# Stability of non-linear systems by means of event-triggered sampling algorithms[*]

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In this paper, the problem of feedback control implementation for nonlinear systems is considered. Some conditions for holding the same control input until an event occurs are derived. With respect to classical approaches, where feedback laws are implemented in a periodically fashion, it is suggested new algorithms to use the same control input. By means of these algorithms, some jumps of the control inputs occur and the nonlinear system becomes hybrid, since it has a mixed discrete/continuous dynamics. Under some assumptions, written in terms of Lyapunov functions, two event-based algorithms are suggested for nonlinear systems. The first algorithm is directly based on the variation of the Lyapunov functions. The last event-based algorithm is based on a selection of the input variables to be updated. The results are particularized to linear control systems and illustrated by numerical simulations of linear and nonlinear control systems.

*Keywords*: Lyapunov functions, nonlinear systems, hybrid systems, asymptotic stability.

## 1. Introduction

Over the years, researches in control of dynamical systems have provided various approaches to design globally asymptotically stabilizing feedbacks. Traditionally, the controller is implemented in the time triggered framework where the sampling for the controller is chosen periodic. The analysis of discrete-time systems has been widely investigated for linear systems (see (Åström and Wittenmark 1997, Chen and Francis 1995) and the references therein). Attempts to extend these results to nonlinear systems were carried out, but the difficulty to obtain a nonlinear discrete-time model is an important obstacle. Some approaches based on an approximation of the system (Nešić and Teel 2004) or a redesign of the control (Nešić and Grüne 2005) where developed but it still remains complex. For linear systems, several studies deal with the robustness of sampled-data controllers with respect to uncertainties in the sampling instants sequence (jitter) and measurement loss (Cervin, Henriksson, Lincoln, Eker and Arzen 2003, Fridman, Seuret and Richard 2004, Fujioka 2009, Seuret 2012). These methods typically ensure the stability of a linear sampled-data system if the sampling period is included in a certain interval. These results are very relevant but they consider the worst situation.

More in the spirit of non regular sampling period, one can find works dealing with the equivalence between controllability and stabilizability of nonlinear systems (Clarke, Ledyaev, Sontag and Subbotin 1997, Marchand and Alamir 2000). In these works, the feedback stabilizes the system whatever a sufficiently fast sampling (for purpose stability) but not too slow (for robustness purposes). The

sampling (even if non regular) is however not depending upon the state as in the present work.

In recent years, an interesting method so-called Event-Based Control suggests to adapt the sampling sequence to some events related to the state of the system. The idea arises in the context of Networked Control Systems (see for example (Hespanha, Naghshtabrizi and Xu 2007, Zampieri 2008)) where systems contain several distributed plants which are connected through a communication network. In this situation, the controlled system works in continuous-time whereas the controller provides a discrete-time input which is hold during a sampling period. It therefore relaxes the periodicity of computations and as a consequence reduces the processor usage in embedded devices or the network bandwidth needs in networked systems. Works on event-based PID have shown the efficiency of the approach with for example reduction of control function calls up to 80% (Durand and Marchand 2009, Årzén 1999). Event-based control approach were further extended to general nonlinear systems in (Tabuada 2007) where an update policy based on the existence of a Lipschitz (at the origin) stabilizing control law and an Input to State Stable Control Lyapunov Function (ISS-CLF) is proposed. Various extensions of the result were done in (Anta and Tabuada 2008, Anta and Tabuada 2010) to polynomial and homogeneous systems. Sontag's general formula for feedback stabilization was extended to event-based stabilization in (Marchand, Durand and Guerrero-Castellanos 2013) with the sole assumption of the existence of a smooth CLF. In both cases and as considered in this paper, the update policy is driven by events issuing from the time derivative of the Lyapunov function. However, contrary to the above references, the notion of *Minimal inter-Sampling Interval* as detailed in (Marchand et al. 2013) is not required since the solutions are intended in the Filipov sense. For this, the problem of the design of an event-triggered algorithm is first rewritten as the stability study of a system with a mixed continuous/discrete dynamics (also called hybrid system), as considered e.g., in (Goebel, Sanfelice and Teel 2012, Prieur, Goebel and Teel 2007, Prieur, Tarbouriech and Zaccarian 2010) in a different context. Using this framework and the Lyapunov theory that is now well known on this kind of nonlinear systems, we compute two new event-triggered algorithms for the implementation of feedback controllers. The first event-triggered algorithm makes a Lyapunov-like function decrease (see Theorem 3.1 below for a precise statement). This algorithm applies to nonlinear control systems for which it is known a (nonlinear) stabilizing controller under weak assumptions, weaker than those required in (Tabuada 2007, Anta and Tabuada 2008, Anta and Tabuada 2010), and are not restricted to affine systems as in (Marchand et al. 2013). Finally, a last algorithm suggests a selection of the input variables to be updated when a suitable Lyapunov-based condition holds.

A preliminary version of this paper has appeared in (Seuret and Prieur 2011) without the proofs and with less results (in particular only two event-triggered algorithms have been considered in (Seuret and Prieur 2011)).

The paper is organized as follows. In Section 2 some materials on hybrid systems are given, and the problem under consideration in this paper is introduced. In Section 3, a synchronized event-triggered algorithm is given for nonlinear control systems. In Section 4, a selected event-triggered algorithm is presented and it is supported by an example of a nonlinear control system borrowed from the literature. Then main results are applied to the linear case in Section 5, and illustrated by an example of linear control system. Section 6 contains some concluding remarks and points out some possible open research lines.

*Notation.* Throughout the article, the sets $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}^n$, $\mathbb{R}^{n \times n}$ and $\mathbb{S}^n$ denote respectively the sets of positive integers, positive scalars, $n$-dimensional vectors, $n \times n$ matrices and symmetric matrices of $\mathbb{R}^{n \times n}$. The notation $|\cdot|$ stands for the Euclidean norm. Given a compact set $\mathscr{A}$, the notation $|x|_{\mathscr{A}} = \min\{|x-y|,\ y \in \mathscr{A}|\}$ indicates the distance of the vector $x$ to the set $\mathscr{A}$. The superscript '$T$' stands for matrix transposition. A function $\mu$ is said to be of class $\mathscr{K}_\infty$ if it is continuous, zero at zero,

strictly increasing and unbounded. The symbols $I$ and $0$ represent the identity and the zero matrices of appropriate dimensions. For a given strictly positive integer $m$, define the set $\mathscr{S}_m = \{1, \ldots, m\}$. For any $j \in \mathscr{S}_m$, define the set $\mathscr{S}_m^j$ of all possible sequences of $j$ distinct elements of $\mathscr{S}_m$.

## 2. Problem formulation

Consider a continuous-time nonlinear system

$$
\begin{aligned}
\dot{x} &= f(x,u), \\
x(t_0) &= \phi_0,
\end{aligned}
\tag{2.1}
$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ stand respectively for the state variable and the input vector, $\phi_0 \in \mathbb{R}^n$ is the initial state and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally Lipschitz function.

