to ξ and the operators

$$\begin{cases} S_k^\gamma u := \widetilde{\chi}_k^\gamma * u = \chi_k(D_x, \gamma) u\,, \\ \Delta_0^\gamma = S_0^\gamma \qquad \text{and} \quad \Delta_k^\gamma = S_\gamma^k - S_\gamma^{k-1} \quad \text{when } k \geq 1. \end{cases} \tag{2.4.2}$$

For all temperate distribution u and $\gamma \geq 1$, one has

$$u = \sum_{k \geq 0} \Delta_k^\gamma u\,. \tag{2.4.3}$$

Note that $\Delta_k^\gamma = 0$ when $\gamma \geq 2^{k+1}$ and the spectrum of $\Delta_k^\gamma u$ (i.e. the support of its Fourier transform), denoted by $\operatorname{spec}(\Delta_k^\gamma u)$, satisfies

$$\operatorname{spec}(\Delta_k^\gamma u) \subset \Big\{ \xi \ : \ 2^{k-1} \leq (\gamma^2 + |\xi|^2)^{1/2} \leq 2^{k+1} \Big\} \tag{2.4.4}$$

For $s \in \mathbb{R}$, let $H^s(\mathbb{R}^n)$ denote the Sobolev space of temperate distributions u such that their Fourier transform $\widehat{u}$ satisfies $(1 + |\xi|^2)^{s/2}\widehat{u} \in L^2(\mathbb{R}^n)$. This space is equipped with the following family of norms :

$$\|u\|_{s,\gamma}^2 := \int (\gamma^2 + |\xi|^2)^s |\widehat{u}(\xi)|^2 \, d\xi\,. \tag{2.4.5}$$

The following propositions immediately follow from the definitions. The important point is that the constants C in (2.4.6) and (2.4.7) do not depend on $\gamma \geq 1$.

Proposition 2.4.1. *Consider $s \in \mathbb{R}$ and $\gamma \geq 1$. A temperate distribution u belongs to $H^s(\mathbb{R}^n)$ if and only if*

i) for all $k \in \mathbb{N}$, $\Delta_k^\gamma u \in L^2(\mathbb{R}^n)$

ii) the sequence $\delta_k = 2^{ks}\|\Delta_k^\gamma u\|_{L^2(\mathbb{R}^n)}$ belongs to $\ell^2(\mathbb{N})$.

Moreover, there is a constant C, independent of $\gamma \geq 1$, such that

$$\frac{1}{C}\|u\|_{s,\gamma}^2 \leq \Big(\sum_k \delta_k^2 \Big)^{1/2} \leq C\|u\|_{s,\gamma}^2 \tag{2.4.6}$$

Proposition 2.4.2. *Consider $s \in \mathbb{R}$, $\gamma \geq 1$ and $R > 0$. Suppose that $\{u_k\}_{k\in\mathbb{N}}$ is a sequence of functions in $L^2(\mathbb{R}^n)$such that*

i) the spectrum of u_k is contained in $\big\{ \frac{1}{R}2^k \leq (\gamma^2 + |\xi|^2)^{1/2} \leq R2^k \big\}$.

ii) the sequence $\delta_k = 2^{ks}\|u_k\|_{L^2(\mathbb{R}^n)}$ belongs to $\ell^2(\mathbb{N})$.

Then $u = \sum u_k$ belongs to $H^s(\mathbb{R}^n)$ and there is a constant C, independent of $\gamma \geq 1$, such that

$$\|u\|_{s,\gamma}^2 \leq C \Big(\sum_k \delta_k^2 \Big)^{1/2} \tag{2.4.7}$$

When $s > 0$, it is sufficient to assume that the spectrum of u_k is contained in $\big\{(\gamma^2 + |\xi|^2)^{1/2} \leq R2^k\big\}$.

We also use the space $W^{1,\infty}(\mathbb{R}^n)$ of functions $u \in L^\infty$ such that $\nabla u \in L^\infty$. It is equipped with the obvious norm. We denote by S_k, Δ_k the usual Paley Littlewood decomposition, which corresponds to the case $\gamma = 0$ in (2.4.2) (2.4.3) (see [Bon], [Mey]). We recall the following results

Proposition 2.4.3. *There is a constant C such that :*
i) for all $u \in L^\infty$ and all $k \in \mathbb{N}$, one has

$$\|S_k u\|_{L^\infty} \leq C\|u\|_{L^\infty} ,$$

ii) for all $u \in W^{1,\infty}$ and all $k \in \mathbb{N}$, one has

$$\|\Delta_k u\|_{L^\infty} \leq C2^{-k}\|u\|_{W^{1,\infty}} , \quad \|u - S_k u\|_{L^\infty} \leq C2^{-k}\|u\|_{W^{1,\infty}} .$$

2.4.2 Paradifferential operators with parameters

Definition 2.4.4. (Symbols) *Let $m \in \mathbb{R}$.*
i) Γ_0^m denotes the space of locally bounded functions $a(x, \xi, \gamma)$ on $\mathbb{R}^n \times \mathbb{R}^n \times [1, \infty[$ which are C^∞ with respect to ξ and such that for all $\alpha \in \mathbb{N}^n$ there is a constant C_α such that

$$\forall (x, \xi, \gamma) , \quad |\partial_\xi^\alpha a(x, \xi, \gamma)| \leq C_\alpha (\gamma + |\xi|)^{m-|\alpha|} \tag{2.4.8}$$

ii) Γ_1^m denotes the space of symbols $a \in \Gamma_0^m$ such that for all j, $\partial_{x_j} a \in \Gamma_0^m$.

iii) For $k = 0, 1$, Σ_k^m is the space of symbols $\sigma \in \Gamma_k^m$ such that there exists $\varepsilon \in]0, 1[$ such that for all (ξ, γ) the spectrum of $x \mapsto a(x, \xi, \gamma)$ is contained in the ball $\big\{|\eta| \leq \varepsilon(\gamma^2 + |\xi|^2)^{1/2}\big\}$.

Consider a C^∞ function $\psi(\eta, \xi, \gamma)$ on $\mathbb{R}^n \times \mathbb{R}^n \times [1, \infty[$ such that
1) there are ε_1 and ε_2 such that $0 < \varepsilon_1 < \varepsilon_2 < 1$ and

$$\begin{aligned} \psi(\eta, \xi, \gamma) &= 1 \quad \text{for } |\eta| \leq \varepsilon_1(\gamma^2 + |\xi|^2)^{1/2} \\ \psi(\eta, \xi, \gamma) &= 0 \quad \text{for } |\eta| \geq \varepsilon_2(\gamma^2 + |\xi|^2)^{1/2} . \end{aligned}$$

2) for all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$, there is $C_{\alpha,\beta}$ such that

$$\forall (\eta, \xi, \gamma) : \quad |\partial_\eta^\alpha \partial_\xi^\beta \psi(\eta, \xi, \gamma)| \le C_{\alpha,\beta}(\gamma + |\xi|)^{-|\alpha|-|\beta|}.$$

For instance one can consider with $N \ge 3$:

$$\psi_N(\eta, \xi, \gamma) = \sum_k \chi(2^{-k+N}\eta, 0)(\chi_k(\xi, \gamma) - \chi_{k-1}(\xi, \gamma)). \tag{2.4.9}$$

We will say that such a function ψ is an *admissible cut-off*. Consider next $G^\psi(\,\cdot\,, \xi, \gamma)$ the inverse Fourier transform of $\psi(\,\cdot\,, \xi, \gamma)$. It satisfies

$$\forall \alpha \in \mathbb{N}^n, \quad \forall (\xi, \gamma) : \quad \|\partial_\xi^\alpha G^\psi(\,\cdot\,, \xi, \gamma)\|_{L^1(\mathbb{R}^n)} \le C_\alpha (\gamma + |\xi|)^{-|\alpha|}, \tag{2.4.10}$$

Proposition 2.4.5. *Let ψ be an admissible cut-off. Then, for all $m \in \mathbb{R}$ and $k = 0, 1$, the operators*

$$a \mapsto \sigma_a^\psi(x, \xi, \gamma) := \int G^\psi(x - y, \xi, \gamma)\, a(y, \xi, \gamma)\, dy \tag{2.4.11}$$

are bounded from Γ_k^m to Σ_k^m.

Moreover, if $a \in \Gamma_1^m$, then $a - \sigma_a^\psi \in \Gamma_0^{m-1}$. In particular, if ψ_1 and ψ_2 are admissible and $a \in \Gamma_1^m$ then $\sigma_a^{\psi_1} - \sigma_a^{\psi_2} \in \Sigma_0^{m-1}$.

Proof. The bounds (2.4.10) imply that the estimates (2.4.8) are preserved by the convolution (2.4.11). Thus $\sigma_a^\psi \in \Gamma_0^m$ if $a \in \Gamma_0^m$. Moreover, $\partial_x \sigma_a^\psi = \sigma_{\partial_x a}^\psi$ and the operator (2.4.11) maps Γ_1^m into itself. On the Fourier side, one has

$$\widehat{\sigma}_a^\psi(\eta, \xi, \gamma) = \psi(\eta, \xi, \gamma)\, \widehat{a}(\eta, \xi, \gamma).$$

Thus, the spectral property is clear and the first part of the proposition is proved.

Using Proposition 1.3 and the spectral property, one shows that

$$\|(a - \sigma_a^\psi)(\,\cdot\,, \xi, \gamma)\|_{L^\infty} \le C\,(\gamma + |\xi|)^{-1} \|a(\,\cdot\,, \xi, \gamma)\|_{W^{1,\infty}}$$

Thus $a - \sigma_a^\psi \in \Gamma_0^{m-1}$ if $a \in \Gamma_1^m$. $\square$

The spectral property implies that the symbols $\sigma \in \Sigma_0^m$ are C^∞ in x and

$$|\partial_x^\beta \partial_\xi^\alpha \sigma(x, \xi, \gamma)| \le C_{\alpha\,\beta}(\gamma + |\xi|)^{m-|\alpha|+|\beta|} \tag{2.4.12}$$

and thus belong to Hörmander's class of symbols $S_{1,1}^m$. The associated operators are

$$P_\sigma^\gamma u(x) := \frac{1}{(2\pi)^n} \int e^{i\xi\cdot x}\, \sigma(x, \xi, \gamma)\, \widehat{u}(\xi)\, d\xi\,. \tag{2.4.13}$$

Using Proposition 2.4.5 we can associate operators to symbols $a \in \Gamma_0^m$. Given an admissible cut-off ψ, define

$$T_a^{\psi,\gamma} u := P^{\gamma}_{\sigma_a^{\psi}} u \tag{2.4.14}$$

Introduce the following terminology:

Definition 2.4.6. *A family of operators $\{P^\gamma\}_{\gamma \geq 1}$ is of order less than or equal to m if for all $s \in \mathbb{R}$, P^γ maps H^s into H^{s-m} and there is a constant C such that*

$$\forall \gamma \geq 1\,,\ \forall u \in H^s(\mathbb{R}^n)\ :\quad \|P^\gamma u\|_{s-m,\gamma} \ \leq\ C\|u\|_{s,\gamma} \tag{2.4.15}$$

Proposition 2.4.7. *i) For all $\sigma \in \Sigma_0^m$, the family of operators P^γ_σ is of order $\leq m$. Moreover, the spectrum of $P^\gamma_\sigma u$ is contained in the set of $\xi \in \mathbb{R}^n$ such that there is ξ' in the spectrum of u such that $|\xi - \xi'| \leq \varepsilon_2(\gamma^2 + |\xi'|^2)^{1/2}$.*

ii) For all admissible cut-off ψ and all $a \in \Gamma_0^m$, the family of operators $T_a^{\psi,\gamma}$ is of order $\leq m$.

iii) If ψ_1 and ψ_2 are admissible and $a \in \Gamma_1^m$, then $T_a^{\psi_1,\gamma} - T_a^{\psi_2,\gamma}$ is of order $\leq m-1$

Proof. (See [Bon]). Using (2.4.3), one obtains that

$$P^\gamma_\sigma = \sum_{k,l} P^\gamma_{\sigma,k} \Delta^\gamma_l$$

where the symbol of $P^\gamma_{\sigma,k}$ is $\big(\chi_k(\xi,\gamma) - \chi_{k-1}(\xi,\gamma)\big)\sigma(x,\xi,\gamma)$. As in [Bon], [Mey], one shows that $P^\gamma_{\sigma,k}$ is bounded from L^2 to L^2 with norm $\leq C2^{km}$ with C independent of k and γ, and the spectrum of $P^\gamma_{\sigma,k} u$ is contained in the set

$$\Big\{(1-\varepsilon)2^{k-1} \leq (\gamma^2 + |\xi|^2)^{1/2} \leq (1+\varepsilon)2^{k+1}\Big\}$$

The conclusion follows from Proposition 1.2. This proves i). The other two parts follow from (2.4.3) and Proposition 1.5.

□

2.4.3 Paraproducts

A function $a(x) \in L^\infty$ can be seen as a symbol in Γ_0^0, independent of (ξ,γ). With ψ given by (2.4.9) with $N = 3$, we define

$$T_a^\gamma\, u := T_a^{\psi,\gamma} u \ =\ \sum_k S_{k-3} a\, \Delta_k^\gamma\, u\,. \tag{2.4.16}$$

Proposition 2.4.7 implies that when $a \in W^{1,\infty}$, taking anaother admissible function ψ would modify the family T_a^γ by a family of operators of order -1. Thus, all the results below do not depend on the specific choice of ψ, but we have to fix one to define the the operators T_a^γ.

Proposition 2.4.8. *i) For all $a \in L^\infty$, T_a^γ is of order ≤ 0.*

ii) There is a constant C such that for all $\gamma \geq 1$ and $a \in W^{1,\infty}$, the mapping $u \mapsto au - T_a^\gamma u$ extends from L^2 to H^1 and

$$\|au - T_a^\gamma u\|_{1,\gamma} \;\leq\; C\|a\|_{W^{1,\infty}}\, \|u\|_{L^2}\,, \tag{2.4.17}$$

In particular, for $u \in H^1$ and $j \in \{1, \dots, n\}$:

$$\begin{aligned} \|au - T_a^\gamma u\|_{L^2} \;&\leq\; \frac{C}{\gamma}\|u\|_{L^2}\,, \\ \|a\partial_{x_j} u - T_a^\gamma \partial_{x_j} u\|_{L^2} \;&\leq\; C\|u\|_{L^2}\,. \end{aligned} \tag{2.4.18}$$

Proof. The first statement is clear from Proposition 2.4.7.

Next write

$$\|au - T_a^\gamma u\|_{1,\gamma} \approx \gamma\|au - T_a^\gamma u\|_{L^2} + \sum_j \|\partial_{x_j}\big(au - T_a^\gamma u\big)\|_{L^2}$$

Thus, the first inequality in (2.4.18) clearly follows from (4.2.17). Moreover, the definition (2.4.16) implies that

$$a\partial_{x_j} u - T_a^\gamma \partial_{x_j} u = \partial_{x_j}\big(au - T_a^\gamma u\big) - (\partial_{x_j} a)u + T^\gamma_{\partial_{x_j} a} u$$

The L^2 norm of last two terms is bounded by $C\|a\|_{W^{1,\infty}}\, \|u\|_{L^2}$, using Proposition 2.4.7 for the last one. Therefore the second estimate in (2.4.18) also follows from (2.4.17).

To prove (2.4.17), start from the identity

$$au - T_a^\gamma u \;=\; \sum_k \Delta_k a\, S^\gamma_{k+2} u = f + g\,. \tag{2.4.19}$$

with

$$\begin{aligned} f &:= \sum_k f_k\,, \qquad f_k := \sum_{|j-k|\leq 2} \Delta_k a\, \Delta_j^\gamma u\,, \\ g &:= \sum_{k\geq 3} \Delta_k a\, S^\gamma_{k-3} u \end{aligned}$$

We first consider f. Propositions 2.4.1 and 2.4.3 imply that

$$\|f_k\|_{L^2} \leq C\, 2^{-k} \|a\|_{W^{1,\infty}}\, \rho_k\,, \quad \rho_k := \sum_{|j-k|\leq 2} \|\Delta_j^\gamma u\|_{L^2}\,.$$

Moreover,

$$\sum_k \rho_k^2 \leq C \sum_j \|\Delta_j^\gamma u\|_{L^2}^2 \;\leq\; C\|u\|_{L^2}^2\,.$$

The spectrum of $\Delta_k a$ is contained in the ball $2^{k-1} \leq |\xi| \leq 2^{k+1}$ and the spectrum of $\Delta_j^\gamma u$ is containded in $2^{j-1} \leq (\gamma^2 + |\xi|^2)^{1/2} \leq 2^{j+1}$. Therefore, the spectrum of f_k in contained in the ball $(\gamma^2 + |\xi|^2)^{1/2} \leq 2^{k+4}$. Hence , Proposition 2.4.2 implies that $f \in H^1$ and

$$\|f\|_{1,\gamma} \leq C\,\|a\|_{W^{1,\infty}} \|u\|_{L^2}\,. \tag{2.4.20}$$

It remains to prove a similar estimate for g. Write $\partial_{x_j} g = g'_j + g''_j$ with

$$g'_j = \sum_k \Delta_k a\, S_{k-3}^\gamma \partial_{x_j} u\,, \qquad g''_j = \sum_k \Delta_k \partial_{x_j} a\, S_{k-3}^\gamma u$$

The spectrum of $\Delta_k a\, S_{k-3}^\gamma \partial_{x_j} u$ is contained in $\{2^{k-2} \leq (\gamma^2 + |\xi|^2)^{1/2} \leq 2^{k+2}\}$. Moreover

$$\|\Delta_k a\, S_{k-3}^\gamma \partial_{x_j} u\|_{L^2} \leq C\|a\|_{W^{1,\infty}} \rho_k\,, \qquad \rho_k := \sum_{j\leq k-3} 2^{j-k} \|\Delta_j^\gamma u\|_{L^2}\,.$$

Since

$$\sum \rho_k^2 \leq C \sum_j \|\Delta_j^\gamma u\|_{L^2}^2 \;\leq\; C\|u\|_{L^2}^2\,,$$

Proposition 2.4.2 implies that

$$\|g'_j\|_{L^2} \leq C\|a\|_{W^{1,\infty}} \|u\|_{L^2}\,.$$

The same estimate holds for $g'_0 = \gamma g$, replacing $\partial_{x_j} u$ by γu.

Therefore, to prove that g satisfies the same estimate (2.4.20) as f, it only remains to show that for all j,

$$\|g''_j\|_{L^2} \leq C\,\|a\|_{W^{1,\infty}} \|u\|_{L^2}\,. \tag{2.4.21}$$

The spectrum of $\Delta_k \partial_{x_j} a\, S_{k-3}^\gamma u$ is contained in $\{2^{k-2} \leq (\gamma^2 + |\xi|^2)^{1/2} \leq 2^{k+2}\}$ and (2.4.21) follows from

$$\sum_k \big\|\Delta_k \partial_{x_j} a\, S_{k-3}^\gamma u\big\|_{L^2}^2 \leq C\|a\|_{W^{1,\infty}}^2 \|u\|_{L^2}^2.$$

This estimate is a consequence of the next two results which therefore complete the proof of Proposition 2.4.8. $\square$

Proposition 2.4.9. *There is a constant C such that for all $b \in L^\infty$ and all sequence v_k in L^2 one has*

$$\int \sum_{k\geq 1} |\Delta_k b(x)|^2 \, |v_k(x)|^2 dx \ \leq \ C\|b\|^2_{L^\infty} \, \|v_*\|^2_{L^2} \tag{2.4.22}$$

with

$$v_*(x) \ := \sup_{k\geq 1} \ \sup_{|y-x|\leq 2^{-k}} |v_k(y)| \, . \tag{2.4.23}$$

Lemma 2.4.10. *Consider $u \in L^2$, $v_k = S^\gamma_k u$ and define v_* by (2.4.23). Then there is a constant C, independent of γ, such that*

$$v_*(x) \leq C u^*(x) \ := \sup_R \frac{1}{R^n} \int_{|y-x|\leq R} |u(y)| dy \tag{2.4.24}$$

In particular, $v_* \in L^2$ and there is a constant C, independent of γ, such that $\|v_*\|_{L^2} \leq C\|u\|_{L^2}$.

In [Co-Me] it is proved that when $b \in BMO$, $\sum_k |\Delta_k b(x)|^2 \otimes \delta_{t=2^{-k}}$ is a Carleson measure which immediately implies (2.4.22). The fact that the maximal function u^* belongs to L^2 when $u \in L^2$ is also a well known result from Harmonic Analysis (see e.g. [Co-Me], [St]). For the sake of completeness, we include a short proof of the estimate (2.4.22) in the easier case when $b \in L^\infty$.

Proof of Proposition 2.4.9

a) We show that for all open set $\Omega \subset \mathbb{R}^d$:

$$\sum_{k>0} \|\Delta_k b\|^2_{L^2(\Omega_k)} \leq C \ \mathrm{meas}(\Omega) \, \|b\|^2_{L^\infty} \, , \tag{2.4.25}$$

where Ω_k denotes the set of points $x \in \Omega$ such that the ball $B(x, 2^{-k}) := \{y \in \mathbb{R}^d \, : \, |x-y| < 2^{-k}\}$ is contained in Ω.

Write $b = b' + b''$ with $b' = b 1_\Omega$. Denote by $I(b)$ the left hand side of (2.4.25). Then $I(b) \leq 2I(b') + 2I(b'')$. Therefore, it is sufficient to prove the inequality separately for b' and b''. One has

$$\sum_k \|\Delta_k b'\|^2_{L^2(\Omega_k)} \leq \sum_k \|\Delta_k b'\|^2_{L^2(\mathbb{R}^d)} \leq \|b'\|^2_{L^2} \leq \|b\|^2_{L^\infty} \, \mathrm{meas}(\Omega) \, .$$

Thus, it remains to prove (2.4.25) for b''.

The kernel of Δ_k is $G_k(x) = 2^{kn}G_0(2^k x)$ where G_0 belongs to the Schwartz'class $\mathcal{S}$. Thus

$$\Delta_k b''(x) = \int 2^{kn} G_0(2^k(x-y))b''(y)\, dy\,.$$

On the support of b'', $y \notin \Omega$ and for $x \in \Omega_l$, the distance $|x-y|$ is larger than 2^{-l}. Thus, for $x \in \Omega_l$

$$|\Delta_k b''(x)| \leq \|b''\|_{L^\infty} \int_{\{|y|\geq 2^{-l}\}} 2^{kn}\, |G_0(2^k y)|\, dy = \|b''\|_{L^\infty}\, g^*_{k-l}$$

with

$$g^*_l = \int_{\{|y|\geq 2^l\}} |G_0(y)|\, dy$$

Let $\Omega'_0 := \Omega_0$ and for $l > 0$, let $\Omega'_l = \Omega_l \setminus \Omega_{l-1}$. Then the pointwise estimate above implies that

$$\|\Delta_k b''\|^2_{L^2(\Omega'_l)} \leq \|b\|^2_{L^\infty} \operatorname{meas}(\Omega'_l)\, \big(g^*_{k-l}\big)^2 \tag{2.4.26}$$

Since $\Omega_k = \bigcup_{l\leq k} \Omega'_l$,

$$\sum_{k\geq 1} \|\Delta_k b''\|^2_{L^2(\Omega_k)} = \sum_{k\geq 1} \sum_{l=0}^{k} \|\Delta_k b''\|^2_{L^2(\Omega'_l)}\,.$$

With (2.4.26), this shows that

$$\sum_{k>0} \|\Delta_k b''\|^2_{L^2(\Omega_k)} \leq \sum_{l\geq 0} \sum_{k\geq l} \|b\|^2_{L^\infty} \big(g^*_{k-l}\big)^2 \operatorname{meas}(\Omega'_l)\,.$$

Since $G_0 \in \mathcal{S}$, the sequence g^*_k is rapidly decreasing and thus in $\ell^2(\mathbb{N})$. Therefore,

$$\sum_{k>0} \|\Delta_k b''\|^2_{L^2(\Omega_k)} \leq C\, \|b\|^2_{L^\infty} \sum_{l\geq 0} \operatorname{meas}(\Omega'_l) = C\, \|b\|^2_{L^\infty} \operatorname{meas}(\Omega)\,.$$

showing that b'' also satisfies (2.4.25).

b) Let $b_k = \Delta_k b$. Then

$$\|b_k v_k\|^2_{L^2} = 2 \int_0^\infty \lambda \|b_k\|^2_{L^2(U_k(\lambda))}\, d\lambda\,, \quad \text{where} \quad U_k(\lambda) = \{|v_k| > \lambda\}$$

For $\lambda > 0$, let $\Omega(\lambda) = \{|v_*| > \lambda\}$. This is the set of points x such that there are $k > 0$ and y such that $|x - y| < 2^{-k}$ and $|v_k(y)| > \lambda$. Thus $\Omega(\lambda)$ is open and if $|v_k(y)| > \lambda$, the ball $B(y, 2^{-k})$ is contained in $\Omega(\lambda)$. This shows that for all k, $U_k(\lambda) \subset \Omega_k(\lambda)$, where the Ω_k's are defined as in (2.4.25) Thus

$$\sum_{k>0} \|b_k\|^2_{L^2(U_k(\lambda))} \leq \sum_{k>0} \|b_k\|^2_{L^2(\Omega_k(\lambda))} \leq C \|b\|^2_{L^\infty} \operatorname{meas}(\Omega(\lambda)) \,,$$

and

$$\sum_{k>0} \|b_k v_k\|^2_{L^2} \leq 2C \|b\|^2_{L^\infty} \int_0^\infty \lambda \operatorname{meas}(\Omega(\lambda))\, d\lambda = C \|b\|^2_{L^\infty} \|v_*\|^2_{L^2}$$

which is (2.4.22). □

Proof of Lemma 2.4.10

S_k^γ is the convolution operator with $\widetilde{\chi}_k^\gamma$, the inverse Fourier transform of $\chi(2^{-k}\xi, 2^{-k}\gamma)$. There is C, independent of γ such that

$$|\widetilde{\chi}_k^\gamma(x)| \;\leq\; C\, 2^{nk}(1 + 2^k|x|)^{-n-1}\,.$$

Thus

$$|v_k(x - x')| \leq \; C 2^{nk} \int (1 + 2^k|y - x'|)^{-n-1}\, |u(x - y)| dy\,.$$

Splitting the domain of integration into annuli $|y| \approx 2^{j-k}$, $j \geq 0$ implies that

$$\sup_{|x'| \leq 2^{-k}} |v_k(x - x')| \;\leq\; C'\, 2^{nk} \sum_{j \geq 0} 2^{-j(n+1)} 2^{n(j-k)} u^*(x)$$

and the lemma follows. □

2.4.4 Symbolic calculus

Theorem 2.4.11. *Consider $a \in \Gamma_1^m$ and $b \in \Gamma_1^{m'}$. Then $ab \in \Gamma_1^{m+m'}$ and $T_a^\gamma \circ T_b^\gamma - T_{ab}^\gamma$ is of order $\leq m + m' - 1$. This extends to matrix valued symbols and operators.*

Remark. The definition of the operators T_a^γ involves the choice of an admissible function ψ. However, Proposition 1.7 implies that the result does not depend on the particular choice of ψ. This is why we do not mention any more the function ψ in the notations.

Proof. Changing ψ if necessary, we can assume that the parameter ε_2 is small enough.

