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Stability of Multihelical Tearing Modes in Shaped Tokamaks

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Abstract

The stability of multihelical tearing modes in tokamaks with shaped cross-section is determined numerically. The method allows inclusion of a large number of singular surfaces resolved with high accuracy. Poloidal and radial couplings are discussed and the convergence is well understood. High poloidal m number modes are found to be unstable for typical equilibria. Completely stable current distributions have been constructed for D-shaped plasmas.

The concept of Δ' (jump in the logarithmic derivative of the perturbed magnetic field at a rational magnetic surface) has dominated resistive stability theory since the pioneer work of Furth, Killeen and Rosenbluth /1/, from Glasser, Greene, Johnson /2/ until recent numerical approaches by Manickam, Grimm and Dewar /3/ and Chu et al. /4/.

Another line of approach started by Furth /5/, continued by Barston /6/ and extended to 2-d geometry by Tasso /7/ and more recently by Tasso and Virtamo /8/ to 3-d perturbations, used the energy method to determine the stability threshold. In particular, the derivation of an energy principle /8/ for 3-d perturbations in tokamaks with shaped cross-section in the tokamak ordering, where integrations over singularities are done in the sense of Cauchy principal values, has permitted the developement of a numerical code /9, 10/ without using local Δ's.

The merits of this method are demonstrated in the stability analysis of D-shaped tokamaks with varying current profiles. Another application to a small-aspect-ratio equilibrium, though it could be in conflict with the tokamak scaling of Ref. /8/, is done in a less rigorous fashion in order to give an indication of the importance of toroidal coupling effects due to the shift of the magnetic surfaces.

The starting point is the quadratic functional derived in Ref. /8/:

$$(1)\qquad \delta W = \int d\tau \left\{ \frac{1}{\mu_o} |\nabla_\perp A|^2 + A^* \frac{dJ(\psi)}{d\psi} (A-\tilde{A}) \right\}$$

A being the z component of the vector potential, which vanishes at the boundary, and ψ, Θ, z a flux coordinate system with straight magnetic field lines. If A is expanded in a Fourier series:

$$(2)\qquad A(\psi,\Theta,z) = \sum_m a_m(\psi)\, e^{i(m\Theta + n2\pi z/L)},$$

then $\tilde{A}$, which is the weighted surface average of A, is given by

$$(3)\qquad A-\tilde{A} = \sum_m \frac{m}{nq(\psi)+m}\, a_m(\psi)\, e^{i(m\Theta + n2\pi z/L)}$$

where n is the toroidal mode number and q denotes the safety factor. This simple representation for $A-\tilde{A}$ prompts the Fourier expansion in equ. (2). The regularity condition for the $a_m(\psi)$ at the origin reads

$$(4)\qquad a_m(\psi=0) = 0 \text{ for } m \neq 0,$$

$$\frac{da_o}{d\sqrt{\psi}}\Bigg)_{\psi=0} = 0.$$

The stability problem reduces to minimizing δW for a given equilibrium. The Galerkin method used in connection with a finite-element representation for the radial dependence of A, i.e. the $a_m(\psi)$ in eq. (2), leads to a Hermitian matrix eigenvalue problem similar to that of ideal MHD computations /11/. The sign of the

lowest eigenvalue λ determines the stability. The band structure of the matrices allows the use of very efficient matrix solvers. The numerical method is explained in more detail in Refs. /9, 10/.

The equilibria are computed numerically with the Garching equilibrium code /12/ for a given current profile $j(\psi)$, a constant longitudinal field B_z and a prescribed boundary, as solutions of the equation for the poloidal flux Ψ

$$(5) \quad \frac{1}{\mu_o} \nabla^2 \psi = j(\psi)$$

Both straight and toroidal equilibria can be produced and fed into the stability analysis. The mapping into flux coordinates is done with the ERATO /11/ algorithm.

The code has been tested for circular cross-sections (single and double tearing modes) and for elliptical cross-sections (external kinks) with moderate ellipticity as described in Refs. /9, 10/. This testing was continued for constant-current equilibria with axis ratio e = b/a of up to 7. The comparison between the numerical and the analytical /13/ results is displayed in Fig. 1. A given m mode is unstable if $nq_L \leq nq \leq m$. Since the conducting wall lies between two confocal ellipses used in Ref. /13/ (for details see Fig. 3 of Ref. /10/), the numerical stability limit is expected to be located between the two analytically calculated marginal points. For this elliptical cross-section

the even and odd modes decouple. Up to 30 even or odd Fourier harmonics are needed in the case of e = 7 to obtain the point of marginal stability from convergence studies. For this extreme ellipticity the broadening of the m = 5 instability region extends from $nq = nq_L = 1.2$ up to $nq = 5$ as demonstrated in Fig. 1. This extreme precision of our results instills strong confidence in the method and leads us to believe that there are no numerical errors left. The difficulties of convergence discussed in Ref. /10/ were only due to inconvenient implementation of the boundary condition for a_o at the origin (see eq. (4)). Now that this has been corrected, much fewer radial finite elements are needed - typically up to 300.