Assume that the system (2.1) is globally asymptotically stabilizable, i.e. that there exist a Lyapunov function $V$ and a state feedback control law $u$ such that the derivative of the Lyapunov function along the trajectories of the closed-loop system is negative definite. This means that:

ASSUMPTION 1. *There exist a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, some functions $\mu_1$, $\mu_2$ and $\mu_3$ in $\mathscr{K}_\infty$ and a continuous controller $u : \mathbb{R}^n \to \mathbb{R}^m$ such that $u(0) = 0$ and, for all $x \in \mathbb{R}^n$,*

$$
\mu_1(|x|) \leqslant V(x) \leqslant \mu_2(|x|),
$$

$$
\nabla V(x).f(x,u(x)) \leqslant -\mu_3(|x|).
$$

This assumption suggests that the control law $u$ has been designed in continuous-time so that the (continuous)-time derivative of a Lyapunov function is negative definite.

In practice, it is not realistic to implement a control law in continuous-time. As the control input is computed on a digital hardware, only a sampled version of the input is implemented in the actuators. Generally speaking, the sampling is chosen periodic and with a small period so that the sampled signal is very close to the continuous one. However the computation of the control values is not done instantaneously. It requires a minimum sampling period which guarantees that the controller is able to compute the correct data on time. Consequently, the use of a small sampling period requires an efficient processor allowing to compute the control value in short time. An alternative solution is to develop an algorithm which triggers the sampling period with respect to the state of the system, as shown in Figure 1. The contribution of this paper is to let the system decide by itself if an update of the control is needed or not.

In order to clarify the notation, a hybrid formulation of the sampled-data system is proposed, using (Goebel et al. 2012, Prieur et al. 2007, Prieur, Tarbouriech and Zaccarian 2013). More precisely the sampled-data system is rewritten as

$$
\begin{cases}
\dot{x} = f(x,s), \\
\dot{s} = 0, \\
\dot{p} = g(x,s,p),
\end{cases}
\quad (x,s,p) \in \mathscr{F},
$$

$$
\begin{cases}
x^+ = x, \\
s^+ \in D(x,s)u(x) + D^-(x,s)s, \\
p^+ = g_0(x,s,p),
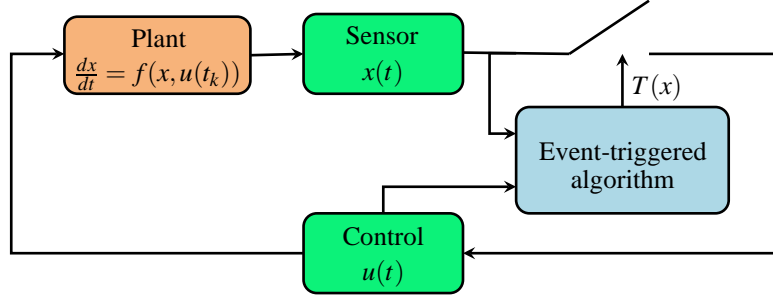\end{cases}
\quad (x,s,p) \in \mathscr{J},
\tag{2.2}
$$

FIG. 1. Control scheme with an event-triggered algorithm.

where $s \in \mathbb{R}^m$ represents the held value of the control input (that is implemented over the sampling interval), $p \in \mathbb{R}^q$ contains additional parameters, $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \rightarrow \mathbb{R}^q$ and $g_0 : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \rightarrow \mathbb{R}^q$ are two continuous functions of appropriate dimensions, and $\mathscr{F}$ and $\mathscr{J}$ are two subsets of $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q$. These sets are respectively called flow set and jump set and are degrees of freedom of the event-triggered algorithm. The function $D : \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^{m \times m}$ is a set-valued map, that takes non-empty values when $(x, s, p) \in \mathscr{J}$, and that is outer semicontinuous and locally bounded[1]. Note that, if it is locally bounded, then it is outer semicontinuous if and only if its graph is closed. Moreover in (2.2), for all $(x, s)$ in $\mathbb{R}^n \times \mathbb{R}^n$, $D^-(x, s)$ denotes $D^-(x, s) = \{I - d, d \in D(x, s)\}$. The design of such a function is proposed in the sequel. The objective of the function $D$ is to select the control input component to be updated. Note that the function $D$ should take values in $2^{\mathbb{R}^{m \times m}}$, that is the set of all subsets of $\mathbb{R}^{m \times m}$. The function $D$ is set-valued because it comes from the regularization of a discontinuous single valued function. Such a regularization is useful to ensure a robustness issue of the stability that will be derived in this paper (see e.g., (Goebel et al. 2012, Chapter 4) for an introduction on generalization of solutions to hybrid systems in connection with perturbations). The robustness with respect to measurement noise or actuation errors follows from general robustness results of asymptotically stable hybrid systems (see e.g., (Goebel et al. 2012, Prieur et al. 2007)).

We recall some basic ingredients on hybrid system theory, and on the notion of solutions to (2.2) (see (Goebel et al. 2012, Prieur et al. 2007)). Due to mixed discrete/continuous dynamics, a solution to (2.2) will be defined on a mixed discrete/continuous time domain. Let us define first the notion of compact hybrid time domain (see (Goebel et al. 2012, Definition 2.3)). A set $E$ is a *compact hybrid time domain* if

$$E = \bigcup_{j=0}^{J-1} \left( [t_j, t_{j+1}], j \right),$$

for some finite sequence of times $0 = t_0 \leqslant t_1 \ldots \leqslant t_J$. It is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots J\})$ is a compact hybrid time domain. A solution $(x, s, p)$ to (2.2) consists of a hybrid time domain dom and functions $x : \text{dom} \rightarrow \mathbb{R}^n$, $s : \text{dom} \rightarrow \mathbb{R}^m$, and $p : \text{dom} \rightarrow \mathbb{R}$ such that $(x, s, p)(t, j)$ is absolutely continuous in $t$ for a fixed $j$ and $(t, j) \in \text{dom}$ satisfying

---

[1] A set-valued mapping $D$ defined on $\mathbb{R}^n$ is *outer semicontinuous* if for each sequence $x_i \in \mathbb{R}^n$ converging to a point $x \in \mathbb{R}^n$ and each sequence $y_i \in D(x_i)$ converging to a point $y$, it holds that $y \in D(x)$. It is *locally bounded* if, for each compact set $K \subset \mathbb{R}^n$ there exists $\mu > 0$ such that $\cup_{x \in K} D(x) \subset B(0, \mu)$, where $B(0, \mu)$ is the ball of radius $\mu$ centered at 0.

(S1) for all $j \in \mathbb{N}$ and almost all $t$ such that $(t,j) \in \mathrm{dom}$,

$$(x(t,j),s(t,j),p(t,j)) \in \mathscr{F}, \ \dot{x}(t,j) = f(x(t,j),s(t,j)),$$
$$\dot{s}(t,j) = 0,$$
$$\dot{p}(t,j) = g(x(t,j),s(t,j),p(t,j)),$$

(S2) for all $(t,j) \in \mathrm{dom}$ such that $(t,j+1) \in \mathrm{dom}$,

$$(x(t,j),s(t,j),p(t,j)) \in \mathscr{J}, \ x(t,j+1) = x(t,j),$$
$$s(t,j+1) \in D(x(t,j),s(t,j))u(x(t,j))$$
$$+D^-(x(t,j),s(t,j))s(t,j),$$
$$p(t,j+1) = g_0(x(t,j),s(t,j),p(t,j)).$$

Then, the solution $(x,s,p)$ is parameterized by $(t,j)$ where $t$ is the ordinary time and $j$ is an independent variable that corresponds to the number of jumps of the solution. This parameterization may be omitted when there is no ambiguity. When the state $x(t,j)$ belongs to the intersection of the flow set and of the jump set, then the solution can either flow or jump.