Let σ_a and σ_b denote the symbols associated to a and b. Thus $T_a^\gamma \circ T_b^\gamma = P_{\sigma_a}^\gamma \circ P_{\sigma_b}^\gamma = P_\sigma^\gamma$ with

$$\sigma(x,\xi,\gamma) := e^{-ix\xi}(P_{\sigma_a}^\gamma \rho_{\xi,\gamma})(x)\,, \quad \rho_{\xi,\gamma}(x) := e^{ix\xi}\sigma_b(x,\xi,\gamma)\,.$$

Proposition 1.7 implies that the spectrum of $\sigma(\,\cdot\,,\xi,\eta)$ is contained in $|\eta| \le 5\varepsilon_2(\gamma^2 + |\xi|^2)^{1/2}$. Thus σ satisfies the spectral property if ε_2 is small enough. In particular, there is an admissible function θ such that

$$\sigma(x,\xi,\gamma) \;=\; \int H(x,y,\xi,\gamma)\,\sigma_b(y,\xi,\gamma)\,dy$$

with

$$H(x,y,\xi,\gamma) \;:=\; \frac{1}{(2\pi)^n}\int e^{i(x-y)\eta}\sigma_a(x,\xi+\eta,\gamma)\theta(\eta,\xi,\gamma)\,d\eta\,.$$

Use Taylor's formula to compute $r := \sigma - \sigma_a\sigma_b$. One has

$$r(x,\xi,\gamma) \;=\; \sum_{j=1}^n \frac{1}{i}\int G_j(x,x-y,\xi,\gamma)\,(\partial_{x_j}\sigma_b)(y,\xi,\gamma)\,dy$$

$$G_j(x,y,\xi,\gamma) \;=\; \frac{1}{(2\pi)^n}\int e^{iy\eta}\left(\int_0^1 \partial_{\xi_j}\sigma_a(x,\xi+t\eta,\gamma)dt\right)\theta(\eta,\xi,\gamma)\,d\eta\,.$$

On the support of θ one has $\gamma + |\xi| \approx \gamma + |\xi + t\eta|$. Using the estimates on the symbols, one obtains that $r \in \Gamma_0^{m+m'-1}$. Because σ and $\sigma_a\sigma_b$ both satisfy the spectral condition, we conclude that $r \in \Sigma_0^{m+m'-1}$. Therefore, $T_a^\gamma \circ T_b^\gamma - P_{\sigma_a\sigma_b}^\gamma = P_r^\gamma$ is of order $\le m + m' - 1$.

On the other hand, Proposition 1.5 implies that $a - \sigma_a \in \Gamma_0^{m-1}$ and $b - \sigma_b \in \Gamma_0^{m'}$. and $ab - \sigma_{ab} \in \Gamma_0^{m+m'-1}$. Thus, $\sigma_a\sigma_b - \sigma_{ab} \in \Sigma_0^{m+m'-1}$ and the theorem follows.

$\square$

Similarly, the next two theorems are extensions of know results ([Bo], [Mey]) to the framework of parameter depending operator.

Theorem 2.4.12. *Consider a matrix valued symbol $a \in \Gamma_1^m$. Denote by $(T_a^\gamma)^*$ the adjoint operator of T_a^γ and by $a^*(x,\xi,\gamma)$ the adjoint of the matrix $a(x,\xi,\gamma)$. Then $(T_a^\gamma)^* - T_{a^*}^\gamma$ is of order $\le m-1$.*

Theorem 2.4.13. *Consider a $N \times N$ matrix symbol $a \in \Gamma_1^m$. Assume that there is constant $c > 0$ such that*

$$\forall (x, \xi, \gamma) : \quad \operatorname{Re} a(x, \xi, \gamma) \geq c(\gamma^2 + |\xi|^2)^{m/2}$$

Then, there is a constant γ_0 such that

$$\forall u \in H^m(\mathbb{R}^n), \ \forall \gamma \geq \gamma_0, : \quad \frac{c}{2} \|u\|_{m/2,\gamma}^2 \leq \operatorname{Re} \left(\!\left(T_a^\gamma u, u \right)\!\right)_{L^2} \tag{2.4.27}$$

2.5 Proof of the main estimate

2.5.1 Paralinearisation

Consider the space $\mathbb{R}^n$ with variables (t, y). We use the paradifferential calculus of section 1 in this space and use the notation T_a^γ for the paraproducts or operators. This calculus directly applies to functions defined on the boundary $\{x_n = 0\}$. We extend it to the interior as follows. When a and u are functions on $\Omega = \mathbb{R}^n \times [0, \infty[$, we still denote by $T_a^\gamma u$ the tangential paraproduct such that for all x_n

$$(T_a^\gamma u)(\,\cdot\,, x_n) \,=\, T^\gamma_{a(\,\cdot\,, x_n)} u(\,\cdot\,, x_n)\,. \tag{2.5.1}$$

More generally, we still call Γ_k^m the space of symbols $a(t, y, x_n, \tau, \eta, \gamma)$ such that the mapping $x_n \mapsto a(\,\cdot\,, x_n)$ is bounded into the space Γ_k^m of Definition 2.4.4. The formula (2.5.1) extends to this case.

Because $\mathcal{A}_n$ is invertible, we can multiply $\mathcal{L}_a^\gamma$ by $\mathcal{A}_n^{-1}(a)$ and therefore we can assume that

$$\mathcal{A}_n \,=\, Id\,. \tag{2.5.2}$$

The coefficients $\widetilde{\mathcal{A}_j} := \mathcal{A}_j(a(x))$, $\widetilde{b}_j := b_j(a(t, y, 0))$ and $\widetilde{M} := M(a(t, y, 0))$ are Lipschitzean with $W^{1,\infty}$ norm dominated by $C(K)$. Therefore, Proposition 2.4.8 implies that

$$\|\widetilde{b}_j \partial_j \psi - T^\gamma_{\widetilde{b}_j} \partial_j \psi\|_0 \leq \, C(K) \|\psi\|_0\,, \quad \|\gamma \widetilde{b}_0 \psi - \gamma T^\gamma_{\widetilde{b}_0} \psi\|_0 \leq \, C(K) \|\psi\|_0\,.$$

Because $\gamma T_c^\gamma = T_{\gamma c}^\gamma$, and $T_c^\gamma \partial_j = i T^\gamma_{c\eta_j}$, this implies that

$$\Big\| \gamma \widetilde{b}_0 \psi + \sum_j \widetilde{b}_j \partial_j \psi - T^\gamma_{i\widetilde{b}} \psi \Big\|_0 \,\leq\, C(K) \|\psi\|_0 \,\leq\, \frac{1}{\gamma} C(K) \|\psi\|_{1,\gamma}\,. \tag{2.5.3}$$

where the symbol $\widetilde{b}$ is given by

$$\widetilde{b}(t,y,\tau,\eta,\gamma) \;=\; b(a(t,y,0),\tau - i\gamma,\eta) \tag{2.5.4}$$

and $b(a,\hat{\tau},\eta) := b_0(a)\hat{\tau} + \sum_{j=1}^{n-1} b_j(a)\eta_j$. Similarly,

$$\|\widetilde{M}v - T^\gamma_{\widetilde{M}}v\|_0 \le \;\frac{1}{\gamma}C(K)\|v\|_0.$$

Thus,

$$\|\mathcal{B}^\gamma_a(\psi,v) \;-\, i\,T^\gamma_{\widetilde{b}}\psi \;-\; T^\gamma_{\widetilde{M}}v\,\|_0 \;\le\; \frac{1}{\gamma}C(K)\Big(\|\psi\|_{1,\gamma} \;+\; \|v_{|x_n=0}\|_0\Big)\,. \tag{2.5.5}$$

In the interior, using the definition (2.5.1), we apply Proposition 2.4.8 for all fixed x_n and then integrate in x_n. This shows that

$$\|\widetilde{\mathcal{A}}_j\partial_j v - T^\gamma_{\widetilde{\mathcal{A}}_j}\partial_j v\|_0 \le\; C(K)\|v\|_0\,, \quad \|\gamma\widetilde{\mathcal{A}}_0 v - T^\gamma_{\gamma\widetilde{\mathcal{A}}_0}v\|_0 \le\; C(K)\|v\|_0\,.$$

Therefore, introducing the tangential symbol $\widetilde{\mathcal{H}}(x,\tau,\eta,\gamma) \;=\; \mathcal{H}(a(x),\tau - i\gamma,\eta)$ with

$$\mathcal{H}(a,\hat{\tau},\eta) \;:=\hat{\tau}\mathcal{A}_0(a) \;+\; \sum_{j=1}^{n-1}\mathcal{A}_j(a)\,\eta_j\,, \tag{2.5.6}$$

we have proved that

$$\|\mathcal{L}^\gamma_a v \;-\partial_n v - iT^\gamma_{\widetilde{\mathcal{H}}}v\|_0 \;\le\; C(K)\,\|v\|_0\,. \tag{2.5.7}$$

Therefore, we have proved :

Proposition 2.5.1. *Let a be a Lipschitzean function on Ω valued in $\mathcal{K}$ satisfying $\|a\|_{W^{1,\infty}} \le K$. Then there is $C(K)$ such that for all $\gamma \ge 1$ and for all $(v,\psi) \in H^1_\gamma(\Omega) \times H^1_\gamma(\omega)$, the estimates* (2.5.5) *and* (2.5.7) *hold.*

Therefore, to prove the a-priori estimate (2.1.9), increasing γ_0 if necessary, it is sufficient to prove the same estimate with

$$\widetilde{\mathcal{L}}^\gamma \;:=\; \partial_n + iT^\gamma_{\widetilde{\mathcal{H}}}\,, \quad \widetilde{\mathcal{B}}^\gamma \;:=\; (iT^\gamma_{\widetilde{b}}\,,\,T^\gamma_{\widetilde{M}}) \tag{2.5.8}$$

in place of $\mathcal{L}^\gamma_a$ and $\mathcal{B}^\gamma_a$ respectively.

2.5.2 Eliminating ψ

Proposition 1.4.3 implies that the vectors $b_j(a)$ are linearly independent and there is $c > 0$ such that for all $a \in \mathcal{K}$ and $(\hat\tau, \eta) \in \mathbb{C} \times \mathbb{R}^n$,

$$c(|\hat\tau| + |\eta|) \le |b(a, \hat\tau, \eta)| \le \frac{1}{c}\big(|\hat\tau| + |\eta|\big)\,. \tag{2.5.9}$$

As in (2.3.5), introduce the projector on $b^\perp$

$$\Pi(a, \hat\tau, \eta)\, h \;=\; h \;-\; \frac{(h, b(a, \hat\tau, \eta))}{|b(a, \hat\tau, \eta)|^2} b(a, \hat\tau, \eta)\,. \tag{2.5.10}$$

The ellipticity (2.5.9) implies that Π is smooth in $(a, \hat\tau, \eta)$ for $a \in \mathcal{K}$, $\hat\tau \in \mathbb{C}$ and $\eta \in \mathbb{R}^n$ provided that $|\hat\tau| + |\eta| \ne 0$. Moreover, it is homogeneous of degree zero in $(\hat\tau, \eta)$. Therefore,

$$\widetilde\Pi(t, y, \tau, \gamma, \eta) \;:=\; \Pi(a(t, y, 0), \tau - i\gamma, \eta).$$

is a symbol in Γ^0_1. Because $\Pi b = 0$, one has $\widetilde\Pi \widetilde b = 0$ and Theorem 2.4.11 implies that

$$\begin{aligned}
&\|T^\gamma_{\widetilde\Pi} T^\gamma_{\widetilde b} \psi\|_0 \;\le\; C(K)\|\psi\|_0 \;\le\; \frac{1}{\gamma} C(K) \|\psi\|_{1,\gamma}\,,\\
&\|T^\gamma_{\widetilde\Pi} T^\gamma_{\widetilde M} v - T^\gamma_{\widetilde\Pi \widetilde M} v\|_0 \;\le\; C(K)\|v_{|x_n=0}\|_{-1,\gamma} \;\le\; \frac{1}{\gamma} C(K) \|v_{|x_n=0}\|_0\,.
\end{aligned} \tag{2.5.11}$$

Therefore, introducing the boundary symbol $B(z) = \Pi(z)M(z)$ as in (2.3.6) and $\widetilde B(x, \tau, \eta, \gamma) := B(a(x), \tau, \eta, \gamma)$, (2.5.11) implies that

$$\|T^\gamma_{\widetilde B} v\|_0 \;\le C(K)\|i\, T^\gamma_{\widetilde b} \psi \,+\, T^\gamma_{\widetilde M} v\,\|_0 + \frac{C(K)}{\gamma}\Big(\|\psi\|_{1,\gamma} + \|v_{|x_n=0}\|_0\Big)\,. \tag{2.5.12}$$

Similarly, introduce the row vector b^*. By (2.5.9), the scalar $p := b^* b$ is homogeneous of degree two and never vanishes on $\mathbb{C} \times \mathbb{R}^{n-1}$. This means that for γ large enough $\widetilde p = p(a(t, y, 0), \tau - i\gamma, \eta) \in \Gamma^2_1$ is elliptic and Theorem 2.4.13 applies. In addition Theorems 2.4.11 and 2.4.12 imply that $T^\gamma_{\widetilde p} - (T^\gamma_{\widetilde b})^* T^\gamma_{\widetilde b}$ is of order less than or equal to one. Adding up, one obtains

$$\begin{aligned}
\|\psi\|^2_{1,\gamma} \;&\le C(K)\, (T^\gamma_{\widetilde p} \psi, \psi)_{L^2} \le\; C(K)\, \|T^\gamma_{\widetilde b} \psi\|^2_0\\
&\le C(K)\Big(\|i\, T^\gamma_{\widetilde b} \psi \,+\, T^\gamma_{\widetilde M} v\,\|^2_0 \,+\, \|v_{|x_n=0}\|^2_0\Big)
\end{aligned} \tag{2.5.13}$$

Using (2.5.5), (2.5.8), (2.5.12) and (2.5.13), we see that the next result implies Theorem 2.1, increasing γ_0 is necessary.

Proposition 2.5.2. *For all $K > 0$, there are $\gamma_0 > 0$ and C such that for all Lipschitzean function a on Ω valued in $\mathcal{K}$ satisfying $\|a\|_{W^{1,\infty}} \leq K$, for all $\gamma \geq \gamma_0$ and for all $(v, \psi) \in H^1_\gamma(\Omega) \times H^1_\gamma(\omega)$, the following estimate holds*

$$\gamma \| v \|_0^2 + \| v_{|x_n=0} \|_0^2 \leq C \left(\frac{1}{\gamma} \| \widetilde{\mathcal{L}}^\gamma v \|_0^2 + \| T^\gamma_{\widetilde{B}} v \|_0^2 \right). \tag{2.5.14}$$

2.5.3 End of the proof of Theorem 2.1.3

We look for a symmetrizer $\mathbf{R}^\gamma$ as a paradifferential operator $T^\gamma_{\widetilde{\mathcal{R}}}$ and use the symbolic calculus to deduce the estimates (2.2.5-6-7-8) for operators from estimates for the symbols.

$$\mathbf{R}^\gamma = \frac{1}{2} \big(T^\gamma_{\widetilde{\mathcal{R}}} + (T^\gamma_{\widetilde{\mathcal{R}}})^* \big) \tag{2.5.15}$$

As above, we choose $\widetilde{\mathcal{R}}(x, \tau, \eta, \gamma) = \mathcal{R}(a(x), \tau, \eta, \gamma)$ with a self adjoint symbol $\mathcal{R}$ and Theorem 1.12 implies that $S^\gamma = T^\gamma_{\widetilde{\mathcal{R}}}$ modulo an operator of order ≤ -1. The symbol $\mathcal{R}$ is given by Theorem 2.4.5. However, we need a more precise version of estimate (2.3.10) which follows from Kreiss' construction. The useful properties of $\mathcal{R}$ are listed in the next theorem.

With notations as in §2.3, we denote by $z = (a, \tau, \eta, \gamma)$ the set of parameters. The component a remains in $\mathcal{U}$ and (τ, η, γ) belongs to $\overline{\mathbb{R}}^{n+1}_+ \backslash \{0\} := \{(\tau, \eta, \gamma) \in \mathbb{R}^{n+1} \backslash \{0\} : \gamma \geq 0\}$.

Theorem 2.5.3 (Kreiss). ([Kr], [Ch-Pi]). *Suppose that Assumptions* 2.1.1 *and* 2.3.2 *are satisfied. There exists a C^∞ function $\mathcal{R}$ on $\mathcal{U} \times \overline{\mathbb{R}}^{n+1}_+ \backslash \{0\}$ with values in the space of self adjoint matrices, homogeneous of degree zero in (τ, η, γ) and such that :*

i) for all compact $\mathcal{K} \subset \mathcal{U}$ there is $c > 0$ such that for all $z = \in \mathcal{K} \times \overline{\mathbb{R}}^{n+1}_+ \backslash \{0\}$

$$\mathcal{R}(z) \ + C B(z)^* B(z) \ \geq \ c \, Id \,, \tag{2.5.16}$$

ii) there are finite sets of C^∞ matrices on $\mathcal{U} \times \overline{\mathbb{R}}^{n+1}_+ \backslash \{0\}$, $\{V_l(\cdot)\}$, $\{H_l(\cdot)\}$ and $\{E_l(\cdot)\}$ such that

$$\operatorname{Im} \mathcal{R}(z) \mathcal{H}(z) \ = \sum_l V_l^*(z) \begin{bmatrix} \gamma H_l(z) & 0 \\ 0 & E_l(z) \end{bmatrix} V_l(z) \,. \tag{2.5.17}$$

Moreover, for all l, V_l is homogeneous of degree zero in (τ, η, γ), $H_l(z)$ is self adjoint and homogeneous of degree zero, and $E_l(z)$ is self adjoint and *homogeneous of degree one. In addition, for all compact $\mathcal{K} \subset \mathcal{U}$ there is a constant $c > 0$ such that for all $z \in \mathcal{K} \times \overline{\mathbb{R}}_+^{n+1} \backslash \{0\}$*

$$\sum_j V_l^*(z) V_l(z) \geq c Id\,, \quad H_l \geq c Id\,, \quad E_l \geq c(|\tau| + |\eta| + \gamma)\,. \tag{2.5.18}$$

Note that the dimension of H_l (and E_l) may depend on l.

Remark. The identity (2.5.17) and the estimates (2.5.18) imply (2.3.10). In the constant coefficient case, (2.3.10) implies the same estimate on the operators. In the variable coefficient case, the analogue is the the *sharp Gårding* inequality. However, this estimate requires that the coefficients are at least C^2. With (2.5.17), the proof uses the usual Gårding inequality, with the great advantage that only one x-derivative is needed for the symbol (see Theorem 2.4.13).

Proof of Proposition 2.5.2.

We show that the assumptions of Lemma 2.2.1 are satisfied.

a) The symbol $\widetilde{\mathcal{R}}$ belongs to Γ_1^0. Thus, Proposition 2.4.7 implies

$$\|\mathbf{R}^\gamma v\|_0 \leq C \|v\|_0\,. \tag{2.5.19}$$

Moreover, $[\partial_n, T^\gamma_{\widetilde{\mathcal{R}}}] = T^\gamma_{\partial_n \widetilde{\mathcal{R}}}$ and $\partial_n \widetilde{\mathcal{R}}$ belong to Γ_0^0. Thus

$$\|[\partial_n, \mathbf{R}^\gamma] v\|_0 \leq C \|v\|_0\,. \tag{2.5.20}$$

b) $\widetilde{\mathcal{H}}$ is a symbol of degree one. Introduce

$$\mathbf{H}^\gamma := i\, T^\gamma_{\mathcal{H}} \tag{2.5.21}$$

Theorems 2.4.11 and 2.4.12 imply that

$$\mathbf{R}^\gamma \mathbf{H}^\gamma + (\mathbf{H}^\gamma)^* \mathbf{R}^\gamma = T^\gamma_{i(\widetilde{\mathcal{R}}\widetilde{\mathcal{H}} - \widetilde{\mathcal{H}}^* \widetilde{\mathcal{R}})} + \mathbf{E}^\gamma$$

where $\mathbf{E}^\gamma$ if of order ≤ 0. Introduce $\mathcal{S} := \operatorname{Im} \mathcal{R}\mathcal{H} = -i(\mathcal{R}\mathcal{H} - \mathcal{H}^*\mathcal{R})$ and $\widetilde{\mathcal{S}} = \mathcal{S}(a(x), \tau, \eta, \gamma) \in \Gamma_1^1$. Denote by F_l the block diagonal matrices with blocks $(\gamma H_l, E_l)$. Then, (2.5.17) and theorems 2.4.11, 2.4.12 imply that

$$\operatorname{Re} \left(\!\left(T^\gamma_{\widetilde{\mathcal{S}}} v, v \right)\!\right) = \sum_l \operatorname{Re} \left(\!\left(T^\gamma_{\widetilde{F}_l} w_l, w_l \right)\!\right) + O(\|v\|_0^2)\,, \quad w_l := T^\gamma_{\widetilde{V}_l} v\,.$$

Theorem 2.4.13 implies that for γ large enough

$$\operatorname{Re} \left(\!\left(T^\gamma_{\widetilde{H}_l} w, w \right)\!\right) \geq c \|w\|_0^2\,, \quad \operatorname{Re} \left(\!\left(T^\gamma_{\widetilde{E}_l} w, w \right)\!\right) \geq c \|w\|^2_{1/2,\gamma} \geq c\gamma \|w\|_0^2\,.$$

Note that $T^\gamma_{\widetilde F_l}$ is block diagonal with blocks $(\gamma T^\gamma_{\widetilde H_l}, T^\gamma_{\widetilde E_l})$. Thus, denoting by (w'_l, w''_l) the components of w_l corresponding to the different blocks, one has

$$\mathrm{Re}\,\big(\!\big(T^\gamma_{\widetilde F_l} w_l, w_l\big)\!\big) \;= \mathrm{Re}\,\big(\!\big(T^\gamma_{\gamma\widetilde H_l} w'_l, w'_l\big)\!\big) + \mathrm{Re}\,\big(\!\big(T^\gamma_{\widetilde E_l} w''_l, w''_l\big)\!\big) \geq c\gamma\|w_l\|_0^2\,.$$

Moreover, $\sum V_l^* V_l$ is elliptic by (2.5.18) and the symbolic calculus implies that

$$\|v\|_0^2 \leq \sum C\|w_l\|_0^2 \,+\, O(\frac{1}{\gamma}\|v\|_0^2)\,.$$

Adding up, this shows that for γ large enough , one has

$$\mathrm{Re}\,\big(\!\big(T^\gamma_{\widetilde{\mathcal S}} v, v\big)\!\big)_{L^2(\Omega)} \geq c\gamma\|v\|_0^2$$

and, increasing γ if necessary,

$$\mathrm{Re}\,\big(\!\big((\mathbf{R}^\gamma\mathbf{H}^\gamma + (\mathbf{H}^\gamma)^*\mathbf{R}^\gamma)v, v\big)\!\big)_{L^2(\Omega)} \leq -c\gamma\|v\|_0^2\,, \tag{2.5.22}$$

c) Theorem 2.4.13 and (2.5.16) imply that for γ large enough,

$$c\|v\|_0^2 \;\leq\; \big(\!\big(\mathbf{R}^\gamma(0)v(0), v(0)\big)\!\big) + \,C\,\mathrm{Re}\,\big(\!\big(T^\gamma_{\widetilde B^*\widetilde B} v(0), v(0)\big)\!\big)\,.$$

Theorems 2.4.11 and 2.4.12 imply that the last term is

$$\|T^\gamma_{\widetilde B} v(0)\|^2 \,+\, O(\|v\|_{-1,\gamma}\|v\|_0) = \|T^\gamma_{\widetilde B} v(0)\|^2 \,+\, \frac{1}{\gamma}O(\|v\|_0^2)$$

Therefore, for γ large enough

$$\big(\!\big(\mathcal{R}^\gamma(0)v(0), v(0)\big)\!\big)_{L^2(\mathbb{R}^n)} + \,C\|T^\gamma_{\widetilde B} v(0)\|^2 \,\geq c\|v(0\|_0^2\,. \tag{2.5.23}$$

d) The estimates (2.5.19-20-22-23) show that the assumptions of Lemma 2.2.1 are satisfied for the equation $\partial_n + \mathbf{H}^\gamma$ and the boundary operator $T^\gamma_{\widetilde B}$. Thus, Lemma 2.2.1 implies the estimate (2.5.14) and the proof of Theorem 2.1.3 is complete.

$\square$

3 Well posedness of the linearized shock front equations

3.1 The main result

In this section we consider the initial-boundary value problem for the linearized shock equations (1.3.6). We prove the existence and uniqueness of a solution $v \in C^0(L^2)$. The equations read

$$
\left\{
\begin{array}{ll}
\mathcal{L}_a v := \displaystyle\sum_{j=0}^{n} \mathcal{A}_j \partial_j v = f \,, & \text{on } x_n > 0 \,, \\
\mathcal{B}_a(\psi, v) := b \,\mathrm{grad}\psi + M v = g \,, & \text{on } x_n = 0 \,, \\
v_{|t=0} = v_0 \,, \quad \psi_{|t=0} = \psi_0 \,, & \text{on } t = 0 \,,
\end{array}
\right.
\tag{3.1.1}
$$

where the coefficients are C^∞ functions of $a = (u^+, u^-, \nabla\Phi)$. We consider $a \in W^{1,\infty}(\Omega)$ valued in a compact set of an open set $\mathcal{U} \subset \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{n+1}$ where Assumptions 1.1.1, 2.1.1 and 2.3.3 are satisfied.

Introduce the notations $\Omega_T^+ =]0, T[\times \mathbb{R}^n_+$ and $\omega_T^+ =]0, T[\times \mathbb{R}^{n-1}$. Consider the problem (3.2.1) with

$$
f \in L^2(\Omega_T^+), \quad g \in L^2(\omega_T^+), \quad v_0 \in L^2(\mathbb{R}^n_+), \quad \psi_0 \in H^{1/2}(\mathbb{R}^{n-1}) \,.
\tag{3.1.2}
$$

Theorem 3.1.1. *Suppose that Assumptions* 2.1.1 *and* 2.3.2 *are satisfied. For all data* (3.1.2) *the initial boundary value problem* (3.1.1) *has a unique solution* $(v, \psi) \in L^2(\Omega_T^+) \times H^1(\omega_T^+)$ *and* $v \in C^0([0, T]; L^2(\mathbb{R}^n_+))$.

Moreover, for all compact subset $\mathcal{K} \subset \mathcal{U}$ *and all real* K, *there are constant* C *and* γ_0 *such that, if* a *takes its values in* $\mathcal{K}$ *and* $\|a\|_{W^{1,\infty}(\Omega_T^+)} \le K$, *the solutions satisfy for all* $\gamma \ge \gamma_0$ *and all* $t \in [0, T]$

$$
\begin{aligned}
& e^{-2\gamma t} \|v(t)\|^2_{L^2(\mathbb{R}^n_+)} + \gamma \|v\|^2_{L^2_\gamma(\Omega_t^+)} + \|v_{|x_n=0}\|^2_{L^2_\gamma(\omega_t^+)} + \|\psi\|^2_{H^1_\gamma(\omega_t)} \\
& \quad \le C \left(\frac{1}{\gamma} \|f\|^2_{L^2_\gamma(\Omega_t^+)} + \|g\|^2_{L^2_\gamma(\omega_t^+)} + \|v_0\|^2_{L^2(\mathbb{R}^n_+)} + \|\psi_0\|^2_{H^{1/2}(\mathbb{R}^{n-1})} \right) .
\end{aligned}
\tag{3.1.3}
$$

The main novelty in Theorem 3.1.1 is the term $\|v(t)\|_{L^2}$ in the left hand side of (3.1.3). This kind of estimate is well known for dissipative boundary conditions, when the energy estimate is given by using $\mathcal{S}$ as a symmetrizer. It is proved in [Ra] for mixed problems with C^∞ coefficients, satisfying the uniform Lopatinski condition. However, the proof in [Ra] does dot easily extend to C^1 coefficients.

The proof is in several steps.

• One considers first the case where the initial data v_0 and ψ_0 vanish. Extending f and g by zero in the past, i.e. for $t < 0$, reduces the problem to solve the equations (2.1.1). On $\Omega = \mathbb{R} \times \mathbb{R}^n_+$, the existence of weak solutions in weighted spaces $e^{\gamma t} L^2(\Omega)$ follows from an a-priori estimate for a dual problem (see [Ch-Pi]). Next one shows that the solution is strong and satisfies the energy estimate of Theorem 2.1.3. In particular, this implies uniqueness. Finally, a classical argument extends the existence and uniqueness result to data and solutions on $\Omega_T =]-\infty, T] \times \mathbb{R}^n_+$.

The energy estimate (2.1.4) gives control of the traces $v_{|x_n=0}$. Knowing this, the usual integration by parts for symmetric systems provides an estimate for $\|v(t)\|_{L^2}$.