The main purpose of this letter is to answer the question of "tearing" stability for general current distributions and general plasma cross-sections. This is by no means a trivial question because destabilizing and stabilizing effects due to the shaping and the external shear are expected to compete. Let us therefore discuss these effects in a straight strongly D-shaped plasma with a bell-shaped current distribution. The contours $s = \sqrt{\psi/\psi_s} =$ const and θ = const are displayed in Fig. 2, and the current profiles together with the shear are shown in Fig. 3. The plasma boundary is taken as fixed for the perturbations, although this could easily be relaxed, as will be done in a future paper. We start with a current

corresponding to that of a circular cylinder with $j(r) = (1-r^2/a^2)^2$, where the ratio of the safety factor on the surface and on the axis is $q_s/q_o = 3$ (see Ref. /9/). The shaping increases the shear near the plasma boundary to a value of $q_s/q_o = 6.3$. With $q_o = 1.15$ this configuration is stable in the large-aspect-ratio toroidal tokamak with respect to Mercier modes as well as to external kinks For tearing modes the marginal points and the mode structure are qualitatively similar to the circular-cross-section case. For the marginal points additional poloidal couplings occur. We find a strong $\tilde{m} = 1$, where $\tilde{m}$ denotes the dominant poloidal component, if the $q = 1$ surface is located inside the plasma. This mode, however, is easily stabilized by increasing the safety factor above one. For $q_o = 1.15$ we are left with unstable $n = 1$, $\tilde{m} = 2$ and $n = 2$, $\tilde{m} = 3$ modes induced by the $q = 2$ and 1.5 surfaces. The higher n modes are stable.

If one tries to flatten the current around $q = 1.5$ and 2.0, as indicated by the broken line in Fig. 3, one excites higher m harmonics. The $n = 1$ mode is now stabilized. The $n = 2$ mode is still unstable with a dominant poloidal component $\tilde{m} = 5$ (instead of 3), resonant at $q = 2.5$, followed by $m = 3$ and 7. The five dominant harmonics of this mode together with their derivatives, are plotted in Fig. 4. The derivatives show strong coupling between a resonant harmonic and several non-resonant ones as $m = 3, 5, 7, 2$ at

nq = 5. This coupling is governed by the geometry as well as by the local gradients of the current and the safety factor. In the computation more than 20 harmonics are used. The n = 2 mode is unstable only if - at least - the m = 3, 4, 5, 6, and 7 components are present. The n = 3 mode is now unstable and is rich in higher harmonics, as can be seen from Fig. 5. Especially the derivatives of harmonics like m = 3, 5, 7 and 9 have large values. The n = 3 mode is unstable only if - at least - the m = 3, 5 and 7 components couple. The convergence behaviour of these n = 2 and 3 modes is very similar to that of the stable case discussed in Fig. 6. To summarize, we have found instabilities with m = 5 and 7 components due to poloidal coupling, which as pure modes are stable in the circular case.

An interesting question is whether there are current distributions with complete stability. Such profiles are known in the circular case from Refs. /14/ and /9/. The current distribution is therefore once more modified to make it less steep between q = 1.5 and 2.0 and steeper for q > 2.0 , as presented by the solid line in Fig. 3. The ratio is now q_s/q_o = 3.7. It turns out that this current distribution is completely stable with eigenfunctions similar to those displayed in Figs. 4 and 5. In order to prove stability for higher toroidal mode numbers such as n = 3, 4, and 5, no fewer than 9 to 20 resonant singular

q-surfaces have to be taken into account. This, in our opinion, is rather remarkable progress in the field.

The accuracy of numerical results also depends on the convergence properties. This is the question of how the lowest eigenvalue λ depends on the number of Fourier components NF and of finite elements NS. Figure 6 displays the convergence behaviour of n = 1, 2 and 3 modes. If the Fourier series is going to be a good representation the influence of higher harmonics is expected to decrease exponentially, i.e. $\lambda \propto \exp(-ct.NF)$. A practical way of judging this is to plot λ versus $1/N_F^2$ and check whether λ levels off for some reasonable NF. This is demonstrated to be the case for a typical example in Fig. 6a. A plot of λ versus any other power of 1/NF displays the same behaviour, but it is more instructive on the quadratic scale. The increase in Fourier harmonics brings new singularities into play. For a given toroidal mode number n, however, their number is finite, but still large enough to necessicate at least an NF of about 20 for n = 5. The steeper slope for increasing n in Fig. 6a enlightens this.

The dependence of λ upon the number of finite elements NS would be like $1/N_S^2$ if expression (1) were regular. This behaviour is found for all external kinks (the singular surfaces being outside the plasma) with smooth $dj/d\psi$ profiles. A rough argument based on the 1/x singularity in eq. (3) leads to a 1/NS dependence. For higher n-values again the coefficient in front of 1/NS is stronger - as seen from

Fig. 6 b - because more singularities are present and the radial structure of the eigenfunction is more complicated.

Up to this point the computations reported are for straight plasmas, which is consistent with the tokamak scaling assumed in Ref. /8/. In order to see the influence of the aspect ratio, we also used toroidal equilibria and identified $j_\phi\ (r,\psi)$ with $j_z(\psi)$ by taking the average on a flux surface, while the Shafranov shift was left untouched. The results of the δW minimization show that these toroidal effects are not significant as long as the aspect ratio is larger than 10. Marginal modes are affected if the aspect ratio goes down to 5. Altogether, the toroidal equilibria seem to stabilize slightly - preferably $\tilde{m} = 3$ modes - and the Shafranov shift increases the poloidal mode coupling.