A solution $(x,s,p)$ to (2.2) is said to be *complete* if its domain is unbounded (either in the $t$-direction or in the $j$-direction), *Zeno* if it is complete but the projection of $\mathrm{dom}$ onto $\mathbb{R}_{\geqslant 0}$ is bounded, and *maximal* if there does not exist another solution $\tilde{x}$ to (2.2) such that $x$ is a truncation of $\tilde{x}$ to some proper subset of its domain. Hereafter, only maximal solutions will be considered. For more details about this hybrid systems framework, we refer the reader to (Goebel et al. 2012, Prieur et al. 2007). The following definition describes the requirements to prove the global asymptotic stability of the solutions to (2.2).

DEFINITION 2.1 Given a closed subset $\mathscr{A}$ of $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q$ the hybrid system (2.2) is said to be

- *stable to $\mathscr{A}$:* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that each solution $(x,s,p)$ to (2.2) with $|(x(0,0),s(0,0),p(0,0))|_{\mathscr{A}} \leqslant \delta$ satisfies $|(x(t,j),s(t,j),p(t,j))|_{\mathscr{A}} \leqslant \varepsilon$ for all $(t,j) \in \mathrm{dom}$;

- *attractive to $\mathscr{A}$:* if every solution $x$ to (2.2) is complete and satisfies

$$\lim_{t+j \to \infty} |(x(t,j),s(t,j),p(t,j))|_{\mathscr{A}} = 0;$$

- *globally asymptotically stable to $\mathscr{A}$:* if it is both stable and attractive to $\mathscr{A}$.

Given an initial condition $(\phi_0, s_0, p_0)$ in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$, and a solution $(x,s,p)$ of (2.2) defined on a hybrid time domain $\mathrm{dom}$, the set of the sampling time instants, when the control input is updated, (plus 0) is denoted $\mathscr{T}$ and is $\{t_j\}$, where $t_j$ is such its domain is written as $\cup_{j \in J}\left([t_j,t_{j+1}] \times \{j\}\right)$. Among other results, we state in this paper some properties on the set $\mathscr{T}$ depending on the choice of the event-triggered algorithm. In particular, in our hybrid systems framework, $\mathscr{T}$ is at most countable.

In this paper, several sets $\mathscr{F}$ and $\mathscr{J}$ and functions $D$ are defined, and thus several event-triggered algorithms are considered. Let the particular case where $p = \tau \in \mathbb{R}$, $D(\cdot) = I$ and such that the dynamics of the system are rewritten, for any $T > 0$, as

$$\begin{cases} \dot{x} = f(x,s), \\ \dot{s} = 0, \qquad (x,s,\tau) \in \mathscr{F}_T, \\ \dot{\tau} = 1, \end{cases}$$

$$\begin{cases} x^+ = x, \\ s^+ = u(x), \qquad (x,s,\tau) \in \mathscr{J}_T, \\ \tau^+ = 0, \end{cases}$$

$$(2.3)$$

where $\mathscr{F}_T$ and $\mathscr{J}_T$ are the following subsets of $\mathbb{R}^n \times \mathbb{R}^m \times [0,T]$:

$$\mathscr{F}_T = \{(x,s,\tau), \tau \leqslant T\},$$

$$\mathscr{J}_T = \{(x,s,\tau), \tau \geqslant T\}. \tag{2.4}$$

As shown in (Goebel et al. 2012), the hybrid model expresses the case of periodic sampling. In this simple algorithm, after each jump, the solution is either at the equilibrium or has to flow. It avoids the existence of Zeno solutions, and also it reduces the complexity when implementing the event-triggered algorithm. Of course, in general, the system (2.3) is not globally asymptotically stable since the update of the control law does not depend on the system position but is done periodically. This motivates us to consider the following problem:

Problem 2.1 Define appropriate sets $\mathscr{F}$ and $\mathscr{J}$ and dynamics of the variable $p$ such that, after each jump of the solutions to (2.2), the solutions have to flow, and such that (2.2) is globally asymptotically stable.

## 3. Synchronized event-triggered algorithm for nonlinear systems

In this section, the set-valued matrix function $D$ is chosen constant and equal to the (singleton given by the) identity matrix $I$. This means that the matrix $D^-$ is equal to the null matrix. Coming back to the definition of the hybrid system (2.2), the dynamics of the system evolving in the jump set becomes

$$\begin{cases} \dot{x} = f(x,s), \\ \dot{s} = 0, & (x,s,p) \in \mathscr{F}, \\ \dot{p} = g(x,s,p), \end{cases}$$

$$\begin{cases} x^+ = x, \\ s^+ = u(x), & (x,s,p) \in \mathscr{J}, \\ p^+ = g_0(x,s,p), \end{cases} \tag{3.1}$$

Using this framework, all the components of variable $s$ may have a jump only when the system enters in the jump set $\mathscr{J}$. We call this algorithm *Synchronized event-triggered algorithm* since the updates of all components of $s$ are achieved simultaneously. The objective is to define some flow and jump sets, based on the decay of the function in continuous-time.

THEOREM 3.1 Under Assumption 1, consider a given function $\mu$ of class $\mathscr{K}_\infty$ such that $\mu(r) < \mu_3(r)$, for all $r > 0$. Consider the flow and jump sets given by

$$\mathscr{F}_1 = \{(x,s), \nabla V(x).f(x,s) \leqslant -\mu(|x|)\},$$

$$\mathscr{J}_1 = \{(x,s), \nabla V(x).f(x,s) \geqslant -\mu(|x|)\}, \tag{3.2}$$

and the associated event-triggered algorithm. Then the system (3.1) with $\mathscr{F} = \mathscr{F}_1$ and $\mathscr{J} = \mathscr{J}_1$ (and without state $p$) is globally asymptotically stable to $\{0\} \times \mathbb{R}^m$. Moreover, for each solution to this hybrid system, at every time when the solution has a jump, either the $x$-component of the state is the origin or the solution has to flow.

*Proof.* The proof of Theorem 3.1 is based on the decreasing property of the function $V$ given by Assumption 1, along the solutions to (3.1), with $\mathscr{F}$ and $\mathscr{J}$ given by (3.2). See (Prieur et al. 2010, Prieur et al. 2013) for analogous ideas for a different problem.