• When the Cauchy data are regular, one can solve (3.1.1) in two steps. First one extends v_0 in $\{x_n < 0\}$ and solve the Cauchy problem without boundary $\mathcal{L}_a v^1 = 0$ with initial data v_0. Next, consider ψ^1 such that $\psi^1_{|t=0} = \psi_0$. The solution of (3.1.1) is $(v, \psi) = (v^1, \psi^1) + (v^2, \psi^2)$ where (v^2, ψ^2) satisfies (3.1.1) with zero initial condition and g replaced by $g - Mv^1 - b\mathrm{grad}\psi^1$. This method requires v_0 to be smooth enough so that the trace of v^1 on the boundary is in L^2. The main new ingredient, is to prove that the solution (v, ψ) satisfies the estimate (3.1.3). This follows from a duality argument. The first step implies that the solutions of the backward initial boundary value dual problem with vanishing initial conditions satisfies $C^0(L^2)$ estimate, which implies (3.1.3).

• The existence of solutions when the initial data is in L^2 follows by a density argument from the existence and the uniform estimate (3.1.3) for smooth data.

3.2 The dual problem

Let $S(u)$ denote the symmetrizer given by Assumption 1.1.1. The symmetrizer of $\mathcal{L}$ is then

$$\mathcal{S} = \begin{bmatrix} S^+ & 0 \\ 0 & S^- \end{bmatrix}, \quad S^\pm := S(u^\pm)\,. \tag{3.2.1}$$

The adjoint operator of $\mathcal{SL}$ is $-\mathcal{SL} + \sum \partial_j(\mathcal{S}\mathcal{A}_j)$. For H^1 functions v and w on $\Omega := \mathbb{R}^n \times]0, \infty[$, one has

$$\begin{aligned}\big(\!\big(\mathcal{SL}v, w\big)\!\big)_{L^2(\Omega)} - \big(\!\big(v, (\mathcal{SL})^* w\big)\!\big)_{L^2(\Omega)} \\ = -\big(\!\big(\mathcal{S}\mathcal{A}_n v_{|x_n=0}, w_{|x_n=0}\big)\!\big)_{L^2(\omega)}\,.\end{aligned} \tag{3.2.2}$$

We determine boundary conditions for $(\mathcal{SL})^*$ such that the boundary term vanishes when v and w satisfy the homogeneous boundary conditions.

Lemma 3.2.1. *There are $N \times 2N$ matrices $M_1(a)$, $R(a)$ and $R_1(a)$ depending smoothly on $a \in \mathcal{K}$, such that for all $(v, w) \in \mathbb{C}^{2N} \times \mathbb{C}^{2N}$,*

$$\big(\mathcal{S}(a)\mathcal{A}_n(a)v, w\big)_{\mathbb{C}^{2N}} \;=\; \big(Mv, R_1 w\big)_{\mathbb{C}^N} + \big(M_1 v, Rw\big)_{\mathbb{C}^N} \tag{3.2.3}$$

Proof. Recall that $Mv = M(v^+, v^-) = M^- u^- - M^+ u^+$ with $M^\pm := A_n^\sharp(u^\pm, \nabla\Phi)$ (see (1.3.3)). Introduce the matrices

$$\begin{aligned}
&Rv = \frac{1}{2\kappa}(S^-)^* v^- - \frac{1}{2\kappa}(S^+)^* v^+ \,,\\
&R_1 v = \frac{1}{2\kappa}(S^-)^* v^- + \frac{1}{2\kappa}(S^+)^* v^+ \,,\\
&M_1 u = M^- u^- + M^+ u^+ \,,
\end{aligned} \tag{3.2.4}$$

with $\kappa = \partial_n\Phi$. They depend smoothly on $a \in \mathcal{U}$. Recall also that $\mathcal{A}_n$ is block diagonal with diagonal terms $\widetilde{A}_n(u^\pm, \nabla\Phi) = \kappa^{-1} M^\pm$ (see (1.4.1)). Thus

$$\big(\mathcal{S}(a)\mathcal{A}_n(a)v, w\big)_{\mathbb{C}^{2N}} \;=\; \frac{1}{\kappa}\big(S^+ M^+ v^+, w^+\big)_{\mathbb{C}^N} + \frac{1}{\kappa}\big(S^- M^- v^-, w^-\big)_{\mathbb{C}^N}$$

and (3.2.3) follows. $\square$

Suppose that $(v, w) \in H^1(\Omega) \times H^1(\Omega)$, $\psi \in H^1(\omega)$ and $Mv_{|x_n=0} + b\nabla\psi = 0$. Then the boundary term in (3.2.2) is

$$\big(\!\big(\mathcal{S}\mathcal{A}_n v, w\big)\!\big)_{L^2(\omega)} = \big(\!\big(M_1 v, Rw\big)\!\big)_{L^2(\omega)} - \big(\!\big(\nabla\psi, b^* R_1 w\big)\!\big)_{L^2(\omega)} \,. \tag{3.2.5}$$

It vanishes when w satisfies the homogeneous dual boundary conditions

$$\begin{cases} Rv = 0\,, \\ \mathrm{div} b^* R_1 v \;=\; 0\,. \end{cases} \tag{3.2.6}$$

Proposition 3.2.2. *The adjoint problem $(\mathcal{S}\mathcal{L})^*$ together with the boundary conditions* (3.2.6) *satisfies the backward uniform Lopatinski condition.*

Proof. The backward Lopatinski condition is the analogue of Definition 1.4.1 when one changes t into $-t$. Then ∂_t is changed into $-\partial_t$ and the weights γ into $-\gamma$. To prove the proposition, it is sufficient to consider the constant coefficient case. Thus, we consider

$$((\mathcal{S}\mathcal{L})^*)^{-\gamma} = (\mathcal{S}\mathcal{L}^\gamma)^* \;=\; -\sum_{j=0}^{n} \mathcal{S}\mathcal{A}_j \partial_j \;+\; \gamma \mathcal{S}\mathcal{A}_0 \,. \tag{3.2.7}$$

and the boundary conditions

$$(\mathcal{B}^*)^{-\gamma} := \left[\begin{array}{l} R \\ \mathrm{div}^{-\gamma} b^* R_1 \end{array}\right] \tag{3.2.8}$$

with $\mathrm{div}^{-\gamma} h := \mathrm{div} h - \gamma h_0$.

Consider now (τ, η, γ) such that $\tau^2 + |\eta|^2 + \gamma^2 = 1$ and $\gamma > 0$. Let $E_+(\tau, \eta, \gamma)$ denote the space (1.4.8) associated to $\mathcal{L}$. It is the space of boundary values of $L^2([0, \infty[)$ solutions of $\mathcal{A}_n \partial_n v + i\mathcal{P}(\tau, \eta) v + \gamma \mathcal{A}_0 v = 0$. Because the matrices $\mathcal{S}\mathcal{A}_j$ are self adjoint, the similar space E_+^* is the space of boundary values of L^2 solutions of $-\mathcal{A}_n \partial_n w - i\mathcal{P}(\tau, \eta) w + \gamma \mathcal{A}_0 w = 0$. For such v and w one has

$$\begin{aligned} \big(\mathcal{S}\mathcal{A}_n u(0), v(0)\big)_{\mathbb{C}^{2N}} = & \big(\!\big(\mathcal{S}(\mathcal{A}_n \partial_n + i\mathcal{P}(\tau, \eta) + \gamma \mathcal{A}_0) v, w\big)\!\big)_{L^2(\mathbb{R}_+)} \\ & + \big(\!\big(v, (\mathcal{S}(\mathcal{A}_n \partial_n + i\mathcal{P}(\tau, \eta) - \gamma \mathcal{A}_0) w\big)\!\big)_{L^2(\mathbb{R}_+)} = 0. \end{aligned}$$

This proves that $E_+^*(\tau, \eta, \gamma)$ is orthogonal to $\mathcal{S}\mathcal{A}_n E_+(\tau, \eta, \gamma)$ for the hermitian structure of $\mathbb{C}^{2N}$. When $\tau = 0$, $\eta = 0$ and $\gamma > 0$ one sees that $\dim E^+$ [resp. $\dim E_+^*$] is the number of positive [resp. negative] eigenvalues of $\mathcal{A}_0^{-1} \mathcal{A}_n$. These dimensions are constant for all (τ, η, γ) as long as $\gamma > 0$. Hence

$$E_+^*(\tau, \eta, \gamma) = \big(\mathcal{S}\mathcal{A}_n E_+(\tau, \eta, \gamma)\big)^{\perp}. \tag{3.2.9}$$

The symbol of $(\mathcal{B}^*)^{-\gamma}$ is

$$\mathcal{B}^*(\tau, \eta, \gamma) = \left[\begin{array}{l} R \\ i\, {}^t e(\tau, \eta, \gamma) b^* R_1 \end{array}\right], \quad e := \left[\begin{array}{c} \tau + i\gamma \\ \eta \end{array}\right].$$

Note that the symbol of the boundary operator $\mathcal{B}^\gamma$ is

$$\mathcal{B}(v, \psi) = M v + i\psi b \overline{e}. \tag{3.2.10}$$

In order to prove that the boundary operator $\mathcal{B}^*$ is an isomorphism from $E_+^*(\tau, \eta, \gamma)$ to $\mathbb{C}^{N+1}$ with uniformly bounded inverse, we consider $(g, \alpha) \in \mathbb{C}^N \times \mathbb{C}$ and show that there is $w \in E_+^*$ such that $\mathcal{B}_+^* w = (g, \alpha)$ and the norm of v is uniformly controlled by $C(|g| + |\alpha|)$.

The uniform Lopatinski condition for $\mathcal{L}$ means that $\mathcal{B}$ an isomorphism from E_+ to $\mathbb{C}^N$ with uniformly bounded inverse. Thus for all $f \in \mathbb{C}^N$, there is $(v, \psi) \in E_+ \times \mathbb{C}$ such that $Mv + i\psi b\overline{e} = f$ and $|v| + |\psi| \le C|f|$. Moreover, (v, ψ) depends linearly on f. This shows that there is a unique $h \in \mathbb{C}^N$ such that for all $(v, \psi) \in E_+ \times \mathbb{C}$

$$\big(Mv + i\psi b\overline{e}, h\big)_{\mathbb{C}^N} = -\big(M_1 v, g\big)_{\mathbb{C}^N} - \varphi \overline{\alpha}. \tag{3.2.11}$$

In addition, $|h| \leq C(|g| + |\alpha|)$. Taking $v = 0$ implies that $i\big(b\overline{e}, h\big) = -\overline{\alpha}$, hence

$$\alpha = i\big(h, b\overline{e}\big) = {}^t e(\tau, \eta, \gamma) b^* h \,. \tag{3.2.12}$$

The definition (3.2.4) of R and R_1, shows that there is a unique $w \in \mathbb{C}^{2N}$ such that

$$Rw = g\,, \quad R_1 w = h\,. \tag{3.2.13}$$

Moreover, w satisfies $|w| \leq C'(|g| + |\alpha|)$. Lemma 3.1.1 and (3.2.11) with $\psi = 0$ imply that for all $v \in E_+$

$$2\big(\mathcal{S}\mathcal{A}_n v, w\big)_{\mathbb{C}^{2N}} = \big(Mv, h\big)_{\mathbb{C}^N} + \big(M_1 v, g\big)_{\mathbb{C}^N} = 0$$

Thus, (3.2.9) implies that $w \in E_+^*$. In addition, (3.2.12) and (3.2.13) show that $\mathcal{B}^* v = (g, \alpha)$. In particular, this shows that $\mathcal{B}^*$ is surjective from $E_+^*(\tau, \eta, \gamma)$ to $\mathbb{C}^{N+1}$. Because $\dim E_+^* = 2N - \dim E_+ = N + 1$, this is an isomorphism. Since the norm of the solution w is uniformly controlled by the norm of (g, α) the inverse is uniformly bounded. $\square$

Proposition 3.1.2 implies the following analogue of Theorem 2.1.3.

Theorem 3.2.3. *Fix a constant $K > 0$. Then there are $\gamma_0 > 0$ and C such that for all Lipschitzean function a on Ω valued in $\mathcal{K}$ satisfying $\|a\|_{W^{1,\infty}} \leq K$, all $\gamma \geq \gamma_0$ and all $w \in H^1(\Omega)$,*

$$\begin{aligned} \gamma \| \, w \,\|_0^2 + \| \, w_{|x_n=0} \,\|_0^2 \; &\leq C\Big(\frac{1}{\gamma} \| \, (\mathcal{S}\mathcal{L}_a^\gamma)^* w \,\|_0^2 \\ &\qquad + \| \, R w_{|x_n=0} \|_0 \, + \|\mathrm{div}^{-\gamma} b^* R_1 w_{|x_n=0}\|_{-1,\gamma}\Big). \end{aligned} \tag{3.2.14}$$

3.3 Existence and uniqueness in weighted spaces

We fix K and we consider a on valued in $\mathcal{K}$ satisfying $\|a\|_{W^{1,\infty}(\Omega)} \leq K$.

Theorem 3.3.1. *There is γ_0 such that for $\gamma \geq \gamma_0$, $f \in L^2_\gamma(\Omega)$ and $g \in L^2_\gamma(\mathbb{R}^n)$, there is a unique pair $(v, \psi) \in L^2_\gamma(\Omega) \times H^1_\gamma(\mathbb{R}^n)$ such that*

$$\mathcal{L} v \,=\, f\,, \quad M v_{|x_n=0} + b \mathrm{grad} \psi \,=\, g\,. \tag{3.3.1}$$

Moreover, (v, ψ) satisfies the energy estimate (2.1.4).

As in §2, we conjugate the equation (3.3.1) by $e^{\gamma t}$. We show that for $\gamma \geq \gamma_0$, the problem

$$\mathcal{L}^\gamma v \;=\; f \in L^2(\Omega)\,, \quad M v_{|x_n=0} + b\mathrm{grad}^\gamma \psi \;=\; g \in L^2(\omega) \tag{3.3.2}$$

has a unique solution $(v,\psi) \in L^2(\Omega) \times H^1(\omega)$ and (v,ψ) satisfies the energy estimate (2.1.9). We first prove the existence of a weak solution.

Proposition 3.3.2. *The problem* (3.3.2) *has a solution* $(v,\psi) \in L^2(\Omega) \times H^{1/2}(\mathbb{R}^n)$.

Note that when $v \in L^2(\Omega)$ and $\mathcal{L}^\gamma v \in L^2(\Omega)$, the trace $v_{|x_n=0}$ is well defined in $H^{-1/2}$ since $\mathcal{A}_n$ is invertible.

Proof. Introduce the space $\mathcal{W}^\gamma$ of functions $w \in H^1(\Omega)$ such that $R w_{|x_n=0} = 0$ and $\mathrm{div}^{-\gamma} b^* R_1 w = 0$. Theorem (3.2.3) implies that there is $v \in L^2(\Omega)$ such that for all $w \in \mathcal{W}^\gamma$

$$\big(\!\!\big(v, (\mathcal{S}\mathcal{L}^\gamma)^* w \big)\!\!\big)_0 \;=\; \big(\!\!\big(f, w \big)\!\!\big)_0 \;+\; \big(\!\!\big(g, R_1 w_{|x_n=0} \big)\!\!\big)_0 \tag{3.3.3}$$

Taking $w \in H^1_0(\Omega)$ shows that $\mathcal{S}\mathcal{L}^\gamma u = f$ in the sense of distributions.

Using tangential mollifiers, one shows that $H^1(\Omega)$ is dense in the space $\{v \in L^2(\Omega)\,;\, \mathcal{L}^\gamma v \in L^2(\Omega)\}$. Therefore the Green's formula (3.2.2) makes sense interpreting the scalar product of the traces as a duality $H^{-1/2} \times H^{1/2}$. Comparing with (3.3.3) and using Lemma 3.2.1 yields

$$\begin{aligned}
\big(\!\!\big(\mathcal{S}\mathcal{A}_n v_{|x_n=0}, w_{|x_n=0} \big)\!\!\big)_{H^{-1/2}\times H^{1/2}} &= -\big(\!\!\big(\mathcal{S}\mathcal{L}^\gamma v, w \big)\!\!\big)_{L^2(\Omega)} + \big(\!\!\big(v, (\mathcal{S}\mathcal{L}^\gamma)^* w \big)\!\!\big)_{L^2(\Omega)} \\
&= \big(\!\!\big(g, R_1 w_{|x_n=0} \big)\!\!\big)_{L^2(\omega)} \\
&= \big(\!\!\big(M v_{|x_n=0}, R_1 w_{|x_n=0} \big)\!\!\big)_{H^{-1/2}\times H^{1/2}}
\end{aligned}$$

In the last equality, we have used that $R w_{|x_n=0} = 0$. Thus

$$\forall w \in \mathcal{W}^\gamma\,, \quad \big(\!\!\big(M v_{|x_n=0} - g, R_1 w_{|x_n=0} \big)\!\!\big)_{H^{-1/2}\times H^{1/2}} \tag{3.3.4}$$

For all $\theta \in H^{1/2}(\omega)$, there is $w \in H^1(\Omega)$ such that

$$R w_{|x_n=0} = 0\,, \quad R_1 w_{|x_n=0} = \theta\,, \quad \|w\|_{H^1(\Omega)} \leq C \|\theta\|_{H^{1/2}(\omega)}\,. \tag{3.3.5}$$

Introduce the space $\mathcal{T}^\gamma$ of functions $\theta \in H^{1/2}(\omega)$, valued in $\mathbb{C}^N$ such that $\mathrm{div}^{-\gamma} b^* \theta = 0$. When $\theta \in \mathcal{T}^\gamma$, and w satisfies (3.3.5), then $w \in \mathcal{W}^\gamma$ and (3.3.4) implies that

$$\forall \psi \in \mathcal{T}^\gamma\,, \quad \big(\!\!\big(M v_{|x_n=0} - g, \psi \big)\!\!\big)_{H^{-1/2}\times H^{1/2}} = 0$$

This implies that there is $\psi \in H^{1/2}$ such that $M v_{|x_n=0} - g = b\mathrm{grad}^\gamma \varphi$. $\square$

Proposition 3.3.3. *Suppose that* $(v, \psi) \in L^2 \times H^{1/2}$ *satisfies* (3.3.2). *Then, there exists a sequence* $(v_\nu, \psi_\nu) \in H^1(\Omega) \times H^1(\omega)$ *such that* $v_\nu \to v$ *in* $L^2(\Omega)$, $\psi_\nu \to \psi$ *in* $H^{1/2}(\omega)$, $f_\nu := \mathcal{L}^\gamma v_\nu \to f$ *in* $L^2(\Omega)$ *and* $g_\nu : M v_{\nu|x_n=0} + b\mathrm{grad}^\gamma \psi_\nu \to g$ *in* $L^2(\omega)$.

Proof. Introduce tangential mollifiers ρ_ν and define $v_\nu := \rho_\nu * v$, $\psi_\nu = \rho_\nu * \psi$. The convergences $f_\nu \to f$ and $g_\nu \to g$ follow from Friedrichs' lemma. □

Proof of Theorem 3.3.1

Consider $(f, g) \in L^2(\Omega) \times L^2(\omega)$ and (v, ψ) a solution of (3.2.2). The energy estimate (2.1.9) proves that the sequence (v_ν, ψ_ν) given by Proposition 3.3.3, is a Cauchy sequence in $L^2(\Omega) \times H^1(\omega)$. Therefore $\psi \in H^1(\omega)$ and the limit (v, ψ) satisfies (2.1.9).

This proves the existence of a solution in $L^2 \times H^1$ and that all the solutions in this space satisfy (2.1.9). This implies uniqueness.

□

The same reasoning applies to the dual problem.

Theorem 3.3.4. *There is* γ_0 *such that for* $\gamma \geq \gamma_0$, $f \in L^2(\Omega)$, $g \in L^2(\mathbb{R}^n)$ *and* $h \in H^{-1}$, *there is a unique* $w \in L^2(\Omega)$ *such that*

$$(\mathcal{S}\mathcal{L}^\gamma)^* w = f, \quad R w_{|x_n=0} = g, \quad \mathrm{div}^{-\gamma} b^* R_1 w_{|x_n=0} = h. \tag{3.3.6}$$

Moreover, w *satisfies the estimate* (3.2.14).

3.4 The Cauchy problem with vanishing initial condition

For $T \in \mathbb{R}$, introduce the notation $\Omega_T = \Omega \cap \{t < T\}$ and $\omega_T = \omega \cap \{t < T\}$.

Theorem 3.4.1. *Suppose that* $f \in L^2_\gamma(\Omega_T)$ *and* $g \in L^2_\gamma(\omega_T)$ *vanish for* $t < 0$. *Then, there is a unique pair* $(v, \psi) \in L^2_\gamma(\Omega_T) \times H^1_\gamma(\omega_T)$ *such that*

$$\mathcal{L} v = f, \quad M v_{|x_n=0} + b\mathrm{grad}\psi = g. \tag{3.4.1}$$

and $(v, \psi) = 0$ *for* $t < 0$. *In addition,* $v \in C^0([0, T]; L^2(\mathbb{R}^n_+))$.

Moreover, there are constants C *and* γ_0 *which depend only on* $\mathcal{K}$ *and* K, *such that for all* $t < T$

$$\begin{aligned} e^{-2\gamma t} \|v(t)\|^2_{L^2(\mathbb{R}^n_+)} + \gamma \|v\|^2_{L^2_\gamma(\Omega_t)} + \|v_{|x_n=0}\|^2_{L^2_\gamma(\omega_t)} + \|\psi\|^2_{H^1_\gamma(\omega_t)} \\ \leq C \left(\frac{1}{\gamma} \|f\|^2_{L^2_\gamma(\Omega_t)} + \|g\|^2_{L^2_\gamma(\omega_t)} \right). \end{aligned} \tag{3.4.2}$$

Lemma 3.4.2. *There is γ_0 such that if $\gamma \geq \gamma_0$, $f \in L^2_\gamma(\Omega)$ and $g \in L^2_\gamma(\omega)$ vanish for $t < T$, the solution $(v, \psi) \in L^2_\gamma(\Omega) \times H^1_\gamma(\omega)$ of* (3.3.1) *vanishes for $t < T$.*

Proof. There is no restriction to assume that $T = 0$.

Introduce a function $\theta \in C^\infty(\mathbb{R})$ such that $\theta(t) = 1$ for $t \leq 0$ and $\theta(t) = e^{-t}$ for $t \geq 1$. Then $\kappa = \theta'/\theta$ is bounded. The energy estimate in Theorem 3.3.1 implies that there is $\gamma_1 > 0$ such that for $\gamma > \gamma_1$ the only solution in $L^2_\gamma \times H^1_\gamma$ of

$$(\mathcal{L} + \kappa \mathcal{A}_0) u = 0\,, \quad M u + b \mathrm{grad} \varphi + \kappa b_0 \varphi = 0 \tag{3.4.3}$$

is $v = 0, \psi = 0$.

Consider $\gamma \geq \sup(\gamma_1, \gamma_0)$ and $(f, g) \in L^2_\gamma(\Omega) \times L^2_\gamma(\omega)$ which vanish for $t < 0$. Then $(f, g) \in L^2_{\gamma'}(\Omega) \times L^2_{\gamma'}(\omega)$ for all $\gamma' \geq \gamma$ and

$$\|f\|_{L^2_{\gamma'}(\Omega)} \leq \|f\|_{L^2_\gamma(\Omega)}\,, \quad \|g\|_{L^2_{\gamma'}(\omega)} \leq \|g\|_{L^2_\gamma(\omega)}\,. \tag{3.4.4}$$

Theorem 3.3.1 implies that for all $j \in \mathbb{N}$ there is $(v_j, \psi_j) \in L^2_{\gamma+j}(\Omega) \times H^1_{\gamma+j}(\omega)$ satisfying the equation (3.3.1). Note that $(\theta(v_{j+1} - v_j), \theta(\psi_{j+1} - \psi_j)) \in L^2_{\gamma+j}(\Omega) \times H^1_{\gamma+j}(\mathbb{R}^n)$ satisfies (3.4.3). Therefore $v_{j+1} = v_j$ and $\psi_{j+1} = \psi_j$ for all j. Denote by (v, ψ) this unique solution. The estimates (2.1.4) and (3.4.4) imply that the norms

$$\sup_j \|v\|_{L^2_{\gamma+j}(\Omega)} < \infty \quad \sup_j \|\psi\|_{L^2_{\gamma+j}(\Omega)} < \infty\,.$$

Therefore v and ψ vanish for $t < 0$. $\square$

Proof of Theorem 3.4.1

a) Existence. Extend f and g by zero for $t > T$. The extensions belong to $L^2_{\gamma'}$ for all γ'. Theorem 3.3.1 immediately implies the existence of a unique solution which satisfies for $\gamma \geq \gamma_0$ large enough

$$\begin{aligned} &\gamma \|v\|^2_{L^2_\gamma(\Omega_T)} + \|v_{|x_n=0}\|^2_{L^2_\gamma(\omega_T)} + \|\psi\|^2_{H^1_\gamma(\omega_T)} \\ &\qquad\qquad \leq \gamma \|v\|^2_{L^2_\gamma(\Omega)} + \|v_{|x_n=0}\|^2_{L^2_\gamma(\omega)} + \|\psi\|^2_{H^1_\gamma(\omega)} \\ &\leq C \left(\frac{1}{\gamma} \|f\|^2_{L^2_\gamma(\Omega)} + \|g\|^2_{L^2_\gamma(\omega)} \right) = C \left(\frac{1}{\gamma} \|f\|^2_{L^2_\gamma(\Omega_T)} + \|g\|^2_{L^2_\gamma(\omega_T)} \right). \end{aligned} \tag{3.4.5}$$

Moreover, Lemma 3.4.2 implies that $(v, \psi) = 0$ for $t < 0$.

b) Uniqueness. Suppose that $(v, \psi) \in L^2_\gamma \times H^1_\gamma$ satisfy

$$\mathcal{L} v = 0\,, \quad M v_{|x_n=0} + b \mathrm{grad} \psi = 0 \tag{3.4.6}$$

for $t < T$ and vanish for $t < 0$. For all $T_1 < T$, introduce $\chi \in C^\infty(\mathbb{R})$ such that $\chi(t) = 1$ for $t \le T_1$ and $\chi(t) = 0$ for $t \ge T$. Then $(\chi v, \chi\psi) \in L^2_{\gamma'}(\Omega) \times H^1_{\gamma'}(\mathbb{R}^n)$ for all γ' and $\mathcal{L}\chi v$ and $M\chi v + b\mathrm{grad}\chi\psi$ vanish for $t < T_1$. Proposition 3.4.2 implies that v and ψ vanish for $t < T_1$ and hence for $t < T$.

c) v is continuous in time. Suppose that $\gamma \ge \gamma_0$ where γ_0 is so large that (3.4.5) holds. we show that there is C such that

$$\|v(t)\|_{L^2} \ \le \ Ce^{\gamma t}\Big(\gamma^{-1/2}\|f\|_{L^2_\gamma(\Omega_t)} + \|g\|_{L^2_\gamma(\omega_t)}\Big). \tag{3.4.7}$$

Proposition 3.3.3 implies that the solution is strong and it is sufficient to prove (3.4.7) for H^1 solutions. The classical energy estimate on Ω_t, using the symmetrizer $\mathcal{S}$, shows that

$$\begin{aligned} &e^{-2\gamma t}\|u(t)\|^2_{L^2(\mathbb{R}^n_+)} + \gamma\|u\|^2_{L^2_\gamma(\Omega_t)} \\ &\qquad\qquad \le C\, \|\mathcal{L}u\|_{L^2_\gamma(\Omega_t)}\|u\|_{L^2_\gamma(\Omega_t)} \ + \ \|u_{|x_n=0}\|^2_{L^2_\gamma(\omega_t)}\,. \end{aligned}$$

the boundary term in the right hand side id controlled by (3.4.5) and (3.4.7) follows. ☐

There a similar result for the dual problem. Denote by $\Omega'_T =]T,\infty[\times\mathbb{R}^n_+$ and $\omega'_T =]T,\infty[\times\mathbb{R}^{n-1}$.