In conclusion, it can be stated that the linear stability problem of multihelical tearing modes in shaped tokamaks with large aspect ratio has been quantitatively solved. Poloidal mode couplings involving practically any number of harmonics can be taken into account. Shaped cross-sections and current dips /9/ are no longer a problem. What is additionally pleasent is the fast execution (about 30 seconds on a CRAY for typical cases discussed) and the simple handling of the code, allowing detailed parameter studies. The open, apparently long-term problem is the finite-pressure, finite-aspect-ratio tokamak, for which we do not see an analogous solution at the moment.

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References

/1/ Furth H.P., Killeen J., Rosenbluth M.N.,
Phys. of Fluids 6, 459 (1963)

/2/ Glasser A.H., Greene J.M., Johnson J.,
Phys. of Fluids 18, 875 (1975)

/3/ Manickam J., Grimm R.C., Dewar R.L.,
Comp.Phys.Com. 24, 355 (1981)

/4/ Chu M.S., Helton F.J., Lee J.K., Greene J.M., Jensen T.H.,
Comp.Phys.Com. 24, 399 (1981)

/5/ Furth H.P., "Propagations and Instabilities",
Stanford University Press, 87 (1963)

/6/ Barston E.M., Phys. of Fluids 12, 2162 (1969)

/7/ Tasso H., Plasma Physics 17, 1131 (1975)

/8/ Tasso H., Virtamo J.T., Plasma Physics 22, 1003 (1980)

/9/ Kerner W., Tasso H., Plasma Physics 24, 97 (1982)

/10/ Kerner W., Tasso H., Comp.Phys.Comm.24, 407 (1981)

/11/ Gruber R., Troyon F., Berger D., Bernard L., Rousset S., Schreiber R., Kerner W., Schneider W., Roberts K.V., Comp.Phys.Com. 21, 323 (1981)

/12/ Lackner K., Comp.Phys.Com. 12, 33 (1976)

/13/ Laval G., Pellat R., Soule J.S., Phys. of Fluids 17, 835 (1974)

/14/ Glasser A.H., Furth H.P., Rutherford P.H., Phys.Rev.Lett. 38, 234 (1977)

Figure Captions

Fig. 1 The unstable nq domain, $nq_L \leq nq \leq m$, versus the elongation of the elliptical cross-section. The lower limit from analytical theory with the wall located on the outer, respectively inner confocal ellipse is given by the solid (——), resp. broken (----) curve and that from the code by the crosses. The crosses on axis for e = 3, 5, and 7 show that there are instabilities for all nq values below m.

Fig. 2 Magnetic surfaces showing lines of constant $s = \sqrt{\psi/\psi_s}$ and the angle Θ belonging to the equilibria with the currents in Fig. 3.

Fig. 3 Current and safety factor profiles of cylinder-like (-.-.-), destabilized (----) and stabilized (——) profiles.

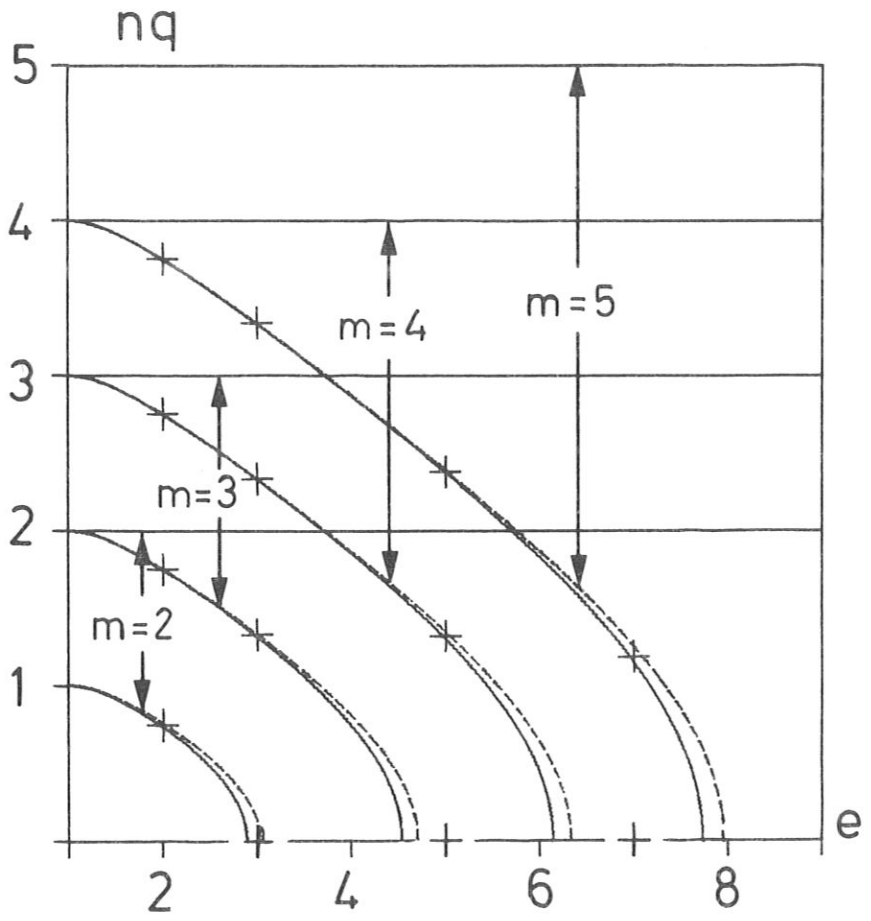
Fig. 4 Radial dependence of the 5 dominant poloidal harmonics and their derivatives of an unstable n = 2 eigenfunction. The broken lines indicate the singular surfaces.