Given a switching time instant $t_0 \in \mathscr{T}$, denoting (with a slight abuse of notation) $x(t_0^+)$ the state after the jump (and similarly for the other variables), using Assumption 1, it yields

$$
\begin{aligned}
\nabla V(x(t_0^+)).f(x(t_0^+), s(t_0^+)) &= \nabla V(x).f(x(t_0^+), u(x(t_0^+))) \\
&\leqslant -\mu_3(|x(t_0^+)|) \\
&\leqslant -\mu(|x(t_0^+)|) - \varepsilon(|x(t_0^+)|),
\end{aligned}
$$

where $\varepsilon(|x(t_0^+)|) = \mu_3(|x(t_0^+)|) - \mu(|x(t_0^+)|)$ is non-negative and equals 0 only if $x(t_0^+)$ is vanishing. Thus, after a jump, two cases may occur:

1) either the $x$-component of the state is at the origin (and the same for the other components), and then the solution remains at the origin;

2) or $x(t_0^+)$ is different to 0. Then $(x(t_0^+), s(t_0^+))$ belongs to $\mathscr{F}_1 \setminus \mathscr{J}_1$ and the solution has to flow.

Consider now $(x, s)$ in $\mathscr{F}_1 \setminus \{0\}$. Then we get

$$
\nabla V(x).f(x, s) = \nabla V(x).(f(x, s) - f(x, u(x))) + \nabla V(x).f(x, u(x)),
$$

and using Assumption 1, we obtain

$$
\nabla V(x).f(x, s) \leqslant -\mu_3(|x|) + \nabla V(x).(f(x, s) - f(x, u(x))).
$$

Then, the solution $(x, s)$ to system (3.1) with $\mathscr{F} = \mathscr{F}_1$ and $\mathscr{J} = \mathscr{J}_1$ stays in $\mathscr{F}_1$ until a state $x = x^*$ (if such a state does exist) defined by

$$
\nabla V(x^*).(f(x^*, s) - f(x^*, u(x^*))) = \mu_3(|x^*|) - \mu(|x^*|).
$$

Two subcases may occur.

2.a) If there exists such $x^*$, then the couple $(x^*, s)$ belongs to $\mathscr{J}_1$, and by definition of $s^+$, $(x^{*+}, s^+)$ belongs to $\mathscr{F}_1$.

2.b) If there does not exist such $x^*$, then the solution to the system (3.1) stays in $\mathscr{F}_1$.

For both cases, the derivative of $V$ is negative while $(x, s)$ is in $\mathscr{F}_1$ and $V$ is constant while $(x, s)$ is in $\mathscr{J}_1$. This implies that the system (3.1) with $\mathscr{F} = \mathscr{F}_1$ and $\mathscr{J} = \mathscr{J}_1$ is stable to $\{0\} \times \mathbb{R}^m$ (as proven in the first part of (Goebel et al. 2012, Theorem 3.18)).

To prove the attractivity of the system (3.1) with $\mathscr{F} = \mathscr{F}_1$ and $\mathscr{J} = \mathscr{J}_1$, let us apply the LaSalle invariance property for hybrid systems (see e.g., (Goebel et al. 2012, Theorem 8.2)). Let us consider a solution to this hybrid system which is included in a level set of the function $V$. Let us show that this solution should be equal to 0.

The solution cannot jump, except if it is at the origin (indeed, if the solution is not at the origin, then, after a jump, the solution has to flow, and thus the value of $V$ has to decrease). Given a solution flowing for all time, due to Assumption 1, the state $x$ cannot stay at the level set of $V$. Thus the solution has to be constant and equal to the origin. Therefore, by (Goebel et al. 2012, Theorem 8.2), the system (3.1) with $\mathscr{F} = \mathscr{F}_1$ and $\mathscr{J} = \mathscr{J}_1$ is globally attractive to $\{0\} \times \mathbb{R}^m$ and therefore it is globally asymptotically stable. This concludes the proof of Theorem 3.1. $\qquad\square$

REMARK 3.1 A main improvement of the proposed method compared for example to (Anta and Tabuada 2010), is that no Input-to-State Stability (ISS) assumption for system (2.1) is needed. More precisely the method that is suggested in (Anta and Tabuada 2010) requires the existence of functions $\alpha$ and $\gamma$ of class $\mathscr{K}_\infty$, such that, for all $x$ in $\mathbb{R}^n$,

$$
\nabla V(x).f(x, u(x + \varepsilon)) \leqslant -\alpha(|x|) + \gamma(|\varepsilon|).
$$

Then the event-triggered algorithm is defined by a condition on the error between the current value of the state $x$ and its *memory m* - that is the value of the state last time the control was updated. $\varepsilon = m - x$ denotes the measurement error. The control is updated as soon as $|\varepsilon| \leqslant \gamma^{-1}(\sigma\alpha(|x|))$ ensuring that way the strict decrease of $V$ for $0 < \sigma < 1$. In the present article, instead of an ISS assumption, only the global asymptotic stability is needed. As remarked in (Sontag 2007), Assumption 1 is weaker than the ISS property, and it is sufficient to define the event-triggered algorithm by the value of the derivative of the Lyapunov function along the trajectories of the system.                                              ∘

REMARK 3.2 Another important issue concerns the possibility that the solution of the system for a given initial condition, never reaches the set $\mathscr{J}_1$. It is the case when that the system is already asymptotically stable without any control (or with a constant control value) and the control law does not need to be updated. This situation is not taken into account in the method proposed in (Anta and Tabuada 2010).

Moreover note that there may exist some Zeno solutions to the hybrid system (3.1) with $\mathscr{F} = \mathscr{F}_1$ and $\mathscr{J} = \mathscr{J}_1$. For such solutions, the attractivity of the origin contained in the conclusion of Theorem 3.1 holds, as the the quantity $t + j$ goes to infinity, as thus as the discrete time $j$ goes to the infinity (since for Zeno solutions, the continuous time $t$ is bounded).                                                ∘

REMARK 3.3 On the other side, there is a drawback of the present method. The derivative of the Lyapunov function $V$ needs to be computed at all time instants to check if the closed-loop system has to flow or to jump.                                                                                ∘

Picking $\mu = 0$ in Theorem 3.1 gives a partial result and allows to design an event-triggered algorithm such that the closed-loop system is globally stable. More precisely we have

PROPOSITION 3.1 Under Assumption 1, consider the flow and jump sets given by

$$\mathscr{F}_1' = \{(x,s),\ \nabla V(x).f(x,s) \leqslant 0\},$$

$$\mathscr{J}_1' = \{(x,s),\ \nabla V(x).f(x,s) \geqslant 0\},$$

and the associated event-triggered algorithm. Then the systems (3.1) with $\mathscr{F} = \mathscr{F}_1'$ and $\mathscr{J} = \mathscr{J}_1'$ is globally stable to $\{0\} \times \mathbb{R}^m$. Moreover, for each solution to this hybrid systems, at every time when the solution has a jump, either the $x$-component of the state is the origin or the solution has to flow.

*Proof.*     The proof follows the lines of Theorem 3.1. More precisely, we may check that by selecting $\mathscr{F} = \mathscr{F}_1'$ and $\mathscr{J} = \mathscr{J}_1'$, and by using Assumption 1, the derivative of the Lyapunov function $V$ is negative while the state of the solution $(x,s)$ of (3.1) is in $\mathscr{F}_1$ and is constant $(x,s)$ is in $\mathscr{J}_1$. This implies, with the first part of the proof of (Goebel et al. 2012, Theorem 3.18)), that the system (3.1) with $\mathscr{F} = \mathscr{F}_1$ and $\mathscr{J} = \mathscr{J}_1$ is stable to $\{0\} \times \mathbb{R}^m$.