Theorem 3.4.3. *There is γ_0 which depends only on K and $\mathcal{K}$ such that for $\gamma \ge \gamma_0$, $f' \in L^2_{-\gamma}(\Omega)$, $g' \in L^2_{-\gamma}(\mathbb{R}^n)$, and $h' \in L^2_{-\gamma}(\mathbb{R}^n)$ there is a unique $v \in L^2_{-\gamma}(\Omega)$ such that*

$$(\mathcal{S}\mathcal{L})^*w \ = \ f'\,, \quad Rw_{|x_n=0} = g'\,, \quad \mathrm{div} b^* R_1 w_{|x_n=0} \ = \ \mathrm{div} h'\,. \tag{3.4.8}$$

When f', g' and h' vanish for $t > T_1$, then the solution v vanishes for $t > T_1$.

Moreover, for all T,

$$\begin{aligned} &\gamma\|w\|^2{}_{L^2_{-\gamma}(\Omega'_T)} + e^{2\gamma T}\|w(T)\|_{L^2_{-\gamma}(\mathbb{R}^n_+)} + \|w_{|x_n=0}\|^2_{L^2_{-\gamma}(\omega'_T)} \\ &\qquad \le C\left(\frac{1}{\gamma}\|f'\|^2_{L^2_{-\gamma}(\Omega'_T)} + \|g'\|_{L^2_{-\gamma}(\omega'_T)} + \|h'\|^2_{L^2_{-\gamma}(\omega'_T)}\right). \end{aligned} \tag{3.4.9}$$

where C depends only on K and $\mathcal{K}$

We have introduced $h \in L^2$, to avoid negative norms for $\mathrm{div} h$.

3.5 The initial boundary value problem

In this section, we prove Theorem 3.1.1.

Lemma 3.5.1. *In addition to* (3.1.2)*, suppose that* $v_0 \in H^1(\mathbb{R}^n_+)$*. Then the problem* (3.1.1) *has a strong solution in* $L^2(\Omega_T^+) \times H^1(\omega_T^+)$ *with* $v_{|x_n=0} \in L^2(\omega_T^+)$*.*

Proof. Extend v_0 in $H^1(\mathbb{R}^n)$ and extend the coefficients of $\mathcal{L}$ on $\{x_n < 0\}$ so that they remain globally Lipschitzean and $\mathcal{L}$ is still hyperbolic symmetric. Thus the Cauchy problem

$$\mathcal{L}v^1 = 0\,, \quad v^1_{|t=0} = v_0 \tag{3.5.1}$$

has a (unique) solution $v^1 \in C^0([0,T], H^1(\mathbb{R}^n))$. In particular, the trace $v^1_{|x_n=0}$ belongs to $L^2(\omega_T^+)$. Next choose $\psi^1 \in H^1(\omega)$ such that $\psi^1_{|t=0} = \psi_0$. Then

$$g^1_{|\omega_T^+} := Mv^1_{|x_n=0} + b\mathrm{grad}\psi^1 \in L^2(\omega_T^+)\,.$$

Introduce $\widetilde{f}$ and $\widetilde{g}$ the extension of f and $g - g^1$ respectively, by zero for $t > T$ and $t < 0$. For all $\gamma \geq \gamma_0$, $\widetilde{f} \in L^2_\gamma(\Omega)$ and $\widetilde{g} \in L^2_\gamma(\omega)$. Thus there is a unique solution $(v^2, \psi^2) \in L^2_\gamma(\Omega) \times H^1_\gamma(\mathbb{R}^n)$ of

$$\mathcal{L}v^2 = \widetilde{f}\,, \quad Mv^2_{|x_n=0} + b\mathrm{grad}\psi^2 = \widetilde{g}\,.$$

Moreover, v^2 and ψ^2 vanish for $t < 0$. In particular, $\psi^2_{|t=0} = 0$ and $v^2_{|t=0} = 0$ in the distribution sense on $\mathbb{R}^n_+$.

It is clear that $(v, \psi) := (v^1, \psi^1) + (v^2, \psi^2)$ satisfies (3.1.1) for $0 \leq t \leq T$. Furthermore, $v_{|x_n=0} \in L^2(\omega_T^+)$ since the trace of v^2 belongs to L^2_γ.

Proposition 3.3.3 implies that there are $(v^2_\nu, \psi^2_\nu) \in H^1(\Omega) \times H^1(\omega)$ which converge to (v^2, ψ^2) in $L^2 \times H^1$ such that $\mathcal{L}^\gamma v^2_\nu \to \widetilde{f}$ and $Mv^2_\nu + b\mathrm{grad}^\gamma \psi^2_n \to \widetilde{g}$. Moreover, because v^2 and ψ^2 vanish for $t < 0$ choosing mollifiers ρ_ν supported in $t < 0$, one can achieve that (v_ν, ψ_ν) vanish in $\{t < 0\}$. Therefore, the sequence $(v_\nu, \psi_\nu) = (v^1, \psi^1) + (v^2_\nu, \psi^2_\nu)$ in $H^1(\Omega_T^+) \times H^1(\omega_T^+)$ satisfies

$$\begin{cases} (v_\nu, \psi_\nu) \to (v, \psi) \quad \text{in } L^2(\Omega_T^+) \times H^1(\omega_T^+)\,, \\ v_{\nu|x_n=0} \to v_{|x_n=0} \quad \text{in } H^1(\omega_T^+)\,, \\ \mathcal{L}v_\nu \to f \quad \text{in } L^2(\Omega_T^+)\,, \\ Mv_{\nu|x_n=0} + b\mathrm{grad}\psi_\nu \to g \quad \text{in } L^2(\omega_T^+)\,, \\ v_{\nu|t=0} = v_0\,, \quad \psi_{\nu|t=0} = \psi_0\,. \end{cases} \tag{3.5.2.}$$

This means that (v, ψ) is a strong solution of (3.1.1). $\square$

Proposition 3.5.2. *In addition to* (3.1.2) *assume that* $v_0 \in H^1(\mathbb{R}^n_+)$. *Then the problem* (3.1.1) *has a unique strong solution. Moreover, there are constants* C *and* γ_0 *which depend only on* K *and* $\mathcal{K}$ *such that for* $\gamma \geq \gamma_0$, *the solutions satisfy*

$$
\begin{aligned}
&\gamma \|v\|^2_{L^2_\gamma(\Omega_T)} + \|v_{|x_n=0}\|^2_{L^2_\gamma(\omega_T)} + \|\psi\|^2_{H^1_\gamma(\omega_T)} \\
&\quad \leq C\Big(\frac{1}{\gamma}\|f\|^2_{L^2_\gamma(\Omega_T)} + \|g\|^2_{L^2_\gamma(\omega_T)} + \|v_0\|^2_{L^2(\mathbb{R}^n_+)} + \|\psi_0\|^2_{H^{1/2}(\mathbb{R}^{n-1})}\Big).
\end{aligned} \tag{3.5.3}
$$

Proof. Consider $w \in H^1(\Omega^+)$ which vanishes for $t > T$. For $v \in H^1(\Omega^+_T)$ one has

$$
\begin{aligned}
&\big(v, (\mathcal{S}\mathcal{L})^*w\big)_{L^2(\Omega^+_T)} - \big(\mathcal{S}\mathcal{L}v, w\big)_{L^2(\Omega^+_T)} \\
&\qquad = \big(\mathcal{S}\mathcal{A}_n v_{|x_n=0}, w_{|x_n=0}\big)_{L^2(\omega^+_T)} + \big(\mathcal{S}\mathcal{A}_0 v_{|t=0}, w_{|t=0}\big)_{L^2(\mathbb{R}^n_+)} .
\end{aligned} \tag{3.5.4}
$$

This extends to the strong solution v given by Proposition 3.5.1. and to the strong solutions w of the dual problem (3.4.8)

$$
(\mathcal{S}\mathcal{L})^*w = f', \quad Rw_{|x_n=0} = g', \quad \mathrm{div} b^* R_1 v_{|x_n=0} = \mathrm{div} h' . \tag{3.5.5}
$$

when $f' \in L^2(\Omega^+)$, $g' \in L^2(\omega^+)$ and $h' \in L^2(\omega^+)$ vanish for $t > T$. Lemma 3.2.1, (3.5.4) and the boundary condition $Mv = g - b\mathrm{grad}\psi$ imply

$$
\begin{aligned}
&\big(\!\big(v, f'\big)\!\big)_{L^2(\Omega^+_T)} - \big(\!\big(M_1 v_{|x_n=0}, g'\big)\!\big)_{L^2(\omega^+_T)} + \big(\!\big(\mathrm{grad}\psi, h'\big)\!\big)_{L^2(\omega^+_T)} \\
&= \big(\!\big(f, w\big)\!\big)_{L^2(\Omega^+_T)} + \big(\!\big(g, R_1 w_{|x_n=0}\big)\!\big)_{L^2(\omega^+_T)} + \big(\!\big(\mathcal{S}\mathcal{A}_0 v_0, w_{|t=0}\big)\!\big)_{L^2(\mathbb{R}^n_+)} \\
&\qquad + \big(\!\big(\mathrm{grad}\psi, h' - b^* R_1 w_{|x_n=0}\big)\!\big)_{L^2(\omega^+_T)} .
\end{aligned} \tag{3.5.6}
$$

Introduce $\psi^1 \in H^1(\omega^+)$ such that $\psi^1_{|t=0} = \psi_0$. Note that the $\mathbb{C}^n$-valued function $h'' := h' - b^* R_1 w_{|x_n=0}$ belongs to $L^2_{-\gamma}(\omega^+)$ for all γ and vanishes for $t > T$. Moreover, the boundary condition in (3.5.5) implies that $\mathrm{div} h'' = 0$. Since $\psi - \psi^1$ belongs to $H^1(\omega^+_T)$ and its trace on $\{t = 0\}$ vanishes, one has

$$
\big(\!\big(\mathrm{grad}(\psi - \psi^1), h''\big)\!\big)_{L^2(\omega^+_T)} = 0 .
$$

Therefore the last term in the right hand side of (3.5.6) is equal to

$$
\big(\!\big(\mathrm{grad}\psi^1, h' - b^* R_1 w_{|x_n=0}\big)\!\big)_{L^2(\omega^+_T)}
$$

and is estimated by

$$
\|\mathrm{grad}\psi^1\|_{L^2_\gamma(\omega^+_T)} \big(\|h'\|_{L^2_{-\gamma}(\omega^+_T)} + C\,\|w_{|x_n=0}\|_{L^2_{-\gamma}(\omega^+_T)}\big) . \tag{3.5.7}
$$

In addition, we note that

$$\|\mathrm{grad}\psi^1\|_{L^2_\gamma} \le \|\mathrm{grad}\psi^1\|_{L^2} \le C\|\psi_0\|_{H^{1/2}}\,. \tag{3.5.8}$$

Theorem 3.4.3 provides estimates for the right hand side of (3.5.5). The estimate (3.4.9) on $\Omega^+ = \Omega'_0$ and (3.5.7) (3.5.8) show that for $\gamma \ge \gamma_0$,

$$\begin{aligned} \Big|\big(v, f'\big)_{L^2(\Omega^+_T)} - \big(M_1 v_{|x_n=0}, g'\big)_{L^2(\omega^+_T)} - \big(\mathrm{grad}\psi, h'\big)_{L^2(\omega^+_T)}\Big| \\ \le C\, N\, N'\ . \end{aligned} \tag{3.5.9}$$

with

$$\begin{aligned} N^2 &:= \frac{1}{\gamma}\|f\|^2_{L^2_\gamma(\Omega'_T)} + \|g\|^2_{L^2_\gamma(\omega'_T)} + \|u_0\|^2_{L^2(\mathbb{R}^n_+)} + \|\varphi_0\|^2_{H^{1/2}(\mathbb{R}^{n-1})} \\ {N'}^2 &:= \frac{1}{\gamma}\|f'\|^2_{L^2_{-\gamma}(\Omega'_T)} + \|g'\|_{L^2_{-\gamma}(\omega'_T)} + \|h'\|^2_{L^2_{-\gamma}(\omega'_T)} \end{aligned}$$

where C depends only on K and $\mathcal{K}$. Since this holds for all test functions (f', g', h') which vanish for $t > T$, this implies (3.4.3).

We have proved that all strong solutions satisfy the energy estimate (3.4.3). Therefore, the strong solution is unique. □

Proposition 3.5.3. *With assumptions as in Proposition* 3.5.2*, the solution* v *belongs to* $C^0([0,T], L^2(\mathbb{R}^n_+)$ *and*

$$\begin{aligned} e^{-2\gamma T}\|v(T)\|^2_{L^2(\mathbb{R}^n_+)} \le C\Big(&\|v_0\|^2_{L^2(\mathbb{R}^n_+)} \\ &+ \|\mathcal{L}v\|_{L^2_\gamma(\Omega^+_T)}\|v\|_{L^2_\gamma(\Omega^+_T)} + \|v_{|x_n=0}\|^2_{L^2_\gamma(\omega^+_T)}\Big). \end{aligned} \tag{3.5.10}$$

Proof. When $v \in H^1(\Omega^+_T)$, the estimate immediately is an immediate consequence of the existence of the symmetrizer $\mathcal{S}$. It extends to strong solutions, i.e. solutions which satisfy (3.5.2)

□

Proof of Theorem 3.1.1

If all the data f, g, v_0, ψ_0 vanish, the extension of a solution (v, ψ) by zero for $t < 0$ is a solution of on Ω_T. Therefore, the uniqueness of the solution on Ω^+_T follows from the uniqueness proved in Theorem 3.4.1.

When $v_0 \in H^1$, the existence of a solution and the estimate (3.1.3) follow from Propositions 3.5.2 and 3.5.3. By density of H^1 in L^2, the estimate shows that the existence extends to data $v_0 \in L^2$ and that the solutions satisfy (3.1.3).

□

4 The existence of multidimensional shocks

In this section, we prove the local solvability of the nonlinear initial boundary value problem (1.2.4), assuming that the uniform stability condition is satisfied. We also prove a continuation theorem, which shows that the shock pattern of one single shock front remains stable as long as the solution remains Lipschitzean.

4.1 The local existence and continuation theorems

Consider the nonlinear shock equation (1.2.4), where u^- is transported to $\{x_n > 0\}$ through changing x_n to $-x_n$.

$$\text{(4.1.1)} \qquad \begin{cases} L(u^\pm, \nabla\Phi)u^\pm = 0\,, & \text{on } x_n > 0\,, \\ b(u^+, u^-)\nabla\varphi + [f_n(u)] = 0\,, & \text{on } x_n = 0 \end{cases}$$

To determine Φ from its trace φ on $\{x_n = 0\}$, we choose

$$\text{(4.1.2)} \qquad \Phi(t, y, x_n) = \kappa x_n + \chi(x_n)\varphi(t, y)\,.$$

with $\chi \in C_0^\infty(\mathbb{R})$ equal to one on $[0, 1]$. We introduce the cut-off function, to work globally in $x_n \geq 0$ and the constant κ is so large that $\partial_n\Phi \geq \kappa/2$.

The set of parameters is $a = (u^+, u^-, \theta)$ where θ is the placeholder for $\nabla\varphi$. We choose a ball $\mathcal{U} \subset \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^n$ such that the Assumptions 1.1.1, 2.1.1 and 2.3.3 are satisfied for $a \in \mathcal{U}$ and $\|\chi'\|_{L^\infty} |\theta| \leq \kappa/2$.

To construct solutions, one uses an iteration scheme, which is much simpler when the boundary conditions are linear. We remark that the uniform stability Assumption implies that $b(u^+, u^-)$ is elliptic. Therefore, for $a \in \mathcal{U}$, there is smooth invertible matrix $W(u^+, u^-)$ such that

$$\text{(4.1.3)} \qquad W(u^+, u^-)b(u^+u^-) = \begin{bmatrix} Id \\ 0 \end{bmatrix} := \underline{b}$$

Next, we note that the mapping $u^- \mapsto W(u^+, u^-)[f_n(u)]$ is a local diffeomorphism.

Lemma 4.1.1. *If $a = (u^+, u^-, \theta)$ satisfies the boundary condition $b(u^+, u^-)\theta + [f_n(u)] = 0$, the differential at a of $u^- \mapsto W(u^+, u^-)[f_n(u)]$ is the mapping $\dot{u} \mapsto -W(u^+, u^-)A_n(u^-, \theta)\dot{u}$.*

Proof. The differential is

$$\dot{u} \mapsto -WA_n(u^-)\dot{u} + (\partial_{u^-}W\dot{u})[f_n(u)]\,.$$

Substitute

$$[f_n u)] = -b\theta = -\sum_j [f_j(u)]\theta_j$$

and use the identities

$$W(u^+, u^-)[f_j(u)] = e_j \quad \Rightarrow \; -WA_j(u^-)\dot{u} + (W'_{u^-}\dot{u})[f_j(u)]$$

to get the desired result. □

Corollary 4.1.2. *Shrinking $\mathcal{U}$ if necessary, there is an invertible change of unknowns $(u^+, u^-) \mapsto H(u^+, u^-)$ such that*

$$W(u^+, u^-)[f_n(u)] \; = \; \underline{M} H(u^+, u^-) \tag{4.1.4}$$

where $\underline{M}$ is a constant matrix.

From now on, we assume that the conclusion of Corollary 4.1.2 holds on $\mathcal{U}$.

Introduce $u = H(u^+, u^-)$. Then $(u^+, u^-, \nabla\varphi)$ valued in $\mathcal{U}$ satisfies the shock equations (4.1.1) (4.1.2), if and only if (u, φ) is solution of a boundary valued problem of the form

$$\begin{cases} \mathcal{L}(u, \nabla\Phi)u = 0\,, & \text{on } x_n > 0\,, \\ \underline{b}\nabla\varphi + \underline{M}u = 0\,, & \text{on } x_n = 0\,, \end{cases} \tag{4.1.5}$$

supplemented with (4.1.2). Note that the hyperbolicity assumptions as well as the uniform stability assumption are invariant under the change of unknowns $(u^+, u^-) \mapsto u$. From there on, we work on the u side, and $\mathcal{U}$ denotes an open subset of $\mathbb{R}^{2N} \times \mathbb{R}^n$ which contains the values of $(u, \nabla\varphi)$.

Assumption 4.1.3. *The system $\mathcal{L}$ is hyperbolic symmetric and for all $(u, \theta) \in \mathcal{U}$, the uniform stability condition is satisfied. Shrinking $\mathcal{U}$ is necessary, we further assume that $\mathcal{U}$ is convex and $(0,0) \in \mathcal{U}$.*

Introduce the notations $\omega_T^+ =]0, T[\times\mathbb{R}^{n-1}$ and $\Omega_T^+ = \omega_T^+ \times]0, +\infty[$.

Definition 4.1.4. *$CH^s(\Omega_T^+)$ denotes the space of distributions u on Ω_T^+ such that for all $j \in \{0, \dots, s\}$, $\partial_t^j u \in C^0([0,T]; H^{s-j}(\mathbb{R}^n_+))$. The space $CH^s(\omega_T^+)$ is defined similarly.*

Theorem 4.1.5 (Local existence). *Consider an integer* $s > \frac{n}{2}+1$*. For all* $u_0 \in H^{s+\frac{1}{2}}(\mathbb{R}^n_+)$ *and* $\varphi_0 \in H^{s+\frac{1}{2}}(\mathbb{R}^{n-1})$*, such that* $(u_0, \nabla\varphi_0)$ *is valued in a compact subset* $\mathcal{K}_0$ *of* $\mathcal{U}$ *and satisfying the compatibility conditions explicited below, there is* $T > 0$ *and a unique solution* $(u, \varphi) \in CH^s(\Omega_T^+) \times H^{s+1}(\omega_T^+)$ *of*

$$\begin{cases} \mathcal{L}(u, \nabla\Phi)u = 0\,, & \text{on } x_n > 0\,, \\ \underline{b}\nabla\varphi + \underline{M}u = 0\,, & \text{on } x_n = 0 \\ u_{|t=0} = u_0\,, & \varphi_{|t=0} = \varphi_0\,, \end{cases} \tag{4.1.6}$$

and Φ *given by* (4.1.2)*.*

In general, the solution of the Riemann Problem for (1.1.1) is expected to develop singularities or fronts for all the eigenvalues. In order to obtain a single shock front, the data must be suitably chosen. The compatibility conditions mentioned above make explicit the conditions to be imposed on the Cauchy data.

Denotes by T_* the maximal time of existence of a smooth solution (u, φ), i.e. the supremum of the set of times T such that (4.1.6) (4.1.2) has a solution in $CH^s(\Omega_T^+) \times H^{s+1}(\omega_T^+)$. This is the maximal time of validity of the pattern of a single shock front separating two smoothly varying states.

Theorem 4.1.6 (Continuation). *Suppose that* $T_* < +\infty$*. Then, either* $(u, \nabla\varphi)$ *does not stay in a compact set of* $\mathcal{U}$*, or*

$$\|\nabla u(t)\|_{L^\infty(\mathbb{R}^n_+)} \to +\infty \quad as \quad t \to T_*\,. \tag{4.1.7}$$

4.2 The compatibility conditions

The compatibility conditions are given by computing Taylor expansions at $t = 0$. The interior equation reads

$$\partial_t u \;=\; -\sum_{j=1}^{n} \mathcal{A}_0^{-1}\mathcal{A}_n(u, \nabla\Phi)\partial_j u\,. \tag{4.2.1}$$

Recall that $\underline{b}$ is given by (4.1.3). Thus, the boundary condition splits into to parts

$$\partial_t\varphi = -\underline{M}_1 u_{|x_n=0}\,, \tag{4.2.2}$$

$$\underline{b}'\partial_y\varphi + \underline{M}'u_{|x_n=0} = 0 \tag{4.2.3}$$

where we have used the notation $b = (b_1, b') \in \mathbb{R} \times \mathbb{R}^{N-1}$. Therefore, the traces $u_j = \partial_t^j u_{|t=0}$ and $\varphi_j = \partial_t^j \varphi_{|t=0}$ are determined inductively by

$$\varphi_{j+1} = \underline{M}_1 u_{j|x_n=0} \tag{4.2.4}$$

$$u_{j+1} = \sum_{p+|k|+|l|+|m| \le j} F_{j,p,k,l}(u, \partial_{x'}\Phi)\,(\partial_{x'}\Phi)_{(l)}(\Phi_1)_{(m)}\, u_{(k)}(\partial_{x'} u_p)\,. \tag{4.2.5}$$

where we have used the notation

$$\text{for } \ k = (k_1, \ldots, k_r)\,, \quad u_{(k)} = u_{k_1} \ldots u_{k_r}\,.$$

and

$$(\Phi_1)_{(m)} = \Phi_{1+m_1} \ldots \Phi_{1+m_r}\,.$$

The coefficients $F_{j,p,k,l,m}$ are C^∞ functions. Moreover, (4.1.2) implies that, for $j \ge 1$,

$$\Phi_j = \chi(x_n)\varphi_j\,. \tag{4.2.6}$$

The multiplicative properties of Sobolev spaces imply the following result (see also §4.5 below).

Lemma 4.2.1. *Suppose that $s > 1+n/2$, $u_0 \in H^s(\mathbb{R}^n_+)$ and $\varphi_0 \in H^{s+\frac{1}{2}}(\mathbb{R}^{n-1})$, (4.2.4), (4.2.5) and (4.2.6) determine $u_j \in H^{s-j}(\mathbb{R}^n_+)$ and $\varphi_j \in H^{s+\frac{1}{2}-j}(\mathbb{R}^{n-1})$, for $j \le s$.*

Proof. Suppose that the result is proved up to $j < s$. Then (4.2.4) implies that

$$\varphi_{j+1} \in H^{s-j-1/2}(\mathbb{R}^{n-1})).$$

Therefore

$$\partial_{x'}\Phi_j\,, \Phi_{j+1} \in H^{s-j-1/2}(\mathbb{R}^n_+))\,. \tag{4.2.7}$$

Because $s > n/2$, $F(u) \in H^s$ when $u \in H^s$ and F is a C^∞ function such that $F(0) = 0$. Moreover, the product $(u_1, \ldots, u_r) \mapsto u_1 \ldots u_r$ is continuous from

$$H^{s-1-k_1}(\mathbb{R}^n_+) \times \ldots \times H^{s-1-k_r}(\mathbb{R}^n_+) \mapsto H^{s-1-j}(\mathbb{R}^n_+)$$

when $s-1 > n/2$ and $k_1 + \ldots k_r \le j$. By induction, with (4.2.7), this implies that

$$u_k \in H^{s-k}\,, \quad \partial_{y,x_n} u_p \in H^{s-1-p}\,,$$
$$\partial_{x'}\Phi_l \in H^{s-l-1/2}\,, \quad \Phi_{1+l} \in H^{s-l-1/2}$$

for $l \le j$, proving that $u_{j+1} \in H^{s-1-j}$. $\square$

Definition 4.2.2. *Consider $s > 1+n/2$. The initial data $u_0 \in H^s(\mathbb{R}^n_+)$ and $\varphi_0 \in H^{s+\frac{1}{2}}(\mathbb{R}^{n-1})$ are compatible to order $\sigma \le s$ when the (u_j, φ_j) satisfy*

$$\partial_y \varphi_j + \underline{M}' u_{j|x_n=0} = 0\,, \quad \text{for} \quad j = 0\,, \ldots, \sigma\,. \tag{4.2.8}$$

The definitions above show that u_j and φ_j are given by nonlinear functions of the derivatives up to order j of u_0 and φ_0. Moreover,

$$u_{j|x_n=0} - (-\mathcal{A}_0^{-1}\mathcal{A}_n)^j \partial_n^j u_{0|x_n=0}$$

only involves the traces $\partial_n^k u_{0|x_n=0}$ for $k < j$ and the derivatives of φ_0. Therefore, (4.2.8) are nonlinear equations for $\partial_n^j u_{0|x_n=0}$ which have a triangular form

$$\underline{M}'(-\mathcal{A}_0^{-1}\mathcal{A}_n)^j \partial_n^j u_{0|x_n=0} = G_j \tag{4.2.9}$$

where G_j depends only on $\partial_n^k u_{0|x_n=0}$ for $k < j$ and the derivatives of φ_0.

Example 1. The first compatibility condition is $\partial_y \varphi_0 + \underline{M}' u_{0|x_n=0} = 0$. It mean that the Rankine-Hugoniot condition is satisfied at time $t = 0$. For planar shocks, u and $\nabla\varphi$ are constant and all the other compatibility conditions are trivially satisfied.

Example 2. Suppose that $(\underline{u}, \underline{\varphi})$ correspond to a planar shock which satisfies the Rankine Hugoniot conditions. If u_0 is C^∞ and $u_0 - \underline{u}$ is flat to infinite order at $x_n = 0$, then $(u_0, \underline{\varphi})$ satisfy the compatibility conditions to infinite order.

Proposition 4.2.3. *Suppose that $u_0 \in H^{s+\frac{1}{2}}(\mathbb{R}^n_+)$ and $\varphi_0 \in H^{s+\frac{3}{2}}(\mathbb{R}^{n-1})$ are compatible to order $s-1$, with $s > 1 + n/2$. Then there is $(u, \varphi) \in H^{s+1}(\Omega) \times H^{s+2}(\omega)$ such that*

$$\text{for } j = 0, \ldots, s-1 \;:\quad \begin{cases} \partial_t^j \mathcal{L}(u, \nabla\Phi) u_{|t=0} = 0\,, \\ \partial_t^j \big(\underline{b}\nabla\varphi + \underline{M} u_{|x_n=0}\big)_{|t=0} = 0\,. \end{cases} \tag{4.2.10}$$

Proof. Consider $(u, \varphi) \in H^{s+1}(\Omega) \times H^{s+2}(\omega)$ such that

$$\partial_t^j u_{|t=0} = u_j \quad \text{and} \quad \partial_t^j \varphi_{|t=0} = \varphi_j \qquad \text{for} \quad j \in \{0, \ldots, s-1\}\,, \tag{4.2.11}$$

where u_j and φ_j are given by Lemma 4.2.1. The equations (4.2.4) (4.2.5) (4.2.6) and (4.2.8) imply (4.2.10). $\square$

The analysis of (4.2.9) implies the following proposition.