Fig. 5 Radial dependence of the 5 dominant poloidal harmonics and their derivatives of an unstable n = 3 eigenfunction. The broken lines indicate the singular surfaces.

Fig. 6 Convergence dependence of the lowest eigenvalue λ

a) versus the number of poloidal harmonics NF in a $1/NF^2$ scale

b) versus the number of radial finite elements NS in a 1/NS scale.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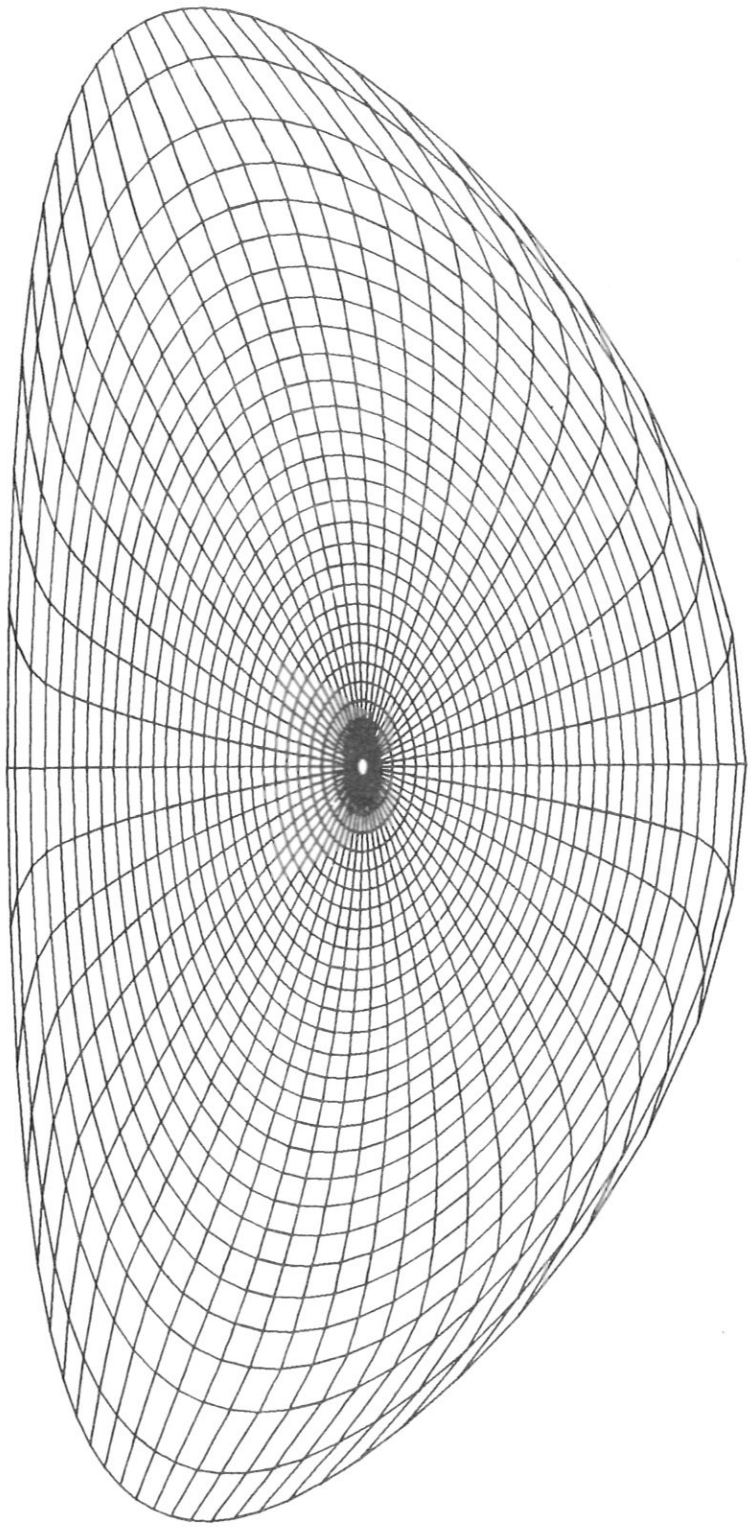
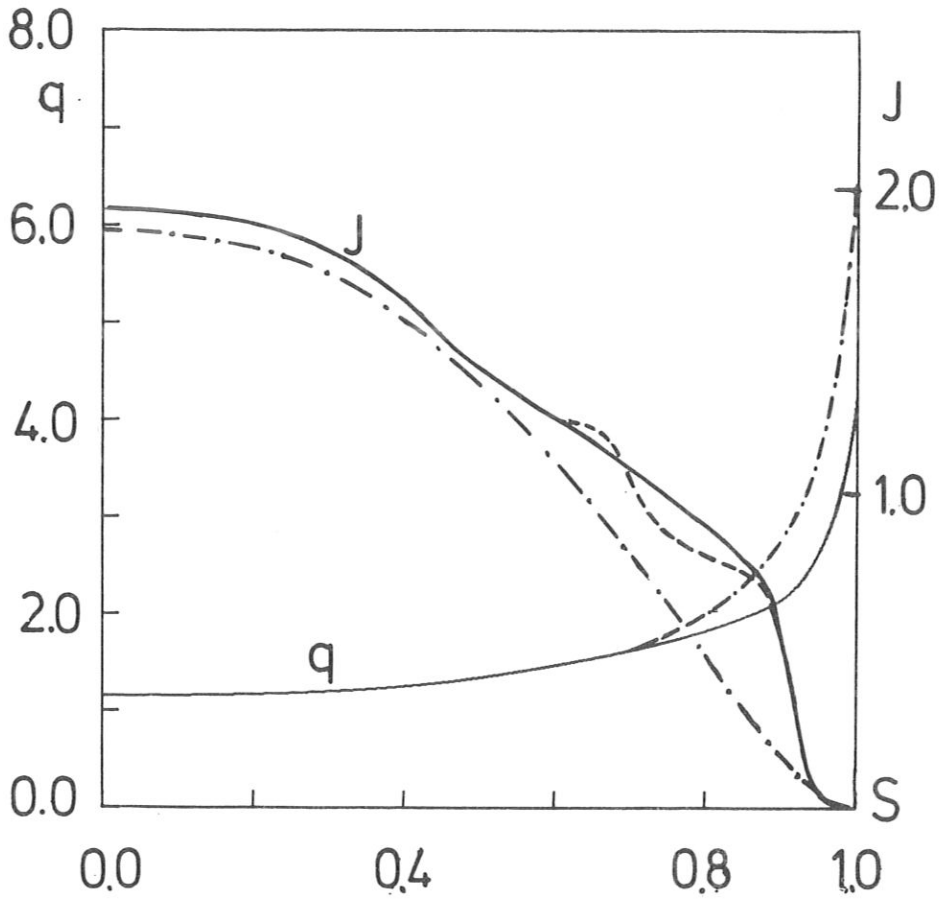
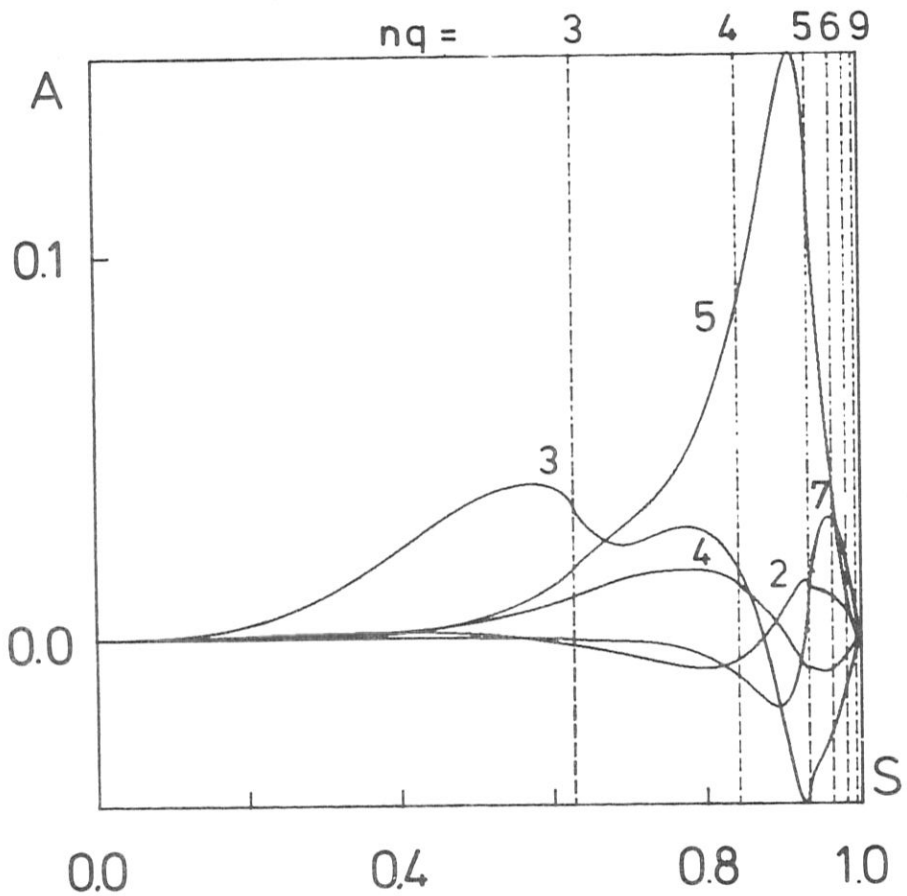
4
3
2
1
m=5
m=4
m=3
m=2
e
2
4
6
8