Finally, using Assumption 1 again, we note that, given a solution of (3.1) with $\mathscr{F} = \mathscr{F}_1'$ and $\mathscr{J} = \mathscr{J}_1'$, after each jump (if such a jump does exist), either the state is the origin or the solution has to flow. This concludes the proof of Proposition 3.1.                                                          □

## 4. Selected event-triggered algorithm for nonlinear systems

### 4.1 *General nonlinear systems*

From now on, the system under consideration is the one defined in (2.2) (without any state $p$). The objective of this section is the design of the matrix function $D$ in order to get a stabilizing event-based algorithm. In comparison with the approach presented in the previous section (where all components of

the input vector are updated at each jump), the event-triggered algorithm is authorized to update only one or several components of the input vector $s$. This problem has already been addressed in (Postoyan, Tabuada, Nesic and Anta 2011) where the matrix function $D$ is a predefined to schedule the control input to be updated. Here the main difference with respect to (Postoyan et al. 2011) is that the matrix $D$ is resulting from an appropriate selection of the control input which depends on the current state of the system. A solution to this problem is described in the sequel for the case of the continuous-time decrease of the Lyapunov function as proposed in Section 3. Consequently, we will also use the same flow and jump sets $\mathscr{F}_1$ and $\mathscr{J}_1$ defined in (3.2) with the appropriate function $\mu$.

In order to propose a simple formulation to this problem, we adopt the following notations.

$$\forall (i,k) \in \{\mathscr{S}_m\}^2, \quad \kappa_i(k) = \begin{cases} 1, & \text{if } k = i, \\ 0, & \text{otherwise.} \end{cases}$$

and the matrices

$$\forall i \in \mathscr{S}_m, \quad D_i = \begin{bmatrix} \kappa_i(1) & \dots & 0 \\ & \ddots & \\ 0 & \dots & \kappa_i(m) \end{bmatrix}.$$

With such a matrix $D_i$ and given $u$ in $\mathbb{R}^m$, $D_i u$ is the vector with all vanishing components, except the $i$-th component which is $u_i$. It allows to update the input of system (2.1) using only one component of $u(x)$. Now given $\lambda \leqslant m$ a positive integer and $\sigma := \{i_1, \dots, i_\lambda\}$ in $\mathscr{S}_m^\lambda$ (where $i_{k_1}$ assumed to be different from $i_{k_2}$ for any $k_1 \neq k_2$), we denote

$$D_\sigma = \sum_{l=1}^{\lambda} D_{i_l} = \begin{bmatrix} \sum_{l=1}^{\lambda} \kappa_{i_l}(1) & \dots & 0 \\ & \ddots & \\ 0 & \dots & \sum_{l=1}^{\lambda} \kappa_{i_l}(m) \end{bmatrix}.$$

When employing this matrix $D_\sigma$, for any $u$ in $\mathbb{R}^m$, $D_\sigma u$ allows to update $\lambda$ components using $u$ (see Theorem 4.1 below for more details).

The set-valued map function $D : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is the Krasovkii's regularization (see e.g., (Goebel et al. 2012, Definition 4.13) for more details on such a regularization) of a discontinuous function $d : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ defined as follows:

For a given state $(x, s)$, $d(x, s) = D_\sigma$ where $\sigma$ is an element of $\mathscr{S}_m^\lambda$ and $\lambda$ in $\{1, \dots, m\}$ are such that

$$\partial V_\lambda(x,s) := \min_{\sigma \in \mathscr{S}_m^\lambda} \{\nabla V(x(t)).f(x(t), D_\sigma u(x(t)) + (I - D_\sigma)s(t))\}, \quad \forall \lambda' \in \{1, \dots, m\}, \tag{4.1a}$$

$$\partial V_\lambda(x,s) \leqslant -\mu(|x|), \tag{4.1b}$$

$$\partial V_{\lambda'}(x,s) > -\mu(|x|), \quad \forall \lambda' < \lambda. \tag{4.1c}$$

As for the previous event-triggered algorithms, with this function $D$, the update of the input vector is done by using Lyapunov inequalities. More precisely, the condition (4.1a) computes, for a given $\lambda$, the minimal value, among all the possible choices when updating only $\lambda$ component(s) of the control input $s$, of the time-derivative of the Lyapunov function $V$.

The second condition (4.1b) and the last one (4.1c) computes the minimal number of necessary updates that are needed to ensure the good sign of the Lyapunov function. With this function $D$, we are in position to prove the following:

THEOREM 4.1 Under Assumption 1, consider a given function $\mu$ of class $\mathscr{K}_\infty$ such that $\mu(r) < \mu_3(r)$, for all $r > 0$. Consider the flow and jump sets given by $\mathscr{F}_1$ and $\mathscr{J}_1$ respectively, defined in (3.2) and the associated event-triggered algorithm and the set-valued map function $D$ defined by (4.1). Then the system (2.2) with $\mathscr{F} = \mathscr{F}_1$ and $\mathscr{J} = \mathscr{J}_1$ (and without state $p$) is globally asymptotically stable to $\{0\} \times \mathbb{R}^m$.

The proof is inspired from Theorem 3.1 and is based on the fact that, after each jump, either the solution to (2.2) with $\mathscr{F} = \mathscr{F}_1$ and $\mathscr{J} = \mathscr{J}_1$ is at the origin, or the solution has to flow. Therefore the proof of Theorem 4.1 is omitted.

### 4.2 *Affine nonlinear systems*

Algorithm 4.1 requires the computation of $\nabla V(x(t)).f(x(t), D_\sigma u(x(t)) + (I - D_\sigma)s(t))$ for all $\lambda$ in $\{1,\ldots,m\}$, that is the effect of any combination between the updated control $u(x(t))$ and its previous value $s(t)$ on the decrease of the Lyapunov function. It asks for $2^m$ evaluations of this derivative (that is as many parts in $\{1,\ldots,m\}$). This may be costly and can highly be simplified in the case of systems that are affine in the control, that is of the following form:

$$\begin{cases} \dot{x} = f_1(x) + f_2(x)s, \\ \dot{s} = 0, \end{cases} \quad (x,s) \in \mathscr{F},$$

$$\begin{cases} x^+ = x, \\ s^+ = u(x), \end{cases} \quad (x,s) \in \mathscr{J}. \tag{4.2}$$

In that case, $\partial V_\lambda(x,s)$ is composed of two terms, the drift being independent from the control value. Hence:

$$\partial V_\lambda(x,s) = \nabla V(x(t))f_1(x(t)) + \min_{\sigma \in \mathscr{S}_m^\lambda} \left\{ \vartheta_1(x)^T \vartheta_2(x,\sigma) \right\}, \tag{4.3}$$

where

$$\vartheta_1(x) := [\nabla V(x(t))f_2(x(t))] \quad \vartheta_2(x,\sigma) := D_\sigma u(x(t)) + (I - D_\sigma)s(t).$$