Proposition 4.2.4. *Suppose that $u_0 \in H^s(\mathbb{R}^n_+)$ and $\varphi_0 \in H^{s+\frac{1}{2}}(\mathbb{R}^{n-1})$ are compatible to order $s-1$, with $s > 1 + n/2$. Then there is a sequence $(u_0^\nu, \varphi_0^\nu) \in H^{s+1}(\mathbb{R}^n_+) \times H^{s+\frac{3}{2}}(\mathbb{R}^{n-1})$ such that (u_0^ν, φ_0^ν) converge to (u_0, φ_0) in $H^s(\mathbb{R}^n_+) \times H^{s+\frac{1}{2}}(\mathbb{R}^{n-1})$ and for all ν, (u_0^ν, φ_0^ν) are compatible to order $s-1$.*

4.3 H^s estimates

To solve the nonlinear equations (4.1.6) we use iterative schemes based on the linearized equations

$$\left\{ \begin{array}{ll} \mathcal{L}(u, \nabla\Phi)v = f\,, & \text{on } x_n > 0\,, \\ \underline{b}\nabla\psi + \underline{M}v = g\,, & \text{on } x_n = 0\,. \end{array} \right. \tag{4.3.1}$$

The main point is to prove uniform Sobolev estimates for the solutions (v, ψ). The basic L^2 estimate is provided by Theorem 3.1.1. The higher order Sobolev estimates are obtained by commuting tangential derivatives and using the equation for estimating the normal derivatives. We use two different H^s estimates. The first one is used to prove the continuation theorem and an existence theorem in H^s for data in $H^{s+\frac{1}{2}}$. The existence Theorem 4.1.5 with data in H^s is proved using Proposition 4.2.4 and the second H^s estimate stated below.

Consider an integer $s > \frac{n}{2} + 1$, a compact set $\mathcal{K} \subset \mathcal{U}$, and a constant K. There is no restriction to assume that $0 \in \mathcal{K}$. As in §2 and 3, $\omega_T :=]-\infty, T] \times \mathbb{R}^{n-1}$ and $\Omega_T := \omega_T \times]0, +\infty[$. For $\gamma > 1$, $H^s_\gamma(\Omega_T) := e^{\gamma t} H^s(\Omega_T)$ is equipped with the norm

$$\|u\|_{H^s_\gamma(\Omega_T)} := \sum_{|\alpha| \le s} \gamma^{s-|\alpha|} \|e^{-\gamma t}\partial^\alpha u\|_{L^2(\Omega_T)}\,. \tag{4.3.2}$$

The spaces $H^s_\gamma(\omega^+_T)$ are defined similarly and equipped with similar norms. Next, introduce the space $CH^s(\Omega_T)$ of distributions u on Ω_T such that $\partial_t^j u \in C^0([0,T]; H^{s-j}(\mathbb{R}^n_+))$ for all $j \in \{0, \dots, s\}$. For $u \in CH^s(\Omega_T)$ introduce the notation

$$|\!|\!| u(t) |\!|\!|_{s,\gamma} := \sum_{|\alpha| \le s} \gamma^{s-|\alpha|} \|\partial^\alpha u(t)\|_{L^2(\mathbb{R}^n_+)} \tag{4.3.3}$$

When

Consider $u \in CH^s(\Omega_T)$ and $\varphi \in H^{s+1}(\omega_T)$ satisfying

$$
\begin{cases}
\forall (t,x)\, (u(t,x), \nabla\varphi(t,y)) \in \mathcal{K}\,, \\
\|\nabla\varphi\|_{W^{1,\infty}(\omega_T)} \le K \quad \text{and} \quad \|u\|_{W^{1,\infty}(\Omega_T)} \le K \\
u = 0 \quad \text{and} \quad \nabla\varphi = 0 \quad \text{for} \quad t < T_0\,.
\end{cases}
\tag{4.3.4}
$$

The last assumption implies that $u \in H^s_\gamma(\Omega_T)$ and $\varphi \in H^{s+1}_\gamma(\omega_T)$ for all γ.

Proposition 4.3.1. *There are C and $\gamma_0 > 0$, which depend only on $\mathcal{K}$ and K, such that for all T, all $\gamma \ge \gamma_0$, all (u, φ) satisfying* (4.3.4) *and all $(v, \psi) \in H^s_\gamma(\Omega_T) \times H^{s+1}_\gamma(\omega_T)$ such that*

$$
\begin{cases}
\mathcal{L}(u, \nabla\Phi) v = f \in H^s_\gamma(\Omega_T) \cap L^\infty(\Omega_T)\,, & \text{on } x_n > 0\,, \\
\underline{b}\nabla\psi + \underline{M} v = g \in H^s_\gamma(\omega_T)\,, & \text{on } x_n = 0\,, \\
v = 0 \quad \textit{and} \quad \psi = 0 & \textit{for} \quad t < T_0\,,
\end{cases}
\tag{4.3.5}
$$

one has $v \in CH^s(\Omega_T)$ and

$$
\begin{aligned}
&\sqrt{\gamma}\|v\|_{H^s_\gamma(\Omega_T)} + e^{-\gamma t}|\!|\!| v(t) |\!|\!|_{s,\gamma} + \|\psi\|_{H^{s+1}_\gamma(\omega_T)} \le \\
&\quad C(K)\Big(\frac{1}{\sqrt{\gamma}}\|f\|_{H^s_\gamma(\Omega_T)} + \|g\|_{H^s_\gamma(\omega_T)}\Big) + \frac{1}{\sqrt{\gamma}}\, C(K) \times \\
&\quad \Big(1 + \|v\|_{W^{1,\infty}(\Omega_T)} + \|f\|_{L^\infty(\Omega_T)}\Big)\Big(\|u\|_{H^s_\gamma(\Omega_T)} + \|\varphi\|_{H^{s+1}_\gamma(\omega_T)}\Big)\,.
\end{aligned}
\tag{4.3.6}
$$

The second a-priori estimate concerns solutions on Ω^+_T. Introduce the notation

$$
|\!|\!| v(t) |\!|\!|_s := \sum_{j=0}^{s} \|\partial_t^j v(t)\|_{H^{s-j}(\mathbb{R}^n_+)}
\tag{4.3.7}
$$

which defines a norm equivalent to $|\!|\!| \cdot |\!|\!|_{s,1}$. Similarly, introduce

$$
|\!|\!| \psi(0) |\!|\!|_{s+\frac{1}{2}} := \sum_{j=0}^{s} \|\partial_t^j \psi(0)\|_{H^{s-j+\frac{1}{2}}(\mathbb{R}^n_+)}
\tag{4.3.8}
$$

Consider $u \in CH^s(\Omega^+_T)$ and $\varphi \in H^{s+1}(\omega^+_T)$ such that

$$
\begin{cases}
\forall (t,x) \in \Omega^+_T\,, \quad (u(t,x), \nabla\varphi(t,y)) \in \mathcal{K}\,, \\
\|\nabla\varphi\|_{W^{1,\infty}(\omega^+_T)} + \displaystyle\sum_{j=0}^{s} \|\partial_t^j \varphi(0)\|_{H^{s-j+1/2}(\mathbb{R}^{n-1})} \le K\,, \\
\|u\|_{W^{1,\infty}(\omega^+_T)} + \displaystyle\sum_{j=0}^{s-1} \|\partial_t^j u(0)\|_{H^{s-j}(\mathbb{R}^{n-1})} \le K\,.
\end{cases}
\tag{4.3.9}
$$

Proposition 4.3.2. *There are C and $T_\flat > 0$, which depend only on $\mathcal{K}$ and K, such that for all $T \le T_\flat$, all (u,φ) satisfying (4.3.9) and all $(v,\psi) \in CH^s(\Omega_T^+) \times H^{s+1}(\omega_T^+)$ such that*

$$\begin{cases} \mathcal{L}(u,\nabla\Phi)v = 0\,, & \text{on } x_n > 0\,, \\ \underline{b}\nabla\psi + \underline{M}v = 0\,, & \text{on } x_n = 0 \end{cases} \tag{4.3.10}$$

one has

$$\begin{aligned} \|v\|_{CH^s(\Omega_T^+)} + \|\psi\|_{H_\gamma^{s+1}(\omega_T^+)} \le \\ TC(K)N_T(v,\psi)\big(1 + \|u\|_{H^s(\Omega_T^+)} + \|\varphi\|_{H^{s+1}(\Omega_T^+)}\big) \\ + C(K)\big(\|\!|v(0)|\!\|_s + \|\!|\psi(0)|\!\|_{s+1/2}\big)\,. \end{aligned} \tag{4.3.11}$$

where

$$N_T(v,\psi) := \|v\|_{W^{1,\infty}(\Omega_T^+)} + \|\nabla\psi\|_{W^{1,\infty}(\omega_T^+)} + \|\!|v(0)|\!\|_s + \|\!|\psi(0)|\!\|_{s+\frac12}\,.$$

The two propositions above are prove in §§4.6-4.7.

4.4 Proof of the main theorems

We fist prove the local existence when the data are $H^{s+\frac12}$ instead of being H^s.

Theorem 4.4.1. *Consider an integer $s > (n+3)/2$, $u_0 \in H^{s+1/2}(\mathbb{R}^n_+)$ and $\varphi_0 \in H^{s+1/2}(\mathbb{R}^{n-1})$, such that the compatibility conditions are satisfied up to order $s-1$ and $(u_0, \nabla\varphi_0)$ is valued in a compact subset $\mathcal{K}_0$ of $\mathcal{U}$. Then there is $T > 0$ and a unique solution $(u,\varphi) \in H^s(\Omega_T^+) \times H^{s+1}(\omega_T^+)$ of (4.1.6).*

Proof.

Proposition 4.2.3 implies that there is $(u^1, \varphi^1) \in H^{s+1}(\Omega) \times H^{s+1}(\omega)$ such that (4.2.10) is satisfied. Introduce a cut-off function $\chi_1(t) \in C^\infty(\mathbb{R})$ such that $\chi_1 = 1$ for $|t|$ small. If the support of χ_1 is small enough, because $\mathcal{U}$ is convex and $0 \in \mathcal{U}$, there exist $\mathcal{K}_0$, K_0 and $T_0 < 0$ such that

$$u^a := \chi_1(t)u \quad , \varphi^a := \chi_1(t)\varphi \tag{4.4.1}$$

satisfy (4.3.4). Introduce

$$\begin{cases} f^a := -\mathcal{L}(u^a, \nabla\Phi^a)u^a\,, & g^a = -(\underline{b}\nabla\varphi^a + \underline{M}u^a)\,, \quad t > 0 \\ f^a := 0 \quad , & g^a = 0\,, \quad t < 0\,, \end{cases} \tag{4.4.2}$$

where Φ^a is associated to φ^a as in (4.1.2). Because u^1 and φ^1 belong to H^{s+1}, the compatibility conditions (4.2.10) imply

$$f^a \in H^s(\Omega), \quad g^a \in H^s(\omega). \tag{4.4.3}$$

With these notations, $u = u^a + v$, $\varphi = \varphi^a + \psi$ is a solution on Ω_T^+ of the initial boundary value problem (4.1.6), if and only if (v, ψ) satisfy

$$\begin{cases} \widetilde{\mathcal{L}}(u^a + v, \nabla\Phi^a + \nabla\Psi)v + \mathcal{E}(u^a, \Phi^a, v, \Psi) = f^a\,, & \text{on } x_n > 0\,, \\ \underline{b}\nabla\psi + \underline{M}v = g^a\,, & \text{on } x_n = 0 \\ v_{|t<0} = 0\,, \qquad \psi_{|t<0} = 0\,, \end{cases} \tag{4.4.4}$$

where

$$\begin{aligned} &\Psi(x) = \chi(x_n)\psi(t,y)\,, \\ &\mathcal{E}(u^a, \Phi^a, v, \Psi) := \Big(\mathcal{L}(u^a + v, \nabla\Phi^a + \nabla\Psi) - \mathcal{L}(u^a, \nabla\Phi^a)\Big)u^a\,. \end{aligned} \tag{4.4.5}$$

Note that f and g vanish for $t < 0$ and that $v = 0$ and $\psi = 0$ satisfy (4.4.4) for $t < 0$.

We solve this equation using Picard's iteration, starting from $v^0 = 0$, $\psi^0 = 0$. We use the following estimate.

Proposition 4.4.2. *There is $T > 0$, such that the solutions (v^ν, ψ^ν) of the iteration scheme*

$$\begin{cases} \widetilde{\mathcal{L}}(u^a + v^\nu, \nabla\Phi^a + \nabla\Psi^\nu)v^{\nu+1} + \mathcal{E}(u^a, \Phi^a, v^\nu, \Psi^\nu) = f^a\,, \\ \underline{b}\nabla\psi^{\nu+1} + \underline{M}v^{\nu+1} = g^a \\ v^{\nu+1}_{|t<0} = 0\,, \qquad \psi^{\nu+1}_{|t<0} = 0\,, \end{cases} \tag{4.4.6}$$

are bounded in $CH^s(\Omega_T) \times H^{s+1}(\omega_T)$ and converge in $CL^2(\Omega_T) \times H^1(\omega_T)$.

Proof of Proposition 4.4.2. We fix $\mathcal{K} \subset \mathcal{U}$ which contains a neighborhood of $\mathcal{K}_0$ and $K > K_0$. We show by induction that there are $T > 0$ and C_1 such that $(u^a + v^\nu, \varphi^a + \psi^\nu)$ satisfy (4.3.4) and the estimates

$$\|v^\nu\|_{CH^s(\Omega_T)} \le C_1\,, \quad \|\psi^\nu\|_{H^{s+1}(\omega_T)} \le C_1\,. \tag{4.4.7}$$

This is satisfied for $\nu = 0$.

Theorem 3.1.1 implies that if (v^ν, ψ^ν) satisfies the induction hypothesis, (4.4.6) has a unique solution in $CL^2 \times H^1$. We use without proof that the solution belongs to $CH^s \times H^{s+1}$ and satisfies the estimates (4.3.6).

Because $u^a \in H^{s+1}$, if $(u^a + v, \varphi^a + \psi)$ satisfies (4.3.4), then $h := \mathcal{E}(u^a, \Phi^a, v, \Psi)$ satisfies

$$
\begin{aligned}
\|h\|_{H^s_\gamma(\Omega_T)} \leq & C(K)\left(\|v\|_{H^s_\gamma(\Omega_T)} + \|\psi\|_{H^{s+1}_\gamma(\omega_T)}\right) + \\
& C(K)\Big(\|v\|_{L^\infty(\Omega_T)} + \|\nabla\psi\|_{L^\infty(\omega_T)}\Big)\|u^a\|_{H^{s+1}_\gamma(\Omega_T)} .
\end{aligned}
\tag{4.4.8}
$$

Therefore, (4.3.6) and the induction hypothesis imply

$$
\begin{aligned}
&\sqrt{\gamma}\|v^{\nu+1}\|_{H^s_\gamma(\Omega_T)} + e^{-\gamma t}|\!|\!| v^{\nu+1}(t)|\!|\!|_{s,\gamma} + \|\psi^{\nu+1}\|_{H^{s+1}_\gamma(\omega_T)} \leq \\
&\qquad \frac{1}{\sqrt{\gamma}}\, C(K)\Big(\|v^\nu\|_{H^s_\gamma(\Omega_T)} + \|\psi^\nu\|_{H^{s+1}_\gamma(\omega_T)}\Big) + \\
&\qquad \frac{1}{\sqrt{\gamma}}\, C(K)\Big(\|v^\nu\|_{W^{1,\infty}(\Omega_T)} + \|\nabla\psi^\nu\|_{W^{1,\infty}(\omega_T)}\Big) + C_1(T,K) .
\end{aligned}
\tag{4.4.9}
$$

where $C_1(T,K)$ involves norms of (u^a, φ^a) on Ω_T^+ and ω_T^+ respectively. In particular, $C_1(T,K) \to 0$ as $T \to 0$. Choose $\gamma = \gamma(K)$ such that the first term in the right hand side is smaller than half the left hand side.

Choosing $T \leq 1/\gamma$, this implies that that there is a constant $C(K)$ such that

$$
\begin{aligned}
&\|v^{\nu+1}\|_{CH^s(\Omega_T)} + \|\psi^{\nu+1}\|_{H^{s+1}(\omega_T)} \leq \\
&\qquad C(K)\Big(\|v^\nu\|_{W^{1,\infty}(\Omega_T)} + \|\psi^\nu\|_{W^{1,\infty}(\omega_T)}\Big) + C_1(T,K) .
\end{aligned}
\tag{4.4.10}
$$

Because, v^ν and ψ^ν vanish for $t < 0$, and because $s > n/2 + 1$, there is $\delta(T)$ such that

$$
\begin{aligned}
&\|v^\nu\|_{W^{1,\infty}(\Omega_T)} + \|\nabla\psi^\nu\|_{W^{1,\infty}(\omega_T)} \leq \\
&\qquad \delta(T)\Big(\|v^\nu\|_{CH^s(\Omega_T)} + \|\psi^\nu\|_{H^{s+1}(\omega_T)}\Big) + C_1(T,K).
\end{aligned}
\tag{4.4.11}
$$

and $\delta(T) \to 0$ as $T \to 0$. Therefore, if T is small enough, this proves that $(u^a + v^{\nu+1}, \varphi^a + \psi^{\nu+1})$ satisfy (4.3.6) and (4.4.7).

$\square$

Proof of Theorem 4.4.1 *(end).*

With this choice of T, the sequence (v^ν, ψ^ν) is bounded in $(CH^s(\Omega_T) \times H^{s+1}(\omega_T)$. Writing the equation satisfied by $(v^{\nu+1} - v^\nu, \psi^{\nu+1} - \psi^\nu)$ and using the L^2 estimate of Theorem 3.1.1, one proves the convergence. The limit (v, ψ) satisfies (4.4.4). v is bounded with values in H^s and continuous with values in $H^{s'}$ for all $s' < s$. Writing down the equations for $\partial^\alpha v$ and using Theorem 3.1.1, one shows that $v \in CH^s(\Omega_T)$.

$\square$

Proof of Theorem 4.1.6.

Suppose that (u, φ) satisfies (4.1.6) on Ω_T^+, is valued in $\mathcal{K}_0 \subset \mathcal{U}$ and

$$\|u\|_{W^{1,\infty}(\Omega_T^+)} \le K_0\,, \quad \|\nabla\varphi\|_{W^{1,\infty}(\omega_T^+)} \le K_0 \tag{4.4.12}$$

and for all $T' < T$

$$u \in CH^s(\Omega_{T'}^+)\,, \quad \varphi \in H^{s+1}(\omega_{T'}^+)\,. \tag{4.4.13}$$

We prove that there is $\tau > 0$ such that (u, φ) extends as a solution on $\Omega_{T+\tau}^+$. This implies Theorem 4.1.6.

Introduce $0 < T_1 < T_1' < T_2' < T_2 < T$ and $\chi_1(t)$ and $\chi_2(t)$ such that

$$\begin{array}{ll} \chi_1(t) = 0 \ \text{ for } \ t \le T_1\,, & \chi_1(t) = 1 \ \text{ for } \ t \ge T_1'\,, \\ \chi_2(t) = 1 \ \text{ for } \ t \le T_2'\,, & \chi_2(t) = 0 \ \text{ for } \ t \ge T_2\,. \end{array}$$

Introduce next $\tilde\chi_1$ and $\tilde\chi_2$ such that

$$\begin{array}{ll} \tilde\chi_1(t) = 0 \ \text{ for } \ t \le 0\,, & \tilde\chi_1(t) = 1 \ \text{ for } \ t \ge T_1\,, \\ \tilde\chi_2(t) = 1 \ \text{ for } \ t \le T_2\,, & \tilde\chi_2(t) = 0 \ \text{ for } \ t \ge T\,. \end{array}$$

Note that

$$\begin{array}{l} \chi_1 = \chi_1\chi_2 + (1 - \chi_2)\,, \quad \tilde\chi_1 = \tilde\chi_1\tilde\chi_2 + (1 - \tilde\chi_2)\,, \\ (1 - \tilde\chi_2) = (1 - \tilde\chi_2)(1 - \chi_2)\,. \end{array}$$

Therefore, $v = (1 - \chi_2)u$ and $\psi = (1 - \chi_2)\varphi$ satisfy

$$\begin{cases} \widetilde{\mathcal{L}}(u^a + (1 - \tilde\chi_2)v, \nabla\Phi^a + \nabla(1 - \tilde\chi_2)\Psi)v = f^a\,, & \text{on } x_n > 0\,, \\ \underline{b}\nabla\psi + \underline{M}v = g^a\,, & \text{on } x_n = 0 \\ v_{|t<0} = 0\,, \qquad \psi_{|t<0} = 0\,, & \end{cases} \tag{4.4.14}$$

with

$$u^a = \tilde\chi_2\tilde\chi_1 u\,, \quad \varphi^a = \tilde\chi_2\tilde\chi_1\varphi\,, \tag{4.4.15}$$

$$\Phi^a(x) = \kappa x_n + \chi(x_n)\varphi^a(t, y)\,, \quad \Psi(x) = \chi(x_n)\psi(t, y) \tag{4.4.16}$$

and by (4.4.13)

$$\begin{array}{l} f := \mathcal{L}(\tilde\chi_1 u, \nabla\tilde\Phi)(\chi_1 u) = \partial_t\chi_1\mathcal{A}_0(\tilde\chi_1, \nabla\tilde\Phi)u \in CH^s(\Omega)\,, \\ g := -\underline{b}_0(\partial_t\chi_2)\varphi \in H^{s+1}(\omega)\,. \end{array} \tag{4.4.17}$$

Note that f and g are supported in $0 \leq t \leq T_1'$.

We consider (4.4.14) as an equation for (v, ψ) and we show that the solution $\big((1-\chi_2)u, (1-\chi_2)\varphi\big)$ extends to $\Omega_{T+\tau}$. Therefore, we consider the iteration scheme

(4.4.18)
$$\begin{cases} \widetilde{\mathcal{L}}(u^a + (1-\tilde\chi_2)v^\nu, \nabla\Phi^a + \nabla(1-\tilde\chi_2)\Psi^\nu)v^{\nu+1} = f^a & \text{on } x_n > 0\,, \\ \underline{b}\nabla\psi^{\nu+1} + \underline{M}v^{\nu+1} = g^a & \text{on } x_n = 0 \\ v_{|t<0} = 0\,, \qquad \psi_{|t<0} = 0\,, \end{cases}$$

together with the equation $\Psi^\nu(x) = \chi(x_n)\psi^\nu(t,y)$. The initial value (v^0, ψ^0) is an arbitrary extension of $\big((1-\chi_2)u, (1-\chi_2)\varphi\big)$ to Ω.

We fix $\mathcal{K} \subset \mathcal{U}$ which contains a neighborhood of $\mathcal{K}_0$ and $K > K_0$. We show by induction that there are $\tau > 0$ and C_1 such that $(u^a+(1-\tilde\chi_2)v^\nu, \varphi^a + (1-\tilde\chi_2)\psi^\nu)$ satisfy (4.3.4) and the estimates

$$\|v^\nu\|_{CH^s(\Omega_T)} \leq C_1\,, \quad \|\psi^\nu\|_{H^{s+1}(\omega_T)} \leq C_1\,. \tag{4.4.19}$$

Theorem 3.1.1 implies that (4.4.18) has a unique solution in $CL^2 \times H^1$. One can show that this solution belongs to $CH^s \times H^{s+1}$ and satisfies the estimates (4.3.6). Therefore (4.4.12) implies that

$$\begin{aligned} \sqrt{\gamma}\|v^{\nu+1}\|_{H^s_\gamma(\Omega_T)} + e^{-\gamma t}{|\!|\!|} v^{\nu+1}(t){|\!|\!|}_{s,\gamma} + \|\psi^{\nu+1}\|_{H^{s+1}_\gamma(\omega_T)} \leq \\ \frac{1}{\sqrt{\gamma}} C(K)\Big(\|v^\nu\|_{H^s_\gamma(\Omega_T)} + \|\psi^\nu\|_{H^{s+1}_\gamma(\omega_T)}\Big) + \\ \frac{1}{\sqrt{\gamma}} C(K)\Big(\|v^\nu\|_{W^{1,\infty}(\Omega_T)} + \|\psi^\nu\|_{W^{1,\infty}(\omega_T)}\Big) + C_1(T,K)\,. \end{aligned} \tag{4.4.20}$$

where $C_1(T,K)$ involves norms of (u^a, φ^a) on Ω_T^+ and ω_T^+ respectively. By uniqueness, one shows by induction that

$$v^{\nu+1}{}_{|t<T} = (1-\tilde\chi_2)u\,, \quad \psi^{\nu+1}{}_{|t<T} = (1-\tilde\chi_2)\varphi\,. \tag{4.4.21}$$

Therefore, (4.4.12) implies that for $\tau \leq 1/\gamma$,

$$\begin{aligned} \|v^\nu\|_{W^{1,\infty}(\Omega_{T+\tau})} + \|\nabla\psi^\nu\|_{W^{1,\infty}(\omega_{T+\tau})} \leq K_1 + \\ C\delta(\tau)e^{\gamma T}\Big(\|v^\nu\|_{CH^s_\gamma(\Omega_T)} + \|\psi^\nu\|_{H^{s+1}_\gamma(\omega_T)}\Big) \end{aligned} \tag{4.4.22}$$

where $\delta(\tau) \to 0$ as $\tau \to 0$. Choosing γ large enough and next τ small enough, (4.4.20) implies by induction that (4.4.19) is satisfied.

The sequence (v^ν, ψ^ν) is bounded in $(CH^s(\Omega_{T+\tau}) \times H^{s+1}(\omega_{T+\tau})$. Using the equation satisfied by $(v^{\nu+1} - v^\nu, \psi^{\nu+1} - \psi^\nu)$ and the L^2 estimate of Theorem 3.1.1, one proves the convergence of this sequence. The limit (v, ψ) satisfies (4.4.18). v is bounded with values in H^s and continuous with values in $H^{s'}$ for all $s' < s$. Writing down the equations for $\partial^\alpha v$ and using Theorem 3.1.1, one shows that $v \in CH^s(\Omega_{T+\tau})$ (see Remark 4.7.2 below).

$\square$

Proof of Theorem 4.1.5.

Consider initial data $(u_0, \varphi_0) \in H^s(\mathbb{R}^n_+) \times H^{s+\frac{1}{2}}(\mathbb{R}^{n-1})$ which satisfy the compatibility conditions up to order $s-1$. By Proposition 4.2.4 there are sequences of smooth initial data (u_0^ν, φ_0^ν), compatible to order $s-1$, which converge to (u_0, φ_0). Theorem 4.4.1 provides a family of solutions (u^ν, φ^ν) of (4.1.5) with initial data (u_0^ν, φ_0^ν). They are defined on $\Omega^+_{T_\nu}$. Moreover, Proposition 4.3.2 implies that there is $T > 0$ such that the sequence (u^ν, φ^ν) is bounded in $CH^s(\Omega^+_{T'_\nu}) \times H^{s+1}(\omega^+_{T'_\nu})$ for $T'_\nu = \min(T, T_\nu)$. Thus, the continuation Theorem 4.1.6 implies that the solutions (u^ν, φ^ν) can be extended to Ω^+_T. One can extract a convergent subsequence and the limit (u, φ) satisfies (4.4.6). $\varphi \in H^{s+1}(\omega_T)$ and u is bounded with values in H^s and continuous with values in $H^{s'}$ for all $s' < s$. One shows that $u \in CH^s(\Omega_{T+\tau})$, see Remark 4.7.2 below.

$\square$

4.5 Nonlinear estimates

The proof of Propositions 4.3.1 and 4.3.2 relies on estimates for nonlinear functions and commutators. Recall the Gagliardo-Nirenberg's inequalities (see e.g. [Maj 3]).

Theorem 4.5.1. *For all $s \in \mathbb{N}$ there is a constant C, such that for all test function u and $\alpha \in \mathbb{N}^n$ such that $|\alpha| \le s$,*

$$\|\partial_x^\alpha u\|_{L^p(\mathbb{R}^n)} \le C \, \|u\|^{1-2/p}_{L^\infty(\mathbb{R}^n)} \|u\|^{2/p}_{H^s(\mathbb{R}^n)}, \quad 2/p \le |\alpha|/s \tag{4.5.1}$$

This estimates holds also on $\mathbb{R}^{n+1}$, on half spaces $\omega_T =]-\infty, T] \times \mathbb{R}^{n-1} \subset \mathbb{R}^n$, on quadrants $\Omega_T := \{x_n > 0\,, t < T\} \subset \mathbb{R}^{n+1}$ and more generally on all Lipschtzean domains. one has similar estimates for the weighted norms (4.3.2) $\|u\|_{H^s_\gamma(\Omega_T)} \approx \|e^{-\gamma t} u\|_{H^s(\Omega_T)}$. Introduce the weighted norm $\|u\|_{L^p_\gamma} = \|e^{-2\gamma t/p} u\|_{L^p}$.