8.0
q
6.0
4.0
2.0
0.0
J
2.0
1.0
S
J
q
0.0
0.4
0.8
1.0


nq =
3
4
5 6 9
A
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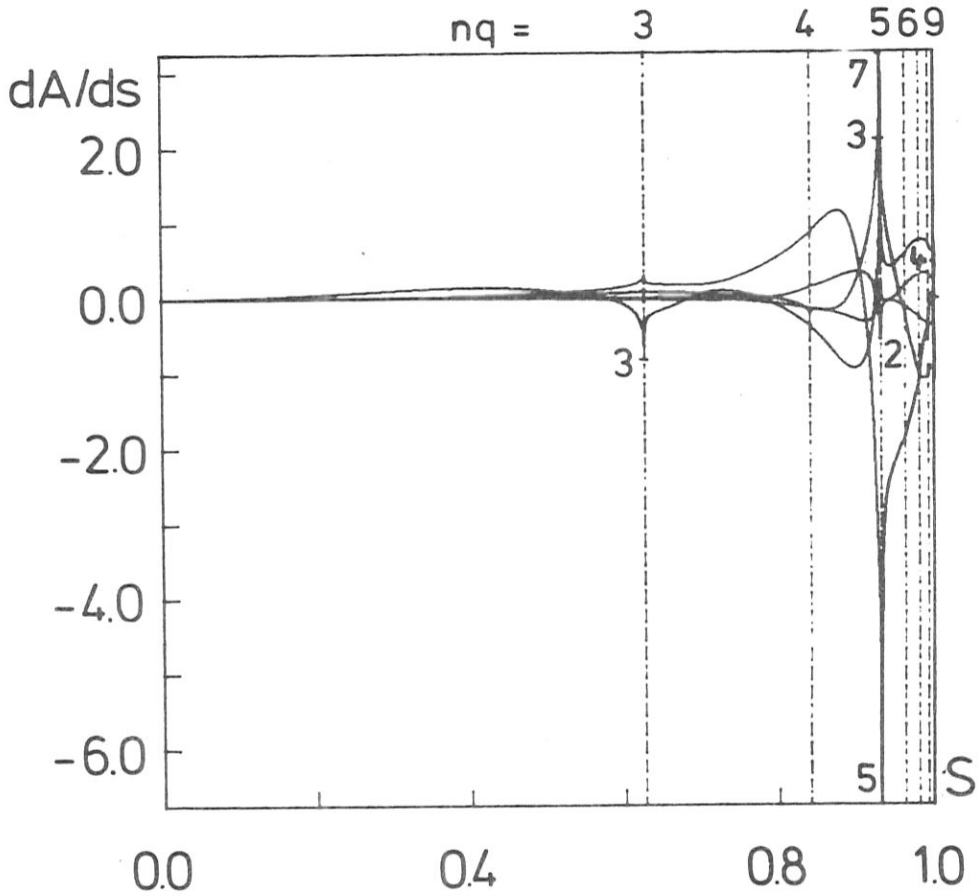
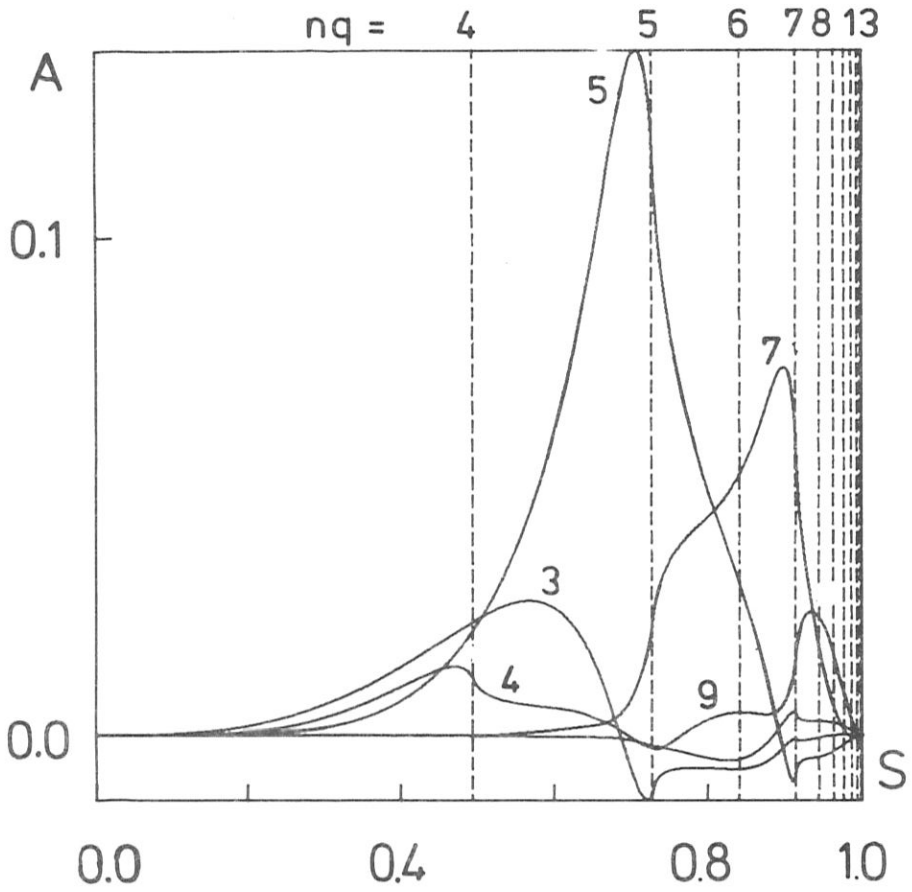
0.0
5
3
7
4
2
S
0.0
0.4
0.8
1.0