The second term to minimize is the scalar product between the two vectors $\vartheta_1(x)$ and $\vartheta_2(x,\sigma)$, and the effects of all components of $\sigma$ on this scalar product are independent. Therefore, the minimum of $\vartheta_1(x)^T \vartheta_2(x,\sigma)$ over $\sigma$ in $\mathscr{S}_m^\lambda$ can be expressed componentwise as follows:

$$\min_{\sigma \in \mathscr{S}_m^\lambda} \{< \vartheta_1(x), \vartheta_2(x,\sigma) >\} = \sum_{i=1}^m \min_{d_i=0,1} \{(d_i u_i + (1 - d_i)s_i)\vartheta_{1_i}(x)\} \tag{4.4}$$

where $u_i$, $s_i$, $\vartheta_{1_i}$ and $\vartheta_{2_i}$ denote the $i^{th}$ term of respectively $u$, $s$, $\vartheta_1$ and $\vartheta_2$. Now, recalling that $D_\sigma$ in $\vartheta_{2_i}(x,\sigma)$ is a diagonal matrix composed of zeros and ones, it follows:

$$\min_{d_i=0,1} \{(d_i u_i + (1 - d_i)s_i)\vartheta_{1_i}(x)\} = \min \{\vartheta_{1_i}(x)u_i(x(t)), \vartheta_{1_i}(x)s_i(t)\} \tag{4.5}$$

The selection algorithm (4.1) becomes therefore much more simple. It can be computed as follows:

1. For $i \in \mathscr{S}_m$, compute the smallest term between $\vartheta_{1_i}(x)u_i(x(t))$ and $\vartheta_{1_i}(x)s_i(t)$ and keep the index only when $\vartheta_{1_i}(x)u_i(x(t)) < \vartheta_{1_i}(x)s_i(t)$. The indexes denote the control component that are would improve the decrease of the Lyapunov function if updated. The set of indexes is a subset of $\mathscr{S}_m$.
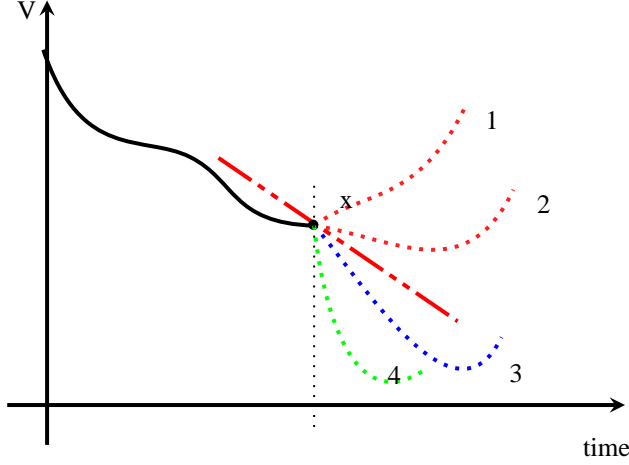
FIG. 2. Effect of the control update on the decrease of the Lyapunov function. In Case 1, the control is not updated. Cases 2 to 4 corresponds to different update strategies. The dashed-dotted red line is of slope $-\mu(|x|)$ that imposes the minimal decrease of the Lyapunov function

2. Sort this set in order to obtain an index set $\mathscr{I}$ defined as follows

$$\mathscr{I} := \{i_1, i_2, \ldots, i_\eta\}, \quad \eta \in \mathscr{S}_m, \tag{4.6a}$$

$$\text{such that:} \quad \vartheta_{1_{i_1}}(x)u_{i_1}(x(t)) \leqslant \vartheta_{1_{i_2}}(x)u_{i_2}(x(t)) \leqslant \cdots \leqslant \vartheta_{1_{i_\eta}}(x)u_{i_\eta}(x(t)) \tag{4.6b}$$

3. Compute the minimal number $\underline{\eta}$ of indexes such that:

$$\sum_{j=1}^{\underline{\eta}} \vartheta_{1_{i_j}}(x)u_{i_j}(x(t)) \leqslant -\mu(|x|) - \nabla V(x(t))f_1(x(t)) - \sum_{\substack{j \in \mathscr{S}_m \\ j \notin \mathscr{I}}} \vartheta_{1_j}(x)s_j(x(t)) \tag{4.7}$$

4. Update the control component $u_{i_1}(x(t))$ to $u_{i_{\underline{\eta}}}(x(t))$.

Following this algorithm (or more precisely its Krasovkii's regularization), as for Theorem 4.1, it is obtained that, after each jump, either the $x$-component of the solution to (2.2) with $\mathscr{F} = \mathscr{F}_1$ and $\mathscr{J} = \mathscr{J}_1$ is at the origin, or the solution has to flow. Therefore we get:

THEOREM 4.2 Under Assumption 1, consider a given function $\mu$ of class $\mathscr{K}_\infty$ such that $\mu(r) < \mu_3(r)$, for all $r > 0$. Consider the flow and jump sets given by $\mathscr{F}_1$ and $\mathscr{J}_1$ respectively, defined in (3.2) and the associated event-triggered algorithm defined by (4.6)-(4.7). Then the system (4.2) with $\mathscr{F} = \mathscr{F}_1$ and $\mathscr{J} = \mathscr{J}_1$ (and without state $p$) is globally asymptotically stable to $\{0\} \times \mathbb{R}^m$.

Figure 2 illustrates the different algorithms. At some points, it may be necessary to update several components so that $\dot{V}$ is lower that the quantity $-\mu(|x|)$. With the algorithm suggested by Theorem 4.2, it is updated the minimal number of inputs so that $\dot{V}$ is lower that the value $-\mu(|x|)$ (Case 3 for this figure), even if by updating more components (as for Case 4 of Figure 2), a lower value for $\dot{V}$ may be obtained.
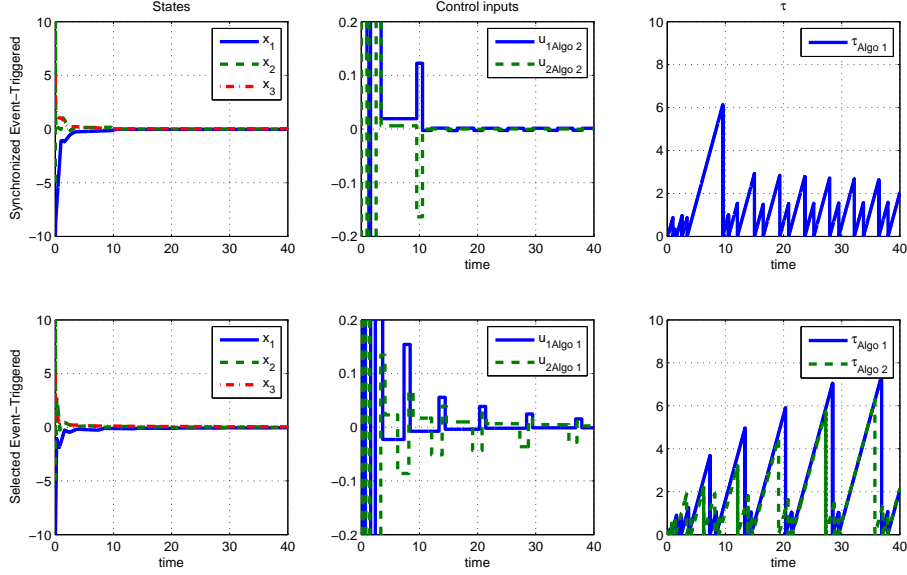
FIG. 3. From the left to right: time-evolution of the states, of the control values and of the $\tau$-variables for the synchronized and the selected algorithms, given respectively by Theorem 3.1 (top), and by Theorem 4.1 (down)