Proposition 4.5.2. *For all $s \in \mathbb{N}$ there is a constant C, such that for all T, $\gamma \geq 1$, $u \in H^s_\gamma(\Omega_T) \cap L^\infty(\Omega_T)$, $l \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$ such that $l + |\alpha| \leq s$,*

$$\gamma^l \|\partial_x^\alpha u\|_{L^p_\gamma(\Omega_T)} \ \leq \ C \, \|u\|^{1-2/p}_{L^\infty(\Omega_T)} \|u\|^{2/p}_{H^s_\gamma(\Omega_T)} \,, \quad 2/p \leq (|\alpha| + l)/s \tag{4.5.2}$$

This estimate also holds on ω_T.

Definition 4.5.3. *A non linear function of order $\leq k$ is a finite sum*

$$\mathcal{F}(u) = F_0(u) + \sum_{l=1}^{k} \sum_{|\alpha_1| + \ldots |\alpha_l| \leq k} F_{l, \alpha_1, \ldots, \alpha_l}(u) \partial_x^{\alpha_1} u \ldots \partial_x^{\alpha_l} u \,, \tag{4.5.3}$$

where the $F_{l,\alpha}(u)$ are l-multilinear mappings which depend smoothly on u and $F_0(0) = 0$.

Proposition 4.5.4. *For all $s \in \mathbb{N}$. and nonlinear function $\mathcal{F}$ of order $k \leq s$, there is function $C(K)$ such that for all T, $\gamma \geq 1$, $u \in H^s_\gamma(\Omega_T) \cap L^\infty(\Omega_T)$, $l \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$ such that $k + l + |\alpha| \leq s$,*

$$\gamma^l \| e^{-2\gamma t/p} \partial_x^\alpha \mathcal{F}(u) \|_{L^p_\gamma(\Omega_T)} \ \leq \ C\big(\|u\|_{L^\infty(\Omega_T)} \big) \|u\|^{2/p}_{H^s_\gamma(\Omega_T)} \tag{4.5.4}$$

where $2/p \leq (k + l + |\alpha|)/s$.

There is a similar estimate for non linear functions of $\varphi \in H^s_\gamma(\omega_T)$.

To prove Proposition 4.3.2 we also need estimates on Ω_T^+ and ω_T^+. However, on these strips, the constants in (4.5.1) and (4.5.2) are not uniform with respect to T, as $T \to 0$. We use the following substitute.

Proposition 4.5.5. *For all integer $s > n/2$, there is C such that for all $T > 0$, $\psi \in H^s(\omega_T^+)$ and $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq s$,*

$$\|\partial^\alpha \psi\|_{L^p(\omega_T^+)} \ \leq \ C \Big(K_{s,T}(\psi)^{1-2/p} \|\psi\|^{2/p}_{H^s(\Omega_T^+)} + K_{s,T}(\psi) \Big) \tag{4.5.5}$$

where $2/p \leq |\alpha|/s$ and

$$K_{s,T}(\psi) \ := \ \|\psi\|_{L^\infty(\omega_T^+)} + \sum_{j=0}^{s-1} \|\partial_t^j \psi(0)\|_{H^{s-j-1/2}(\mathbb{R}^{n-1})} \,. \tag{4.5.6}$$

Proof. There is $\psi_1 \in H^s(\mathbb{R}^n)$ such that $\partial^j \psi_{1|t=0} = \partial_t^j \psi_{|t=0}$ for all $j \leq s - 1$. In addition,

$$\|\psi_1\|_{H^s(\mathbb{R}^n)} \leq C \sum_{j=0}^{s-1} \|\partial_t^j \psi(0)\|_{H^{s-j-1/2}(\mathbb{R}^{n-1})} \,.$$

Because $s > n/2$, the Sobolev imbedding theorem implies that $\psi_1 \in L^\infty(\mathbb{R}^n)$ and $\partial^\alpha \psi_1 \in L^p(\mathbb{R}^n)$ for $2/p = |\alpha|/s$. Thus $\psi_2 := \psi - \psi_1$ belongs to $H^s(\omega_T^+)$ and

$$\|\psi_2\|_{L^\infty(\omega_T^+)} \le C K_{s,T}(\psi)\,, \quad \|\psi_2\|_{H^s(\omega_T^+)} \le \|\psi\|_{H^s(\omega_T^+)} + C K_{s,T}(\psi)\,.$$

Because $\partial^j \psi_{2|t=0} = 0$ for all $j \le s-1$, the extension of ψ_2 by zero for $t < 0$ belongs to $H^s(\omega_T)$. Applying Theorem 4.5.1 to ψ_2 yields (4.5.5). $\square$

There is an analogue for functions u in Ω_T^+.

Proposition 4.5.6. *For all integer $s > n/2$, there is C such that for all $T > 0$, $u \in CH^s(\Omega_T^+)$ and $\alpha \in \mathbb{N}^{n+1}$ such that $|\alpha| \le s$,*

$$\|\partial^\alpha u\|_{L^p(\Omega_T^+)} \;\le\; C\Big(\widetilde{K}_{s,T}(u)^{1-2/p} \|u\|^{2/p}_{H^s(\Omega_T^+)} + \widetilde{K}_{s,T}(u)\Big) \tag{4.5.7}$$

where $2/p \le |\alpha|/s$ and

$$\widetilde{K}_{s,T}(u) \;:=\; \|u\|_{L^\infty(\Omega_T^+)} + \sum_{j=0}^{s-1} \|\partial_t^j u(0)\|_{H^{s-j}(\mathbb{R}^n_+)}\,. \tag{4.5.8}$$

Proof. Extending $\partial^j u_{|t=0}$ on $\{x_n < 0\}$ one constructs $u_1 \in H^{s+\frac{1}{2}}(\mathbb{R}^{n+1})$ such that $\partial^j u_{1|t=0} = \partial_t^j u_{|t=0}$ for all $j \le s-1$ and

$$\|u_1\|_{H^{s+\frac{1}{2}}(\mathbb{R}^{n+1})} \le C \sum_{j=0}^{s-1} \|\partial_t^j u(0)\|_{H^{s-j}(\mathbb{R}^n_+)}\,.$$

Because $s + 1/2 > (n+1)/2$, the Sobolev imbedding imply that $u_1 \in L^\infty(\mathbb{R}^{n+1})$ and $\partial^\alpha u_1 \in L^p(\mathbb{R}^{n+1})$ for $2/p = |\alpha|/s$. Thus $u_2 := u - u_1$ belongs to $H^s(\Omega_T^+)$ and

$$\|u_2\|_{L^\infty(\Omega_T^+)} \le C\widetilde{K}_{s,T}(\psi)\,, \quad \|u_2\|_{H^s(\Omega_T^+)} \le \|u\|_{H^s(\Omega_T^+)} + C\widetilde{K}_{s,T}(u)\,.$$

Moreover, $\partial^j u_{2|t=0} = 0$ for all $j \le s-1$. Thus, the extension of u_2 by zero for $t < 0$ belongs to $H^s(\Omega_T)$ and Theorem 4.5.1 applies to u_2, implying (4.5.7). $\square$

Corollary 4.5.7. *For all $s > n/2$. and nonlinear function $\mathcal{F}$ of order $\le k \le s$, there is function $C(K)$ such that for all $T > 0$, $u \in CH^s(\Omega_T^+)$,*

$$\|\mathcal{F}(u)\|_{L^p(\Omega_T^+)} \;\le\; C\big(\widetilde{K}_{s,T}(u)\big)\Big(\|u\|_{H^s(\Omega_T^+)} + \widetilde{K}_{s,T}(u)\Big)^{2/p} \tag{4.5.9}$$

where $2/p \le k/s$.

Thee is a similar estimate for non linear functions of $\varphi \in H^s(\omega_T^+)$.

Remark 4.5.8. The estimates (4.5.7) and (4.5.9) are satisfied for all $u \in H^s(\Omega_T^+) \cap L^\infty(\Omega_T^+)$ such that $\partial^j u_{|t=0} \in H^{s-j}(\mathbb{R}^n_+)$ for all $j \le s-1$.

4.6 Proof of the H^s_γ estimates

Consider a compact set $\mathcal{K} \subset \mathcal{U}$, a constant K, times $T > T_0$ and $(u, \varphi) \in CH^s(\Omega_T) \times H^{s+1}(\omega_T)$ which satisfy (4.3.4). We consider a solution (v, ψ) of the linearized equations (4.3.5) and we show that the a-priori estimate (4.3.6) holds. Because the hyperbolicity assumptions and the stability assumptions are invariant by change of unknowns, we deduce from section 3 the following result.

Proposition 4.6.1. *Consider data $f \in L^2(\Omega_T)$ and $g \in L^2(\omega_T)$ which vanish for $t < T_0$. Then the equation (4.3.5) has a unique solution $(v, \psi) \in L^2(\Omega_T) \times H^1(\omega_T)$ and $v \in C^0(]-\infty, T]; L^2(\mathbb{R}^n_+))$.*

Moreover there are constants C and γ_0 such that for all $T > 0$, $\gamma \ge \gamma_0$, and $t \in [0, T]$

$$
\begin{aligned}
&e^{-2\gamma t}\|v(t)\|^2_{L^2(\mathbb{R}^n_+)} + \gamma\|v\|^2_{L^2_\gamma(\Omega_t)} + \|v_{|x_n=0}\|^2_{L^2_\gamma(\omega_t)} + \|\psi\|^2_{H^1_\gamma(\omega_t)} \\
&\qquad\qquad \le C\left(\frac{1}{\gamma}\|f\|^2_{L^2_\gamma(\Omega_t)} + \|g\|^2_{L^2_\gamma(\omega_t)}\right).
\end{aligned}
\tag{4.6.1}
$$

Proof of Proposition 4.3.1

a) To prove Proposition 4.3.1, we first estimate the tangential derivatives. Introduce the tangential norms

$$
\|u\|^2_{H^{s,0}_\gamma(\Omega_T)} := \int_0^\infty \|u(\,\cdot\,, x_n)\|^2_{H^s_\gamma(\omega_T)} dx_n\,. \tag{4.6.2}
$$

Introduce $v_\alpha = \partial^\alpha_{t,y} v$ and $\psi_\alpha = \partial^\alpha_{t,y}\psi$. Then

$$
\begin{cases}
\mathcal{L}(u, \nabla\Phi) v_\alpha = f_\alpha := \mathcal{A}_n \partial^\alpha_{t,y}(\mathcal{A}_n^{-1} f) - \mathcal{A}_n\big[\partial^\alpha_{t,y}\,,\, \mathcal{A}_n^{-1}\mathcal{L}\big] v \\
\underline{b}\nabla\psi_\alpha + \underline{M} v_\alpha = g_\alpha := \partial^\alpha_{t,y} g\,.
\end{cases}
\tag{4.6.3}
$$

Proposition 4.6.1 implies that

$$
\begin{aligned}
&e^{-2\gamma t}\|v_\alpha(t)\|^2_{L^2(\mathbb{R}^n_+)} + \gamma\|v_\alpha\|^2_{L^2_\gamma(\Omega_t)} + \|v_{\alpha|x_n=0}\|^2_{L^2_\gamma(\omega_t)} + \|\psi_\alpha\|^2_{H^1_\gamma(\omega_t)} \\
&\qquad\qquad \le C\left(\frac{1}{\gamma}\|f_\alpha\|^2_{L^2_\gamma(\Omega_t)} + \|g_\alpha\|^2_{L^2_\gamma(\omega_t)}\right).
\end{aligned}
\tag{4.6.4}
$$

We now estimate $\|f_\alpha\|_{L^2_\gamma}$. We use the Gagliardo-Nirenberg estimates of Proposition 4.5.2 on ω_T for fixed x_n and then we integrate in x_n. The norm of the first term in the right hand side of (4.6.3) is smaller than

$$C(K)\|f\|_{H^s_\gamma(\Omega_T)} + C(K)\|f\|_{L^\infty(\Omega_T)}\big(\|u\|_{H^s_\gamma(\Omega_T)} + \|\nabla\Phi\|_{H^s_\gamma(\omega_T)}\big)\,. \tag{4.6.5}$$

The commutator is a sum of terms

$$\mathcal{F}_l(u)\mathcal{G}_k(\nabla\Phi)\partial^\beta_{t,y}v \tag{4.6.6}$$

where $\mathcal{F}_l$ [resp. $\mathcal{G}_k$] is a nonlinear function of degree l [resp. k] of u [resp. $\nabla\Phi$], and $k+l+|\beta|\le s+1$, $k+l\ge 1$ and $|\beta|\ge 1$. Proposition 4.5.4 and the estimates (4.3.4) imply that for $l\ge 1$ and $k\ge 1$,

$$\|\mathcal{F}_l(u(\,\cdot\,,x_n))\|_{L^p_\gamma(\omega_T)} \le C(K)\|u(\,\cdot\,,x_n)\|^{2/p}_{H^s_\gamma(\omega_T)}\,,\quad 2/p\le (l-1)/(s-1)$$

$$\|\mathcal{G}_k(\nabla\Phi(\,\cdot\,,x_n))\|_{L^q_\gamma(\omega_T)} \le C(K)\widetilde{\chi}(x_n)\,\|\varphi\|^{2/q}_{H^{s+1}_\gamma(\omega_T)}\,,\quad 2/q\le (k-1)/(s-1)\,,$$

where $\widetilde{\chi}\in C^\infty_0(\mathbb{R})$. When $k=0$ or $l=0$, one uses that

$$\|\mathcal{F}_0(u)\|_{L^\infty}\le C(K)\,,\quad \|\mathcal{G}_0(\nabla\Phi)\|_{L^\infty}\le C(K)\,.$$

Because $\beta\neq 0$, Proposition 4.5.2 implies that

$$\|\partial^\beta_{t,y}v(\,\cdot\,,x_n)\|_{L^r_\gamma(\omega_T)} \le C\|v\|^{1-2/r}_{W^{1,\infty}(\Omega_T)}\|v(\,\cdot\,,x_n)\|^{2/r}_{H^s_\gamma(\omega_T)}$$

where $2/r\le(|\beta|-1)/(s-1)$.

Because $k+l\ge 1$ and $|\beta|\ge 1$, one can choose p,q and r such that $2/p=2/q+2/r=(k+l-1+|\beta|-1)/(s-1)=1$. Thus, adding up, this implies that the L^2_γ-norm of the commutator is smaller than

$$C(K)\|v\|_{H^{s,0}_\gamma(\Omega_T)} + C(K)\Big(\|u\|_{H^s_\gamma(\Omega_T)} + \|\varphi\|_{H^{s+1}_\gamma(\Omega_T)}\Big)\|v\|_{W^{1,\infty}(\Omega_T)}\,. \tag{4.6.7}$$

With (4.6.4) and (4.6.5), this implies that for $\gamma\ge\gamma_0(K)$

(4.6.8)

$$\begin{aligned}
&e^{-\gamma t}|\!|\!|v(t)|\!|\!|_{s,\gamma,0} + \sqrt{\gamma}\|v\|_{H^{s,0}_\gamma(\Omega_T)} + \|v_{|x_n=0}\|_{H^s_\gamma(\omega_T)} + \|\psi_\alpha\|_{H^{s+1}_\gamma(\omega_T)} \le\\
&\frac{1}{\sqrt{\gamma}}\,C(K)\big(1+\|v\|_{W^{1,\infty}(\Omega_T)}+\|f\|_{L^\infty(\Omega_T)}\big)\big(\|u\|_{H^s_\gamma(\Omega_T)}+\|\varphi\|_{H^{s+1}_\gamma(\omega_T)}\big)\\
&\qquad\qquad + \frac{1}{\sqrt{\gamma}}\,C(K)\big(\|f\|_{H^s_\gamma(\Omega_T)} + \|g\|_{H^s(\omega_T)}\big)\,.
\end{aligned}$$

where

$$\| v(t) \|_{s,\gamma,0} := \sum_{|\alpha|\le s} \gamma^{s-|\alpha|} \|\partial^\alpha_{t,y} v(t)\|_{L^2(\mathbb{R}^n_+)} \,.$$

b) The normal derivatives are estimated using the equation

$$\partial_n v = \mathcal{A}_n^{-1}\Big(f - \sum_{j=0}^{n} \mathcal{A}_j \partial_j v \Big) \,. \tag{4.6.9}$$

Therefore,

$$\partial_x^\alpha v \;=\; \sum_{|\alpha'|=|\alpha|} \mathcal{C}_\alpha(u, \nabla\Phi) \partial^{\alpha'}_{t,y} v \;+\; E_\alpha + F_\alpha \tag{4.6.10}$$

where E_α is a sum of terms (4.6.6) with $l + k + |\beta| \le |\alpha|$ with $|\beta| < |\alpha|$, and F_α a sum of terms

$$\mathcal{F}_l(u) \mathcal{G}_k(\nabla\Phi) \partial_x^\beta f \tag{4.6.11}$$

with $l + k + |\beta| \le |\alpha| - 1$. The L^2_γ norm of $\partial_n^k v$ is easily obtained form Proposition 4.5.4.

The estimate of the CL^2 norm is more delicate. The first term w in the right hand side of (4.6.10) satisfies

$$\gamma^{s-|\alpha|} \|w(t)\|_{L^2(\mathbb{R}^n_+)} \le C(K) \| v(t) \|_{s,\gamma,0}$$

Next we remark that $\partial_t E_\alpha$ is a sum of terms (4.6.6) with $l + k + |\beta| \le |\alpha| + 1$. with $|\beta| \le |\alpha|$. Therefore, as in part a) above, one can estimate $\gamma^{s-|\alpha|+1} \|E_\alpha\|_{L^2_\gamma}$ and $\gamma^{s-|\alpha|} \|\partial_t E_\alpha\|_{L^2_\gamma}$ by (4.6.7). Therefore, $e^{-\gamma t} \gamma^{s-|\alpha|+\frac12} \|E_\alpha(t)\|_{L^2(\mathbb{R}^n_+)}$ is also bounded from above by (4.6.7).

Similarly, $\partial_t F_\alpha$ is a sum of terms (4.6.11) with $l + k + |\beta| \le |\alpha|$. Thus $\gamma^{s-|\alpha|+1} \|F_\alpha\|_{L^2_\gamma}$, $\gamma^{s-|\alpha|} \|\partial_t F_\alpha\|_{L^2_\gamma}$ and $e^{-\gamma t} \gamma^{s-|\alpha|+\frac12} \|F_\alpha(t)\|_{L^2(\mathbb{R}^n_+)}$ are estimated by (4.6.5).

With (4.6.8), this implies (4.3.6) and Proposition 4.3.1 is proved.

□

4.7 H^s estimates for the initial-boundary value problem

Given $\mathcal{K}$ and K, consider $T > 0$, $u \in CH^s(\Omega^+_T)$ and $\varphi \in H^{s+1}(\omega^+_T)$ which satisfy (4.3.9). Choosing $T = 1/\gamma$, Theorem 3.1.1. implies

Proposition 4.7.1. *Consider data*

$$f \in L^2(\Omega_T^+), \quad g \in L^2(\omega_T^+), \quad v_0 \in L^2(\mathbb{R}^n_+), \quad \psi_0 \in H^{1/2}(\mathbb{R}^{n-1})\,. \tag{4.7.1}$$

The initial boundary value problem

$$\begin{cases} \mathcal{L}(u, \nabla\Phi)v = f\,, & \text{on } x_n > 0\,, \\ \underline{b}\nabla\psi + \underline{M}v = g\,, & \text{on } x_n = 0 \\ v_{|t=0} = u_0\,, & \psi_{|t=0} = \varphi_0\,. \end{cases} \tag{4.7.2}$$

has a unique solution $(v, \psi) \in L^2(\Omega_T^+) \times H^1(\omega_T^+)$ *and* $v \in C^0([0,T]; L^2(\mathbb{R}^n_+))$.

Moreover, there are constants C *and* T_0 *which depend only on* $\mathcal{K}$ *and* K *such that, if* $T \le T_0$*, the solution* (v, ψ) *satisfies for all* $t \in [0, T]$

$$\begin{aligned} &\|v(t)\|^2_{L^2(\mathbb{R}^n_+)} + \frac{1}{T}\|v\|^2_{L^2(\Omega_t^+)} + \|v_{|x_n=0}\|^2_{L^2(\omega_t^+)} + \|\psi\|^2_{H^1(\omega_t^+)} \\ &\le C\left(T\|f\|^2_{L^2(\Omega_t^+)} + \|g\|^2_{L^2(\omega_t^+)} + \|v_0\|^2_{L^2(\mathbb{R}^n_+)} + \|\psi_0\|^2_{H^{1/2}(\mathbb{R}^{n-1})}\right). \end{aligned} \tag{4.7.3}$$

Proof of Proposition 4.3.2

Consider $(v, \psi) \in CH^s(\Omega_T^+) \times H^{s+1}(\omega_T^+)$ a solution of (4.3.10), i.e. a solution of (4.7.2) with $f = 0$ and $g = 0$. We prove that the estimate (4.3.11) is satisfied.

a) *Tangential derivatives.* For $|\alpha| \le s$, $v_\alpha = \partial^\alpha_{t,y} v$ and $\psi_\alpha = \partial^\alpha_{t,y}\psi$ satisfy (4.6.3). If $T \le T_0$, Proposition 4.7.1 implies that

$$\begin{aligned} &\sup_{t\in[0,T]} \|v_\alpha(t)\|^2_{L^2(\mathbb{R}^n_+)} + \frac{1}{T}\|v_\alpha\|^2_{L^2(\Omega_t^+)} + \|\psi_\alpha\|^2_{H^1(\omega_T^+)} \\ &\qquad\qquad \le C\left(T\|f_\alpha\|^2_{L^2(\Omega_t^+)} + \|v(0)\|^2_s + \|\psi(0)\|^2_{s+\frac{1}{2}}\right). \end{aligned} \tag{4.7.4}$$

Because $g = 0$, the boundary source term g_α in (4.6.3) vanishes. Similarly, the commutator f_α is the sum of the terms in (4.6.6). Corollary 4.5.7 and (4.3.9) imply that

$$\begin{aligned} &\|f_\alpha\|_{L^2(\Omega_T^+)} \le \\ &\quad C(K)\Big(\|v\|_{H^{s,0}(\Omega_T^+)} + N_T(v,\psi)\big(1 + \|u\|_{H^s(\Omega_T^+)} + \|\varphi\|_{H^{s+1}(\Omega_T^+)}\big)\Big)\,, \end{aligned} \tag{4.7.5}$$

Summing (4.7.4) over α, and choosing T_0 so small that $T_0 C(K) \le 1/2$, yields

$$\begin{aligned} &\|v\|_{CH^s(\Omega_T^+)} + \|\psi\|_{H^{s+1}_\gamma(\omega_T^+)} + \frac{1}{T}\|v\|^2_{H^{s,0}(\Omega_t^+)} \le \\ &\qquad TC(K)N_T(v,\psi)\big(1 + \|u\|_{H^s(\Omega_T^+)} + \|\varphi\|_{H^{s+1}(\Omega_T^+)}\big) \\ &\qquad\qquad + C(K)\big(\|v(0)\|_s + \|\psi(0)\|_{s+\frac{1}{2}}\big)\,. \end{aligned} \tag{4.7.6}$$

b) *Normal derivatives.* We proceed as for the proof or Proposition 4.3.1, part b), using Corollary 4.5.7 in place of Proposition 4.5.4.

$\square$

Remark 4.7.2. In the proof of Proposition 4.3.1 [resp. Proposition 4.3.2] it is sufficient to assume that u and v belong to $H^s_\gamma(\Omega_T)$ [resp. $(u, v) \in H^s(\Omega_T) \cap W^{1,\infty}(\Omega_T)$ and $\|(u, v)(0)\|_s < \infty$]. The estimates (4.6.7) (4.7.5) are satisfied under these assumptions and the equations (4.6.3) make sense. Moreover, Theorem 3.1.1 implies that $\partial^\alpha_{t,y} v \in CL^2$ (see also Propositions 6.6.1 and 6.7.1). Estimating the normal derivatives implies that $v \in CH^s$.

5 Stability of Weak Shocks

In this section we study the uniform stability of weak shocks. We concentrate on planar shocks and our main concern is to make precise the loss of stability when the strength of the shock tends to zero. We refer to [Mét 1] for details.

5.1 Statement of the result

Consider the linearized shock equations (1.3.8) around a constants (u^+, u^-) and $\Phi = x_n + \sigma t$.

$$
\text{(5.1.1)} \qquad \begin{cases} L^\pm v^\pm := \displaystyle\sum_{j=0}^{n-1} A_j^\pm \partial_j v^\pm + \widetilde{A}_n^\pm \partial_n v^\pm = f^\pm & \text{on} \quad \pm x_n > 0\,, \\ B(\psi, v) := \displaystyle\sum_{j=0}^{n-1} b_j \partial_j \psi - \widetilde{A}_n^+ v^+ + \widetilde{A}_n^- v^- = g & \text{on} \quad x_n = 0 \end{cases}
$$

with $A_j^\pm = A_j(u^\pm)$, $\widetilde{A}_n^\pm = A_n^\pm - \sigma A_0^\pm$ and $b_j = [f_j(u)]$. The constant states $u^\pm$ and σ are chosen close to solutions of the Rankine-Hugoniot equations

$$
\text{(5.1.2)} \qquad [f_n(u)] = \sigma\, [f_0(u)]\,.
$$

We still assume that the hyperbolicity Assumption 1.1.1 is satisfied. We consider the eigenvalues $\lambda_1(u, \xi') \leq \ldots \leq \lambda_N(u, \xi')$ of the matrix

$$
\text{(5.1.3)} \qquad A(u, \xi') := \sum_{j=1}^{n} \xi_j A_0^{-1}(u) A_j(u)\,.
$$

They are real and positively homogeneous of degree one in $\xi' = (\xi_1, \ldots, \xi_n)$. Note also that $\lambda_1(u, -\xi') = -\lambda_N(u, \xi')$, etc until $\lambda_N(u, -\xi') = -\lambda_1(u, \xi')$.

Assumption 5.1.1. *For u in a neighborhood $\mathcal{U}$ of $\underline{u} \in \mathbb{R}^N$ and ξ in a neighborhood of $\underline{\xi}' = (0, \ldots, 0, 1) \in \mathbb{R}^n$, $\lambda_1(u, \xi')$ is a simple and genuinely nonlinear eigenvalue. Moreover, the hessian matrix*

$$
\left\{ \frac{\partial^2 \lambda_1}{\partial \xi_j \partial \xi_k} \right\}_{1 \leq j,k \leq n-1}
$$

is definite negative.

REMARKS. 1. Changing x_n into $-x_n$ changes ∂_n into $-\partial_n$ and thus ξ_n into $-\xi_n$. Hence, all the results are valid if one considers λ_N instead of λ_1, changing the sign condition on the Hessian matrix. More generally, one should be able to consider anyone of the eigenvalues λ_k, assuming a suitable block structure condition. This is done in [Mét 1] under the assumption of strict hyperbolicity.

2. Because the eigenvalue is simple, λ_1 is a smooth function of (u, ξ') and the matrix (5.1.3) is well defined. In addition, the cone of directions of hyperbolicity is convex and therefore the Hessian matrix is necessarily nonpositive. Thus our assumption is a nondegeneracy condition.

3. In the case of Euler's equations, the eigenvalues are $u \cdot \xi'$ and $u \cdot \xi' \pm c|\xi'|$. The non degenerate eigenvalues are $u \cdot \xi' \pm c|\xi'|$ and they satisfy Assumption 5.1.1.