nq =
3
4
5 6 9
dA/ds
2.0
0.0
-2.0
-4.0
-6.0
7
3
3
2
5
S
0.0
0.4
0.8
1.0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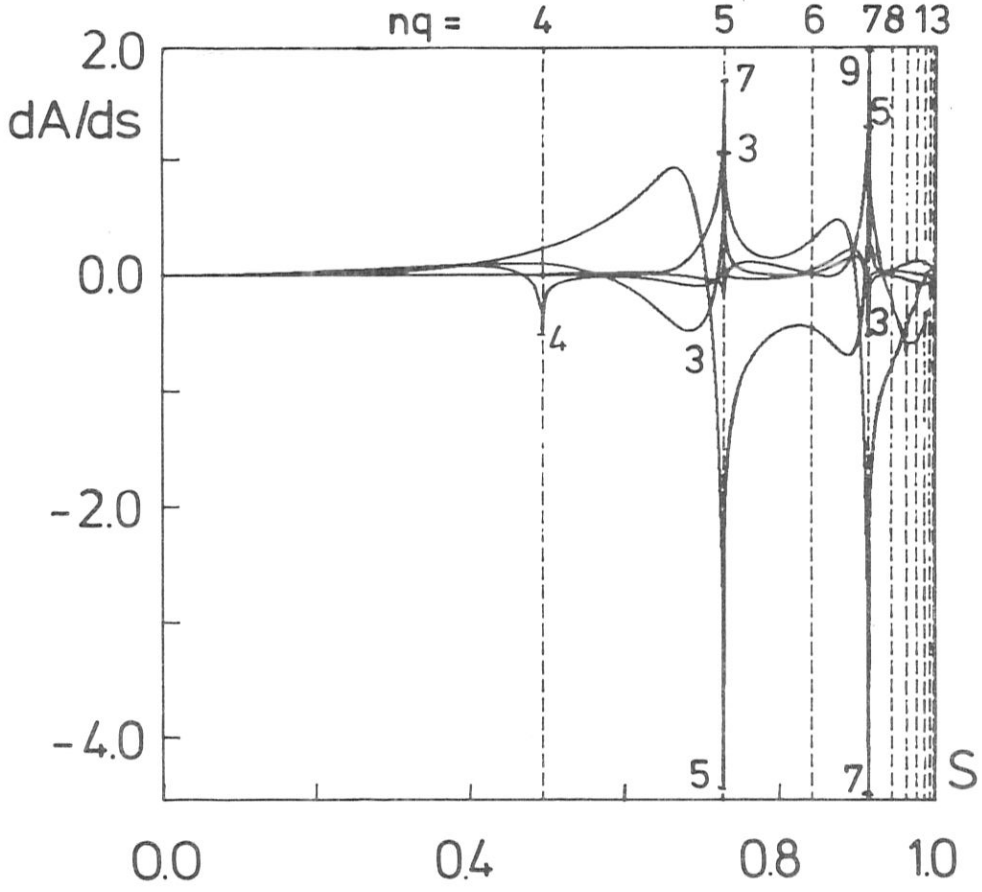
0.0
5
7
3
4
9
S
0.0
0.4
0.8
1.0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dA/ds
0.0
-2.0
-4.0
7
9
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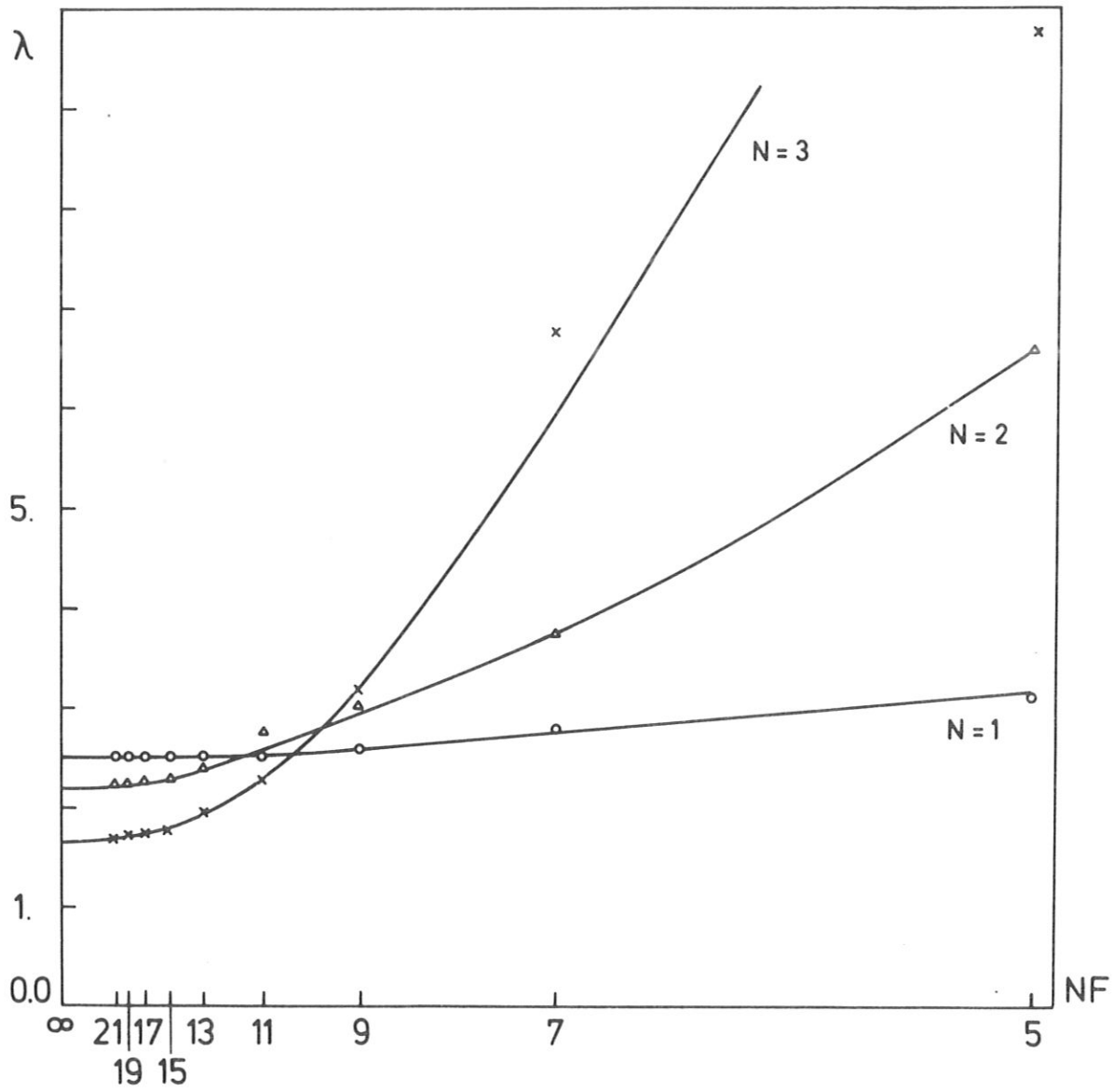
3
4
3
5
7
3
S
0.0
0.4
0.8
1.0