### 4.3 *Nonlinear example*

Consider the following nonlinear system borrowed from (Anta and Tabuada 2010, Byrnes and Isidori 1989):

$$\dot{x}_1 = u_1, \ \dot{x}_2 = u_2, \ \dot{x}_3 = x_1 x_2, \tag{4.8}$$

where $(x_1, x_2, x_3)$ and $(u_1, u_2)$ stand respectively for the state and for the control. A stabilizing controller is computed in (Byrnes and Isidori 1989). It is given by, for all $(x_1, x_2, x_3)$ in $\mathbb{R}^3$,

$$\begin{cases} u_1(x_1, x_2) = -x_1 x_2 - 2x_2 x_3 - x_1 - x_3, \\ u_2(x_1, x_2) = 2x_1 x_2 x_3 + 3x_3^2 - x_2. \end{cases} \tag{4.9}$$

A Lyapunov function for this system is computed in the same reference. It is defined by, for all $(x_1, x_2, x_3)$ in $\mathbb{R}^3$,

$$V(x) = (x_1 + x_3)^2/2 + (x_2 - x_3^2)^2/2 + x_3^2.$$

Thus Assumption 1 holds.

The simulation results for the synchronous and selected algorithms are shown in Figure 3 where two different numerical simulations are done: 1) the event-triggered algorithm which are considered in Theorem 3.1, 2) the event-triggered algorithm provided in Theorem 4.1. The figure contains the state, the input and the variable $\tau$ defined in (2.3) and the variables $\tau_i$ representing the sampling of each control input. The following parameters have been selected $\mu(x) = 10^{-3}|x|^2 + 10^{-3}|x|^4$, $\lambda = 0.2$. The initial conditions are $x_1(0) = -10$, $x_2(0) = -5$ and $x_3(0) = 5$.

In Figure 3, it can be seen that for a simulation of 40$s$, the synchronized event-triggered algorithm requires $2 \times 18 = 36$ updates of control inputs, while the selected event-triggered algorithm requires

only 27. This shows that the number of updates of the control inputs can be significantly reduced by the use of an adequate selection policy of the control inputs.

It is important to stress that the selection of updating the first or the second control inputs are not defined in advanced as in (Postoyan et al. 2011).

## 5. Application to linear systems

Consider now a linear system of the form

$$\dot{x} = Ax + Bu, \tag{5.1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ stand for the state variable and the input vector. The matrices $A$ and $B$ are constant and known and of appropriate dimensions. Let us assume that the pair $(A, B)$ is controllable. Then the proposed control law for this system is $u = Kx$ where $K$ in $\mathbb{R}^{m \times n}$ is such that the matrix $A + BK$ is Hurwitz. There also exist a positive scalar $\alpha$ and a symmetric positive definite matrix $P$ so that

$$P(A + BK) + (A + BK)^T P < -2\alpha P. \tag{5.2}$$

Thus Assumption 1 holds with $V(x) = x^T Px$ and $u(x) = Kx$ for all $x \in \mathbb{R}^n$. Rewriting the closed-loop system in a hybrid framework, we get:

$$\begin{cases} \begin{cases} \dot{x} = Ax + Bs, \\ \dot{s} = 0, \end{cases} & (x, s) \in \mathscr{F}, \\[2ex] \begin{cases} x^+ = x, \\ s^+ = Kx, \end{cases} & (x, s) \in \mathscr{J}. \end{cases} \tag{5.3}$$

By noting that

$$\nabla V(x) f(x, s) = \begin{bmatrix} x \\ s \end{bmatrix} \Pi_1 \begin{bmatrix} x \\ s \end{bmatrix}$$

where $\Pi_1 = \begin{bmatrix} PA + A^T P + 2\bar{\alpha}P & PB \\ B^T P & 0 \end{bmatrix}$, $\bar{\alpha} \in (0, \alpha)$, the following result follows readily from Theorem 3.1:

PROPOSITION 5.1  Assume there exist a symmetric positive definite matrix $P$ in $\mathbb{R}^{n \times n}$, a matrix $K$ in $\mathbb{R}^{n \times m}$ and a positive scalar $\alpha$ satisfying (5.2). Consider $\bar{\alpha} \in (0, \alpha)$ and the flow and jump sets defined by

$$\mathscr{F}_{1L} = \left\{ (x, s), \begin{bmatrix} x \\ s \end{bmatrix}^T \Pi_1 \begin{bmatrix} x \\ s \end{bmatrix} \leqslant 0 \right\},$$

$$\mathscr{J}_{1L} = \left\{ (x, s), \begin{bmatrix} x \\ s \end{bmatrix}^T \Pi_1 \begin{bmatrix} x \\ s \end{bmatrix} \geqslant 0 \right\}.$$

Then the system (5.3) with the event-triggered algorithm derived from $\mathscr{F} = \mathscr{F}_{1L}$ and $\mathscr{J} = \mathscr{J}_{1L}$ is globally asymptotically stable to $\{0\} \times \mathbb{R}^m$. Moreover, for each solution to this hybrid system, at all time when the solution has a jump, either the state is the origin or the solution has to flow.

REMARK 5.1 In (Fiter, Hetel, Perruquetti and Richard 2011), an LMI based mapping approach is proposed in addition to Proposition 5.1. The authors introduce the notion of maps. The idea is to divide the state space into appropriate sectors. Then off-line calculations allows to attribute a sampling period to each sector. By this mean, it is possible to create a map composed of sectors such that, when the system enters in the jump set, the controller only has to find in which sector the state of the system belongs to choose the appropriate sampling period. The interest of this method is that this mapping approach avoid the computation of the test function embedded in the controller.                    ○

$$
\begin{cases}
\dot{x} = Ax + Bs, \\
\dot{s} = 0, \qquad (x, s, v, \tau) \in \mathscr{F}, \\
\dot{\tau} = 1,
\end{cases}
$$
$$
\begin{cases}
x^+ = x, \\
s^+ = Kx, \quad (x, s, v, \tau) \in \mathscr{J}. \\
\tau^+ = 0,
\end{cases}
\tag{5.4}
$$

### 5.1  *Comments on the linear case*

The event-triggered algorithms which are exposed above do not provide any information of the duration while a control law is held. In the sequel, a complementary analysis is provided for the case of linear systems to give an upper-bound and a lower bound of the holding times. Consider now the hybrid representation of system (5.1)

$$
\begin{cases}
\dot{x} = Ax + Bs, \\
\dot{s} = 0, \qquad (x, s, v, \tau) \in \mathscr{F}, \\
\dot{\tau} = 1,
\end{cases}
$$
$$
\begin{cases}
x^+ = x, \\
s^+ = Kx, \quad (x, s, v, \tau) \in \mathscr{J}. \\
\tau^+ = 0,
\end{cases}
\tag{5.5}
$$

This hybrid system is essentially the same as system (5.3), except that a timer $\tau$ has been added to the dynamics of the system. Let $\chi \in \mathbb{R}^n$ be the value of $x$-component in the system (5.5) with the event-triggered algorithm derived from $\mathscr{F} = \mathscr{F}_{1L}$ and $\mathscr{J} = \mathscr{J}_{1L}$, at an instant when the system is jumping, i.e. $\chi = x(t_j)$ for some $t_j \in \mathscr{T}$. In the case of linear sampled-data systems, the relations between $\chi$, $x$ and $s$ are given by

$$
x(t_k + \tau) = \Gamma(\tau)\chi, \; s(t_k + \tau) = K\chi,
$$

where $\Gamma(\tau) = e^{A\tau} + \int_0^\tau e^{A(\theta - \tau)} d\theta BK$. For the sake of simplicity, we will denote

$$
X(\chi, \tau) = (\Gamma(\tau)\chi, \, K\chi),
$$

for any given $\chi \in \mathbb{R}^n$ and $\tau \in \mathbb{R}^+$. Based on these linking relations, bounds on the inter sampling times can be provided. This is stated in the sequel.