We now assume that the discontinuity (u^+, u^-, σ) is close to a weak 1-shock, meaning that u^+ is close to the Rankine Hugoniot curve associated to λ_1 and starting at u^-. Let $r(u, \xi')$ denote the right eigenvector associated to the eigenvalue λ_1 such that

$$r(u, \xi') \cdot \nabla_u \lambda_1(u, \xi') \;=\; 1\,. \tag{5.1.4}$$

Assumption 5.1.2. *Fix a constant C_0 and a compact subset $\mathcal{K} \subset \mathcal{U}$. We consider data (u^+, u^-, σ) such that $u^- \in \mathcal{K}$ and there exits $\varepsilon > 0$ such that*

$$\begin{aligned} &|u^+ \,-\, (u^- \,-\, \varepsilon r(u^-))| \;\le\; C_0\, \varepsilon^2\,, \\ &|\sigma - (\lambda_1(u^-, \underline{\xi}') \,-\, \varepsilon/2| \;\le\; C_0\, \varepsilon^2\,. \end{aligned} \tag{5.1.5}$$

Note that the condition $\varepsilon > 0$ is the Lax's entropy condition (1.5.11). With (5.1.5), it implies that the shock conditions (1.5.1) are satisfied if ε is small enough.

We denote by $\Omega_\pm$ the half space $\{\pm x_n > 0\}$ and we recall the definition of the weighted norms

$$\|v\|^2_{L^2_\gamma} \;=\; \int e^{-2\gamma t} |v|^2 dx\,, \tag{5.1.6}$$

$$\|\psi\|^2_{H^1_\gamma} \;:=\gamma\, \|v\|^2_{L^2_\gamma} + \|\nabla\psi\|^2_{L^2_\gamma} \tag{5.1.7}$$

Theorem 5.1.3. *If ε is small enough, the linearized equation* (5.1.1) *is uniformly stable. More precisely, there are $\varepsilon_0 > 0$, γ_0 and C such that for all $\varepsilon \in]0, \varepsilon_0]$, all $\gamma \geq \gamma_0$ and all $v^{\pm} \in H^1_\gamma(\Omega_\pm)$ and $\psi \in H^1_\gamma(\mathbb{R}^n)$, one has*

$$
\begin{aligned}
\text{(5.1.8)} \qquad &\sqrt{\gamma}\|v\|_{L^2_\gamma(\Omega)} + \sqrt{\varepsilon}\|v_{|x_n=0}\|_{L^2_\gamma(\mathbb{R}^n)} + \sqrt{\varepsilon}\|\psi\|_{L^2_\gamma(\mathbb{R}^n)} + \varepsilon\|\psi\|_{H^1_\gamma(\mathbb{R}^n)} \\
&\qquad \leq C\left(\frac{1}{\sqrt{\gamma}}\|\, Lv \,\|_{L^2_\gamma(\Omega)} + \frac{1}{\sqrt{\varepsilon}}\|\, B(\psi, v)\,\|_{L^2_\gamma(\mathbb{R}^n)}\right).
\end{aligned}
$$

In this estimate, the norm $\|v\|$ stands for $\|v^+\| + \|v^-\|$ and we use a similar notation for the traces and Lv.

Note that Theorem 5.1.3 extends to variable $(u^+, u^-, \nabla\Phi)$ (see [Mét 2]). The proof in the constant coefficient case given below is an introduction to the general case.

5.2 Structure of the equations

In this section, we choose bases which simplify the analysis of the equations.

Lemma 5.2.1. *For all $u \in \mathcal{U}$, there is an invertible matrix V which depends smoothly on u, such that the operator $\widetilde{L} = V^{-1}A_0^{-1}LV$ has the following structure :*

$$
\text{(5.2.1)} \qquad \widetilde{L} = \widetilde{G}\partial_n + P, \quad P = \partial_t + \sum_{j=1}^{n-1} \widetilde{A}_j \partial_j
$$

with

$$
\text{(5.2.2)} \qquad \widetilde{G}(u) = \begin{bmatrix} \lambda_1(u, \underline{\xi}') - \sigma & 0 \\ 0 & G'(u) \end{bmatrix},
$$

$$
\text{(5.2.3)} \qquad P(u, \tau, \eta) = \begin{bmatrix} \tau + \nu(u, \eta), \cdot\eta & {}^t\ell(u, \eta) \\ \ell(u, \eta) & P'(u, \tau, \eta) \end{bmatrix}
$$

where $\widetilde{G}(u)$ and $P(u, \tau, \eta)$ are real and symmetric. Moreover, $\ell(\eta) \neq 0$ for all $\eta \neq 0$.

Proof. **a)** Let S denote the symmetrizer given by Assumption I.1.1. Because SA_0 is definite positive, one can define the positive definite square root

$S_0 = (SA_0)^{1/2}$. The matrices $S_0 A_0^{-1} A_j S_0^{-1} = S_0^{-1} S A_j S_0^{-1}$ are symmetric. Therefore there is an orthogonal matrix U such that

$$G := U^{-1} S_0 A_0^{-1} A_n S_0^{-1} = \begin{bmatrix} \lambda_1(u, \underline{x}') & 0 \\ 0 & A_n' \end{bmatrix} \tag{5.2.4}$$

Since $\widetilde{A}_n = A_n - \sigma A_0$, this implies that $\widetilde{G} := U^{-1} S_0 A_0^{-1} \widetilde{A}_n S_0^{-1} U$ has the form (5.2.2) with $G' = A_n' - \sigma$.

We perform the change of unknowns

$$v = V \widetilde{v}, \quad \text{with} \quad V := (SA_0)^{-1/2} U . \tag{5.2.5}$$

Then the equations for v are equivalent to

$$\widetilde{L}\widetilde{v} := \partial_t \widetilde{v} + \sum_{j=1}^{n-1} \widetilde{A}_j \partial_j \widetilde{v} + \widetilde{G} \partial_n \widetilde{v} = V^{-1} A_0^{-1} f := \widetilde{f} \tag{5.2.6}$$

with $\widetilde{A}_j = V^{-1} A_0^{-1} A_j V = U^{-1} S_0^{-1} S A_j S_0^{-1/2} U$ symmetric.

b) We now prove that $\ell(\eta)$ does not vanish when $\eta \neq 0$. We do not mention explicitly the dependence on u which appears only as a parameter. Differentiating the identity

$$(A(\xi) - \lambda_1(\xi')) r(\xi') := \Big(\sum_{j=1}^{n} \xi_j A_0^{-1} A_j - \lambda_1(\xi') \Big) r(\xi') = 0 ,$$

implies that for $j \in \{1, \dots, n-1\}$,

$$\big(A_0^{-1} A_j - \partial_{\xi_j} \lambda_1(\xi') \big) r(\xi') = -(A(\xi') - \lambda_1(\xi')) \partial_{\xi_j} r(\xi') \tag{5.2.7}$$

Let $e = (1, 0, \dots, 0)$ denote the first vector of the canonical basis in $\mathbb{R}^N$. Then, (5.2.1) shows that $r(\underline{\xi}) = \alpha V e$ with $\alpha \neq 0$. Thus, multiplying (5.2.7) by $\alpha^{-1} V^{-1}$ and evaluating at $\xi' = \underline{\xi}'$, implies that the first column of $\widetilde{A}_j$ is

$$\widetilde{A}_j e = \partial_{\xi_j} \lambda_1(\underline{\xi}') \, e - (G - \lambda_1(\underline{\xi}')) r_j , \quad r_j := \alpha^{-1} V^{-1} \partial_{\xi_j} r(\underline{\xi}') . \tag{5.2.8}$$

Differentiating (5.2.7) once more, multiplying on the left by ${}^t e V^{-1}$ and noticing that ${}^t e V^{-1} (A(\underline{\xi}') - \lambda_1(\underline{\xi}')) = {}^t e (G - \lambda_1(\underline{\xi}')) V^{-1} = 0$ yields

(5.2.9)

$$\begin{aligned} \alpha \partial^2_{\xi_j \xi_k} \lambda_1(\underline{\xi}') = {}^t e V^{-1} \big(A_0^{-1} A_j - \partial_{\xi_j} \lambda_1(\underline{\xi}') \big) \partial_{\xi_k} r(\underline{\xi}') \\ + {}^t e V^{-1} \big(A_0^{-1} A_k - \partial_{\xi_k} \lambda_1(\underline{\xi}') \big) \partial_{\xi_j} r(\underline{\xi}') \big) \end{aligned}$$

Because $\widetilde{A}_j$ is symmetric, (5.2.8) implies that

$$ {}^t e V^{-1}(A_0^{-1}A_j - \partial_{\xi_j}\lambda_1(\underline{\xi})) = {}^t e(\widetilde{A}_j - \partial_{\xi_j}\lambda_1(\underline{\xi}))V^{-1} = -{}^t r_j(G - \lambda_1(\underline{\xi}'))V^{-1} . $$

Hence, substituting $\partial_{\xi_j} r(\underline{\xi}') = \alpha V r_j$ in (5.2.9) and using (5.2.4) yields

$$ \partial^2_{\xi_j\xi_k}\lambda(\underline{\xi}') = -2r_j(G - \lambda_1(\underline{\xi}'))r_k = -2r'_j(A'_n - \lambda)r'_k \tag{5.2.10} $$

r'_j denotes the set of the $N-1$ last components of r_j. We can now compute the $N-1$ last components of the first column of P. By (5.2.8) it is

$$ \ell(\eta) = \sum_{j=1}^{n-1} \eta_j(A'_n - \lambda_1(\underline{\xi}'))r'_j \tag{5.2.11} $$

and (5.2.10) implies that

$$ {}^t\ell(\eta)\,(A'_n - \lambda_1(\underline{\xi}'))\,\ell(\eta) \;=\; -2\sum_{j,k}\eta_j\eta_k\,\partial^2_{\xi_j\xi_k}\lambda(\underline{\xi}) . $$

When $\eta \neq 0$, this is strictly positive by Assumption 5.1.1, and thus $\ell(\eta) \neq 0$. $\square$

We apply the lemma to u^+ and to u^- and use the notation $V^\pm = V(u^\pm)$ and $\widetilde{L}^\pm = \widetilde{L}_{u^\pm}$.

Proposition 5.2.2. *There are $\varepsilon_0 > 0$ and $C > 0$ such that for all (u^+, u^-, σ) which satisfy Assumption 2.1 with $\varepsilon < \varepsilon_0$, there are invertible matrices $W^\pm$ such that $\|W^\pm\| \leq C$, $\|(W^\pm)^{-1}\| \leq C$ and*

$$ (W^\pm)\widetilde{L}^\pm W^\pm \;=\; J^\pm\partial_n \,+\, Q^\pm\,, \quad Q^\pm = \sum_{j=0}^{n-1} Q_j^\pm\partial_j \tag{5.2.12} $$

with

$$ \widetilde{J}^\pm = \begin{bmatrix} \mp\varepsilon & 0 \\ 0 & Id \end{bmatrix}, \quad Q^\pm(\tau,\eta) \;=\; \begin{bmatrix} q_1^\pm(\tau,\eta) & {}^t q'^\pm(\eta) \\ q'^\pm(\eta) & Q'^\pm(\tau,\eta) \end{bmatrix}, \tag{5.2.13} $$

the $Q_j^\pm$ are symmetric, Q_0 is definite positive and

i) $q_1^\pm$ are linear functions of (τ,η) and $|\partial_\tau q_1| \geq 1/C$.

ii) $\hat{q}^\pm$ are functions of η only and $|q'^\pm(\eta)| \geq |\eta|/C$.

Proof. Because $\lambda_1$is the smallest eigenvalue, Assumption 5.1.2 implies that for ε small enough,

$$\begin{aligned} &\lambda_1(u^\pm, \underline{\xi'}) - \sigma = \mp\varepsilon\, b^\pm\,, \quad \text{with } 1/4 < b < 3/4 \\ &G'(u^\pm) = A'_n(u^\pm) - \sigma \geq c\, Id\,. \end{aligned} \tag{5.2.14}$$

Introduce

$$W^\pm := \begin{bmatrix} (b^\pm)^{-1/2} & 0 \\ 0 & (G'^\pm)^{-1/2} \end{bmatrix} \tag{5.2.15}$$

The operator $(W^\pm)\widetilde{L}^\pm W^\pm$ has clearly the form indicated. Moreover, $Q_j = (W^\pm)\widetilde{A}^\pm_j W^\pm$ is symmetric since $\widetilde{A}_j$ is symmetric. In addition ,

$$\begin{aligned} &q_1^\pm(\tau,\eta) = (b^\pm)^{-1}(\tau + \nu(u^\pm, \eta)), \\ &q'^\pm(\eta) \;=\; (G'^\pm)^{-1/2}\ell(u^\pm,\eta)(b^\pm)^{-1/2} \end{aligned} \tag{5.2.16}$$

and the properties $i)$ and $ii)$ follow.

□

The new boundary conditions after the change of unknowns

$$v^\pm \;=\; V^\pm W^\pm w^\pm \tag{5.2.17}$$

are

$$J^+ w^+ -\; \varepsilon X\psi = M^+ M_- J^- w^- \;- M^+ g \tag{5.2.18}$$

with

$$\begin{aligned} &M^+ = (W^+)^{-1}(V^+)^{-1}(A_0^+)^{-1}\,, \quad M_- = A_0^- V^- W^-\,, \\ &\varepsilon X \;=\; M^+ \sum_{j=0}^{n-1} [f_j(u)]\partial_j\,. \end{aligned} \tag{5.2.19}$$

Proposition 5.2.3. *There are $\varepsilon_0 > 0$ and $C > 0$ such that for all (u^+, u^-, σ) satisfying Assumption* 2.1 *with $\varepsilon < \varepsilon_0$, there is $\beta \in \mathbb{R}$ with $1/C \leq |\beta| \leq C$ and such that*

$$M := M^+ M_- = Id + O(\varepsilon)\,, \tag{5.2.20}$$

$$X = \underline{X} + \varepsilon Y\,, \quad \text{with} \quad \underline{X}(\tau,\eta) = \beta \begin{bmatrix} q_1^+(\tau,\eta) \\ q'^+(\eta) \end{bmatrix}\,. \tag{5.2.21}$$

Proof. One has $V^+ - V^- = V(u^+) - V(u^-) = O(\varepsilon)$. Moreover, Assumption 5.1.2 implies that $b^\pm = 1/2 + O(\varepsilon)$ and therefore $W^+ - W^- = O(\varepsilon)$. Thus $M = Id + O(\varepsilon)$.

Assumption 1.2 implies that $[f_j(u)] = -\varepsilon A_j(u^+) r(u^+) + O(\varepsilon^2)$. Thus,

$$M^+[f_j(u)] = -\varepsilon (W^+)^{-1}(V^+)^{-1}(A_0^+)^{-1}A_j^+ r^+ + O(\varepsilon^2)$$

Recall that $r^+ = \alpha V^+ e$ where e is the first vector of the canonical basis in $\mathbb{R}^N$ and $\alpha \neq 0$. Thus, with notations of Lemma 2.1,

$$X(\tau, \eta) = \alpha (W^+)^{-1} P^+(\tau, \eta) e + \varepsilon Y(\tau, \eta)\,.$$

The explicit form (5.2.15) of W^+ and the relations (5.2.16) imply the second equation in (5.2.20) with $\beta = \alpha\sqrt{b^+}$.

$\square$

5.3 Several reductions

Consider problems of the form

$$\text{(5.3.1)} \qquad \begin{cases} J^\pm \partial_n w^\pm + Q^\pm w^\pm = f^\pm & \text{on} \quad \pm x_n > 0\,, \\ J^+ w^+ - \varepsilon X \psi = M J^- w^- + g & \text{on} \quad x_n = 0 \end{cases}$$

where Q and X satisfy the conclusions of Propositions 2.2 and 2.3. We denote by w_1 the first component of w and q' the collection of the $N-1$ other components. The next result implies Theorem 5.1.3 and improves the smoothness of the w' component of the traces.

Theorem 5.3.1. *There are $\varepsilon_0 > 0$, γ_0 and C such that for all $\varepsilon \in]0, \varepsilon_0]$, all $\gamma \geq \gamma_0$ and all $w^\pm \in H^1_\gamma(\Omega_\pm)$ and $\psi \in H^1_\gamma(\mathbb{R}^n)$, one has*

(5.3.2)

$$\sqrt{\gamma}\|w\|_{L^2_\gamma(\Omega)} + \|w'_{|x_n=0}\|_{L^2_\gamma(\mathbb{R}^n)} + \sqrt{\varepsilon}\|w_{1|x_n=0}\|_{L^2_\gamma(\mathbb{R}^n)} + \sqrt{\varepsilon}\|\psi\|_{L^2_\gamma(\mathbb{R}^n)} + \varepsilon\|\psi\|_{H^1_\gamma(\mathbb{R}^n)} \leq C\left(\frac{1}{\sqrt{\gamma}}\|f\|_{L^2_\gamma(\Omega)} + \|g'\|_{L^2_\gamma(\mathbb{R}^n)} + \frac{1}{\sqrt{\varepsilon}}\|g_1\|_{L^2_\gamma(\mathbb{R}^n)}\right).$$

5.3.1 Reduction to a one sided problem

First, we take advantage that λ_1 is the smallest eigenvalue. Because J^- is positive, the problem on $x_n < 0$ is well posed *without boundary conditions*. Multiplying the first equation by $e^{-2\gamma t} w^-$ and integrating by parts, implies the following estimate.

Proposition 5.3.2. *On* $\Omega_{=}\{x_n < 0\}$*, one has*

$$\sqrt{\gamma}\|w^-\|_{L^2_\gamma(\Omega)} + \|w'^-_{|x_n=0}\|_{L^2_\gamma(\mathbb{R}^n)} + \sqrt{\varepsilon}\|w^-_{1\ |x_n=0}\|_{L^2_\gamma(\mathbb{R}^n)} \leq C\frac{1}{\sqrt{\gamma}}\|f^-\|_{L^2_\gamma(\Omega)}. \tag{5.3.3}$$

Therefore, we are left with a boundary value problem for (w^+, ψ) and we consider $\widetilde{g} = MJ^-w^- + g$ as a boundary data. Because $M = Id + O(\varepsilon)$, note that

$$\begin{aligned} \|\widetilde{g}'\|_{L^2_\gamma(\mathbb{R}^n)} + \frac{1}{\sqrt{\varepsilon}}\|\widetilde{g}_1\|_{L^2_\gamma(\mathbb{R}^n)} &\leq \|g'\|_{L^2_\gamma(\mathbb{R}^n)} + \frac{1}{\sqrt{\varepsilon}}\|g_1\|_{L^2_\gamma(\mathbb{R}^n)} \\ &+ C\Big(\|w'^-_{|x_n=0}\|_{L^2_\gamma(\mathbb{R}^n)} + \sqrt{\varepsilon}\|w^-_{1\ |x_n=0}\|_{L^2_\gamma(\mathbb{R}^n)}\Big). \end{aligned} \tag{5.3.4}$$

Thus, it is sufficient to prove the estimate (5.3.2) for (w^+, ψ) with $f^+ = J^+\partial_n w^+ + Q^+ w^+$ and $\widetilde{g} = J^+ w^+_{|x_n=0} + \varepsilon X\psi$.

5.3.2 Estimates for ψ

Propositions 5.2.3 and 5.2.2 imply that X is an elliptic first order system for ε small enough. Therefore, the boundary equation implies that

$$\varepsilon\|\psi\|_{H^1_\gamma(\mathbb{R}^n)} \leq C\left(\|Jw_{|x_n=0}\|_{L^2_\gamma(\Omega)} + \|\widetilde{g}\|_{L^2_\gamma(\mathbb{R}^n)}\right). \tag{5.3.5}$$

Moreover, the first equation is of the form

$$\varepsilon X_1\psi \ = \ -\widetilde{g}_1 \ + \ \varepsilon w_1^+ \tag{5.3.6}$$

Propositions 2.2 and 2.3 imply that the coefficient of ∂_t in the vector field X_1 does not vanish. Therefore, multiplying (5.3.6) by $e^{-2\gamma t}\psi$ and integrating by parts yields

$$\sqrt{\varepsilon}\|\psi\|_{L^2_\gamma(\mathbb{R}^n)} \leq C\left(\sqrt{\varepsilon}\|w^+_{1\ |x_n=0}\|_{L^2_\gamma(\mathbb{R}^n)} + \frac{1}{\sqrt{\varepsilon}}\|g_1\|_{L^2_\gamma(\mathbb{R}^n)}\right). \tag{5.3.7}$$

Thus, the estimates for ψ follow from the estimates for the traces of w^+.

5.3.3 Interior estimates for w^+

On $\{x_n > 0\}$, the analogue of (5.3.3) is obtained by multiplication of the equation by $e^{-2\gamma t}w^-$ and integration by parts. This yields the following estimate.

$$
\begin{aligned}
\sqrt{\gamma}\|w^+\|_{L^2_\gamma(\Omega)} &+ \sqrt{\varepsilon}\,\|w_1^+{}_{|x_n=0}\|_{L^2_\gamma(\mathbb{R}^n)} \\
&\leq C\left(\frac{1}{\sqrt{\gamma}}\|f^+\|_{L^2_\gamma(\Omega)} + \|w'^+_{|x_n=0}\|_{L^2_\gamma(\mathbb{R}^n)}\right).
\end{aligned}
\tag{5.3.8}
$$

Therefore, to prove Theorem 5.3.1, Proposition 5.3.2 and the estimates (5.3.4) (5.3.5) (5.3.7) and (5.3.8) show that is sufficient to prove that

$$
\begin{aligned}
\|w'^+_{|x_n=0}\|_{L^2_\gamma(\mathbb{R}^n)} &+ \sqrt{\varepsilon}\|w_1^+{}_{|x_n=0}\|_{L^2_\gamma(\mathbb{R}^n)} \\
&\leq C\left(\frac{1}{\sqrt{\gamma}}\|f^+\|_{L^2_\gamma(\Omega)} + \|g'\|_{L^2_\gamma(\mathbb{R}^n)} + \frac{1}{\sqrt{\varepsilon}}\|g_1\|_{L^2_\gamma(\mathbb{R}^n)}\right).
\end{aligned}
\tag{5.3.9}
$$

One can reduce a little further the analysis, considering the unique solution $z \in H^1_\gamma$ of the dissipative boundary value problem

$$
\begin{cases}
J^+\partial_n z + Q^+ z = 0 & \text{on } x_n > 0\,, \\
z' = g' & \text{on } x_n = 0
\end{cases}
\tag{5.3.10}
$$

The estimate (5.3.8) implies that

$$
\sqrt{\varepsilon}\,\|z_{1|x_n=0}\|_{L^2_\gamma(\Omega)} + \|z'_{|x_n=0}\|_{L^2_\gamma(\mathbb{R}^n)} \leq C\|g'\|_{L^2_\gamma(\mathbb{R}^n)}.
\tag{5.3.11}
$$

Then, $w = w^+ - z$ satisfies

$$
\begin{cases}
J^+\partial_n w + Q^+ w = f^+ & \text{on } x_n > 0\,, \\
J^+ w - \varepsilon X\psi = \widetilde{g} & \text{on } x_n = 0
\end{cases}
\tag{5.3.12}
$$

with $\widetilde{g}' = 0$ and $\widetilde{g}_1 = g_1 - \varepsilon z_1$. Therefore, (5.3.11) and Theorem 5.3.1 follow from the next estimate.

Proposition 5.3.3. *There are $\varepsilon_0 > 0$, γ_0 and C such that for all $\varepsilon \in]0, \varepsilon_0]$, all $\gamma \geq \gamma_0$ and all $w \in H^1_\gamma(\Omega_\pm)$ and $\psi \in H^1_\gamma(\mathbb{R}^n)$ satisfying* (5.3.12)*, one has*

$$
\begin{aligned}
\|w'_{|x_n=0}\|_{L^2_\gamma(\mathbb{R}^n)} &+ \sqrt{\varepsilon}\|w_{1|x_n=0}\|_{L^2_\gamma(\mathbb{R}^n)} \\
&\leq C\left(\frac{1}{\sqrt{\gamma}}\|f^+\|_{L^2_\gamma(\Omega)} + + \frac{1}{\sqrt{\varepsilon}}\|\widetilde{g}\|_{L^2_\gamma(\mathbb{R}^n)}\right).
\end{aligned}
\tag{5.3.13}
$$

5.3.4 Fourier-Laplace analysis

So far the analysis applies as well to variable coefficient equations. Now we really use the constant coefficient assumption. We introduce the unknowns $e^{-\gamma t} w^+$ and we perform a tangential Fourier transformation. In the variable coefficient case, the substitute for this Fourier analysis is a suitable paradifferential calculus, as is Part II (see [Mét 1]). Let $\widehat{w}$, $\widehat{f}$ and $\widehat{g}$ denote the Fourier transform with respect to the variables (t, y) of $e^{-\gamma t} w$, $e^{-\gamma t} f^+$ and $e^{-\gamma t} \widetilde{g}$ respectively. The equations (5.3.12) are transformed into

$$\begin{cases} J\partial_n \widehat{w} + i\, Q^\gamma \widehat{w} = \widehat{f} & \text{on } x_n > 0\,, \\ J\widehat{w} - i\,\varepsilon X^\gamma \widehat{\psi} = \widehat{g} & \text{on } x_n = 0 \end{cases} \tag{5.3.14}$$

with

$$J := \begin{bmatrix} -\varepsilon & 0 \\ 0 & Id \end{bmatrix}, \quad \begin{array}{l} Q^\gamma(\tau, \eta) = Q(\tau - i\gamma, \eta)\,, \\ X^\gamma(\tau, \eta) = X(\tau - i\gamma, \eta). \end{array} \tag{5.3.15}$$

By Plancherel's Theorem, it is sufficient to prove the following estimate.

Proposition 5.3.4. *There are $\varepsilon_0 > 0$ and C such that for all $\varepsilon \in]0, \varepsilon_0]$, all $\gamma > 0$, all $(\tau, \eta) \in \mathbb{R}^n$ and all $\widehat{w} \in H^1_\gamma(\mathbb{R}_+)$ and $\widehat{\psi} \in \mathbb{C}$, one has*

$$|\widehat{w}'(0)| + \sqrt{\varepsilon}\, |\widehat{w}_1(0)| \le C \left(\frac{1}{\sqrt{\gamma}} \|\widehat{f}\|_0 + \frac{1}{\sqrt{\varepsilon}} |\widehat{g}| \right). \tag{5.3.16}$$

In this estimate, $\|\cdot\|_0$ denotes the L^2 norm on $[0, \infty[$.

5.4 Proof of the main estimate.

The proof of (5.3.16) is microlocal, i.e. the symmetrizer depends on (τ, η). The analysis depends on wether $|q_1(\tau - i\gamma, \eta)|$ is large or small with respect to $|q'(\eta)|$.

5.4.1 Case I

Proposition 5.4.1. *For all $\delta > 0$, there are ε_0, γ_0 and C such that the estimate* (5.3.16) *is satisfied for all (τ, η) such that*

$$|q_1(\tau - i\gamma, \eta)| \ge \delta |\eta|\,. \tag{5.4.1}$$

Proof. For simplicity, we denote by w, ψ etc the unknowns. When (5.4.1) is satisfied one can use the standard symmetrizer : multiply (5.3.12) by $\overline{w}$ and take the real part. Because of the symmetry of the matrices Q_j, this yields

$$-\frac{1}{2}\big(Jw(0), w(0)\big) + \gamma\big(\!\big(Q_0 w(x_n), w(x_n)\big)\!\big)_0 = \operatorname{Re}\big(\!\big(f(x_n), w(x_n)\big)\!\big)_0 \tag{5.4.2}$$

where $(\!(\cdot,\cdot)\!)_0$ denotes the scalar product in $L^2([0,\infty[;\mathbb{C}^N)$ and $(\cdot,\cdot)$ the scalar product in $\mathbb{C}^N$. The boundary term is

$$-\big(Jw(0), w(0)\big) = \varepsilon|w_1(0)|^2 - |w'(0)|^2 .$$

Because Q_0 is definite positive, this implies that there is $c_0 >$ such that

$$\varepsilon|w_1(0)|^2 + |w'(0)|^2 + c_0\gamma\|w\|_0^2 \leq 2|w'(0)|^2 + \frac{1}{c_0\gamma}\|f\|_0^2 \tag{5.4.3}$$

The boundary conditions read

$$\begin{cases} -\varepsilon w_1(0) = g_1 + \varepsilon X_1(\tau - i\gamma, \eta)\psi , \\ \quad w'(0) = g' + \varepsilon X'(\tau - i\gamma, \eta)\psi . \end{cases} \tag{5.4.4}$$

Propositions 5.2.2 and 5.2.3 imply that there is $c > 0$ such that

$$|\underline{X}(\hat\tau, \eta)| \geq c|(\hat\tau, \eta)| , \quad \hat\tau := \tau - i\gamma . \tag{5.4.5}$$

Moreover, since $\underline{X}'$ depends only on η, the assumption (5.4.1) implies that

$$|\underline{X}_1(\hat\tau, \eta)| \geq c'\delta|\underline{X}'(\hat\tau, \eta)|$$

Since $X = \underline{X} + \varepsilon Y$, this implies that there are $\varepsilon_0 > 0$ and C such that for all $\varepsilon \in]0, \varepsilon_0]$ and all $(\hat\tau, \eta)$ which satisfies (5.4.1), one has

$$|X'(\hat\tau, \eta)| \leq C|X_1(\hat\tau, \eta)| \tag{5.4.6}$$

Thus

$$|w'(0)| \leq |g'| + C|g_1 + \varepsilon w_1(0)| \leq C|g| + C\varepsilon|w_1(0)| .$$

Substituting in (5.4.3), this shows that

$$\varepsilon|w_1(0)|^2 + |w'(0)|^2 + c_0\gamma\|w\|_0^2 \leq C|g|^2 + C\varepsilon^2|w_1(0)|^2 + \frac{1}{c_0\gamma}\|f\|_0^2$$

which implies (5.3.16) if ε is small enough.