λ
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1.
0.0
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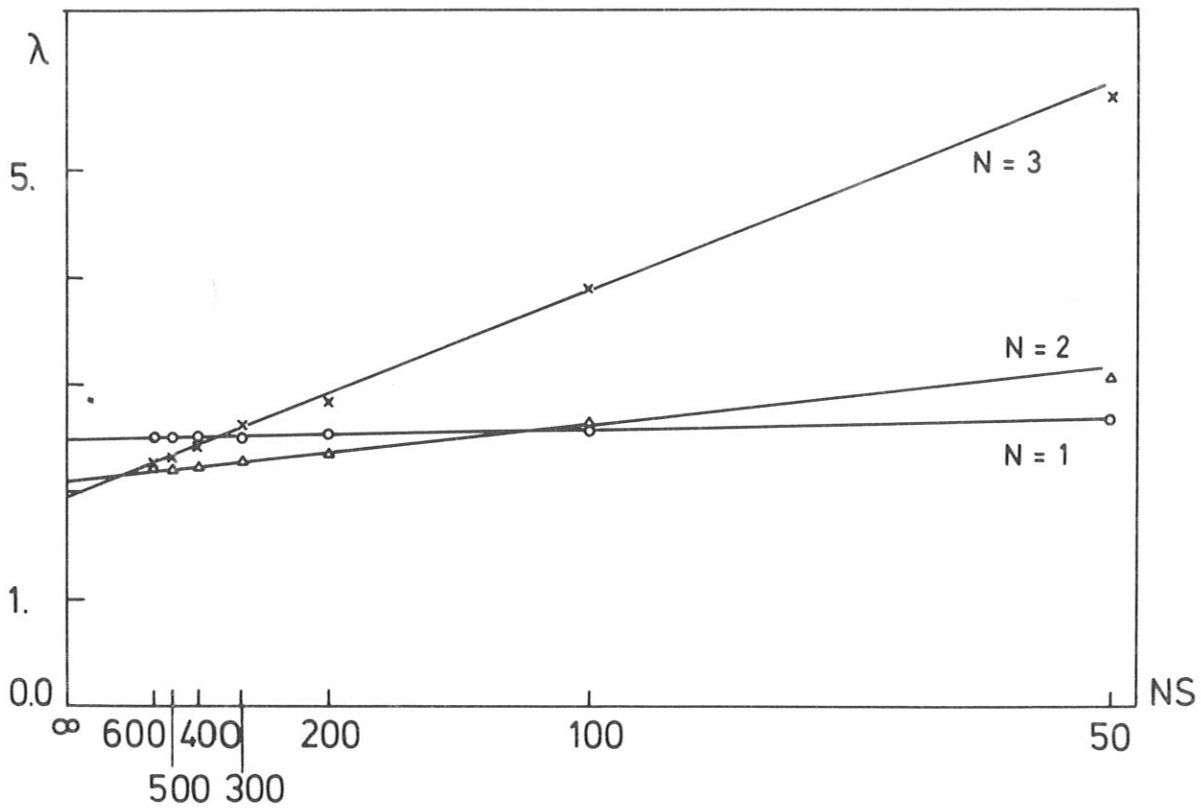
N = 2
N = 1
∞
21
19
17
15
13
11
9
7
5
NF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N = 2
N = 1
NS
∞
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500
300
200
100
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