PROPOSITION 5.2 Consider the linear system (5.5) with the event-triggered algorithm derived from $\mathscr{F} = \mathscr{F}_{1L}$ and $\mathscr{J} = \mathscr{J}_{1L}$. Then, the difference between two successive sampling instants is included in

the interval $[T_m, T_M]$ defined as follows

$$T_{max} = \max_{\rho \in \mathbb{R}^n, |\rho|=1} \left\{ \max_{X(\rho,\tau) \in \mathscr{F}_{1L}} \tau \right\},$$

$$T_{min} = \min_{\rho \in \mathbb{R}^n, |\rho|=1} \left\{ \max_{X(\rho,\tau) \in \mathscr{F}_{1L}} \tau \right\}.$$

*Proof.*   Consider any state $\chi$ in $\mathbb{R}^n$ for a solution to system (5.3) with the event-triggered algorithm derived from $\mathscr{F} = \mathscr{F}_{1L}$ and $\mathscr{J} = \mathscr{J}_{1L}$ when a jump is occurring. By simple computations, it is clear that if, for any $\tau > 0$, $X(\chi, \tau)$ belongs to $\mathscr{F}_{1L}$ then $X(\chi/|\chi|, \tau)$ also belongs to $\mathscr{F}_{1L}$. Then, from the definition of $T_{min}$ and $T_{max}$, the next update will happen between these two bounds.          □

### 5.2   *Linear example*

Consider the linear system (5.1) and the control $u = Kx$ studied in (Naghshtabrizi, Hespanha and Teel 2008, Zhang, Branicky and Phillips 2001) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ -0.1 \end{bmatrix}, K = \begin{bmatrix} 3.75 \\ 11.5 \end{bmatrix}^T. \tag{5.6}$$

Several robust stability conditions dedicated to the previous example of sampled-data systems can be found in (Fridman 2010, Fujioka 2009, Oishi and Fujioka 2009, Seuret 2009). In these articles, the main idea is to provided the largest upper-bound $T$ so that the closed-loop system is stable for any asynchronous samplings whose period is lying in $[0, T]$. It was shown in (Seuret 2009) that the system remains stable with the upper-bound $T = 1.729$.

To provide an efficient event-triggered algorithm, the Lyapunov matrix is taken from (Seuret 2009) with $T = 0.2$ and

$$P = \begin{bmatrix} 21.213 & 10.843 \\ 10.843 & 20.666 \end{bmatrix}, \quad \alpha = 0.17.$$

Figure 4 shows the simulations results of the closed system using the continuous-time controller, and the event-driven control algorithm provided in Proposition 5.1 with $\bar{\alpha} = 10^{-3}$ and $\lambda = 0.15$. The control algorithm requires 12 sampling instants over a simulation time of 20sec.

Using Proposition 5.2, the algorithm leads to the following bounds on difference between two successive sampling instants $T_{min} = 0.978$ and $T_{max} = 6.96$. The event-triggered algorithm allows solutions for which the length between two successive sampling instants is greater than the upper-bound obtained using robust approaches from (Fridman 2010, Fujioka 2009, Oishi and Fujioka 2009, Seuret 2009, Seuret 2012). This shows the main interest of the proposed method.

## 6. Conclusions

In this paper, using a Lyapunov-like function, three event-triggered algorithms are designed. It is assumed that a stabilizing controller for the continuous control system is given, and these algorithms suggest an implementation method, alternative to the classical periodic implementation method. The event-triggered algorithms require to study a closed-loop system with a mixed discrete/continuous dynamics (namely this is a hybrid system). Some numerical simulations illustrate the stability properties for nonlinear and linear control systems.

In a forecoming work, the performance issue should be analyzed. It is remarked that the event-triggered algorithms have a different performance. The first one seems to ensure a good speed of convergence on numerical simulations, whereas the second event-triggered algorithm allows less jumps
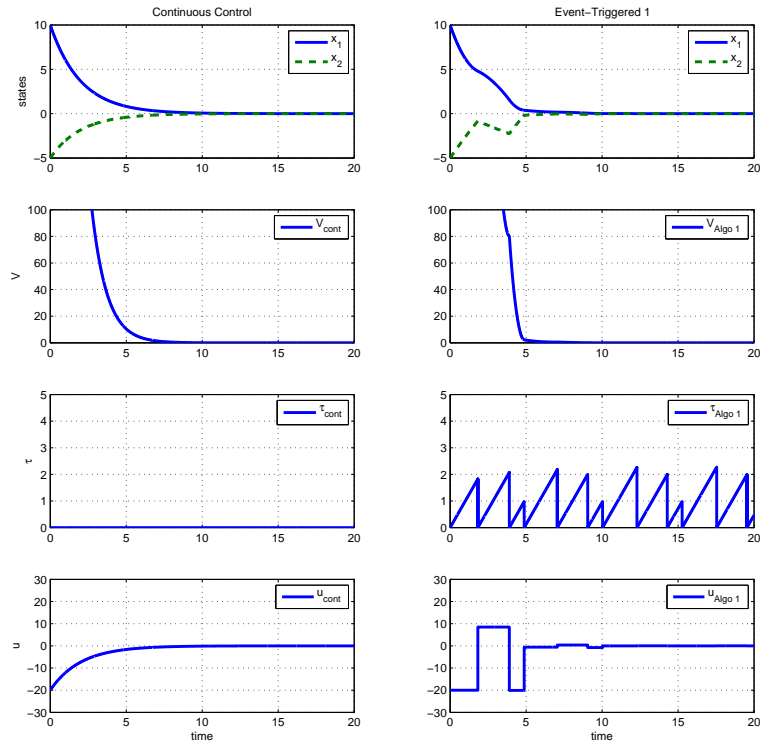
FIG. 4. From the top to down: time-evolution of the states, of the *V*-function, of the *τ*-variables and of the control values (when closing the loop with the continuous control, or with the algorithm of Proposition 5.1, or with the algorithm of Proposition 5.2, from left to right)

and thus needs to compute less often the control variables. The advantages and disadvantages of each algorithm will be studied more precisely in a forecoming work, for a theoretical point of view (e.g. by estimating a priori the number of switches), or on applications (to understand which algorithm is better depending on the application).

### Acknowledgement

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