$\square$

5.4.2 Case II, microlocal decoupling

Proposition 5.4.2. *There are $\delta > 0$, $\varepsilon_0 > 0$ and C such such that the estimate* (5.3.16) *is satisfied for all (τ, η) such that*

$$|q_1(\tau - i\gamma, \eta)| \le \delta|\eta| \,. \tag{5.4.7}$$

We first analyze the algebraic structure of the equations near points where $\varepsilon = 0$ and $q_1 = 0$.

Lemma 5.4.3. *There are $\delta > 0$, $\varepsilon_0 > 0$ and C such that for all (u^+, u^-, σ) satisfying Assumption* 2.1 *with $\varepsilon \in]0, \varepsilon_0]$, all (τ, γ, η) satisfying* (5.4.7) *there is an invertible matrix H such that $|H| + |H^{-1}| \le C$ and*

$$HJ^tH = J\,, \quad HQ^\gamma(\tau,\eta)^tH = \begin{bmatrix} \rho & \alpha & 0 \\ \alpha & \mu & 0 \\ 0 & 0 & R'' \end{bmatrix} . \tag{5.4.8}$$

Moreover H is smooth and homogeneous of degree one in (τ, γ, η). H is real when $\gamma = 0$ and

$$|{}^teH - {}^te| \le C\varepsilon\,, \tag{5.4.9}$$

where $e = (1, 0, \dots, 0)$ is the first vector of the canonical basis of $\mathbb{C}^N$.

Proof. The idea is to put the matrix $J^{-1}Q$ in a block diagonal form. It has unbounded eigenvalues as ε tends to zero. The block R'' will correspond to the bounded eigenvalues.

Because q_1 has real coefficients and the coefficient of τ does not vanish, (5.4.7) implies that $|\tau| \le C|\eta|$ and $\gamma \le C\delta|\eta|$. Therefore, by homogeneity, we can assume in the proof that $|\eta| = 1$, $|\tau| \le C$ and $\gamma \le C\delta$.

a) Proposition 5.2.2 implies that one can choose a vector basis e_2 parallel to $(0, q'(\eta))$. Thus there is a real orthogonal matrix H_1 which depends smoothly on the parameters and keeps e invariant such that

$$H_1J^tH_1 = J\,, \quad Q_1 := H_1Q^tH_1 = \begin{bmatrix} q_1 & a_1 & 0 \\ a_1 & \mu_1 & {}^t\ell \\ 0 & \ell & R''_1 \end{bmatrix} \tag{5.4.10}$$

Moreover, $a_1(\eta) = |q'(\eta)| \ge c$.

For $z \in \mathbb{C}$, consider the matrix $zJ + Q_1$, which depends on the parameters $(\varepsilon, \tau, \gamma, \eta)$. For $\varepsilon = 0$, $q_1 = 0$ and $|\eta| = 1$, the equations $(zJ + Q_1)h = 0$ read

$$a_1h_2 = 0, \quad a_1h_1 + (z + \mu_1)h_2 + {}^t\ell h'' = 0, \quad h_2\ell + (z + R''_1)h'' = 0\,.$$

Thus $zJ + Q_1$ is invertible, unless $-z$ is an eigenvalue of R_1''. Introduce a circle Γ in the complex plane, which contains all the eigenvalues of $R_1''(\eta)$ for all unitary η. There are $\delta > 0$, $\varepsilon_0 > 0$ and C, such that for all $z \in \Gamma$, all $\varepsilon \in]0\varepsilon_0$, all (τ, γ, η) with $|\eta| = 1$ and satisfying (5.4.7), the matrix $zJ + Q_1$ is invertible and the norm of the inverse is less than C. For these (τ, γ, η), one can define

$$\Pi = \frac{1}{2i\pi}\int_\Gamma (zJ + Q_1)^{-1} J\, dz\,, \quad \Pi^\sharp = \frac{1}{2i\pi}\int_\Gamma J(zJ + Q_1)^{-1}\, dz \tag{5.4.11}$$

The integrals do not depend on the circle Γ, provided that that the integrand remains defined and holomorphic for z between the two circles. Using the identity

$$(zJ + Q_1)^{-1} J (z'J + Q_1)^{-1} = \frac{1}{z' - z}\big((zJ + Q_1)^{-1} - (z'J + Q_1)^{-1}\big)$$

and integrating over two nearby circles implies that Π and $\Pi^\sharp$ are projectors in $\mathbb{C}^N$. Furthermore

$$J\Pi = \Pi^\sharp J\,, \quad Q\Pi = \Pi^\sharp Q\,. \tag{5.4.12}$$

Because Q_1 is symmetric, $\Pi^\sharp$ is the transposed matrix ${}^t\Pi$. Moreover, when $\gamma = 0$, the matrix Q is real, and therefore $\Pi = \overline{\Pi}$ is also a real matrix.

b) Let E [resp $E^\sharp$] denote the range of Π [resp $\Pi^\sharp$]. Let E_0 [resp $E_0^\sharp$] denote the kernel of Π [resp $\Pi^\sharp$]. Thus, one has the splittings

$$\mathbb{C}^N = E_0 \oplus E\,, \quad \mathbb{C}^N = E_0^\sharp \oplus E^\sharp\,. \tag{5.4.13}$$

When $\varepsilon = q_1 = 0$, one can compute explicitly the projectors. In the basis where (5.4.10) holds, their matrix is

$$\Pi = \begin{bmatrix} 0 & 0 & -{}^t\ell/a_1 \\ 0 & 0 & 0 \\ 0 & 0 & Id \end{bmatrix}\,, \quad \Pi^\sharp = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\ell/a_1 & 0 & Id \end{bmatrix} \tag{5.4.14}$$

In this case, $\dim E_0 = \dim E_0^\sharp = 2$ and this remains true for $\varepsilon \le \varepsilon_0$ and q_1 satisfying (5.4.7).

The intertwining relations (5.4.12) imply that both J and Q map E_0 into $E_0^\sharp$ and E into $E^\sharp$. J also maps E_1 into itself

Introduce next the space E_1 generated by the first two vectors of the canonical basis. Then (5.4.14) shows that when $\varepsilon = q_1 = 0$,

$$\mathbb{C}^N = E_1 \oplus E\,, \quad \mathbb{C}^N = E_1 \oplus E^\sharp \tag{5.4.15}$$

with uniformly bounded projectors. This remains true for ε and q_1 small. J maps E_1 to E_1 and E to $E^\sharp$. J is invertible except when $\varepsilon = 0$ and in this case its kernel is contained in E_1. This shows that J maps E onto $E^\sharp$ and, that its inverse is uniformly bounded. Hence, there is a constant C such that

$$\forall h \in E\,, \quad |h| \le C|Jh|\,. \tag{5.4.16}$$

In addition, because $\Pi^\sharp = {}^t\Pi$, E_0 [resp. $E_0^\sharp$] is the orthogonal of $E^\sharp$ [resp. E] for the *bilinear* duality ${}^t h \cdot h$ in $\mathbb{C}^N$. Thus

$$\forall h \in E_0, \forall k \in E^\sharp\,, \quad {}^t k\cdot h = 0 \quad \text{and} \quad \forall h \in E, \forall k \in E_0^\sharp\,, \quad {}^t k\cdot h = 0 \tag{5.4.17}$$

Moreover, since the projectors are real when $\gamma = 0$, we remark that the bilinear form ${}^t h \cdot h$ is non degenerate both on $E_0 \times E_0^\sharp$ and on $E \times E^\sharp$, for ε small and δ small

c) Introduce the decompositions of e in the direct sums (5.4.13) : $e = r + s$ and $e = r^\sharp + s^\sharp$. Because $Je = -\varepsilon e$, one has $Jr = -\varepsilon r^\sharp$ and $Js = -\varepsilon s^\sharp$. Note that (5.4.16) implies that $|s| \le C\varepsilon$. With (5.4.17) this implies that ${}^t r^\sharp \cdot r = 1 - {}^t s^\sharp \cdot s = 1 + O(\varepsilon)$. Therefore, for ε small, one can renormalize the vectors r and $r^\sharp$ to find vectors r_1 and $r_1^\sharp$, depending smoothly on the parameters and such that

$$r_1 = e + O(\varepsilon)\,, \quad {}^t r_1^\sharp \cdot r_1 = 1 \quad Jr_1 = -\varepsilon\, {}^t r_1^\sharp\,. \tag{5.4.18}$$

Because the bilinear form ${}^t h \cdot h$ is non degenerate on $E_0 \times E_0^\sharp$, one can find $r_2 \in E_0$ and $r_2^\sharp \in E_0^\sharp$ such that

$${}^t r_2^\sharp \cdot r_2 = 1 \quad {}^t r_2^\sharp \cdot r_1 = 1 \quad {}^t r_1^\sharp \cdot r_2 = 1\,. \tag{5.4.19}$$

The vectors can be chosen real when $\gamma = 0$. Because J is symmetric, ${}^t(Jr_2)\cdot r_1 = {}^t r_2 \cdot (Jr_1) = -\varepsilon\, {}^t r_2 \cdot r_1^\sharp = 0$. Thus $Jr_2 \in E_0^\sharp$ must be proportional to $r_2^\sharp$ and $Jr_2 = \kappa r_2^\sharp$. For $\varepsilon = q_1 = 0$, $r_1 = e$ and one can choose $r_2 = r_2^\sharp = e_2$, the second vector of the canonical basis. Thus, for ε and δ small the coefficient κ remains larger than a fixed positive number. Changing r_2 and $r_2^\sharp$ into $r_2/\sqrt{\kappa}$ and $\sqrt{\kappa} r_2^\sharp$ respectively, this proves that one can choose r_2 and $r_2^\sharp$ satisfying (5.4.19) and

$$Jr_2 = r_2^\sharp \tag{5.4.20}$$

J is an isomorphism from E to $E^\sharp$ which is the identity when $\varepsilon = q_1 = 0$. Thus, there are orthonormal basis in E for the bilinear form ${}^t(Jh) \cdot k$. Let

$(r_3^\sharp, \ldots, r_N^\sharp)$ be such a basis, which can be chosen real when $\gamma = 0$. Then $(r_3^\sharp, \ldots, r_N^\sharp)$ with $r_j^\sharp = Jr_j$ is a dual basis in $E^\sharp$.

Let K [resp $K^\sharp$ denote the matrix whose j-th column is the vector of the components of r_j [resp. $r_j^\sharp$] in the canonical basis. The orthogonality implies that ${}^tK^\sharp K = Id$. The matrices of J and Q_1 in the new bases are $(K^\sharp)^{-1}JK$ and $(K^\sharp)^{-1}Q_1K$ respectively. The lemma follows with $H = {}^tKH_1$. Indeed, H is real when $\gamma = 0$ and ${}^tHe = K^tH_1e = Ke = r_1 = e + O(\varepsilon)$. The block structure of HQ^tH follows from (5.4.12) and the identity $HJ^tH = J$ follows from the choice of the bases.

$\square$

Suppose that (5.4.7) is satisfied. In particular, $|\tau| + \gamma \leq C|\eta|$.In the equation (5.3.12) we perform the change of unknowns : $\widehat{w} = {}^tHw$. The equations become

$$
\begin{cases} -\varepsilon\partial_n w_1 + i\,\rho w_1 + i\,a w_2 = f_1\,, \\ \quad \partial_n w_2 + i\,a w_1 + i\,\mu\, w_2 = f_2\,, \\ \quad \partial_n w'' + i\,R w'' = f''\,. \end{cases} \tag{5.4.21}
$$

and the new boundary conditions are

$$
\begin{cases} -\varepsilon w_1(0) = g_1 + i\,\varepsilon\, Z_1\psi\,, \\ \quad w_2(0) = g_2 + i\,\varepsilon\, Z_2\psi\,, \\ \quad w''(0) = g'' + i\,\varepsilon\, Z''\psi\,. \end{cases} \tag{5.4.22}
$$

with $Z(\tau, \eta, \gamma) = H(\tau, \eta, \gamma)X(\tau - i\gamma, \eta)$. Propositions 5.2.2 and 5.2.3 imply that $X(\tau - i\gamma, \eta) = \beta Q(\tau - i\gamma, \eta)e + O(\varepsilon|\eta|$. Lemma 5.4.3 implies that $e = {}^tHe + O(\varepsilon)$ and therefore $Z = \beta\, HQ^tH + O(\varepsilon|\eta|)$. Thus,

$$
Z_1 = \beta\rho + O(\varepsilon|\eta|)\,, \quad Z_2 = \beta a + O(\varepsilon|\eta|)\,, \quad Z'' = O(\varepsilon|\eta|)\,. \tag{5.4.23}
$$

Because $|a| \geq c|\eta|$, $|Z_2| \geq c|\eta|/2$ when (5.4.7) is satisfied and ε is small enough. Therefore, the boundary conditions are equivalent to

$$
\begin{cases} \varepsilon Z_2 w_1(0) + Z_1 w_2(0) = Z_1 g_2 - Z_2 g_1 \\ \varepsilon\psi = i(g_2 - w_2(0))/Z_2\,, \\ w''(0) = g'' + i\,\varepsilon\, Z''\psi\,. \end{cases} \tag{5.4.24}
$$

They imply that

$$
|w''(0)| \leq |g''| + C\varepsilon|\eta||\psi|\,, \quad \varepsilon|\eta||\psi| \leq C(|g_2| + |w_2(0)|)\,. \tag{5.4.25}
$$

Therefore, to prove Proposition 5.4.2 it is sufficient to prove the estimate (5.3.16) for the solution (w_1, w_2) of

$$\left\{\begin{array}{ll} -\varepsilon\partial_n w_1 + i\,\rho w_1 + i\,a w_2 = f_1\,, & \\ \partial_n w_2 + i\,a w_1 + i\,\mu\,w_2 = f_2\,, & \text{on } x_n > 0\,, \\ \varepsilon Z_2 w_1(0) + Z_1 w_2(0) = g\,. & \end{array}\right. \tag{5.4.25}$$

Proposition 5.4.4. *There are $\delta > 0$, $\varepsilon_0 > 0$ and C such such that for all (τ, η) satisfying* (5.4.7)*, the solutions of* (5.4.25) *satisfy*

$$|w_2(0)|^2 + \varepsilon\,|w_1(0)|^2 \le C\left(\gamma^{-1}\|f\|_0^2 + \varepsilon^{-1}|g|^2\right)\,. \tag{5.4.26}$$

Example. A model example which leads to a system of the form (5.4.25) is

$$\left\{\begin{array}{ll} -\varepsilon\partial_x w_1 + \partial_t w_1 + \partial_y w_2 = f_1\,, & \\ \partial_x w_2 + \partial_t w_2 + \partial_y w_1 = f_2\,, & \text{on } x > 0\,, \\ \varepsilon\partial_y w_1 + \partial_t w_2 = g\,, & \text{on } x > 0\,. \end{array}\right.$$

5.4.3 The 2×2 boundary value problem

In this section we prove Proposition 5.4.4. Introduce the notation

$$Q(\tau, \eta, \gamma) := \left[\begin{array}{cc} \rho & a \\ a & \mu \end{array}\right].$$

We will construct a symmetrizer based on the following matrix.

$$S(\tau, \eta, \gamma) := \left[\begin{array}{cc} 1 & -2b \\ 2\varepsilon^{-1}\overline{b} & 1 - 2\varepsilon^{-1}|b|^2 \end{array}\right], \quad b := (\rho + \varepsilon\overline{\mu})/a\,. \tag{5.4.27}$$

Lemma 5.4.5. *i) SJ is self adjoint and*

$$(SJw, w) = \varepsilon|w_1|^2 + |w_2|^2 - 2\varepsilon^{-1}|bw_2 + \varepsilon w_1|^2\,. \tag{5.4.28}$$

ii) If δ and ε_0 are small enough there is $c > 0$ such that when (5.4.7) *is satisfied, one has*

$$\mathrm{Im}\,(SQw, w) \ge c\gamma\big(|w_1|^2 + \varepsilon^{-1}|w_2|^2\big)\,. \tag{5.4.29}$$

Proof. i)

$$SJ = \left[\begin{array}{cc} -\varepsilon & -2b \\ -2\overline{b} & 1 - 2\varepsilon^{-1}|b|^2 \end{array}\right]$$

is self adjoint and (5.4.28) follows by direct computation.

ii)

$$SQ = \begin{bmatrix} -\rho & a - 2b\mu \\ a - 2\overline{b}\overline{\mu} & \varepsilon^{-1}\big(2\overline{b}a - (2|b|^2 - \varepsilon)\mu\big) \end{bmatrix}$$

The choice of b has been made so that the left lower term has no singular term in $\varepsilon^{-}1$ and is the conjugate of the right upper term when $\gamma = 0$. More precisely, b is so chosen that $\overline{b}\rho - |b|^2 a = -\varepsilon\overline{b}\overline{\mu}$. Thus

$$\operatorname{Im} SQ = \begin{bmatrix} -\operatorname{Im}\rho & \operatorname{Im} a \\ \operatorname{Im} a & \varepsilon^{-1}\operatorname{Im} m \end{bmatrix}, \quad m := 2\overline{b}a - (2|b|^2 - \varepsilon)\mu\,.$$

Lemma 5.4.3 implies that

$$\rho = q_1 + O(\varepsilon) \quad \text{and} \quad \operatorname{Im}\rho = 0 \text{ when } \gamma = 0\,. \tag{5.4.30}$$

Thus $\operatorname{Im}\rho - \operatorname{Im} q_1$ vanishes both when $\varepsilon = 0$ and when $\gamma = 0$. Because the functions are smooth, this implies that $\operatorname{Im}\rho - \operatorname{Im} q_1 = O(\varepsilon\gamma)$ when $|\eta| = 1$. By homogeneity, this is true for all η, provided that (5.4.7) holds. Moreover, q_1 is a linear function of $(\tau - i\gamma, \eta)$ thus $\operatorname{Im} q_1 = -\gamma\partial q_1/\partial\tau$. Therefore, Proposition 5.2.2 implies that there is $c > 0$ such that

$$-\operatorname{Im}\rho \ \geq \ c\,\gamma\,. \tag{5.4.31}$$

Because a is real when $\gamma = 0$, one has

$$\operatorname{Im} a = O(\gamma)\,. \tag{5.4.32}$$

Next,

$$m = 2\overline{\rho}a/\overline{a} \ + \ \Big(\varepsilon + 2\varepsilon a/\overline{a} - 2|b|^2 - \varepsilon\Big)\mu$$

Note that μ and a are real when $\gamma = 0$ and, thanks to (5.4.30), $\rho = O\big((\delta + \varepsilon)|\eta|\big)$ when (5.4.7) holds. Thus $b = O\big(\delta + \varepsilon\big)$ and

$$\begin{aligned} \operatorname{Im} m &= -2\operatorname{Im}\rho\operatorname{Re}\frac{a}{\overline{a}} \ + 2\operatorname{Re}\rho\operatorname{Im}\frac{a}{\overline{a}} \ + \ O\big((\delta^2 + \varepsilon)\gamma\big) \\ &= -2\operatorname{Im}\rho\operatorname{Re}\frac{a}{\overline{a}} \ + \ O\big((\delta + \varepsilon)\gamma\big) \end{aligned}$$

The real part of $a\overline{a}$ is larger than a positive constant and (5.4.31) implies that if δ and ε_0 are small enough, there is $c > 0$ such that

$$\operatorname{Im} m \geq c\gamma\,. \tag{5.4.33}$$

With (5.4.31) and (5.4.32), this implies (5.4.29). $\square$

Corollary 5.4.6. *With δ and ε_0 as in Lemma* 4.5*, there is $c > 0$ such that for $\alpha > 0$ small enough, one has*

$$\text{(5.4.34)} \quad \operatorname{Im}\Big(S(Q - i\alpha\gamma\varepsilon^{-1}J)w, w\Big) \ \geq\ c\gamma(|w_1|^2 + \varepsilon^{-1}|w_2|^2 + \alpha\varepsilon^{-2}|bw_2|^2)$$

Proof of Proposition 5.4.4

Fix the parameters δ, ε_0 and α such that the estimates of Lemma 4.5 and Corollary 5.4.6 are satisfied. We use the symmetrizer $e^{-2x_n\alpha\gamma/\varepsilon}S$. Equivalently, introduce $\widetilde{w} = e^{-x_n\alpha\gamma/\varepsilon}w$ and $\widetilde{f} = e^{-x_n\alpha\gamma/\varepsilon}f$. The transformed equations are

$$\text{(5.4.35)} \qquad \begin{cases} J\partial_n\widetilde{w} \,+\, i\,(Q - i\varepsilon^{-1}\alpha\,\gamma J)\widetilde{w} \,=\, \widetilde{f}\,, \\ \varepsilon\widetilde{w}_1(0) \,+\, \widetilde{b}\widetilde{w}_2(0) \,=\, 0\,, \end{cases}$$

where $\widetilde{b} := Z_1/Z_2$. Multiply the equation by $-S$ and take the real part of the scalar product with $\widetilde{w}$. With (5.4.28) and (5.4.34) one obtains

$$\text{(5.4.36)} \quad \begin{aligned} \varepsilon|\widetilde{w}_1(0)|^2 + |\widetilde{w}_2(0)|^2 \,+\, c\,\gamma(\|\widetilde{w}_1\|_0^2 + \varepsilon^{-1}\|\widetilde{w}_2\|_0^2 + \alpha\varepsilon^{-2}\|b\widetilde{w}_2\|_0^2) \\ \leq\ \varepsilon^{-1}|bw_2(0) + \varepsilon w_1(0)|^2 \,-\, 2\mathrm{Re}\,\big(\!\big(S\widetilde{f}, \widetilde{w}\big)\!\big)_0\,. \end{aligned}$$

Because b is bounded, the definition (5.4.27) of S implies that

$$\text{(5.4.37)} \qquad \Big|\big(\!\big(S\widetilde{f}, \widetilde{w}\big)\!\big)_0\Big| \ \leq\ C\,\|\widetilde{f}\|_0\Big(\|\widetilde{w}\|_0 + \varepsilon^{-1}\|b\widetilde{w}_2\|_0\Big)\,.$$

In addition, (5.4.23) and the ellipticity of a imply that

$$\widetilde{b} = b + O(\varepsilon)\,.$$

Thus the boundary condition implies that $bw_2(0) + \varepsilon w_1(0) = O(\varepsilon)w_2$ and

$$\text{(5.4.38)} \qquad \varepsilon^{-1}|bw_2(0) + \varepsilon w_1(0)|^2 \ \leq\ \varepsilon|w_2(0)|^2\,.$$

Therefore, for ε small enough, (5.4.36) implies that there are constants c' and C' such that

$$\text{(5.4.39)} \quad \varepsilon|\widetilde{w}_1(0)|^2 + |\widetilde{w}_2(0)|^2 \,+\, c'\,\gamma(\|\widetilde{w}\|_0^2 + \varepsilon^{-2}\|b\widetilde{w}_2\|_0^2) \leq\ C'\,\gamma^{-1}\,\|\widetilde{f}\|_0^2\,.$$

Remark that $\|\widetilde{f}\|_0^2 \leq \|f\|_0^2$ and that $\widetilde{w}(0) = w(0)$. Thus (5.4.39) implies (5.4.26) and Proposition 4.4 is proved.

References

[Al] S.Alinhac, *Existence d'ondes de raréfaction pour des systèmes quasi-linéaires hyperboliques multi-dimensionnels*, Comm. in Partial Diff. Equ., 14 (1989) pp 173-230.

[Bo] J.M.Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann.Sc.E.N.S. Paris, 14 (1981) pp 209-246.

[Ch-Pi] J.Chazarain - A.Piriou, *Introduction à la théorie des équations aux dérivées partielles linéaires*, Gauthier-Villars, Paris 1981 & *Introduction to the Theory of Linear Partial Differential equations*, North Holland, Amsterdam, 1982

[Co-Me] R.Coifman - Y.Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque 57, (1978).

[Cor] A.Corli, *W*eak shock waves for second-order multi-dimensional systems, Boll. Unione Mat. Ital., VII. Ser., B 7, No.3, (1993) pp 493-510.

[Fr-Me] J.Francheteau-G.Métivier *Existence de chocs faibles pour des systèmes quasi-linéaires hyperboliques multidimensionnels.* Astérisque 268 (2000) and C.R.AC.Sc. Paris, 327 Série I (1998) pp 725-728.

[Kr] H.O.Kreiss, *Initial boundary value problems for hyperbolic systems*, Comm. on Pure and Appl.Math., 23 (1970) pp 277-298.

[Maj 1] A. Majda, *The stability of multidimensional shocks*, Memoirs of the Amer.Math.Soc. 275 (1983).

[Maj 2] A. Majda, *The Existence of multidimensional shocks*, Memoirs of the Amer.Math.Soc. 281 (1983).

[Maj 3] A. Majda, *Compressible fluid flow and systems of conservation laws*, Appl.Math.Sc. 53, Springer Verlag (1984).

[Maj-Tho] A. Majda - E.Thomann, *Multi-dimensional shock fronts for second order wave equation*, Comm. Partial Diff.Equ., 12 (1987) pp 777-828.

[Ma-Ra] F.Massey - J.Rauch, *Differentiability of solutions to hyperbolic boundary value problems*, Trans. Amer.Math.Soc., 189 (1974) pp 303-318.

[Mét 1] G.Métivier , *Stability of multidimensional weak shocks*, Comm. Partial Diff.Equ., 15 (1990) pp 983-1028.

[Mét 2] G.Métivier, *Interaction de deux chocs pour un système de deux lois de conservation, en dimension deux d'espace.*, Trans. Amer.Math.Soc., 296 (1986) pp 431-479.

[Mét 3] G.MÉTIVIER, *Ondes soniques*, J.Math.Pures et appl., 70 (1991) pp 197-268.

[Mét 4] G.MÉTIVIER, *The block structure condition for symmetric hyperbolic systems.* Bull. London Math. Soc,, 32 (2000) pp 689-702.

[Mey] Y.MEYER, *Remarques sur un théorème de J.M.Bony*, Supplemento al Rendiconti der Circolo Matematico di Palermo, Serie II, No1, 1981.

[Mo] A.MOKRANE, *Problèmes mixtes hyperboliques non linéaires*, Thèse, Université de Rennes 1, 1987.

[Ral] J.RALSTON, *Note on a paper of Kreiss*, Comm.Pure and Appl.Math., 24 (1971) pp 759-762.

[Rau] J.RAUCH, L^2 *is a continuable initial condition for Kreiss' mixed problems*, Comm.Pure and Appl.Math., 25 (1972) pp 265-285.

[Ste] E.STEIN, *Singular Integrals and Differentiability Propseties of Functions*, Princeton Univ. Press, Princeton, 1970.